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CONTACT PROBLEMS IN ELASTICITY AND THE SOLUTION FOR A THICK LAYER ON A WINKLER TYPE FOUNDATION

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ABSTRACT

This study presents many aspects of contact problems and presents a solution for the determination of the stresses and displacements in an arbitrarily thick layer that is resting on an elastic (Winkler type) foundation. The contact problems discussed are the nonlinear bending of beams, the stress in a thin plate resting on an elastic foundation, the stress in a half plane because of surface punches and pressures, the stress in a half space because of surface punches and pressures, and the stress in a layer resting on a rigid base because of surface punches and pressures. The problem of determining the stresses of a thick layer resting on an elastic foundation is then solved for a general surface pressure distribution. Numerical results are presented for the particular case of a surface pressure uniformly distributed over a circle.

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CONTACT PROBLEMS IN ELASTICITY AND THE SOLUTION FOR A THICK LAYER ON A WINKLER TYPE FOUNDATION*

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SUMMARY

This study presents many aspects of contact problems and presents the solution to the problem of an arbitrarily loaded thick layer resting on an elastic (Winkler type) foundation. When solving stress problems, the forces are usually transmitted by contact between two bodies. If the method of contact influences the stress, the problem is called a contact problem. Here, such problems are discussed in order of difficulty, beginning with the simplest. The first type of contact problem considered is the nonlinear bending of beams. The solutions to the two-dimensional and three-dimensional problems are presented. Finally, the stress in a layer that is resting on a rigid foundation is determined.

The problem of a thick layer resting on an elastic foundation is then solved to determine the effect of plate thickness on the stress distribution. The solution is given in terms of infinite integrals that are virtually impossible to integrate by hand. To plot the solution, the integrals are integrated numerically on a computer. Numerous plots are given for a particular surface loading which consists of a pressure that is uniformly distributed over a circle.

INTRODUCTION

This study has two main purposes. The first purpose is to present numerous examples of the many aspects of contact problems in elasticity. The second and more important purpose is to solve the problem of a thick layer resting on an elastic (Winkler type) foundation, the surface of which is subjected to any prescribed normal loading. Contact problems are of practical importance in the design of building foundations, ball bearings, clamps, or any machine part which involves forced contact between two objects. The first section of the paper gives a brief history of the proposed solutions for the plane contact problems. The following four sections deal with aspects of punch problems; the solutions are presented, but the derivations are omitted. However, the derivation of the solution to the problem of a thick plate resting on an elastic foundation is presented in detail in a later section because this solution is considered to be the main contribution of the paper.

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SYMBOLS

A arbitrary constants of integration associated with $\bar{u}_1$

$$A_n = -\frac{1}{\sqrt{\pi}} \frac{(n/2 - 1/2)!}{(1 - \nu^2)(n/2 - 1)!} a_n$$

a boundary of punch

B arbitrary constant of integration associated with $\bar{w}_0$

b thickness of elastic layer resting on rigid foundation

c distance from the origin to the force

D plate rigidity

d layer thickness

E modulus of elasticity

F total resultant force of pressure exerted on a surface

$$F_1(m, \tau) = \int_0^\infty \frac{w \cosh wJ_0(mw)J_1(\tau w)}{2w + \sinh(2w)} dw$$

$$F_2(m, \tau) = \frac{1}{m} \int_0^\infty \frac{\cosh wJ_1(mw)J_1(\tau w)}{2w + \sinh 2w} dw$$

$f(t_0)$ displacement under the punch for two-dimensional problems

$f(t_0)_1$ constant part of deflection

$f(t_0)_2$ varying part of deflection

$\bar{f}_\nu(\epsilon)$ Hankel transform of $f(r)$ of the order ν

$$G = \frac{\mu}{2hK(1 - \nu)}$$

$$G_1(m, \tau) = \int_0^{\infty} \frac{\sinh wJ_0(mw)J_1(\tau w)}{2w + \sinh 2w} dw$$

$$G_2(m, \tau) = \frac{1}{m} \int_0^{\infty} \frac{\sinh wJ_1(mw)J_1(\tau w)}{2w + \sinh 2w(w)} dw$$

$g(t)$	representation of solution of the Fredholm equation
$H_i(\ )$	ith order Hankel transform
h	half of plate thickness
I	cross section area moment of inertia
$I(\psi, \beta, \gamma)$	integrals of Lipschitz-Hankel type involving products of Bessel functions
J_ν	Bessel function of the first kind, order ν
K	modulus of foundation
K_0	Bessel function of the second kind
k	dummy variable
kei	Kelvin function
ker	Kelvin functions
L	length of beam
l	characteristic length
M	bending moment on beam
m	dummy variable
P	concentrated normal load
P_0	total vertical load
$\overline{P}_0$	zeroth order Hankel transform of the surface pressure
p	uniformly distributed pressure
$p(t_0)$	normal load as a function of boundary horizontal coordinate

q normal load distribution on the surface

R radius of curvature of rigid body

Re real part of any function

r radial (polar) coordinate

$T(t_0)$ shear loading on the boundary

t dummy variable

t_0 x-coordinate of any point on the surface of an elastic body

u displacement in x-direction (Cartesian coordinates) or in r-direction (polar coordinates)

v displacement in y-direction

w displacement in z-direction

x, y, z axes of Cartesian coordinate system

α angle between the vertical and the point of contact of a beam and a rigid body

β dummy variable

γ dummy variable

γ_{xy} shear strain on x-face in y-direction

δ beam displacement in the vertical direction

δ_B deflection at point B (fig. 1)

δ_R Dirac delta function

δ_1 deflection of segment DB considered as simple cantilever beam (fig. 1)

δ_2 deflection at point B that results from the slope at point D (fig. 1)

δ_3 deflection at point B representing the vertical distance of point D from the horizontal tangent at point A (fig. 1)

ϵ deflection of a beam

$\epsilon_{x, y, z}$ normal strain in x-, y-, or z-direction

η dummy variable equal to $2\xi h$

θ angular (cylindrical) coordinate

κ constant equal to $3 - 4\nu$

λ Lame constant

μ Lame constant (shear modulus)

ν Poisson ratio

ξ dummy variable

ρ radial (cylindrical) coordinate

σ_r normal stress on radial face in radial direction

σ_{rz} shear stress on radial face in z-direction

$\sigma_{r\theta}$ shear stress on radial face in θ-direction

σ_x normal stress on x-face in x-direction

σ_{xy} shear stress on x-face in y-direction

σ_y normal stress on y-face in y-direction

σ_z normal stress on z-face in z-direction

σ_θ normal stress on θ-face in θ-direction

σ_ϕ normal stress on ϕ-face in ϕ-direction

τ dummy variable

Φ complex function of z

$\chi(t)$ operational function of t

ψ dummy variable

$\psi(\zeta)$ operational function of ζ

ω any argument of H

Operators:

$(\bar{\ })$ complex conjugate of a function

$(')$ derivative

∇^2 Laplacian operator

HISTORICAL BACKGROUND

Sadovskii (ref. 1) solved several problems dealing with the pressure of a rigid body on an elastic semi-infinite plane. He considered the pressure of a plane-base punch on an elastic semiplane, and he also considered the case of infinitely many punches. The general method of solving plane contact problems by the Cauchy integrals was given by Muskhelishvili in reference 2. Here, the case was established that the required function of a complex variable is regular everywhere, including the edges of the contact area. (Muskhelishvili's Cauchy integral solutions are presented in a later section.) Begiashvili (ref. 3) generalized the problem of one punch to solve a problem involving any number of punches. Lomidze (ref. 4) solved the problem concerning the pressure of a system of jointed ground foundations. Klubin (ref. 5) determined the stresses arising inside a heavy elastic semiplane. A number of frictionless punch problems for a semiplane were considered by Shtaerman in reference 6.

Numerous problems of elastic equilibrium, both two-dimensional and three-dimensional, have been examined and the exact solutions obtained by means of the Fourier, Mallin, and Hankel transforms. Until recently, the Fourier transform was the one most widely used. Filon's (ref. 7) investigation of the state of stress in an infinite strip with a known exterior force was probably the first solution in the theory of elasticity solved by the use of Fourier integrals. Sneddon (ref. 8) investigated the elastic equilibrium of an infinite strip and, in particular, the case of a half plane. The application of the Fourier transform to the solution of the first fundamental problem for the half plane is discussed in the monograph of Novozhilov (ref. 9). With the use of the Fourier integrals in bipolar coordinates, the exact solutions of plane contact problems can be derived for a single punch and also for similar mixed problems of stresses in an infinite body weakened by a plane slit. The simplest of these problems was examined by Sneddon (ref. 10), and the solution was worked in Cartesian coordinates and led to a set of dual integral equations. A solution of the Boussinesq problem with the help of the Hankel transform is given in the books by Sneddon (ref. 8) and Tranter (ref. 11). Uflyand (ref. 12) obtained an exact solution of the Boussinesq problem by the use of the Papkovich-Neuber function of the mixed problem for an elastic layer, on one boundary of which are given the external forces and, on the other, the displacements.

Padfield and Sida (ref. 13) determined the elastic stresses that exist in a layer depressed symmetrically by two flat cylindrical punches. Muki (ref. 14) solved numerous asymmetric problems for a semi-infinite solid and a thick slab. Nelson (ref. 15) considered the problem of a thick plate loaded axisymmetrically. Greenwood (ref. 16) computed the stresses produced in the midplane of a slab by pressures applied symmetrically to its surfaces. The solution for the stresses in a plate that has been pressed between two spheres was given by Tu and Gazis (ref. 17).

Many punch problems have been solved for the case when the punch is rigidly connected to the elastic body, when friction is present, or when two elastic bodies are in contact; but because they are not illustrated in this paper, they will not be mentioned here. The material cited in this section is not intended to be a complete bibliography on contact problems but rather a limited list of sources for information on specific contact problems.

ELEMENTARY CONTACT PROBLEMS

This section on elementary contact problems is presented as a natural starting point in the presentation of certain aspects of punch problems arising in the strength of materials. All simplifying assumptions of the theory of beams are taken into account. However, not all contact problems can be solved by this approach, and only the simpler cases are considered. The concept of contact plays an important role in the solution of problems involving the elementary theory of beams and plates because the concept changes the constraints of the problem. The problems presented here are the nonlinear bending of a cantilever beam, the nonlinear bending of a simply supported beam, and the problem of a thin plate resting on an elastic foundation. The solution to the last problem will be compared to the problem of a thick layer resting on an elastic foundation.

The governing equation for the bending of beams is given by

$$\frac{M}{EI} = \frac{1}{R} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \tag{1}$$

or, because dy/dx is usually small, equation (1) can be approximated as

$$\frac{M}{EI} \cong \frac{d^2y}{dx^2} \tag{2}$$

Nonlinear Bending of a Cantilever Beam

The problem is to determine the free-end (point B) deflection of a cantilever beam that is gradually forced into contact with a rigid circular surface (AC) (fig. 1). The solution to this problem may be found in reference 18 and is given as

$$\delta_B = \delta_1 + \delta_2 + \delta_3 = \frac{L^2}{2R} - \frac{(EI)^2}{6P_0^2R^3} \tag{3}$$

where δ_1 = the deflection of segment DB considered as a simple cantilever beam

δ_2 = the deflection at point B caused by the slope at point D

δ_3 = the deflection at point B representing the vertical distance of point D from the horizontal tangent at point A

Nonlinear Bending of a Simply Supported Beam

The problem is to determine the angle α, which defines the positions of the points of contact between the elastic beam and the rigid circular surfaces (fig. 2). For small angles, the result was presented in reference 18 as

$$\alpha = \frac{P_0(L - 2R\alpha)^2}{16EI} \tag{4}$$

Solutions for the problems involving the nonlinear bending of both the cantilever beam and the simply supported beam are obtained from the condition that at the points of contact the deflection curve is tangent to the supporting surfaces.

Thin Plate Resting on an Elastic Foundation

The problem is to determine the vertical displacement equation of a thin, infinitely extended plate that is resting on an elastic foundation and whose surface is subjected to a concentrated load (fig. 3). The reaction of the elastic foundation is assumed to be proportional to the deflection in the z-direction w and is given by Kw.

