FILTERING FOR LINEAR DISTRIBUTED PARAMETER SYSTEMS

by

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1. Introduction

The systems to be considered are described by parabolic equations with 'white noise' inputs. We are interested in conditions which guarantee that the solution $U(x,t)$, a random surface, has certain smoothness properties, and also in the smoothness properties of the conditional expectation $E[U(x,t)|$ given data up to $t]$. Such results are developed in [1], [2] using the Sobolev imbedding theorem.

First, some of these results will be stated. A system model (first boundary value problem) is discussed in Section 2, Lemma 3. The noisy observations for this problem have the form (7). Lemma 4 proves the smoothness of the conditional mean and covariance, and Theorem 1 gives the form of the optimal filter. Section 3 considers a second boundary value problem (16) with surface observations of the form (B6). Lemma 5 proves the smoothness of the solution to (16), and Theorem 2 gives the form of the optimal filter.

Smoothness Results on Random Surfaces. Let z_t be a normalized Wiener process, D a bounded open domain in E^n with closure $\bar{D}$ and a continuous and piecewise uniformly differentiable boundary and write $\bar{R} = \bar{D} \times [0,T]$. Let $D_t = \partial/\partial t$, $D_i = \partial/\partial x_i$, $D_i^\ell = \partial^\ell/\partial x_j^\ell$. Let $f(x,t)$ be a stochastic process on $\bar{D} \times [0,T] = \bar{R}$. The parenthesis in $(D_i f(x,t))$, denotes the 'mean square' derivative of $f(x,t)$ with respect to x_i, if it exists. Define the norm

$$\|g(x)\|_{W_{\ell,p}(\bar{D})} = \sum_{k=0}^{\ell} \sum_{\ell_1+\cdots+\ell_n=k} \|D_1^{\ell_1}\cdots D_n^{\ell_n} g(x)\|_{L_p(\bar{D})}. \qquad (1)$$

where $\psi \in L_p(\bar{D})$ means that $\int_{\bar{D}} |\psi(x)|^p dx \equiv \|\psi\|^p_{L_p(\bar{D})} < \infty$. References [1] and [2], from which Lemmas 1 and 2 are taken give conditions on the expectations of integrals of powers of the 'mean square' derivatives, which guarantee that $f(x,t)$ has a w.p.l. continuous version on $\bar{R}$, and perhaps several continuous derivatives with respect to components of x. The proof of Lemma 1 is contained in [2].

Lemma 1. *Let the boundary ∂D of D have the property that any line intersects it only finitely often. Let the functions*

$$\begin{gathered} \alpha(x,t,s),\ \{D_i\alpha(x,t,s)\},\ \{D_iD_j\alpha(x,t,s)\}, \\ \{D_iD_jD_k\alpha(x,t,s)\},\ \{D_iD_jD_kD_\ell\alpha(x,t,s)\} \end{gathered} \qquad (*)$$

be defined on $\bar{D} \times [0,T] \times [0,T] = \bar{R} \times [0,T]$, continuous in (x,t) for each s, and bounded (in absolute value) by a square integrable function of s. Let f be any function in the set $(*)$, and let $z(t)$ be a Wiener process. Then $\int_0^T f^2(x,t,s)ds \le M < \infty$ for some real number M, and $\int_0^t f(x,t,s)dz_s$ can be defined to be a separable and measurable process with parameter (x,t). There is a null set N and a separable and measurable version of $\int_0^t \alpha(x,t,s)dz_s = \psi(x,t)$ which, for $\omega \notin N$, is continuous in (x,t) and has three continuous (in (x,t)) derivatives with respect to the components of x. These derivatives are equal to continuous (for $\omega \notin N$), separable and measurable versions of $\int_0^t D_i\alpha(x,t,s)dz_s$, $\int_0^t D_iD_j\alpha(x,t,s)dz_s$, $\int_0^t D_iD_jD_k\alpha(x,t,s)dz_s$, respectively.

Let in addition, for some real numbers $K < \infty$, $\beta > 0$,

$$E\left\{\int_0^{t+\Delta} f(x,t+\Delta,s)dz_s - \int_0^t f(x,t,s)dz_s\right\}^2$$

$$= \int_0^t [f(x,t+\Delta,s) - f(x,t,s)]^2 ds + \int_t^{t+\Delta} f^2(x,t+\Delta,s)ds \leq K\Delta^\beta, \qquad (**)$$

where f is any member of $(*)$. Let g be any member of the first three sets of $(*)$. Then the continuous version (for $\omega \notin N$) of $\int_0^t g(x,t,s)dz_s = \phi(x,t)$ is Holder continuous on $\bar{R}$, i.e., there is some $K(\omega) < \infty$ w.p.1. and a real $\gamma > 0$ so that

$$|\phi(x+\delta,t+\Delta) - \phi(x,t)| \leq K(\omega)[|\Delta|^\gamma + |\delta|^\gamma],$$

where $|\cdot|$ refers to the Euclidean norm.

