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OUTLINE OF RELATIVISTIC COSMOLOGY FOR COSMIC GAMMA-RAY STUDIES

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In the general theory of relativity, Einstein set out to identify the gravitational field with the geometry of space-time itself. He derived an equation using differential geometry which identified the curvature of space-time with the matter-energy density contained in it. Expressed in tensor form, the Einstein field equations are written

$$G_{\mu\nu} - \frac{1}{2}\left(G g_{\mu\nu} - 2\lambda g_{\mu\nu}\right) = 8\pi\gamma T_{\mu\nu} \qquad (1)$$

where $G_{\mu\nu}$ is the Einstein-Ricci tensor, G is its trace, $g_{\mu\nu}$ is the metric tensor which is defined by the equation

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu \qquad (2)$$

μ and ν are tensor indices which take on four values (one time-like, three space-like), Λ is the cosmological constant, γ is the gravitational constant and $T_{\mu\nu}$ is the energy-momentum tensor, given by

$$T_{\mu\nu} = \left(\frac{p}{c^2} + \rho\right)\beta_\mu\beta_\nu + \frac{p}{c^2} g_{\mu\nu} \qquad (3)$$

for a macroscopic body. In equation (1), p is the pressure, ρ, the mass density and β_μ and β_ν are components of the comoving velocity vector within the mass.

For an isotropic and homogeneous universe, the metric tensor has the Robertson-Walker form in dimensionless spherical coordinates, i.e.,

$$g_{\mu\nu} = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & iR(t)(1-k\eta^2)^{-1/2} & 0 & 0 \\ 0 & 0 & iR(t)\eta & 0 \\ 0 & 0 & 0 & iR(t)\eta\sin\theta \end{pmatrix} \qquad (4)$$

so that

$$ds^2 = c^2dt^2 - R^2(t)\Big[(1-k\eta^2)^{-1}d\eta^2 + \eta^2(d\theta^2 + \sin^2\theta\, d\phi^2)\Big] \qquad (5)$$

where $R(t)$ is a time-dependent scale factor to be determined by solving the field equations (Eq. 1), and k is the curvature constant which has the values (-1, 0, + 1).

Using the Robertson-Walker metric, equation (1) simplifies into the relations

$$\frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} + \frac{8\pi\gamma p}{c^2} = -\frac{kc^2}{R^2} + \Lambda c^2 \tag{6}$$

and

$$\frac{\dot{R}^2}{R^2} - \frac{8\pi\gamma\rho}{3} = -\frac{kc^2}{R^2} + \frac{\Lambda c^2}{3} \tag{7}$$

In the present state of the universe, $\rho \gg p/c^2$ and equations (6) and (7) are usually solved for the zero-pressure approximation.

The Robertson-Walker line element given in equation (5) can also be written in the form

$$ds^2 = c^2 dt^2 - R^2(t)\, du^2 \tag{8}$$

where

$$du^2 = \frac{d\eta^2}{1-k\eta^2} + \eta^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \tag{9}$$

The dimensionless length measured along the radial direction is then given by

$$u = \int_0^{\eta} \frac{d\eta}{\sqrt{1-k\eta^2}} = \begin{cases} \sin^{-1}\eta & \text{for } k=+1, \\ \eta & \text{for } k=0, \\ \sinh^{-1}\eta & \text{for } k=-1. \end{cases} \tag{10}$$

Photons travel along geodesics which obey the same relation as in special relativity, viz.,

$$ds^2 = 0 \tag{11}$$

which, from equation (8) is equivalent to the condition

$$u = c\int_{t_e}^{t_r} \frac{dt}{R(t)} \tag{12}$$

with t_e being the time when the photon was emitted and t_r being the time when the photon was received.

The emitting and receiving points are embedded in the metric so that the distance between them is changing by the scale factor $R(t)$; the dimensionless metric distance, u , is a constant. If we therefore consider two sucessive wave crests of a light ray as being emitted at times t_e and $t_e + \Delta t_e$ respectively and being received at times t_r and $t_r + \Delta t_r$, then

$$\int_{t_e}^{t_r} \frac{dt}{R(t)} = \int_{t_e+\Delta t_e}^{t_r+\Delta t_r} \frac{dt}{R(t)} = u = \text{constant}. \tag{13}$$

Thus

$$\int_{t_e+\Delta t_e}^{t_r+\Delta t_r} \frac{dt}{R(t)} - \int_{t_e}^{t_r} \frac{dt}{R(t)} = \int_{t_r}^{t_r+\Delta t_r} \frac{d(t)}{R(t)} - \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{R(t)}$$

$$= \frac{\Delta t_r}{R(t_r)} - \frac{\Delta t_e}{R(t_e)} = 0 \,, \tag{14}$$

or

$$\frac{\Delta t_r}{R(t_r)} = \frac{\Delta t_e}{R(t_e)}$$

Since the wavelength of the emitted wave is $c\Delta t_e$ and that of the wave when received is $c\Delta t_r$, equation (14) the wavelength is shifted by the amount

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e} = \frac{\Delta\lambda}{\lambda} = \frac{R(t_r) - R(t_e)}{R(t_e)}$$

or

$$\frac{R(t_r)}{R(t_e)} = 1 + z \qquad (15)$$

We have observed this shift in the spectral lines of distant galaxies as always being toward longer wavelength so that $R(t_r) > R(t_e)$ From this evidence, it has therefore been deduced that our universe is expanding with time.