The constant K, expressed in units of pressure per unit deflection, is called the modulus of the foundation. The general equation to be solved is in the form

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = \frac{q - Kw}{D} \tag{5}$$

where q is the normal load distribution on the surface, h is half the plate thickness, and D is the plate rigidity given by $D = 4Eh^2/12(1 - \nu^2)$. For q equal to zero everywhere except at the origin and also for the deflection vanishing for large values of r, the solution for equation (5) is given by

$$w = \frac{-P_0 l^2}{2\pi D}\ \mathrm{kei}(r/l) \tag{6}$$

where $l = (D/K)^{1/4}$, and the subgrade reaction is

$$\sigma_z = \frac{-P_0 \mathrm{kei}(r/l)}{2\pi l^2} \tag{7}$$

In equation (7), kei(x) is defined by the relations

$$K_0(x\sqrt{\pm i}) = \mathrm{ker}(x) \pm \mathrm{kei}(x) \tag{8}$$

where K_0 is a Bessel function of the second kind. This type of problem was originally solved by Hertz (ref. 19), and his data are presented in graphic form in figure 4.

TWO-DIMENSIONAL PROBLEMS OF A HALF PLANE

This section involves solutions for both the first fundamental problem and the punch problem in a two-dimensional half plane. However, before proceeding, a distinction will be made between the first fundamental problem and the punch problem. The first fundamental problem involves the determination of the elastic equilibrium of a body when the pressure on its surface is a prescribed function (fig. 5). The second fundamental problem involves the determination of the elastic equilibrium of a body when the displacements on its surface are prescribed. The mixed boundary problem is one in which the surface displacements are prescribed on only a portion of the boundary while the surface pressures are prescribed elsewhere. The problem of the punch is a special case of the mixed boundary problem in that the surface displacements are prescribed on a portion of the boundary, but the normal stresses are equal to zero everywhere else (fig. 6).

Two different but equivalent methods are used to solve the first fundamental problem and the punch problem. The solutions derived by both the Cauchy integral and the integral transform will be listed preceding the appropriate section. The examples that are presented consist of a point force on the boundary, a uniformly distributed load on part of the boundary, a straight horizontal-base punch problem, and a wedge-base punch problem.

The basic equations of the classical (infinitesimal) theory of elasticity that govern the problems solved here are given as follows in Cartesian coordinates.

1. Equilibrium (with body forces neglected)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{9a}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{9b}$$

2. Compatibility

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = 2\,\frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} \tag{10}$$

where $\epsilon_x = \frac{\partial u}{\partial x}$, $\epsilon_y = \frac{\partial v}{\partial y}$, and $\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$.

3. Stress-strain

$$\sigma_x = \lambda\left(\epsilon_x + \epsilon_y\right) + 2\mu\epsilon_x \tag{11a}$$

$$\sigma_{xy} = 2\mu\gamma_{xy} \tag{11b}$$

$$\sigma_y = \lambda\left(\epsilon_x + \epsilon_y\right) + 2\mu\epsilon_y \tag{11c}$$

First Fundamental Problem for Half Plane

Let $p\left(t_0\right)$ be the pressure distribution on the surface, and let t_0 denote the x-coordinate when $y = 0$ (fig. 7). The following formulas (ref. 10) are obtained by applying Fourier transforms to the partial differential equations of the problem, by observing boundary conditions, and by inverting the transform.

$$\sigma_x = -\frac{2y}{\pi}\int_{-\infty}^{\infty}\frac{(x - t)^2 p(t)dt}{\left[(x - t)^2 + y^2\right]^2} \tag{12}$$

$$\sigma_{xy} = -\frac{2y^2}{\pi}\int_{-\infty}^{\infty}\frac{(x - t)p(t)dt}{\left[(x - t)^2 + y^2\right]^2} \tag{13}$$

$$\sigma_y = -\frac{2y^3}{\pi}\int_{-\infty}^{\infty}\frac{p(t)dt}{\left[(x - t)^2 + y^2\right]^2} \tag{14}$$

$$u = -\frac{1}{2\mu\pi}\int_{-\infty}^{\infty} p(t)\left[(1 - 2\nu)\tan^{-1}\left(\frac{x - t}{y}\right) - \frac{(x - t)y}{(x - t)^2 + y^2}\right]dt \tag{15a}$$

$$v = -\frac{1}{\pi\mu}\int_{-\infty}^{\infty} p_1(t)\left\{\frac{(1 - \nu)(x - t)}{(x - t)^2 + y^2} + \frac{(x - t)y^2}{\left[(x - t)^2 + y^2\right]^2}\right\}dt \tag{15b}$$

where $p_1(t) = \int_{-\infty}^{x} p(t)dt$.

However, if the Cauchy integral method is used and the known equations (ref. 2) are utilized, the following formulas are obtained.

$$\sigma_y - i\sigma_{xy} = \Phi(z) - \Phi(\bar{z}) + (z - \bar{z})\overline{\Phi}'(\bar{z}) \tag{16}$$

$$\sigma_x + \sigma_y = 2[\mathrm{Re}\Phi(z)] \tag{17}$$

$$2\mu \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (3 - 4\nu)\Phi(z) + \Phi(\bar{z}) - (z - \bar{z})\overline{\Phi}'(\bar{z}) \tag{18}$$

where (') denotes differentiation.

For the first fundamental problem, the following equation is obtained.

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{p(t_0) - T(t_0)dt_0}{(t_0 - z)} \tag{19}$$

where $p(t_0)$ and $T(t_0)$ are the normal and shear loading on the boundary, respectively.

Concentrated force on the boundary. - For the case of a concentrated force acting on the boundary (fig. 8), the following formulas are obtained for the first fundamental *problem of a half plane.*

$$\sigma_x = -\frac{2P(x - c)^2 y}{\pi\left[(x - c)^2 + y^2\right]^2} \tag{20}$$

$$\sigma_{xy} = -\frac{2P(x - c)y^2}{\pi\left[(x - c)^2 + y^2\right]^2} \tag{21}$$

$$\sigma_y = -\frac{2Py^3}{\pi\left[(x - c)^2 + y^2\right]^2} \tag{22}$$

Equations (20), (21), and (22) can be obtained by the Cauchy integral equations (16), (17), and (19) or by the integral transform equations (12), (13), (14), and (15). The solution to this problem is important because with it, the solution for any distribution of a load can be obtained by simple integration.

Uniformly distributed load on a part of the boundary. - The solution for the problem of a concentrated load on a boundary proves to be useful in solving the problem of the uniformly distributed load on a part of the boundary. The solution to the latter problem may be obtained by integration of equations (20), (21), and (22) with respect to x in the limits from -a to a, where -a and a are the extremes of the loading in the x-direction. The solution may also be obtained directly from equations (12) to (15) or from equations (16), (17), and (19). The solution is given as

$$\sigma_x = -\frac{p}{\pi}\left[\tan^{-1}\left(\frac{y}{a - y}\right) - \tan^{-1}\left(\frac{y}{a + x}\right)\right] + \frac{2pay\left(x^2 - y^2 - a^2\right)}{\pi\left[\left(x^2 + y^2 - a^2\right)^2 + 4a^2y^2\right]} \tag{23}$$

$$\sigma_y = -\frac{p}{\pi}\left[\tan^{-1}\left(\frac{y}{a - x}\right) - \tan^{-1}\left(\frac{y}{a + x}\right)\right] - \frac{2pay\left(x^2 - y^2 - a^2\right)}{\pi\left[\left(x^2 + y^2 - a^2\right)^2 + 4a^2y^2\right]} \tag{24}$$

$$\sigma_{xy} = \frac{4paxy^2}{\pi\left[\left(x^2 + y^2 - a^2\right)^2 + 4a^2y^2\right]} \tag{25}$$

Punch Problems for the Half Plane

In solving punch problems, as in solving the first fundamental problems, the basic equations (9), (10), and (11) must be satisfied, but a different approach is necessary because the boundary conditions are different.

There are two classes of punch problems. The first class involves the type in which the surface of contact is known and does not change in the process of loading. The second is the class in which the surface of contact is to be found from the solution of the problem. A typical example of the first type is the flat-bottom punch (fig. 9).

In the second type of punch problem, the surface contact depends on the properties of the bodies in contact, on the profile of the punch, and on the applied force. The solution obtained by integral transform (ref. 8) is given by

$$f\left(t_0\right) = f\left(t_0\right)_1 + f\left(t_0\right)_2 \tag{26}$$

where $f(t_0)_1$ is the constant part of the deflection and

$$f(t_0)_2 = \sum_{n=0}^{n} a_n \left(\frac{t_0}{a}\right)^n \tag{27}$$

The pressure under the punch is given as

$$p(t_0) = -\frac{Ef(t_0)_1\left[1 - \left(\frac{t_0}{a}\right)^2\right]^{-1/2}}{2a\left(1 - \nu^2\right)\ln 2}$$

$$+ E\sum A_n\left\{\left[1 - \left(\frac{t_0}{a}\right)^2\right]^{-1/2} - n\left(\frac{t_0}{a}\right)^{n-1}\int_0^{a/x} u^{n-1}\left(u^2 - 1\right)^{-1/2} du\right\} \tag{28}$$

where $A_n = -\frac{1}{\sqrt{\pi}}\frac{(n/2 - 1/2)!\, a_n}{\left(1 - \nu^2\right)(n/2 - 1)!}$. By the Cauchy integral method (ref. 2), the solution for the second type of punch problem is also given by

$$p(t_0) = \frac{\mu}{\pi(1 - \nu)\sqrt{(t_0 - a)(b - t_0)}}\int_a^b \frac{\sqrt{(t - a)(b - t)}f'(t)}{t - t_0} dt + \frac{\frac{P_0}{\pi}}{\sqrt{(t_0 - a)(b - t_0)}} \tag{29}$$

where a and b are the extreme points of contact, $f'(t)$ is the derivative of the punch profile, and P_0 is the total vertical load.

Cylinder indenting a semiplane. - The problem of a cylinder indenting a semiplane is an example of a punch problem in which the surface varies (fig. 10). The equation for the profile of a cylinder is $y \cong t_0^2/2R = f(t_0)$, and the contact pressure is, therefore, found to be

$$P(t_0) = \frac{2\mu\left(a^2 - 2t_0^2\right)}{R(\kappa + 1)\sqrt{a^2 - t_0^2}} + \frac{P_0}{\pi\sqrt{a^2 - t_0^2}} \tag{30}$$

To find the value of a as a function of P_0, the contact pressure is observed to be zero at $P(-a)$ and $P(+a)$. Therefore, $\lim_{t_0 \to a} P(t_0) = 0$, so that $a = \pm\left[\frac{P_0 R(\kappa + 1)}{2\pi\mu}\right]^{1/2}$. Now the pressure $P(t_0)$ is completely determined.

Wedge-shaped punch. - Another example of a punch with varying surface of contact is a wedge-shaped punch (fig. 11). The profile of the punch is given as $f(t_0) = b + \epsilon(1 - t_0/a)$ with $0 \leq t_0 \leq a$. Here, the method of solution is exactly the same as for the last problem, except that in this problem a corner exists at $t_0 = 0$; therefore, the singularity appears at that point. The pressure under the punch is given as

$$P(t_0) = -\frac{E\epsilon}{\pi\left(1 - \nu^2\right)} \cosh^{-1}\left(\frac{a}{t_0}\right) \tag{31}$$

In the classical (infinitesimal) theory of elasticity, the deflections and their partial derivatives are assumed to be small. Therefore, the squares and products of these quantities are neglected when deriving the basic elasticity equations. In this problem and in that of the flat-base punch, there are points on the contact surface where $\partial^2 v/\partial x^2$ (change in slope in the x-direction) is by no means negligible; therefore, at those points, singularities exist.

FIRST FUNDAMENTAL AND PUNCH PROBLEMS FOR THE HALF SPACE

The solutions to the first fundamental problem and the punch problem for a three-dimensional half space are presented in this section. The equations for these general solutions were derived by the use of the Hankel transform and are presented in the appropriate section. The first fundamental problems considered involve a concentrated force acting normal to the boundary and a uniform pressure distributed over a circle.

The punch problems that are presented involve a flat-base cylindrical punch and a conical-base punch. Cylindrical coordinates will be used throughout this section.

The solution to the axisymmetrical stress problem must satisfy the following basic equations of classical theory of elasticity, as given in cylindrical coordinates.