Lemma 2. Let $f(x,t)$ be a process on $\bar{R}$, which is continuous in probability together with its 'mean square' derivatives up to order ℓ on $\bar{R}$. Let $p\ell > n$, $p > 1$, and suppose that† for $0 \leq s \leq t \leq T$,

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† Recall that (2) is equivalent to

$$E[\int_D |(D_1^{\ell_1} \cdots D_n^{\ell_n}\{f(x,t) - f(x,s)\})|^p dx]^{q/p} \leq K|t-s|^{1+\alpha}$$

for all $\ell_1+\cdots+\ell_n \leq k \leq \ell$, for $0 \leq s \leq t \leq T$.

$$E\|f(\cdot,t) - f(\cdot,s)\|^q_{W_{\ell,p}(\overline{D})} \leq K|t-s|^{1+\alpha} \qquad (2)$$

for some real $K < \infty$ and $1 \leq q < \infty$ and $\alpha > 0$. Then there is a w.p.l. continuous version of $f(\cdot,\cdot)$ on $\overline{R} \times [0,T]$, and the version is Holder continuous in t, uniformly in x, w.p.l.

If $0 < m < \ell - n/p$, then the 'mean square' derivatives of order $\leq m$ have continuous versions on $\overline{R}$ w.p.l., and $f(x,t)$ has w.p.l. a continuous version whose first m x-derivatives coincide with the 'mean square' derivatives.

For proof, see Theorem 4 in [1].

2. Filtering for a Stochastic First Boundary Value Problem

System Model. The first system with which we will deal has the representation[†]

$$dU(x,t) = [\mathcal{L}U(x,t) + \int k(y,x,t)U(y,t)dy]dt + \sigma(x,t)dz, \qquad (4)$$

where

$$\mathcal{L} = \sum a_{ij}(x,t)D_iD_j + \sum b_i(x,t)D_i \qquad (5)$$

and (A1) - (A7) hold.

(A1) ∂D (the boundary of D) has a local representation with holder continuous 4^{th} derivatives.

(A2) The coefficients of $\mathcal{L}$, and their first two derivatives are Holder continuous in $\bar{R}$.

(A3) $\sum a_{ij}\xi_i\xi_j \geq K \sum \xi_i^2$ for some real $\infty > K > 0$.

(A4) σ and its first four x-derivatives are Holder continuous on $\bar{R}$.

(A5) σ and $\mathcal{L}\sigma$ go to zero as $x \to \partial D$.

(A6) $k(y,x,t)$ is bounded, measurable and Holder continuous in x,t, uniformly in y, and $k(y,x,t) \to 0$ as $x \to \partial D$.

(A7) $U(x,0)$ is Gaussian for each x, has a bounded variance, Holder

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[†]For notational simplicity, we let the 'driving term' be $\sigma(x,t)dz$. It could be $\sum \sigma_i(x,t)dz_i$, where the z_i are independent. See Lemma 2.2, [2].

continuous second derivatives, and $U(x,0)$ and $\mathcal{L}U(x,0) \to 0$ as $x \to \partial D$. $U(x,0)$ is independent of z_t and of w_t (to be introduced below).

In [2], Lemmas 1 and 2 are applied to (4) to give it a precise definition and

Lemma 3.† (See [2], Lemma 3.2 for proof.) Assume (A1) - (A7). Then there is a random function $U(x,t)$ on $(0,T] \times \bar{D}$ so that a version (for $\omega \notin N$, a null set) of the uniformly (in $(0,T] \times D$) 'mean square' continuous functions

$$U(x,t),\ (D_i U(x,t)),\ldots,\ (D_i D_j D_k U(x,t)) \tag{6}$$

are continuous on $(0,T] \times \bar{D}$ w.p.1.; these versions of the 'mean square' derivatives are true derivatives. $U(x,t)$ and $\mathcal{L}U(x,t) \to 0$ as $x \to \partial D$ (for $\omega \notin N$), $U(x,t) \to U(x,0)$ (for $\omega \notin N$, and uniformly in x) as $t \to 0$. The first three sets of (6) are Holder continuous in t, for $\omega \notin N$. $U(x,t)$ is a Markov process (with values in a state space of functions with Holder continuous second derivatives). The members of (6) are Gaussian, and have uniformly bounded variances. The variances of $U(x,t)$ and of $\mathcal{L}U(x,t)$ tend to zero as $x \to \partial D$. $U(x,t)$ is non-anticipative with respect to the z_t process and the Ito differential of $U(x,t)$ satisfies (4). $U(x,t)$ satisfies the condition (2) of Lemma 2, for $m = 3$, $\ell = 4$, and all large p, and some finite q and $\alpha > 0$. $(D_i D_j D_k D_\ell U(x,t))$ is also uniformly 'mean square' continuous in $(0,T] \times R$.

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† The smoothness in (A1), (A2), (A4) gives a $U(x,t)$ with continuous third x-derivatives, hence Holder continuous second derivatives. In the control problem in [2], we wanted $U(x,t)$ to have Holder continuous second derivatives. If only continuous second derivatives are required, then the differentiability requirements in (A1), (A2), (A4) can be reduced by 1.