Gamma-Ray Fluxes

Let us now consider the effect of cosmological factors in calculating gamma-ray fluxes emitted at large redshifts, z. The number of photons received per second is reduced by a factor $R(t_e)/R(t_r)$ from the number produced per second at time, t_e. We consider here gamma rays produced in particle collisions between two components having densities $n_a(t_e)$ and

$n_b(t_e)$ respectively. We specify the differential photon intensity produced per collision as

$$G(E_\gamma) \qquad (cm^2 \cdot sec \cdot sr \cdot GeV \cdot cm^{-6})^{-1}$$

Then the differential photon flux received at t_r is given by

$$dF_r = \frac{4\pi\, n_a(t_e)\, n_b(t_e)\, G(E_{\gamma,e})\, dE_{\gamma,e}\, dV_e\, dt_e}{4\pi R^2(t_r)\, u^2} \tag{16}$$

where the numerator represents the photon flux emitted at t_e, and the denominator indicates the fact that at t_r this flux is evenly distributed over a spherical wavefront of radius $R(t_r)$

We now define the three dimensional length

$$dl = R(t)\, du \tag{17}$$

so that the volume element

$$dV_e = dl\left[R^2(t_e)\, u^2\, d\Omega\right] \tag{18}$$

Since $$dt_e = \left[R(t_e)/R(t_r)\right] dt_r$$

and $$dE_{\gamma,e} = \left[R(t_r)/R(t_e)\right] dE_{\gamma,r} \tag{19}$$

because the energy of a gamma-ray is inversely proportional to its wavelength, we may substitute (18) and (19) into (17) and obtain

$$dF_r = n_a(t_e)\,n_b(t_e)\,G\{[R(t_r)/R(t_e)]E_{\gamma,r}\} \times \frac{4\pi R^2(t_e)\,u^2\,d\Omega\,d\ell\,dE_{\gamma,r}\,dt_r}{4\pi R^2(t_r)\,u^2} \tag{20}$$

By making use of equation (15) and dropping the subscript, r , since we only measure gamma-rays when they are received, equation (20) reduces to

$$\frac{dF}{d\Omega\,dt\,dE_\gamma} \equiv dI = \frac{n_a(z)\,n_b(z)\,G[(1+z)E_\gamma]\,d\ell}{(1+z)^2}$$

$$= \frac{n_a(z)\,n_b(z)\,G[(1+z)E_\gamma]}{(1+z)^2}\left(\frac{d\ell}{dz}\right)dz\,. \tag{21}$$

Derivation of $d\ell/dz$:

Equation (21) is quite useful in evaluating the metagalactic gamma-ray spectra from various high-energy interactions. The results are obtained from numerical integration of the relation

$$I(E_\gamma) = \int_0^{z_{MAX}} dz\, n_a(z)\, n_b(z)\, \frac{G[(1+z)E_\gamma]}{(1+z)^2} \left(\frac{d\ell}{dz}\right) \tag{22}$$

Therefore, in order to utilize equation (22), we must determine the factor, $d\ell/dz$, by solution of the field equations (6) and (7) for particular cases. As we shall discuss later, the two most important cases to consider are the Einstein-de Sitter model and the low density model. We shall now evaluate $d\ell/dz$ for these models.

A. The Einstein-de Sitter model:

This model corresponds to a Euclidean three-space with $k = \Lambda = p = 0$ Equation (7) then reduces to

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi\gamma\rho}{3} \tag{23}$$

In this case, our space is a Euclidean three-sphere containing a constant mass-energy, M, so that

$$\rho = \frac{3M}{4\pi R^3} \tag{24}$$

From (23) and (24) we find

$$R(t) = (3\gamma t)^{2/3} \tag{25}$$

In these equations, it is common to define the Hubble parameter

$$H \equiv \frac{\dot{R}}{R} \tag{26}$$

so that from (25) and (26) we obtain

$$H = \frac{2}{3} t^{-1} \tag{27}$$

It is also common to lable quantities associated with the present epoch ($z = 0$) by a subscript 0 . The quantity, H_0 , is then referred to as the Hubble constant. From equations (15), (25) and (27) we then find

$$R(z) = \frac{R_0}{1+z} \tag{28}$$

and $$t(z) = \frac{t_0}{(1+z)^{3/2}}$$

From equation (8), (11), and (17), we obtain

$$\frac{dl}{dz} = -c\,\frac{dt}{dz} \tag{29}$$

and from equations (27) - (29) we then obtain

$$\frac{dl}{dz} = \frac{cH_0^{-1}}{(1+z)^{5/2}} \tag{30}$$

B. The low density model:

This model holds when $\Lambda = 0$ and $\frac{8\pi\gamma\rho R^2}{3c^2} \ll 1$. Thus, equation (7) reduces to

$$\frac{\dot{R}^2}{R^2} = -\frac{kc^2}{R^2} \tag{31}$$

and therefore $\dot{R} =$ const. and $R = Kt$ where $K =$ const. Therefore

$$t(z) = \frac{t_0}{1+z}$$

$$H(z) = H_0(1+z) \tag{32}$$

$$H_0 = t_0^{-1}$$

and
$$\frac{dt}{dz} = -\frac{t_0}{(1+z)^2}$$

From (29) and (32) we then find

$$\frac{dl}{dz} = \frac{cH_0^{-1}}{(1+z)^2} \tag{33}$$