1. Equilibrium (with body forces neglected)

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{32a}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \tag{32b}$$

2. Compatibility

$$\nabla^2 \sigma_r - \frac{2}{r^2}\left(\sigma_r - \sigma_\theta\right) + \frac{1}{1+\nu}\frac{\partial^2}{\partial r^2}\left(\sigma_r + \sigma_\theta + \sigma_z\right) = 0 \tag{33a}$$

$$\nabla^2 \sigma_\theta + \frac{2}{r^2}\left(\sigma_r - \sigma_\theta\right) + \frac{1}{1+\nu}\frac{\partial}{\partial r}\left(\sigma_r + \sigma_\theta + \sigma_z\right) = 0 \tag{33b}$$

where $\nabla^2 = \frac{\partial}{\partial r^2} + \frac{\partial}{r\,\partial r} + \frac{\partial^2}{\partial z^2}$

3. Stress-strain

$$\sigma_r = \left[(\lambda + 2\mu)\frac{\partial}{\partial r} + \frac{\lambda}{r}\right]u + \lambda\frac{\partial w}{\partial z} \tag{34a}$$

$$\sigma_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{34b}$$

$$\sigma_z = \lambda\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)u + (\lambda + 2\mu)\frac{\partial w}{\partial z} \tag{34c}$$

$$\sigma_\theta = \left(\lambda\frac{\partial}{\partial r} + \frac{\lambda + 2\mu}{r}\right)u + \lambda\frac{\partial w}{\partial z} \tag{34d}$$

The Hankel transform is used extensively in these sections. Therefore, it becomes necessary to define this transform and its inversion formula. The Hankel transform of the function $f(r)$ given on the positive real line $0 < r < \infty$ is defined by the integral

$$\bar{f}_\nu(\xi) = \int_0^\infty f(r)J_\nu(\xi r)r\,dr \tag{35}$$

where $0 \leq \xi < \infty$, $\nu > -1/2$ and $J_\nu(x)$ is the Bessel function of the first kind. If the function $f(r)$ is continuous in parts in any finite interval belonging to the region $(0, \infty)$ and if the integral $\int_0^\infty |f(r)|\sqrt{r}\,dr$ converges, then the Hankel transform exists.

If the function $f(r)$ also satisfies the Dirichlet conditions in any open interval $0 < r < s$, the Hankel inversion formula is

$$f(r) = \int_0^\infty \bar{f}_\nu(\xi)J_\nu(\xi r)\xi\,d\xi \tag{36}$$

where $0 < r < \infty$.

First Fundamental Problem for a Half Space

By first applying the Hankel transform to the basic elasticity equations, then applying the appropriate boundary conditions, and finally using the inversion formula, the following solution for the first fundamental problem of a half space (ref. 10) can be obtained. The problem is illustrated in figure 12.

If $p(r)$ is the axisymmetric pressure distribution on the surface $z = 0$ and if the shear stresses are zero on this surface, then the equations for the solution of the problem in the half space are given by the following conditions and by equation (37). If $f(r) = -p(r)/2\mu$ and the total load $P_0 = 4\pi\mu\int_0^\infty rf(r)dr$ and $\bar{f}_0(\xi) = H_0[f(r); r \rightarrow \xi]$ where $H_i(\ )$ denotes the ith order Hankel transform, then

$$u_r(r, z) = -H_1\left\{[(1 - 2\nu) - \xi z]\xi^{-1}\bar{f}(\xi)e^{-\xi z}; \xi \rightarrow r\right\} \tag{37a}$$

$$w_z(r, z) = H_0\left\{[2(1 - \nu) + \xi z]\xi^{-1}\bar{f}(\xi)e^{-\xi z}; \xi \rightarrow r\right\} \tag{37b}$$

$$\sigma_{rz}(r, z) = -2\mu z H_1\left[\xi\bar{f}(\xi)e^{-\xi z}; \xi \rightarrow r\right] \tag{37c}$$

$$\sigma_z(r, z) = -2\mu H_0\left[(1 + \xi z)\bar{f}(z)e^{-\xi z}; \xi \rightarrow r\right] \tag{37d}$$

$$\sigma_r(r, z) = -2\mu H_0\left[(1 - \xi z)\bar{f}(\xi)e^{-\xi z}; \xi \rightarrow r\right] + \frac{2\mu}{r} H_1\left[(1 - 2\nu)\xi^{-1} - z\bar{f}(\xi)e^{-\xi z}\right] \tag{37e}$$

$$\sigma_\theta(r, z) = -4\mu\nu H_0\left[\bar{f}(\xi)e^{-\xi z}; \xi \rightarrow r\right] - \frac{2\mu}{r} H_1\left\{\left[(1 - 2\nu)\xi^{-1} - z\right]\bar{f}(\xi)e^{-\xi z}; \xi \rightarrow r\right\} \tag{37f}$$

Concentrated force acting normal to a half space. - As an example of the first fundamental problem, the problem of a concentrated force acting normal to a half space (Boussinesq problem) will be presented. The problem is illustrated in figure 13. Because the force is localized at the origin, the expression $f(r) = \frac{\delta(r)}{r} k$ may be used where $\delta(r)$ is the Dirac delta function.

Choose k so that the equation $P_0 = 4\pi\mu\int_0^\infty rf(r)dr$ is satisfied. Note that $k = \frac{P_0}{4\pi\mu}$ and, therefore, $f(r) = \frac{P_0\delta(r)}{4\pi\mu r}$; thus, $\bar{f}_0(\xi) = \frac{P_0}{4\pi\mu}$. From the preceding equations, it can be shown that

$$u_r = \frac{-P_0}{4\pi\mu}\left\{\frac{(1 - 2\nu)}{r}\left[1 - \frac{z}{\left(r^2 + z^2\right)^{1/2}}\right] - \frac{zr}{\left(r^2 + z^2\right)^{3/2}}\right\} \tag{38a}$$

$$w_z = \frac{P_0}{4\pi\mu}\left[\frac{2(1 - \nu)}{\sqrt{r^2 + z^2}} + \frac{z^2}{\left(r^2 + z^2\right)^{3/2}}\right] \tag{38b}$$

$$\sigma_{rz}(r, z) = -\frac{3P_0 z^2 r}{2\pi\left(r^2 + z^2\right)^{5/2}} \tag{38c}$$

$$\sigma_z(r, z) = -\frac{3P_0 z^3}{2\pi\left(r^2 + z^2\right)^{5/2}} \tag{38d}$$

$$\sigma_\theta(r, z) = \frac{P_0(1 - 2\nu)}{2\pi}\left[\frac{z}{\left(r^2 + z^2\right)^{3/2}} - \frac{1}{r^2}\left(1 - \frac{z}{\sqrt{r^2 + z^2}}\right)\right] \tag{38e}$$

$$\sigma_r(r, z) = \frac{P_0}{2\pi}\left[\frac{1 - \nu}{r^2}\left(1 - \frac{z}{\sqrt{r^2 + z^2}}\right) - \frac{3r^2 z}{\left(r^2 + z^2\right)^{5/2}}\right] \tag{38f}$$

Uniform pressure over a circle. - The solution for a uniform pressure distributed over a circle may be obtained by simple integration of the concentrated force solution (eq. (38)) or by the application of equation (37). This problem is illustrated in figure 14. The case of the surface pressure distributed uniformly over a circle of radius a is represented by

$$p(r) = -pH(a - r) \tag{39}$$

where $H(\omega) = 1$ when $\omega \geq 0$ and $H(\omega) = 0$ when $\omega < 0$. The total load P_0 is equal to $\pi a^2 p$, or $p = \frac{P_0}{\pi a^2}$, and equation (39) becomes $p(r) = -\frac{P_0}{\pi a^2} H(a - r)$. Therefore, $f(r) = \frac{P_0 H(a - r)}{2\pi a^2 \mu}$, and $\overline{f}_0(\xi) = \frac{P_0 J_1(a\xi)}{2\pi a \mu \xi}$.

If the following notation is introduced $I(\psi, \beta, \gamma) = \int_0^\infty J_\psi(a\xi) J_\beta(r\xi)\xi^\gamma e^{-\xi z} d\xi$, then

$$u_r(r, z) = -\frac{P}{2\pi a \mu}(1 - 2\nu)I(1, 1, -1) - zI(1, 1, 0) \tag{40a}$$

$$w_z(r, z) = \frac{P}{2\pi a \mu}\ 2(1 - \nu)I(1, 0, -1) + zI(1, 0, 0) \tag{40b}$$

$$\sigma_{rz}(r, z) = -\frac{Pz}{\pi a} I(1, 0, 1) \tag{40c}$$

$$\sigma_{zz} = -\frac{P}{\pi a} I(1, 0, 0) + zI(1, 0, 1) \tag{40d}$$

$$\sigma_r = -\frac{P}{\pi a} I(1, 0, 0) - zI(1, 0, 1) + \frac{P}{\pi a}(1 - 2\nu)I(1, 1, -1) - zI(1, 1, 0) \tag{40e}$$

$$\sigma_\theta = -\frac{2P\nu}{\pi a} I(1, 0, 0) - \frac{P}{\pi a r}(1 - 2\nu)I(1, 1, -1) - zI(1, 1, 0) \tag{40f}$$

Numerical values for $I(\psi, \beta, \gamma)$ may be found in reference 20.

Punch Problem for the Half Space

When solving the punch problem for the half space, as in solving the first fundamental problem, the partial differential equations of elasticity (eqs. (32), (33), and (34)) must be satisfied. The boundary condition for the punch problem is a known displacement under the punch and zero normal stress elsewhere on the boundary, as illustrated in figure 15. By transforming the equations, using the boundary condition, and inverting the results, Sneddon obtained the following solution for the axisymmetric punch problem (ref. 10). If the displacement under the punch is $w(r, 0)$, then

$$w(r, 0) = b + f(r/a) \tag{41}$$

where $0 \leq r \leq a$, b is as yet unspecified, and $f(r/a)$ is a prescribed function; also,

$$\sigma_z(r, 0) = 0 \tag{42}$$

where $r > a$.

Define the functions $\chi(t)$ and $\psi(\zeta)$ such that

$$\chi(t) = \frac{2b}{\pi} + \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{xf(x)}{\sqrt{t^2 - x^2}} dx \tag{43}$$

where $x = r/a$, and

$$\psi(\zeta) = \int_0^1 \chi(t)\cos(\zeta t)dt \tag{44}$$

Then the total P_0 is given as

$$P_0 = \frac{2\pi}{a(1 - \nu)} \int_0^\infty \zeta\psi(\zeta)d\zeta \int_0^a rJ_0(\zeta r/a)dr \tag{45}$$

The pressure directly under the punch is given as

$$\sigma_z(r, 0) = \frac{\mu}{a(1 - \nu)} \left[\frac{\chi(1)}{\sqrt{1 - x^2}} - \int_x^1 \frac{\chi'(t)}{t^2 - x^2} dt \right] \tag{46}$$

When the pressure under the punch is known, the problem may be solved by the method given in the first part of this section.

Flat-base cylindrical punch. - As an example of the type of punch whose surface area does not vary with load application, the punch in the form of a flat-base cylinder (fig. 16) will be considered. For the horizontal-base punch, $w_z(r, 0) = \epsilon$ where $0 \leq r \leq a$. By equations (41) and (42), $b = \epsilon$ and $f(r/a) = 0$. Therefore, by equation (43), it can be seen that $\chi(t) = \frac{2\epsilon}{\pi}$; and by equations (44) and (45), the total load on the punch is $P_0 = \frac{4\mu a\epsilon}{(1 - \nu)}$. From this, the equality $\epsilon = \frac{P_0(1 - \nu)}{4\mu a}$ is obtained. The pressure under the punch is given by equation (46) as

$$\sigma_z(r, 0) = \frac{2\mu\epsilon}{\pi(1 - \nu)\sqrt{a^2 - r^2}} \tag{47}$$

If the expression for ϵ is substituted into this last equation, the pressure under the punch will be obtained as a function of the applied load.

$$\sigma_z(r, 0) = \frac{P_0}{2\pi a\sqrt{a^2 - r^2}} \tag{48}$$

Conical-base punch. - An example of the punch problem in which the surface of contact varies is the conical-base punch, illustrated in figure 17. For this problem, $w_z(r, 0) = b + \epsilon - \frac{\epsilon r}{a}$ where $r < a$. Let $m = r/a$; then $w_z(r, 0) = b + \epsilon(1 - m)$, and therefore $f(m) = \epsilon(1 - m)$. The total load P_0 as a function of ϵ then becomes

$$P_0 = \frac{4\mu a\pi\epsilon}{(1 - \nu)} \tag{49}$$

and

$$b = -f(0) - \int \frac{f'(m)dm}{\sqrt{1 - m^2}} \tag{50}$$

$$b = \epsilon\left(\frac{\pi}{2} - 1\right) \tag{51}$$

$$\chi(t) = \frac{2b}{\pi} + \frac{2\epsilon}{\pi}\left(1 - \frac{\pi}{2}t\right) \tag{52}$$

$$\chi'(t) = -\epsilon \tag{53}$$

$$\sigma_z(r, 0) = -\frac{\mu\epsilon}{(1 - \nu)a} \cosh^{-1}(a/r) \tag{54}$$

where $0 < r < a$; or equivalently

$$\sigma_z(r, 0) = -\frac{P_0}{4a^2\pi} \cosh^{-1}(a/r) \tag{55}$$

where $0 < r < a$. Note the similarity between the curves for the pressure under the punches of the two-dimensional case and those of the three-dimensional axisymmetric case.