The Filtering Problem. Let $\tilde{w}_t$ be a normalized Wiener process independent of the z_t process, and suppose that

(A8) $H(x,t)$ is a vector-valued function which is defined and continuous on $\bar{R}$.

(A9) $B(t)$ is continuous on $[0,T]$ and $B(t)B'(t) = \Sigma_t$ is strictly positive definite on $[0,T]$.

Define $w_s = \int_0^s B(\tau)d\tilde{w}_\tau = \int_0^s \Sigma_\tau^{1/2} d\tilde{w}_\tau$. Suppose that the data

$$y(s) = \int_0^s [\int H(x,\tau)U(x,\tau)dx]d\tau + \int_0^s B(\tau)d\tilde{w}_\tau \equiv \int_0^s h_\tau d\tau + w_s, \quad s \leq t \quad (7)$$

is available at time t. All introduced σ-algebras are assumed to be complete with respect to whatever measures are imposed on them; let[†] $\mathcal{F}_t$ be the minimal σ-algebra determined by $y(s)$, $s \leq t$. Let μ_1 be the measure determined by the processes $U(x,s)$, $s \leq t$ and $dy(s) = h_s ds + dw_s$, $s \leq t$, and μ_o the measure determined by the processes $U(x,t)$ and $dy(s) = dw_s$, $s \leq t$. Let E_i^t denote the expectation with respect to μ_i, and conditioned on $\mathcal{F}_t$.

Define

$$M(x,t) = E_1^t U(x,t)$$

$$P(x,y,t) = E_1^t (U(x,t) - M(x,t))(U(y,t) - M(y,t))$$

$$= E_1 (U(x,t) - M(x,t))(U(y,t) - M(y,t)).$$

[†] To be more precise, let Ω be a function space with generic element $\omega = (\omega',\omega'')$, where ω' is a member of the space of bounded functions on $\bar{R}$, and ω'' is a member of the space of bounded functions on $[0,T]$, with values in the Euclidean m-dimensional space E^m, where m is the dimension of w_t and y_t. The terminology is used later. See part 1° of the proof of Theorem 1.

Let $\mathcal{L}_x P(x,y,t)$ denote $\mathcal{L}$ operating on $P(x,y,t)$ as a function of x. Lemma 4 proves that there is a version of the estimate $M(x,t)$ which is, w.p.1., as smooth as the signal $U(x,t)$.

Lemma 4. Assume (A1) - (A9). Then (excluding a null set independent of (x,t)) there are on $(0,T] \times \bar{D}$ continuous versions of the first four sets of the continuous in quadratic mean functions

$$M(x,t),(D_iM(x,t)),(D_iD_jM(x,t)),(D_iD_jD_kM(x,t))(D_iD_jD_kD_\ell M(x,t)); \qquad (8)$$

also $M(x,t) \to M(x,0)$ as $t \to 0$ (and also in quadratic mean) and the first three sets of 'mean square' derivatives are true derivatives and $E_1^t \mathcal{L}U(x,t) = \mathcal{L}M(x,t)$; also $M(x,t)$ and $\mathcal{L}M(x,t) \to 0$ as $x \to \partial D$, and the first three sets of (7) have Holder continuous versions. $P(x,y,t)$ has continuous third derivatives in the components of x and y on $(0,T] \times D$, and $P(x,y,t)$ and $\mathcal{L}_x P(x,y,t)$ and $\mathcal{L}_y P(x,y,t) \to 0$ as $x \to \partial D$ or $y \to \partial D$. $P(x,y,t) \to P(x,y,0)$ as $t \to 0$.

Proof. $M(x,t)$ and the $(D_iM(x,t)),\ldots,(D_iD_jD_kD_\ell M(x,t))$ exist and are 'mean square' continuous in x-uniformly in (x,t) in $(0,T] \times D$. Also $E_1^t(D_iU(x,t)) = (D_iM(x,t))$ w.p.1. (as well as for the next three derivatives) for each x,t in $(0,T] \times D$. These assertions easily follow from estimates of the following type: let e_i be the i^{th} coordinate direction in E^n, where n is the dimension of x. Then

$$E_1\left|\frac{M(x+e_i\Delta,t) - M(x,t)}{\Delta} - E_1^t(D_iU(x,t))\right|^2$$

$$= E_1 \left| E_1^t \left\{ \frac{U(x+e_i\Delta,t) - U(x,t)}{\Delta} - (D_i U(x,t)) \right\} \right|^2$$

$$\leq E_1 \left| \frac{U(x+e_i\Delta,t) - U(x,t)}{\Delta} - (D_i U(x,t)) \right|^2 \to 0 \qquad (*)$$

as $\Delta \to 0$, uniformly for x,t in $D \times (0,T]$.

Furthermore, $M(x,t)$ also satisfies the estimates (**) and (***)

$$E_1|M(x,t)|^u = E_1|E_1^t U(x,t)|^u \leq E_1|U(x,t)|^u$$

$$E_1|(D_i D_j D_k D_\ell M(x,t))|^u \leq E_1|(D_i D_j D_k D_\ell U(x,t))|^u \qquad (**)$$

and

$$E_1|M(x,t) - M(x,s)|^u = E_1|E_1^t U(x,t) - E_1^s U(x,s)|^u \qquad (***)$$

$$\leq KE_1|E_1^t(U(x,t) - U(x,s))|^u + KE_1|E_1^t U(x,t) - E_1^s U(x,t)|^u$$

$$\leq KE_1|U(x,t) - U(x,s)|^u + \epsilon_u(x,t,s).$$

For $u = 2$, $\epsilon_u(x,t,s) \leq K_1|t-s|$ (hence for $u = 2r$, $\epsilon_u(x,t,s) \leq K_r|t-s|^r$). The inequality on $\epsilon_u(x,t,s)$ follows from the Gaussianness, and the uniform positive definiteness of $\sum_s$, which gives an upper limit on the rate at which information can be collected.

Estimates (*), (**), (***) imply that the $M(x,t),\ldots,(D_i D_j D_k D_\ell M(x,t))$ are uniformly 'mean square' continuous in $(0,T] \times D$.

The statements concerning the continuity of $M(x,t)$ and its derivatives then follow from Lemmas 2 and 3 since by the estimates (**), (***) (and obvious similar estimates for $(D_i M(x,t)),\ldots,(D_i D_j D_k D_\ell M(x,t))$, if $U(x,t)$ satisfies Lemma 2 for $m = 3$, so does $M(x,t)$. Since (***) is valid for $s = 0$, $M(x,t) \to M(x,0)$.

$M(x,t)$ and $\mathcal{L}M(x,t) \to 0$ as $x \to \partial D$ since both $U(x,t)$ and $\mathcal{L}U(x,t)$ and their variances $\to 0$ as $x \to \partial D$.

The asserted smoothness of $P(x,y,t)$ and its boundary properties follow from the continuity in quadratic mean of the elements of (8), (8'), $\mathcal{L}M(x,t)$ and $\mathcal{L}U(x,t)$.

$$U(x,t),\ldots,(D_i D_j D_k D_\ell U(x,t)) \tag{8'}$$

(see, for example, Loeve [6], Sec. 34.2 for the type of calculations which are required.) Q.E.D.

<u>Theorem 1.</u> <u>Assume</u> (A1) - (A9). <u>Then there is a version of</u> $M(x,t)$ <u>which has the Itô differential</u> w.p.1.

$$\begin{aligned} dM(x,t) = {} & [\mathcal{L}M(x,t) + \int k(\xi,x,t)M(\xi,t)d\xi]dt + \\ & [dy - \int H(\xi,t)M(\xi,t)d\xi]'\Sigma_t^{-1}[\int H(\xi,t)P(\xi,x,t)d\xi] \end{aligned} \tag{9}$$

<u>and, for this version,</u> w.p.1., $M(x,t)$ <u>and</u> $\mathcal{L}M(x,t) \to 0$ <u>as</u> $x \to \partial D$. <u>Furthermore,</u> $P(x,y,t)$ <u>satisfies</u>

$$\begin{aligned} P_t(x,y,t) = {} & [\mathcal{L}_x + \mathcal{L}_y]P(x,y,t) \\ & + \int k(y,\xi,t)P(x,\xi,t)d\xi + \int k(\xi,x,t)P(\xi,y,t)d\xi \\ & + \sigma(x,t)\sigma(y,t) - [\int H(\xi,t)P(x,\xi,t)d\xi]'\Sigma_t^{-1}[\int H(\xi,t)P(\xi,y,t)d\xi]. \end{aligned} \tag{10}$$

$P(x,y,t)$, $\mathcal{L}_x P(x,y,t)$ and $\mathcal{L}_y P(x,y,t) \to 0$ as $x \to \partial D$.

Proof. For the sake of keeping a framework which will allow a generalization (not proved here) to non-linear systems, we take a slightly more general approach than necessary. The non-linear problem for ordinary stochastic Ito equations was treated in [3], however here, we follow a slightly different approach, due to Zakai [4], which gives the result under weaker conditions than those required in [3].

1°. μ_o and μ_1 are absolutely continuous with respect to one another and $d\mu_1/d\mu_o = \exp R_t$

$$R_t = -\frac{1}{2}\int_o^t h_s' \Sigma_s^{-1} h_s \, ds + \int_o^t h_s' \Sigma_s^{-1} dy_s.$$

Next, following Zakai [4], note that if $E_1|f(\omega,t)| < \infty$, then (see Loeve, [5], Sec. 24.4)

$$E_1^t f(\omega,t) = \frac{E_o^t f(\omega,t)(d\mu_1/d\mu_o)}{E_o^t(d\mu_1/d\mu_o)} = \frac{E_o^t f(\omega,t)\exp R_t}{E_o^t \exp R_t} \quad . \tag{11}$$