FIRST FUNDAMENTAL AND PUNCH PROBLEMS FOR AN ELASTIC LAYER RESTING ON A RIGID FOUNDATION

In this section, the solutions are sought for the first fundamental and the punch problems for an elastic layer resting on a rigid foundation. The basic configuration for this problem is shown in figure 18. The equations of the classical theory of elasticity (eqs. (32), (33), and (34)) must be satisfied by the solution for an elastic layer.

The solutions presented in this section were derived by the use of Hankel transforms. Two examples are presented for the elastic layer resting on a rigid foundation. The first example is that of a uniform pressure distributed over a circle, and the second example consists of a flat-ended cylindrical punch being forced into the free surface of the layer.

First Fundamental Problem for an Elastic Layer Resting on a Rigid Foundation

For the first fundamental problem, the Hankel transform is applied to the basic elasticity equation, the appropriate boundary conditions are used, and the resulting expressions are inverted to give the following solution for the elastic layer (ref. 21). If an elastic slab is resting on a rigid foundation with the surface loading axisymmetric about z equal to p(r), then the following equations may be used to solve the problem.

Let $m = r/b$, $p(r) = -2\mu f(r/b)$, $\sigma_{r\theta} = 0$, and let $\bar{f}_0(w)$ be the notation for the zeroth order Hankel transform of $f(r/b)$. Then

$$w_z(mb, b) = -4(1 - \nu)b \int_0^\infty \frac{\sinh^2 w\bar{f}_0(w)J_0(mw)}{2w + \sin\,(2w)}\,dw \tag{56a}$$

$$\sigma_z(mb, 0) = -4\mu \int_0^\infty \frac{\boxed{w\bar{f}_0(w)}\,[\sinh(w) + w\cosh(w)]\,J_0(mw)}{2w + \sinh(2w)}\,dw \tag{56b}$$

$$\left(\sigma_r + \sigma_\theta + \sigma_z\right)_{z=0} = -8(1 - \nu)\mu \int_0^\infty \frac{w\bar{f}_0(w)\sinh wJ_0(mw)}{2w + \sinh(2w)}\,dw \tag{56c}$$

$$\begin{aligned}\left(\sigma_r - \sigma_\theta\right)_{z=0} &= -4\mu \int_0^\infty \frac{w\bar{f}_0(w)[(1 - 2\nu)\sinh(w) - w\cosh(w)]\,J_0(mw)}{2w + \sinh(2w)}\,dw \\ &\quad + \frac{8\mu b}{r} \int_0^\infty \frac{\bar{f}_0(w)[(1 - 2\nu)\sinh(w) - w\cosh(w)]\,J_1(mw)}{2w + \sinh(2w)}\,dw\end{aligned} \tag{56d}$$

For the case of a load uniformly distributed over a circle (fig. 19), if $H(\omega) = 1$ when $\omega > 0$ and $H(\omega) = 0$ when $\omega < 0$ (where ω is any argument of H), then

$p(r) = -P/\pi a^2 H(a - r)$ and

$$p(r) = \sigma_z(r, b) = -\frac{PH(a - r)}{\pi a^2} = -2\mu f(r/a) \tag{57}$$

Let $m = r/b$; then $f(m) = \dfrac{PH(a/b - m)}{2\pi a^2 \mu}$ so that $\bar{f}_0(w) = \dfrac{P\tau J_1(\tau w)}{2\pi a^2 \mu w}$ where $\tau = a/b$. Substituting $\bar{f}_0(w)$ into the list of equations,

$$\sigma_r = \frac{2P\tau}{\mu \pi a^2}\left[F_1 - G_1 - F_2 + (1 - 2\nu)G_2\right] \tag{58a}$$

$$\sigma_\theta = -\frac{2P\tau}{\mu \pi a^2}\left[2\nu G_1 - F_2 + (1 - 2\nu)G_2\right] \tag{58b}$$

$$\sigma_z = -\frac{2P\tau\left(F_1 + G_1\right)}{\mu \pi a^2} \tag{58c}$$

where F_1, G_1, F_2, G_2 denote the integrals defined by the equations

$$F_1(m, \tau) = \int_0^\infty \frac{w \cosh(w) J_0(mw) J_1(\tau w)}{2w + \sinh(2w)}\, dw \tag{59a}$$

$$G_1(m, \tau) = \int_0^\infty \frac{\sinh w J_0(mw) J_1(\tau w)}{2w + \sinh(2w)}\, dw \tag{59b}$$

$$F_2(m, \tau) = \frac{1}{m}\int_0^\infty \frac{\cosh w J_1(mw) J_1(\tau w)}{2w + \sinh(2w)}\, dw \tag{59c}$$

$$G_2(m, \tau) = \frac{1}{m}\int_0^\infty \frac{\sinh w J_1(mw) J_1(\tau w)}{2w + \sinh(2w)(w)}\, dw \tag{59d}$$

Numerical values for equation (59) for various values of m and τ are given in reference 22. Data for Sneddon's solution for contact pressure between a layer and the rigid foundation are presented in figure 20.

Punch Problem for an Elastic Layer Resting on a Rigid Foundation

When $0 \leq r < a$, the equation $u_z(r,b) = -D - w_0(r/a)$ applies to the flat-ended cylindrical punch as shown in figure 21. The depth of penetration is $D = \epsilon$, $w_0(x) = 1$. It can then be shown that $\chi(t) = \frac{\epsilon}{\pi b(1 - \nu)} g(t)$ where $g(t)$ represents the solution of the Fredholm equation.

$$g(t) - \int_0^1 g(u)K(t,u)du = 1 \tag{60}$$

where $0 \leq t < 1$ and where $K(t,u) = \frac{2}{\pi}\int_0^{\infty} k(w)\cos(uw)\cos(tw)dw$ and $k(w) = \frac{2w + 1 - e^{-2w}}{2w + \sinh(2w)}$. In terms of the function $g(t)$, the total load on the punch is given as

$$P_0 = \frac{2\mu a^2\epsilon}{b(1 - \nu)} \int_0^1 g(t)dt \tag{61}$$

The integral equation (60) has been solved numerically by Lebedev and Uflyand (ref. 23). The ratio

$$\frac{P_0(1 - \nu)}{4a\mu\epsilon} = \frac{a}{2b} \int_0^1 g(t)dt \tag{62}$$

has been computed for a range of values of the ratio a/b; the results are given in the following table.

a/b	0	0.5	1.0	1.5	2.0
$\frac{P_0(1-\nu)}{4a\mu\epsilon}$	1.0	1.51	2.20	2.95	3.72

SOLUTION FOR THE PROBLEM OF A THICK LAYER RESTING ON AN ELASTIC (WINKLER TYPE) FOUNDATION

This section concerns the determination of the stresses and displacements in a thick layer that is resting on an elastic (Winkler type) foundation and whose surface is loaded by any known axisymmetric pressure distribution (fig. 22). This problem is of practical importance in the design of airport runways, building foundations, seals, concrete roads, or any arbitrarily thick layer resting on an elastic foundation. Because of the generality of this problem, its solution also may be applied to the design of layers resting on rigid foundations and to the design of very thick layers that approach the half space.

The method used in the solution of this problem is that of the Hankel transform which consists of transforming both the boundary conditions and the governing partial differential equations. The transformation of the partial differential equations reduces them to ordinary differential equations which, along with the transformed boundary conditions, may be solved by elementary means. Thus, the expressions for the transforms of the desired solution can be obtained, and by inversion of the transforms, the solution itself can be obtained.

Considering bending only, Hertz (ref. 19) solved the problem of a thin, infinitely extended plate resting on an elastic foundation. Sneddon (ref. 21) obtained the solution to the problem of a thick layer while considering all the components of stress, but he considered the layer to be resting on a rigid foundation. It is believed that the problem solved in this section is of a more general nature than either of the previously mentioned problems and, therefore, will reduce to Hertz's solution for small plate thicknesses and to Sneddon's solution for an infinitely stiff foundation modulus.

The Winkler type of foundation is one in which the contact pressure, which exists between a rigid punch and the foundation, is proportional to the surface displacement of the foundation $\sigma_z = Kw$. No confusion should arise between the Winkler foundation and the semispace foundation because the contact pressure for the latter is given in the complicated expression of equation (46).

The problem solution of a layer resting on a Winkler type foundation must satisfy the equations of classical elasticity (eqs. (32), (33), and (34)). For the case of symmetry, the displacement components are (u, 0, w), and the nonvanishing stress components are σ_r, σ_θ, σ_z, and σ_{rz}. The boundary conditions for this problem are

$\sigma_{rz} = 0$ and $\sigma_z = -p(r)$, where $z = +h$; and $\sigma_{rz} = 0$ and $\sigma_z = Kw$, where $z = -h$. The Navier equations reduce to the following forms.

$$2(1 - \nu)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + (1 - 2\nu)\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial r \partial z} = 0 \tag{63}$$

$$(1 - 2\nu)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + 2(1 - \nu)\frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = 0 \tag{64}$$

The first order Hankel transform of the radial component of displacement will be defined as

$$\bar{u}_1(\xi, z) = \int_0^\infty ru(r, z)J_1(\xi r)dr \tag{65}$$

Similarly, the zeroth order Hankel transform of the z component of displacement will be defined as

$$\bar{w}_0(\xi, z) = \int_0^\infty rw(r, z)J_0(\xi r)dr \tag{66}$$

By first multiplying both sides of equation (63) by $rJ_1(\xi r)$ and then integrating with respect to r from zero to infinity, and finally using the well-known properties (ref. 8) of Hankel transforms, the original equation becomes

$$\left[(1 - 2\nu)D^2 - 2(1 - \nu)\xi^2\right]\bar{u}_1 - \xi D\bar{w}_0 = 0 \tag{67}$$

where $D = \frac{\partial}{\partial z}$. Also, by multiplying equation (64) by $rJ_0(\xi r)$ and then integrating with respect to r from zero to infinity, the original equation becomes

$$\left[2(1 - \nu)D^2 - (1 - 2\nu)\xi^2\right]\bar{w}_0 + \xi D\bar{u}_1 = 0 \tag{68}$$

The solution to equations (67) and (68) will be taken in the following form.

$$\bar{u}_1 = A_1 \sinh(\xi z) + A_2 \cosh(\xi z) + A_3 z\xi \sinh(\xi z) + A_4 z\xi \cosh(\xi z) \tag{69}$$

$$\bar{w}_0 = B_1 \sinh(\xi z) + B_2 \cosh(\xi z) + B_3 z\xi \sinh(\xi z) + B_4 z\xi \cosh(\xi z) \tag{70}$$

After substituting these expressions into equations (67) and (68) and noting that for any z they must be true, the following relation is found among the constants:

$$A_4 = -B_3 \tag{71a}$$

$$A_3 = -B_4 \tag{71b}$$

$$A_2 = -B_1 + (-3 + 4\nu)B_4 \tag{71c}$$

$$A_1 = -B_2 + (-3 + 4\nu)B_3 \tag{71d}$$

Therefore, the transform of u and w may be expressed in terms of only the B's as

$$\bar{u}_1 = -B_2 + (-3 + 4\nu)B_3 \sinh(\xi z) + -B_1 + (-3 + 4\nu)B_4 \cosh(\xi z)$$

$$- B_4 z\xi \sinh(\xi z) - B_3 z\xi \cosh(\xi z) \tag{72}$$

$$\bar{w}_0 = B_1 \sinh(\xi z) + B_2 \cosh(\xi z) + B_3 z\xi \sinh(\xi z) + B_4 z\xi \cosh(\xi z) \tag{73}$$

The shear stress σ_{rz} expressed in terms of displacement is $\sigma_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)$. The first order Hankel transform of σ_{rz} yields

$$\bar{\sigma}_{rz} = \int_0^\infty r\sigma_{rz} J_1(\xi r)dr = \mu\left(D\bar{u}_1 - \xi\bar{w}_0\right) \tag{74}$$

Therefore,

$$\bar{\sigma}_{rz} = \mu\xi\Big\{-\Big[(3 - 4\nu)B_3 + B_2\Big]\cosh(\xi z) - \Big[(3 - 4\nu)B_4 + B_1\Big]\sinh(\xi z)$$
$$- B_4[\sinh(\xi z) + z\xi\cosh(\xi z)] - B_3[\cosh(\xi z) + z\xi\sinh(\xi z)]$$
$$- B_1\sinh(\xi z) - B_2\cosh(\xi z) - B_3 z\xi\sinh(\xi z) - B_4 z\xi\cosh(\xi z)\Big\} \quad (75)$$

The boundary condition on σ_{rz} at $z = \pm h$ is that $\sigma_{rz} = 0$ and, therefore, $\bar{\sigma}_{rz} = 0$ also. The following equations are obtained by using the latter boundary condition and equation (75).

$$\Big[4(1 - \nu)B_3 + 2B_2\Big]\cosh(\xi h) + \Big[4(1 - \nu)B_4 + 2B_1\Big]\sinh(\xi h)$$
$$+ 2B_4 h\xi\cosh(\xi h) + 2B_3\xi h\sinh(\xi h) = 0 \quad (76)$$

$$\Big[4(1 - \nu)B_3 + 2B_2\Big]\cosh(\xi h) - \Big[4(1 - \nu)B_4 + 2B_1\Big]\sinh(\xi h)$$
$$- 2B_4 h\xi\cosh(\xi h) + 2B_3\xi h\sinh(\xi h) = 0 \quad (77)$$

Addition of equations (76) and (77) yields

$$\Big[2(1 - \nu)B_3 + B_2\Big]\cosh(\xi h) + B_3\xi h\sinh(\xi h) = 0 \quad (78)$$

Subtracting equation (77) from equation (76) results in the following equation.

$$\Big[2(1 - \nu)B_4 + B_1\Big]\sinh(\xi h) + B_4\xi h\cosh(\xi h) = 0 \quad (79)$$

From equations (78) and (79), the following relationships among the B's can be obtained.

$$B_2 = -B_3[2(1 - \nu) + \xi h\tanh(\xi h)] \quad (80)$$

$$B_1 = -B_4[2(1 - \nu) + \xi h\coth(\xi h)] \quad (81)$$

The expressions for $\bar{u}_1$ and $\bar{w}_0$ may be written in terms of B_3 and B_4 as follows.

$$\bar{u}_1 = [2\nu - 1 + \xi h \tanh(\xi h)]B_3 \sinh(\xi z) + [2\nu - 1 + \xi h \coth(\xi h)]B_4 \cosh(\xi z)$$
$$-B_4 \xi z \sinh(\xi z) - B_3 \xi z \cosh(\xi z) \tag{82}$$

$$\bar{w}_0 = -[2(1 - \nu) + \xi h \coth(\xi h)]B_4 \sinh(\xi z) - [2(1 - \nu) + \xi h \tanh(\xi h)]B_3 \cosh(\xi z)$$
$$+ B_3 \xi z \sinh(\xi z) + B_4 \xi z \cosh(\xi z) \tag{83}$$

Also,

$$\sigma_z = \frac{2\mu(1 - \nu)\partial w}{(1 - 2\nu)\partial z} + \frac{2\mu\nu}{1 - 2\nu}\left(\frac{\partial u}{\partial r} + \frac{1}{r}\right)u \tag{84}$$

Therefore,

$$\bar{\sigma}_{z_0} = \frac{2\mu\left[(1 - \nu)\bar{w}_0' + \xi\nu\bar{u}_1\right]}{1 - 2\nu} \tag{85}$$

When $\bar{u}_1$ and $\bar{w}_0$ are substituted into equation (85), $\bar{\sigma}_{z_0}$ becomes

$$\bar{\sigma}_{z_0} = -2\mu\xi\Big\{[1 + \xi h \coth(\xi h)]B_4 \cosh(\xi z) + [1 + \xi h \tanh(\xi h)]B_3 \sinh(\xi z)$$
$$-\xi B_3 z \cosh(\xi z) - \xi B_4 z \sinh(\xi z)\Big\} \tag{86}$$

The boundary conditions on σ_z are as follows: where $z = +h$, $\sigma_z = -p(r)$ and, therefore, $\bar{\sigma}_{z_0} = -\bar{P}_0$; and where $z = -h$, $\sigma_z = Kw$ and, therefore, $\bar{\sigma}_{z_0} = K\bar{w}_0$. Here, $\bar{P}_0$ is the zeroth order Hankel transform of $p(r)$. By applying the last expression for $\bar{\sigma}_{z_0}$

(eq. (86)) to the boundary conditions, the following two equations are obtained.

$$\overline{P}_0 = 2\mu\xi\Big\{[1 + \xi h \coth(\xi h)]\cosh(\xi h)B_4 + [1 + \xi h \tanh(\xi h)]\sinh(\xi h)B_3$$
$$- \xi h \cosh(\xi h)B_3 - \xi h \sinh(\xi h)B_4\Big\} \tag{87}$$

$$K\Big\{[2(1 - \nu) + \xi h \coth(\xi h)]\sinh(\xi h)B_4 - [2(1 - \nu) + \xi h \tanh(\xi h)]$$
$$\times \cosh(\xi h)B_3 + \xi h \sinh(\xi h)B_3 - \xi h \cosh(\xi h)B_4\Big\} =$$
$$-2\mu\xi\Big\{[1 + \xi h \coth(\xi h)]B_4 - [1 + \xi h \tanh(\xi h)]\sinh(\xi h)B_3$$
$$+ \xi h \cosh(\xi h)B_3 - \xi h \sinh(\xi h)B_4\Big\} \tag{88}$$

Equations (87) and (88) are two equations which can be solved simultaneously so that the constants B_3 and B_4 may be determined as

$$B_3 = \frac{-\overline{P}_0}{\mu\xi}\left\{\frac{\{2K(1 - \nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]\}\cosh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]}\right\} \tag{89}$$

$$B_4 = \frac{-\overline{P}_0}{\mu\xi}\left\{\frac{\{2K(1 - \nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]\}\sinh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]}\right\} \tag{90}$$

From the previous expressions for the relation among the B's, values for B_2 and B_3 are determined as

$$B_2 = \frac{\overline{P}_0}{\mu\xi}\left\{\frac{2K(1 - \nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]}\right.$$
$$\left.\times [2(1 - \nu)\cosh(\xi h) + \xi h \sinh(\xi h)]\right\} \tag{91}$$

$$B_1 = \frac{\overline{P}_0}{\mu\xi}\left\{\frac{2K(1-\nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]}\right.$$

$$\left. \times [2(1-\nu)\sinh(\xi h) + \xi h\cosh(\xi h)]\right\} \qquad (92)$$

Recalling the relation among the A's and the B's, the following expressions are obtained for the A's.

$$A_1 = \frac{-\overline{P}_0}{\mu\xi}\left\{\frac{\{2K(1-\nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]\}\cosh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]}\right.$$

$$\left. \times [2\nu - 1 + \xi h\tanh(\xi h)]\right\} \qquad (93)$$

$$A_2 = \frac{-\overline{P}_0}{\mu\xi}\left\{\frac{\{2K(1-\nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]\}\sinh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]}\right.$$

$$\left. \times [2\nu - 1 + \xi h\coth(\xi h)]\right\} \qquad (94)$$

$$A_3 = \frac{\overline{P}_0}{\mu\xi}\left\{\frac{2K(1-\nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]}\right\}\sinh(\xi h) \qquad (95)$$

$$A_4 = \frac{\overline{P}_0}{\mu\xi}\left\{\frac{2K(1-\nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]}\right\}\cosh(\xi h) \qquad (96)$$

By substituting equations (89) to (96) into equations (82), (83), (84), and (86), the following expressions are obtained for $\bar{u}_1$, $\bar{w}_0$, $\bar{\sigma}_{z_0}$, and $\bar{\sigma}_{rz_1}$.

$$\bar{u}_1 = -\{[2\nu - 1 + \xi h \tanh(\xi h)]\sinh(\xi z) - \xi z \cosh(\xi z)\}$$

$$\times \frac{\bar{P}_0}{\mu\xi}\left\{\frac{\{2K(1-\nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]\}\cosh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]}\right\}$$

$$-\{[2\nu - 1 + \xi h \coth(\xi h)]\cosh(\xi z) - \xi z \sinh(\xi z)\}$$

$$\times \frac{\bar{P}_0}{\mu\xi}\left\{\frac{\{2K(1-\nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]\}\sinh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]}\right\} \tag{97}$$

$$\bar{w}_0 = -\{-[2(1-\nu) + \xi h \coth(\xi h)]\sinh(\xi z) + \xi z \cosh(\xi z)\}$$

$$\times \frac{\bar{P}_0}{\mu\xi}\left\{\frac{\{2K(1-\nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]\}\sinh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]}\right\}$$

$$-\{-[2(1-\nu) + \xi h \tanh(\xi h)]\cosh(\xi z) + \xi z \sinh(\xi z)\}$$

$$\times \frac{\bar{P}_0}{\mu\xi}\left\{\frac{\{2K(1-\nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]\}\cosh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]}\right\} \tag{98}$$

$$\bar{\sigma}_{z_0} = 2\{[1 + \xi h \coth(\xi h)]\cosh(\xi z) - \xi z \sinh(\xi z)\}$$

$$\times \frac{\bar{P}_0\{2K(1-\nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]\}\sinh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]}$$

$$+ \{[1 + \xi h \tanh(\xi h)]\sinh(\xi z) - \xi z \cosh(\xi z)\}$$

$$\times \frac{\bar{P}_0\{2K(1-\nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]\}\cosh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]} \tag{99}$$

$$\bar{\sigma}_{rz_1} = -\{2\xi[h \cosh(\xi z)\tanh(\xi h) - z \sinh(\xi z)]\}$$

$$\times \frac{\bar{P}_0\left\{2K(1 - \nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]\right\}\cosh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]}$$

$$+ \{2\xi[h \coth(\xi h)\sinh(\xi z) - z \cosh(\xi z)]\}$$

$$\times \frac{\bar{P}_0\left\{2K(1 - \nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]\right\}\sinh(\xi h)}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]} \tag{100}$$

The preceding expressions for the Hankel transforms of u, w, σ_z, and σ_{rz} only need be inverted to obtain the desired solutions. The Hankel inversion formula is

$$f(r, z) = \int_0^\infty \bar{f}_\nu(\xi, z)J_\nu(\xi r)\xi \, d\xi \tag{101}$$

where $0 < r < \infty$. Therefore,

$$u(r, z) = \int_0^\infty \bar{u}_1 J_1(\xi r)\xi \, d\xi \tag{102}$$

$$w(r, z) = \int_0^\infty \bar{w}_0 J_0(\xi r)\xi \, d\xi \tag{103}$$

$$\sigma_z = \int_0^\infty \bar{\sigma}_{z_0} J_0(\xi r)\xi \, d\xi \tag{104}$$

$$\sigma_{rz} = \int_0^\infty \bar{\sigma}_{rz_1} J_1(\xi r)\xi \, d\xi \tag{105}$$

Substituting equations (97) to (100) into the appropriate inversion formula yields the desired solution in terms of any prescribed surface pressure $p(r)$.