2°. Write

$$F_t = [\mathcal{L}U(x,t) + \int k(y,x,t)U(y,t)dy].$$

Then, w.p.1., by virtue of Lemma 4,

$$E_1^t F_t = \mathcal{L}M(x,t) + \int k(y,x,t)M(y,t)dy.$$

In (11), let $f(\omega,t) = U(x,t)$. Both $U(x,t)$ and $\exp R_t$ are stochastic integrals and

$$d[\exp R_s] = (\exp R_s) h_s' \Sigma_s^{-1} dy_s .$$

Then Ito's lemma, applied to (11) yields

$$E_1^t U(x,t) = M(x,t) = \tag{13}$$

$$\frac{E_o^t[U(x,0) + \int_o^t \exp R_s (F_s ds + \sigma(x,s)dz_s)][\int_o^t U(x,s)(\exp R_s) h_s' \Sigma_s^{-1})dy_s]}{E_o^t[1 + \int_o^t (\exp R_s) h_s' \Sigma_s^{-1} dy_s]}$$

$$\equiv \frac{A_t}{B_t},$$

$$A_t = E_o^t \int_o^t dU(x,s) \exp R_s ds + E_o^t U(x,0)$$

$$= E_o^t \int_o^t [U(x,s)(\exp R_s) h_s' \Sigma_s^{-1} dy_s + (\exp R_s)(F_s ds + \sigma(x,s)dz_s)] + E_o^t U(x,0)$$

and where $dydz = 0$ is used to eliminate the $(dU(x,s))(d \exp R_s)$ term from A_t.

As in Kushner [3] or Zakai [4], it can be shown below that,[†] w.p.1.

[†]The demonstration of (14), by the method of [3] requires more stringent conditions on R_s and U, then by the method of [4].

The method of [4] is applicable under the conditions of the hypothesis of Theorem 1.

The method of [3] may also be applied, by applying it to a suitable sequence of bounded $F_s^\epsilon, h_s^\epsilon$ which converges to F_s and h_s in probability.

$$E_o^t \int_o^t (\exp R_s)[F_s ds + \sigma(x,s)dz_s] = \int_o^t [E_o^s (\exp R_s)F_s]ds \tag{14}$$

$$E_o^t \int_o^t U(x,s)[(\exp R_s)h_s' \Sigma_s^{-1}]dy_s = \int_o^t [E_o^s U(x,s)(\exp R_s)h_s' \Sigma_s^{-1}]dy_s$$

$$E_o^t \int_o^t (\exp R_s)h_s' \Sigma_s^{-1} dy_s = \int_o^t [E_o^s (\exp R_s)h_s' \Sigma_s^{-1}]dy_s$$

where the second integrals are well-defined w.p.l. Assuming (14) now, we proceed exactly as in [3] and get

$$dM(x,t) = \frac{dA_t}{B_t} - \frac{A_t dB_t}{B_t^2} + \frac{A_t(dB_t)^2}{B_t^3} - \frac{(dA_t)(dB_t)}{B_t^2} \tag{15}$$

where

$$(dB_t)^2 = E_o^t[(\exp R_t)h_t']\Sigma_t^{-1}[E_o^t(\exp R_t)h_t],$$

$$(dA_t)(dB_t) = [E_o^t U(x,t)(\exp R_t)h_t']\Sigma_t^{-1}[E_o^t (\exp R_t)h_t].$$

(9) is obtained by substituting in (12) and using the fact ((11)) that $E_o^t f = [E_o^t f \exp R_t]/E_o^t \exp R_t$.

3^o. Similarly, dP is calculated from the expression

$$dP(x,y,t) = dE_1^t U(x,t)U(y,t) - dM(x,t)M(y,t).$$

To get $dE_1^t U(x,t)U(y,t)$, repeat the procedure starting with (11), where we now let $f(\omega,t) = U(x,t)U(y,t)$, and use the w.p.l. equalities

$$
\begin{aligned}
E_1^t U(x,t)\mathcal{L}_y U(y,t) &= M(x,t)\mathcal{L}_y M(y,t) \\
&\quad + E_1^t(U(x,t) - M(x,t))\mathcal{L}_y(U(y,t) - M(y,t)) \\
&= M(x,t)\mathcal{L}_y M(y,t) + \mathcal{L}_y P(x,y,t).
\end{aligned}
$$

The details are straightforward and are omitted. Q.E.D.

3. The Second Boundary Value Problem

Now, we consider the equation

$$dU(x,t) = [\mathcal{L}U(x,t) - f(x,t)]dt - \sigma(x,t)dz, \tag{16a}$$

$$\begin{aligned} U_V(x,t) + \beta(x,t)U(x,t) &= g(x,t) + v(x,t)r(t) \\ \mathcal{L} &= \sum a_{ij}(x,t)D_iD_j + \sum b_i(x,t)D_i \end{aligned} \tag{16b}$$

where $U_V(x,t)$ is the co-normal derivative† $\partial U/\partial V = \lim_{\substack{y \to x \\ y \in D}} \frac{\partial U(y,t)}{\partial V(x)}$ at x on ∂D, and (B1) - (B8) are assumed.

(B1) $\sum a_{ij}(x,t)\xi_i\xi_j \geq K \sum \xi_i^2$ for some real $K > 0$.

(B2) $a_{ij}(x,t)$ and $b_i(x,t)$ are Holder continuous in R.

(B3) $f(x,t)$ is continuous, and Holder continuous in x, uniformly in t.

(B4) ∂D has a local representation with Holder continuous derivatives.

(B5) Real-valued $g(x,t)$ and row-vector valued $v(x,t)$ are continuous on $\bar{R}$ and r is the Gaussian random process satisfying $dr = A(t)rdt + G(t)d\tilde{z}$, where $\tilde{z}_t$ is independent the z_t and w_t processes introduced earlier, and of $U(x,0)$. $A(t)$ and $G(t)$ are bounded continuous functions.

(B6) The observations $dy = [\int_{\partial D} H(\xi,t)U(\xi,t)dS_\xi]dt + dw$ are taken, where $H(\xi,t)$ is continuous on $\partial D \times [0,T]$, and w_t is independent of $U(x,0)$, and dS_ξ is the differential surface measure on ∂D. Also Σ_t satisfies (A9), where $dw = \Sigma_t^{1/2}d\tilde{w}$, and $\tilde{w}_t$ is a normalized Wiener process.

†$V(x)$ is the co-normal direction at the point x on ∂D

(B7) Denote $\alpha(x,t,s) = \int_D \Gamma(x,\xi; t,s)\sigma(\xi,s)d\xi$, where Γ is the fundamental solution of $D_t U = \mathcal{L}U$. Let $\sigma(\xi,s)$ be uniformly continuous. Let $\gamma(x,t,s)$ represent either $\alpha(x,t,s)$, $D_i\alpha(x,t,s)$ or $D_iD_j\alpha(x,t,s)$. Let, uniformly in $\bar{R}$,

$$\int_t^{t'} \gamma^2(x,t',\tau)d\tau + \int_0^t [\gamma(x,t',\tau) - \gamma(x,t,\tau)]^2 d\tau \leq K|t'-t|^{\beta} \qquad (17)$$

for some real K and $\beta > 0$. Let $D_iD_jD_k\gamma(x,t,s)$ satisfy (17) uniformly for x,t,t' in any compact subset of $D \times [0,T]^2$.

(B8) Let $U(x,0)$ be differentiable w.p.1., and let $a_{ij}(x,0)$ be continuously differentiable in some neighborhood of ∂D.

Lemma 5. Assume (B1) - (B8). Then there is a random function $U(x,t)$ which has a version with the following properties, w.p.1. (where the null set doesn't depend on x,t).

(a) $U(x,t)$ is continuous† on $\bar{R}$ (also in quadratic mean); $(D_iU(x,t))$ is continuous on compact subsets of $\bar{D} \times (0,T]$ (also in quadratic mean).

(b) The $(D_iD_jU(x,t))$ are continuous on compact subsets of $D \times (0,T]$.

(c) $U(x,t)$ has an Itô differential which satisfies (16a), for $t > 0$.

(d) $U(x,t)$ satisfies the boundary condition (16b), and $U(x,t) \to U(x,0)$ as $t \to 0$.

(e) The variances of $U(x,t)$, $(D_iU(x,t))$ (in compact subsets of $\bar{D} \times (0,T]$) and $(D_iD_jU(x,t))$ (in compact subsets of $D \times (0,T]$) are

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† $D_iU(x,t)$ on ∂D is defined as $\lim_{\substack{y \to x \\ y \in D}} D_iU(x,t)$.

uniformly bounded.

(f) $U(x,t)$ is non-anticipative with respect to the z_t and $\tilde{z}_t$ processes.

Proof. The treatment in Friedman [7], (Theorem 2, p. 144 and Corollary 2, p. 147) will be followed, with the few modifications required by the stochastic nature of the problem taken into account. Define

$$\gamma(x,t) = \int_0^t dz_s \alpha(x,t,s), \quad \gamma_i(x,t) = \int_0^t dz_s D_i \alpha(x,t,s),$$
$$\gamma_{ij}(x,t) = \int_0^t dz_s D_i D_j \alpha(x,t,s), \quad \gamma_{ijk}(x,t) = \int_0^t dz_s D_i D_j D_k \alpha(x,t,s).$$

Let $k(x,t) = \int_0^t dz_s \rho(x,t,s)$. Then

$$\begin{aligned} Ek^2(x,t) &= \int_0^t dt \rho^2(x,t,s) \\ Ek^{2n}(x,t) &= K_n [Ek^2(x,t)]^n, \text{ for some real } K_n \\ E[k(x,t') - k(x,t)]^2 &= \int_t^{t'} ds \rho^2(x,t',s) + \int_0^t ds [\rho(x,t',s) - \rho(x,t,s)]^2. \end{aligned} \tag{18}$$

Note also that $\gamma_i(x,t)$ is the 'mean square' derivative of $\gamma_o(x,t)$ with respect to the i^{th} coordinate of x in $D \times (0,T]$, and $\gamma_{ijk}(x,t)$ is the 'mean square' derivative of $\gamma_{jk}(x,t)$ with respect to the i^{th} coordinate of x in $D \times (0,T]$.