$$u(r, z) = -\int_0^\infty \{[2\nu - 1 + \xi h \tanh(\xi h)] \sinh(\xi z) - \xi z \cosh(\xi z)\}$$

$$\times \frac{J_1(\xi r)\overline{P}_0\left\{2K(1 - \nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]\right\}\cosh(\xi h)d\xi}{\mu\left\{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]\right\}}$$

$$- \int_0^\infty \{[2\nu - 1 + \xi h \coth(\xi h)] \cosh(\xi z) - \xi z \sinh(\xi z)\}$$

$$\times \frac{J_1(\xi r)\overline{P}_0\left\{2K(1 - \nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]\right\}\sinh(\xi h)d\xi}{\mu\left\{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]\right\}} \tag{106}$$

$$w(r, z) = -\int_0^\infty \{-[2(1 - \nu) + \xi h \coth(\xi h)] \sinh(\xi z) + \xi z \cosh(\xi z)\}$$

$$\times \frac{J_0(\xi r)\overline{P}_0\left\{2K(1 - \nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]\right\}\sinh(\xi h)d\xi}{\mu\left\{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]\right\}}$$

$$- \int_0^\infty \{-[2(1 - \nu) + \xi h \tanh(\xi h)] \cosh(\xi z) + \xi z \sinh(\xi z)\}$$

$$\times \frac{J_0(\xi r)\overline{P}_0\left\{2K(1 - \nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]\right\}\cosh(\xi h)d\xi}{\mu\left\{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]\right\}} \tag{107}$$

$$\sigma_z(r, z) = 2\int_0^\infty \{[1 + \xi h \coth(\xi h)] \cosh(\xi z) - \xi z \sinh(\xi z)\}$$

$$\times \frac{J_0(\xi r)\overline{P}_0\xi\left\{2K(1 - \nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]\right\}\sinh(\xi h)d\xi}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]}$$

$$+ 2\int_0^\infty \{[1 + \xi h \tanh(\xi h)] \sinh(\xi z) - \xi z \cosh(\xi z)\}$$

$$\times \frac{J_0(\xi r)\overline{P}_0\xi\left\{2K(1 - \nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]\right\}\cosh(\xi h)d\xi}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]} \tag{108}$$

$$\sigma_{rz}(r, z) = -\int_0^{\infty} [2\xi h \cosh(\xi z)\tanh(\xi h) - 2\xi z \sinh(\xi z)]$$

$$\times \frac{J_1(\xi r)\xi\overline{P}_0\left\{2K(1-\nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]\right\}\cosh(\xi h)d\xi}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]}$$

$$-\int_0^{\infty} [2\xi h \coth(\xi h)\sinh(\xi z) - 2\xi z \cosh(\xi z)]$$

$$\times \frac{J_1(\xi r)\xi\overline{P}_0\left\{2K(1-\nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]\right\}\sinh(\xi h)d\xi}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1-\nu)[4\xi h + \sinh(4\xi h)]} \qquad (109)$$

NUMERICAL RESULTS ON THE CALCULATION OF CONTACT STRESS BETWEEN A LAYER AND ITS ELASTIC FOUNDATION

In the preceding section, equations were derived for displacements and stresses in an elastic layer resting on an elastic foundation and subject to any prescribed axisymmetric surface pressure. A comparison will be made between these equations and those derived by Sneddon (ref. 21) for a layer resting on a rigid foundation. The equations derived in the last section will be used to determine the distribution of contact pressure between a layer and its foundation for the case of the surface pressure uniformly distributed over a circle. The equations derived in the last section involve infinite integrals which, for the purpose of plotting the results, were numerically integrated. Numerous examples are solved for different foundation moduli and different circle radii. The results are illustrated in figure 23.

The following equation was presented by Sneddon (ref. 21) for the contact stress between an elastic layer and its rigid foundation.

$$\sigma_z = -\frac{2}{d^2}\int_0^{\infty} \frac{[\eta \cosh(\eta) + \sinh(\eta)]\eta\overline{P}(\eta/d)J_0(\rho\eta)d\eta}{2\eta + \sinh(2\eta)} \qquad (110)$$

where d is the layer thickness and $\rho = r/d$. The equation for the stress in the z-direction for a thick layer resting on an elastic foundation was solved in the preceding section and was given in equation (108). The following expression can be obtained for $\lim_{K\to\infty} \sigma_z(r, z)$ by dividing both the numerator and the denominator by $K(1 - \nu)$ and taking the limit as K approaches infinity.

$$\lim_{K\to\infty} \sigma_z(r, z) = -4\int_0^{\infty} J_0(\xi r)\overline{P}_0\xi\left\{\frac{\{[1 + \xi h \coth(\xi h)]\cosh(\xi z) - \xi z \sinh(\xi z)\}}{4\xi h + \sinh(4\xi h)}\right.$$

$$\times \cosh^2(\xi h)\sinh(\xi h)$$

$$\left. + \frac{\{[1 + \xi h \tanh(\xi h)]\sinh(\xi z) - \xi z \cosh(\xi z)\}\sinh^2(\xi h)\cosh(\xi h)}{4\xi h + \sinh(4\xi h)}\right\} d\xi \tag{111}$$

The equation for contact stress for $z = -h$ is given by the preceding equation.

$$\lim_{K\to\infty} \sigma_z(r, -h) = -2\int_0^{\infty} J_0(\xi r)\overline{P}_0\xi\left[\frac{\sinh(2\xi h) + (2\xi h)\cosh(2\xi h)}{4\xi h + \sinh(4\xi h)}\right] d\xi \tag{112}$$

Let $\eta = 2\xi h$; therefore, $\xi = \eta/2h$ and $d\xi = d\eta/2h$. Also, let $\rho = r/2h$ and $\propto = a/2h$. Then

$$\lim_{K\to\infty} \sigma_z(r, -h) = -2\int_0^{\infty} J_0\left(\frac{\eta r}{2h}\right)\frac{\overline{P}_0\eta}{2h}\left[\frac{\sinh(\eta) + \eta \cosh(\eta)}{2\eta + \sinh(2\eta)}\right]\frac{d\eta}{2h} \tag{113}$$

$$\lim_{K\to\infty} \sigma_z(r, -h) = -\frac{1}{2h^2}\int_0^{\infty} J_0\left(\frac{\eta r}{2h}\right)\overline{P}_0\eta\left[\frac{\sinh(\eta) + \eta \cosh(\eta)}{2\eta + \sinh(2\eta)}\right] d\eta \tag{114}$$

Because h is only half the layer thickness $[h = (1/2)d]$, the following equation is obtained in terms of d.

$$\lim_{K\to\infty} \sigma_z(r, -h) = -\frac{2}{d^2}\int_0^{\infty} J_0(\rho\eta)\overline{P}_0(\eta/d)\eta\left[\frac{\sinh(\eta) + \eta \cosh(\eta)}{2\eta + \sinh(2\eta)}\right] d\eta \tag{115}$$

This solution is the same as the one obtained by Sneddon.

Derivation of the Contact Equation

In this section, curves for the contact pressure between the layer and its elastic foundation will be given for a surface loading of a uniform pressure distributed over a circle (fig. 23). The following material presents a step-by-step derivation of the contact pressure equation to be numerically integrated.

By using the expression for $\sigma_z(r, z)$ in equation (108) and evaluating it for $z = -h$, the following equation is obtained.

$$\sigma_z(r, -h) = 2\int_0^\infty J_0(\xi r)\overline{P}_0\xi \left\{ \frac{[\sinh(\xi h)\cosh(\xi h) + \xi h]\{2K(1 - \nu)\cosh^2(\xi h) + \mu\xi[\sinh(2\xi h) - 2\xi h]\}}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]} + \frac{[-\sinh(\xi h)\cosh(\xi h) + \xi h]\{2K(1 - \nu)\sinh^2(\xi h) + \mu\xi[\sinh(2\xi h) + 2\xi h]\}}{2\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[4\xi h + \sinh(4\xi h)]} \right\} d\xi \tag{116}$$

When terms are collected and the hyperbolic identities are used, equation (116) reduces to

$$\sigma_z(r, -h) = \int_0^\infty \frac{J_0(\xi r)\overline{P}_0\xi\{K(1 - \nu)[\sinh(2\xi h) + 2\xi h \cosh(2\xi h)]\}d\xi}{\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[2\xi h + \sinh(2\xi h)\cosh(2\xi h)]} \tag{117}$$

If $p(r)$ is described as $p(r) = p$ where $0 < r < a$ and $p(r) = 0$ where $r > a$, then $\overline{P}_0 = \frac{a}{\xi} J_1(a\xi)p$. By inserting this into equation (117), the following expression for $\sigma_z(r, -h)$ is obtained.

$$\sigma_z(r, -h) = ap\int_0^\infty \frac{J_0(\xi r)J_1(a\xi)\{K(1 - \nu)[\sinh(2\xi h) + 2\xi h \cosh(2\xi h)]\}d\xi}{\mu\xi\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - K(1 - \nu)[2\xi h + \sinh(2\xi h)\cosh(2\xi h)]} \tag{118}$$

Divide the numerator and the denominator by $K(1 - \nu)$ so that

$$\sigma_z(r, -h) = ap\int_0^{\infty} \frac{J_0(\xi r)J_1(a\xi)[\sinh(2\xi h) + 2\xi h \cosh(2\xi h)]\,d\xi}{\frac{\mu\xi}{K(1 - \nu)}\left[(2\xi h)^2 - \sinh^2(2\xi h)\right] - [2\xi h + \sinh(2\xi h)\cosh(2\xi h)]} \tag{119}$$

When the following substitutions $\rho = r/2h$, $\alpha = a/2h$, and $\eta = 2h\xi$ are made, equation (115) may be written as

$$\sigma_z(r, -h) = \frac{ap}{2h}\int_0^{\infty} \frac{J_0(\rho\eta)J_1(\alpha\eta)[\sinh(\eta) + \eta \cosh(\eta)]\,d\eta}{\frac{\mu\eta}{2hK(1 - \nu)}\left[(\eta)^2 - \sinh^2(\eta)\right] - [\eta + \sinh(\eta)\cosh(\eta)]} \tag{120}$$

Let $G = \frac{\mu}{2hK(1 - \nu)}$ and $p = F/\pi a^2$; therefore,

$$\sigma_z(r, -h) = \frac{F}{4\pi h^2 \alpha}\int_0^{\infty} \frac{J_0(\rho\eta)J_1(\alpha\eta)[\sinh(\eta) + \eta \cosh(\eta)]\,d\eta}{G\eta\left[\eta^2 - \sinh^2(\eta)\right] - [\eta + \sinh(\eta)\cosh(\eta)]} \tag{121}$$

The dimensional quantity $\left(\frac{\sigma_z \pi h^2}{F}\right)_{r,-h}$ has been plotted in figure 23 for various values of α, ρ, and G.

Discussion of Analytical Results

The curves in figure 23 show the distribution of contact stress σ_z at $z = -h$ for various foundation moduli and circle radii. Curves for σ_z, σ_r, σ_{rz}, u, and w could have been plotted for any $|z| \leq h$. Because $w = \sigma_z/K$ at the contact surface, the curves also represent the deflection in the z-direction and, therefore, permit the visualization of the deflection at that surface.

Because equation (121) is rather complicated to evaluate, it was necessary to use numerical integration techniques. The equation was evaluated on the Univac 1108 computer by the use of the Gaussian integration formula. Because the limits of integration are zero to infinity, it is necessary to perform the integration repeatedly with limits of zero to 10n for $n = 1, 2, 3, \ldots,$ until an increase in n would cause less than 0.00001 increase in the accuracy when compared to the last integration. Each calculated point of each curve $a = 0$, $a = h$, $a = 2h$ for each value of K is computed in this way.

A few statements must be made about the curves in general. Note that each curve crosses the $r/2h$ axis. This means that the elastic foundation is actually put in tension at certain places and that the displacement in the z-direction actually becomes positive. This fact is not shown in the curves presented by Sneddon (ref. 21) for a rigid foundation. As a further check on the validity of the solution, any of the curves presented in this section can be used to show that the total reaction on the bottom layer is equal to the total applied load F; this must be the case for the true solution.

A few statements should be made about particular curves. For either a soft foundation or for a thin plate (fig. 23(a)), the contact stress is almost independent of the area of load distribution. For slightly larger foundation moduli (figs. 23(b) and 23(c)), the contact stress is affected by the circle radius only in the near vicinity of the load. For higher ratios of $2h(1 - \nu)K/\mu$, the contact stress is significantly affected at all points along the radius. By distributing the surface load over a larger area, the contact stress decreases in intensity but also is spread over a larger area, as shown by the curves. As the ratio $2h(1 - \nu)K/\mu$ increases, these curves approach the curves for a layer resting on a rigid foundation (ref. 21). When $K = \infty$, the curves correspond exactly to those presented by Sneddon (ref. 21) for a layer on a rigid foundation.