Then, by the estimates (18), (B7) and Lemma 2, there is a version of $\gamma_o(x,t)$ which (w.p.1.) is continuous on $\bar{R}$; it has continuous derivatives $D_i \gamma_o(x,t) = (D_i \gamma_o(x,t)) = \gamma_i(x,t)$ on $\bar{R}$ and continuous second derivatives $D_iD_j\gamma_o(x,t) = \gamma_{ij}(x,t) = (D_iD_j\gamma_o(x,t))$ in compact subsets of $D \times [0,T]$. Furthermore, for $(x,t) \in \partial D \times (0,T]$, $\partial/\partial V(x) = \sum \varphi_i(x)D_i$, where φ_i are Holder continuous. Hence, the function

$$\frac{\partial}{\partial V(x)} \int_0^t dz_s \alpha(x,t,s) \equiv \gamma_V(x,t)$$

also has a continuous w.p.l. version on $\partial D \times (0,T]$, and in fact, can be identified with $\int_0^t dz_s \frac{\partial\alpha(x,t,s)}{\partial V(x)}$. Next $D_t\alpha(x,t,s) = \mathcal{L}\alpha(x,t,s)$, $s < t$, $x \notin \partial D$, and, by (B7), $\int_0^t (D_t\alpha(x,t,s))^2 ds \le K < \infty$ on $\bar{R}$. Also $\alpha(x,t,s)$ is continuous on $\bar{R}$ and tends to $\sigma(x,t)$ as $s \uparrow t$. Hence $d \int_0^t \alpha(x,t,s)dz_s = \sigma(x,t)dz_t + [\int_0^t \alpha_t(x,t,s)dz_s]dt = \sigma(x,t)dz_t + \mathcal{L}\int_0^t \alpha(x,t,s)dz_s dt$.

From what has been said the function $F(x,t)$ defined by

$$\begin{aligned} F(x,t) = & \int_D \frac{\partial\Gamma(x,\xi;t,0)}{\partial V(x)} U(\xi,0)d\xi - \int_0^t d\tau \int \frac{\partial\Gamma(x,\xi;t,\tau)}{\partial V(x)} f(\xi,\tau)d\xi \\ & + \beta(x,t)\int_D \Gamma(x,\xi;t,0)U(\xi;0)d\xi \\ & - \beta(x,t)\int_0^t d\tau\int_D \Gamma(x,\xi;t,\tau)f(\xi,\tau)d\tau - \beta(x,t)\gamma_o(x,t) - g(x,t) \\ & - v(x,t)r(t) \end{aligned}$$

is continuous and uniformly bounded w.p.l. on $\partial D \times (0,T]$ (see Friedman [7], p. 145, where continuity is shown for a similar deterministic problem). Then, there is a continuous (and uniformly bounded (w.p.l.) solution on $\partial D \times [0,T]$ to the equation (see Friedman [7], eqn. 3.6, p. 145))

$$\varphi(x,t) = 2\int_0^t d\tau \int_{\partial D\times[0,T]} [\frac{\partial\Gamma(x,\xi;t,\tau)}{\partial\nu(x)} + \beta(x,t)\Gamma(x,\xi;t,\tau)]\varphi(\xi,\tau)dS_\xi + 2F(x,t)$$

where dS_ξ is the differential surface measure on ∂D. Finally, (see [7], Theorem 2, p. 144 and Corollary 2, p. 147), it is evident that the function

$$U(x,t) = \int_0^t d\tau\int_{\partial D} \Gamma(x,\xi;t,\tau)\varphi(\xi,\tau)dS_\xi + \int_D \Gamma(x,\xi;t,0)U(\xi,0)d\xi - \gamma_0(x,t) - \int_0^t d\tau\int \Gamma(x,\xi;t,\tau)f(\xi,\tau)d\xi$$

has the properties required. In particular, $F(x,t)$ is a non-anticipative functional of the z_t and $\tilde{z}_t$ processes, which implies that $\varphi(x,t)$ and, in turn, $U(x,t)$, are also non-anticipative. Q.E.D.

Now, redefine μ_1 to be the measure determined by $U(x,s)$, $s \leq t$, and dy_s given by (B6) for $s \leq t$, and $dr(s)$, $s \leq t$, given by (B5). Let μ_0 be the measure determined by $U(x,s)$, $r(s)$, $s \leq t$, and $w(s)$, $s \leq t$.

Let $R(t)$ denote the vector $E_1^t r(t)$, $P_R(t)$ denote the covariance matrix $E_1^t(r(t) - E_1^t r(t))\ (r(t) - E_1^t r(t))'$ and $P_{MR}(x,t)$ denote the covariance $E_1^t(U(x,t) - E_1^t U(x,t))(r(t) - E_1^t r(t))$.