When the numerical values for the curve $2h(1 - \nu)/\mu = 1000$ and those for $K = \infty$ are studied, there appears to be only a 0.24-percent difference. Therefore, it can be concluded that if the ratio $2h(1 - \nu)K/\mu$ is greater than 1000, an error of no more than 0.24 percent will be encountered by the use of Sneddon's solution for a rigid foundation.

CONCLUSION

Elementary problems that involve contact problems in elasticity have been presented, primarily to clarify the term "contact problem." In addition, equations have been derived for computation of the stresses and displacements in a two-dimensional half plane, in a three-dimensional half space, and in a layer resting on a rigid foundation.

A method for determination of the stresses and displacements in a thick layer that is surface loaded and that rests on an elastic foundation has been developed in a step-by-step manner. This has provided not only exposure to the procedure used to derive the equations but also the equations to be used for computational purposes. Then, these equations were used to compute values for the special case of uniform pressure distribution over a circle, and the resultant curves were plotted. The following observations were made from the curves: (1) the elastic foundation is actually placed in tension at various points on the contact surface, (2) for a soft foundation, the contact pressure is almost independent of the area of load distribution, (3) for a larger foundation modulus, the contact pressure is affected by the area of load distribution only in the vicinity of the load, and (4) for a very large foundation modulus, the contact stress approaches that of a layer on a rigid foundation.

Two topics are suggested for future work involving variations of the problem of a layer resting on an elastic foundation. These topics are the indentation of a layer with a rigid punch and the presence of a variable foundation modulus.

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NASA-S-69-150

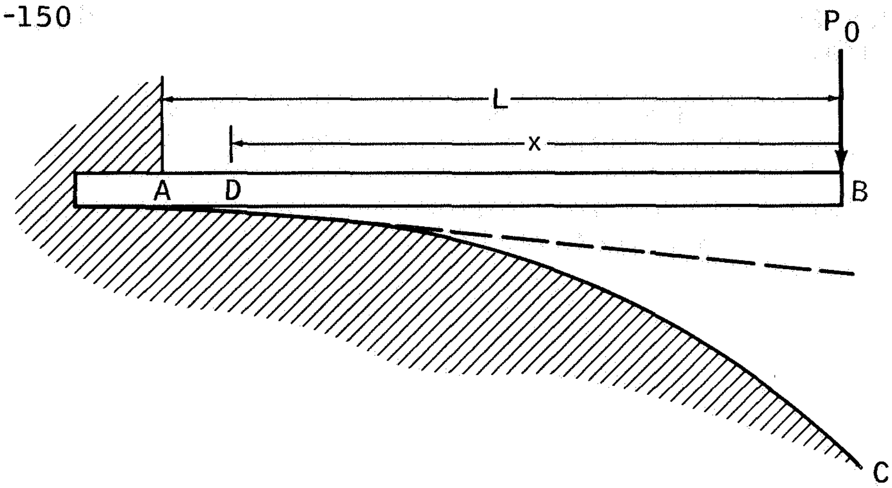


Figure 1. - Nonlinear bending of a cantilever beam.

NASA-S-69-142

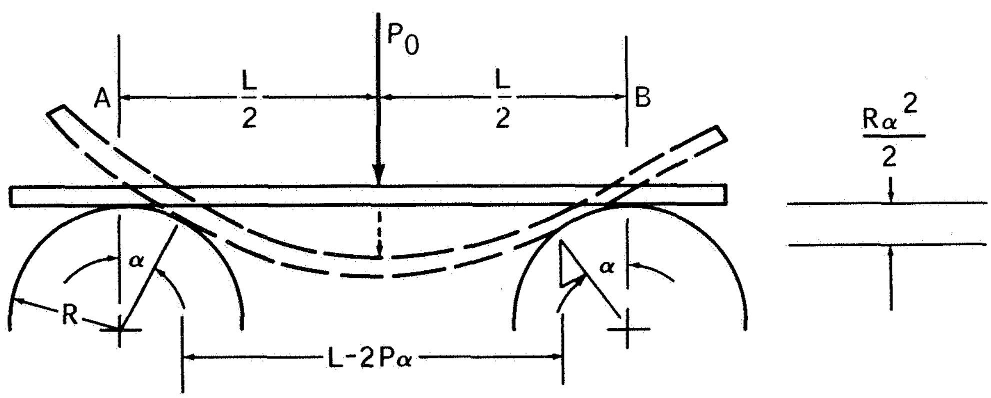


Figure 2. - Nonlinear bending of a simply supported beam.

NASA-S-69-151

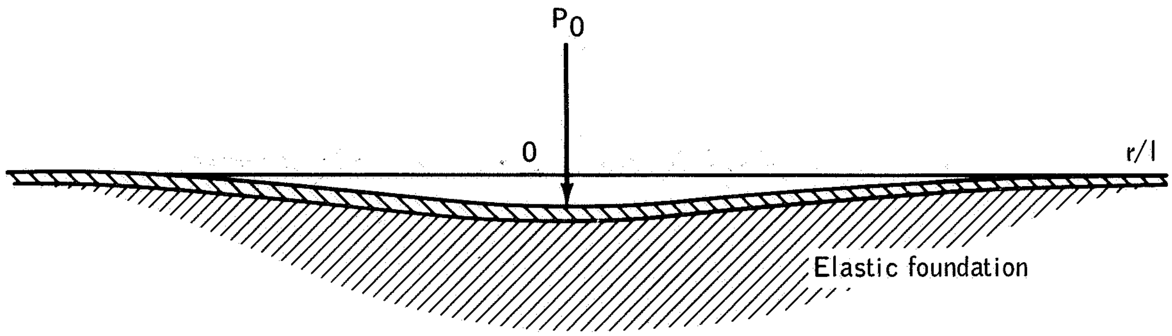


Figure 3. - Thin plate resting on an elastic foundation.

NASA-S-69-154

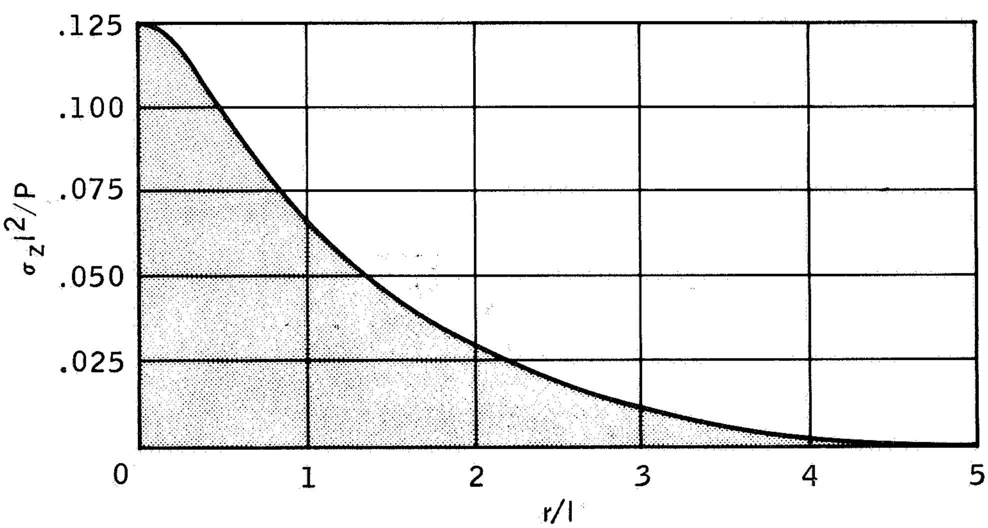


Figure 4. - Pressure on bottom of thin plate resting on an elastic foundation and subjected to a concentrated load P.

NASA-S-69-152

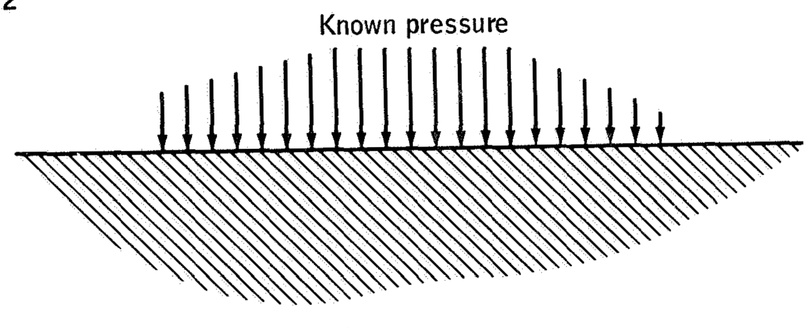


Figure 5. - First fundamental problem.

NASA-S-69-163

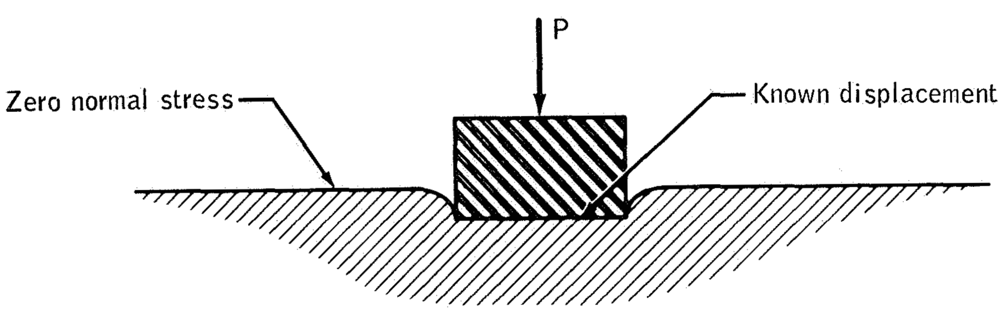


Figure 6. - Punch problem.

NASA-S-69-161

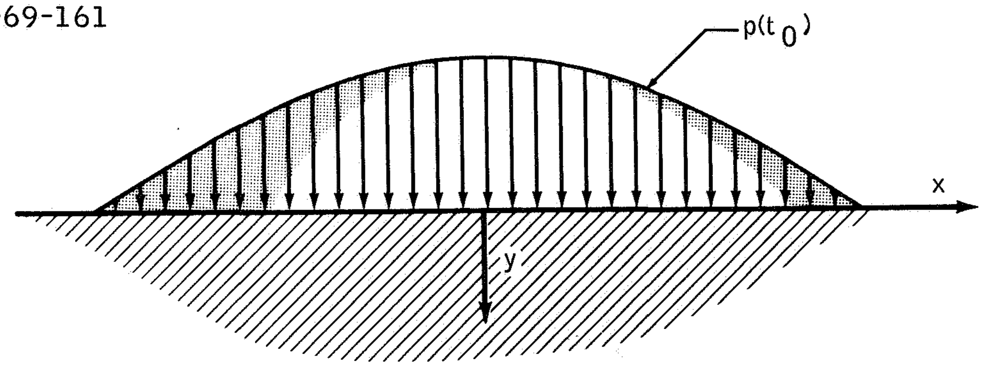


Figure 7. - Half plane loaded by a surface pressure.

NASA-S-69-141

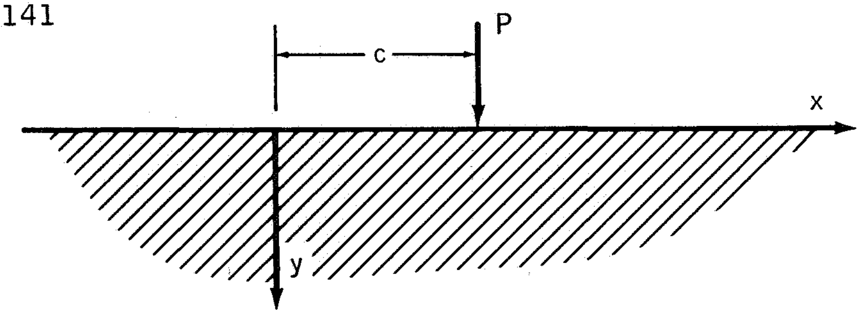


Figure 8. - Half plane loaded by a concentrated normal force.