<u>Theorem 2</u>. <u>Assume</u> (B1) - (B8). <u>Then there is a version of</u> $M(x,t)$ <u>such that</u> w.p.l.: $M(x,t)$ <u>and its first 'mean square'</u> <u>(or true)</u> <u>derivative are continuous</u> w.p.l. <u>on</u> $\bar{R}$ <u>and</u> $\bar{D}\times(0,T]$, <u>resp.</u> <u>The second 'mean square'</u> <u>(or true)</u> <u>derivatives of</u> $M(x,t)$ <u>are continuous in</u> $D\times(0,T]$ <u>and</u> $M(x,t)$ <u>has an Itô differential which satisfies</u>

$$dM(x,t) = [\mathcal{L}M(x,t) - f(x,t)]dt + [dy - \int_{\partial D} H(\xi,t)M(\xi,t)dS_\xi]'\Sigma_t^{-1}[\int_{\partial D} H(\xi,t)P(\xi,x,t)dS_\xi]. \tag{19a}$$

Also

$$\frac{\partial M(x,t)}{\partial V(x)} + \beta(x,t)M(x,t) = g(x,t) + R(x,t) \tag{19b}$$

$$dR(t) = AR(t)dt + [dy - \int_{\partial D} H(\xi,t)M(\xi,t)dS_\xi]'\Sigma_t^{-1}[\int_{\partial D} H(\xi,t)P_{MR}(\xi,t)dS_\xi \tag{20}$$

$$\dot{P}(x,y,t) = (\mathcal{L}_x + \mathcal{L}_y)P(x,y,t) + \sigma(x,t)\sigma(y,t) - [\int_{\partial D} H(\xi,t)P(x,\xi,t)dS_\xi]'\Sigma_t^{-1}[\int_{\partial D} H(\xi,t)P(\xi,y,t)dS_\xi] \tag{21}$$

$$\dot{P}_R(t) = A'P_R(t) + P_R(t)A + G(t)G'(t) - - [\int_{\partial D} H(\xi,t)P_{MR}(\xi,t)dS_\xi]'\Sigma_t^{-1}[\int_{\partial D} H(\xi,t)P_{MR}(\xi,t)dS_\xi]$$

$$\dot{P}_{MR}(x,t) = \mathcal{L}P_{MR}(x,t) + AP_{MR}(x,t) - [\int_{\partial D} H(\xi,t)P(x,\xi,t)dS_\xi]'\Sigma_t^{-1}[\int_{\partial D} H(\xi,t)P_{MR}(\xi,t)dS_\xi]$$

$P(x,y,t)$ satisfies the boundary conditions for (x,t) on $\partial D \times (0,T]$,

$$\frac{\partial P_y(x,y,t)}{\partial V(x)} + \beta(x,t)P(x,y,t) = v(x,t)P_{MR}(x,t)$$

and $P_{MR}(x,t)$ satisfies the (vector) boundary conditions

$$\frac{\partial P_{MR}(x,t)}{\partial V(x)} + \beta(x,t)P_{MR}(x,t) = v(x,t)P_R(t).$$

Proof. The details are very similar to those of Theorem 1 and Lemma 4 and are omitted. Only the boundary conditions will be discussed. By Lemma 5, and a result similar to that of Lemma 4, it is easy to show that there is a version of $M(x,t)$ so that (w.p.1.) $M(x,t)$ is continuous on $\bar{R}$ (and in quadratic mean) $(D_i M(x,t)) = D_i M(x,t)$ is continuous on $\bar{D} \times (0,T]$ (and in quadratic mean). Similarly, for $x \in \partial D$, it can be shown that $\partial M(y,t)/\partial V(x)$ and $\partial U(y,t)/\partial V(x)$ are continuous in quadratic mean on $\bar{D} \times (0,T]$ (as functions of (x,y,t)). Then $E_1^t(\partial U(y,t)/\partial V(x))$ $= \partial M(y,t)/\partial V(x)$, where the last term is defined by $\lim_{y \to x, y \in D} \partial M(y,t)/\partial V(x) \equiv \partial M(x,t)/\partial V(x)$. Also $\lim_{y \to x, y \in D} \partial U(y,t)/\partial V(x)$ satisfies (w.p.1.)

$$E_1^t\left[\frac{\partial U(x,t)}{\partial V(x)} + \beta(x,t)U(x,t) - v(x,t)r(t) - g(x,t)\right]$$
$$= \frac{\partial M(x,t)}{\partial V(x)} + \beta(x,t)M(x,t) - v(x,t)R(t) - g(x,t).$$

The equation

$$E_1^t[U(y,t) - M(y,t)]\left[\frac{\partial U(x,t)}{\partial V(x)} + \beta(x,t)U(x,t) - v(x,t)r(t) - g(x,t)\right] = 0$$

implies

$$\frac{\partial P(x,y,t)}{\partial V(x)} + \beta(x,t)P(x,y,t) - v(x,t)P_{RM}(y,t) = 0$$

Also

$$E_1^t[r(t) - R(t)][\frac{\partial U(x,t)}{\partial V(x)} + \beta(x,t)U(x,t) - v(x,t)r(t) - g(x,t)] = 0$$

implies

$$\frac{\partial P_{MR}(x,t)}{\partial V(x)} + \beta(x,t)P_{MR}(x,t) - v(x,t)P_R(t) = 0.$$

End of details.

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