NASA-S-69-153

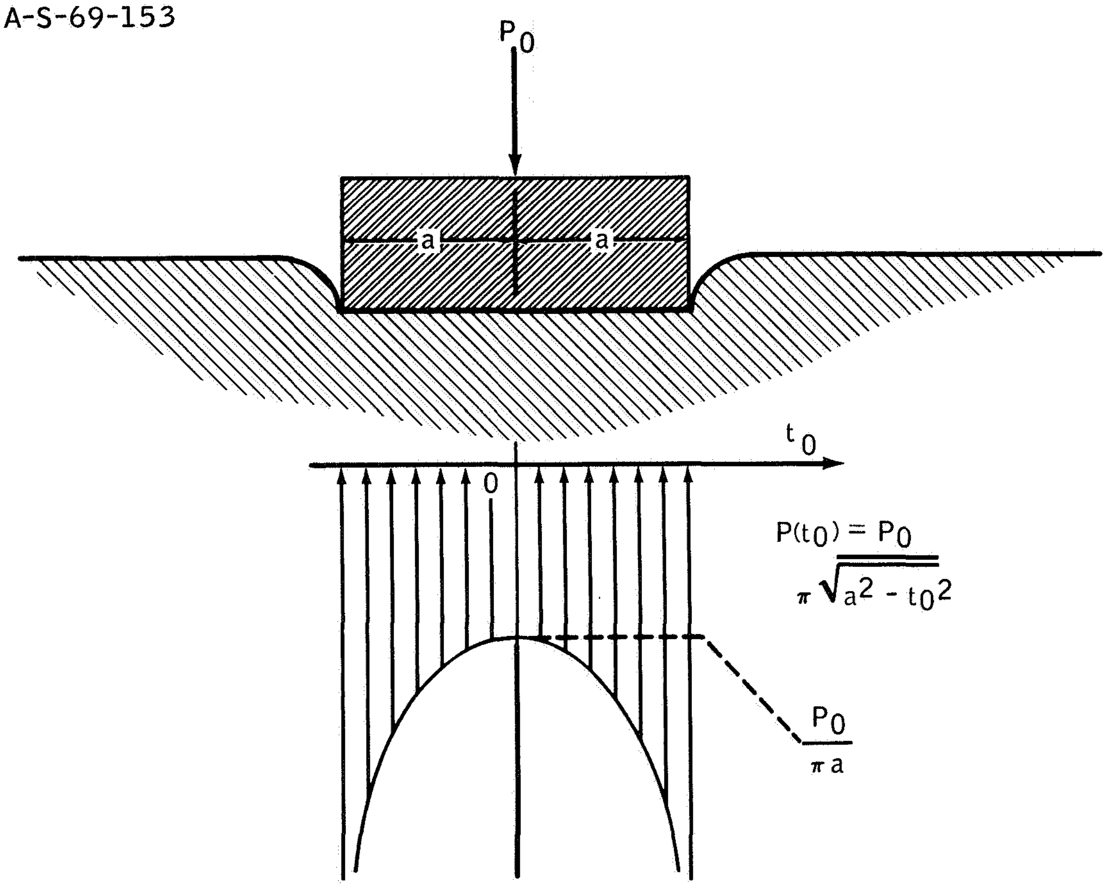


Figure 9. - Contact pressure between half plane and flat bottom punch.

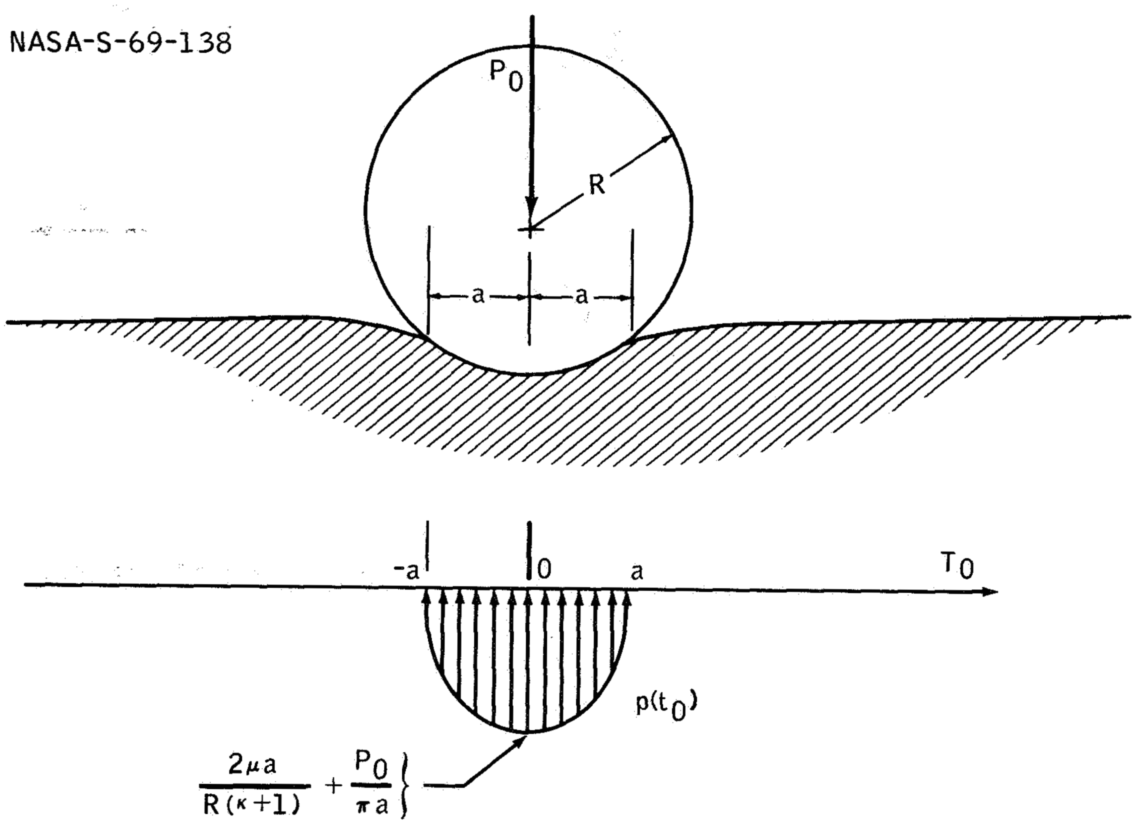


Figure 10. - Contact pressure between half plane and a cylinder.

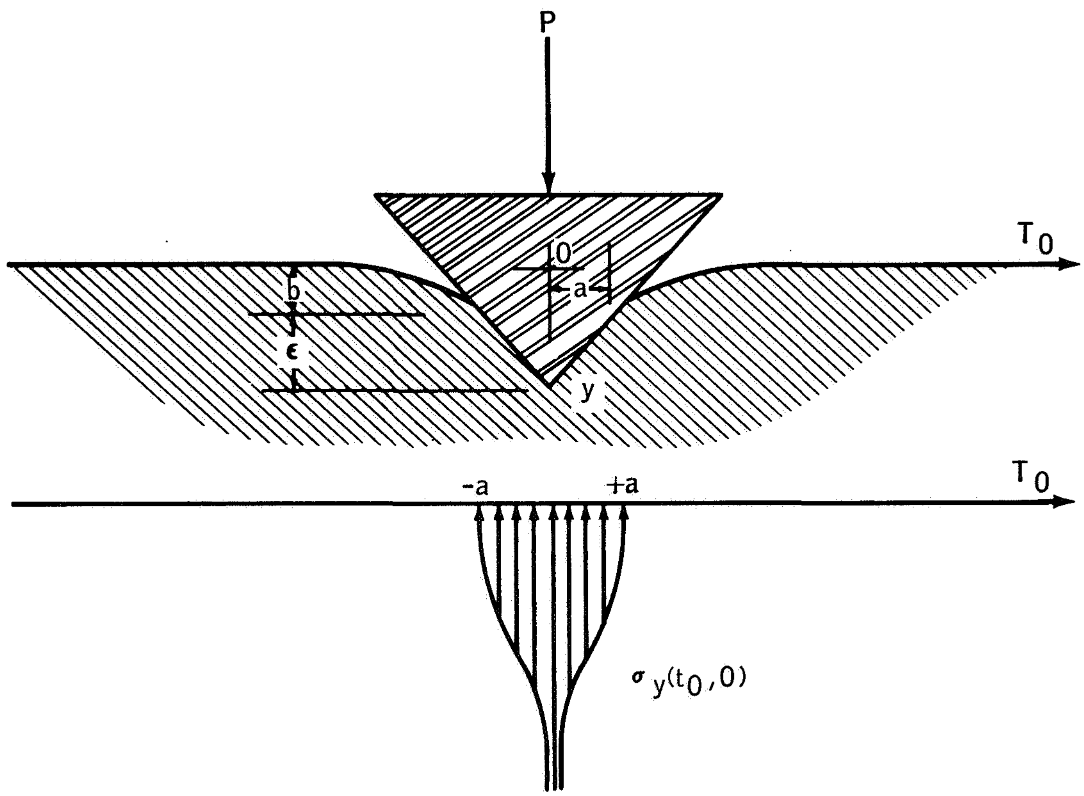


Figure 11.- Contact pressure between a half plane and a wedge-shaped punch.

NASA-S-69-160

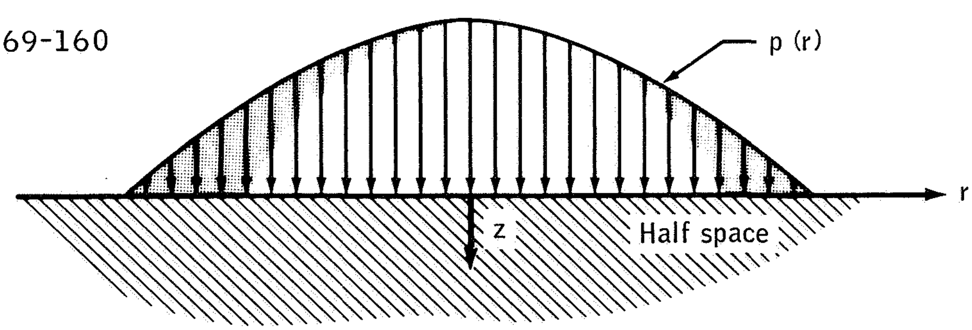


Figure 12.- Half space loaded by a surface pressure.

NASA-S-69-156

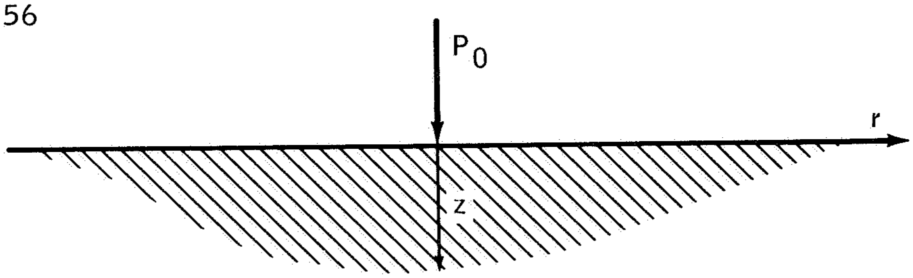


Figure 13. - Half space loaded by a concentrated normal force.

NASA-S-69-164

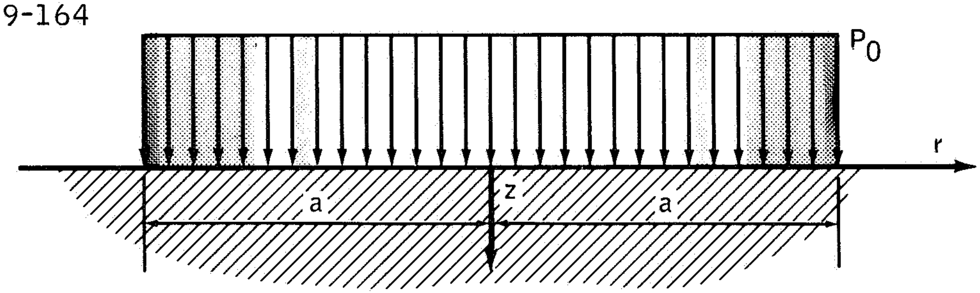


Figure 14. - Half space loaded by a pressure uniformly distributed over a circle.

NASA-S-69-157

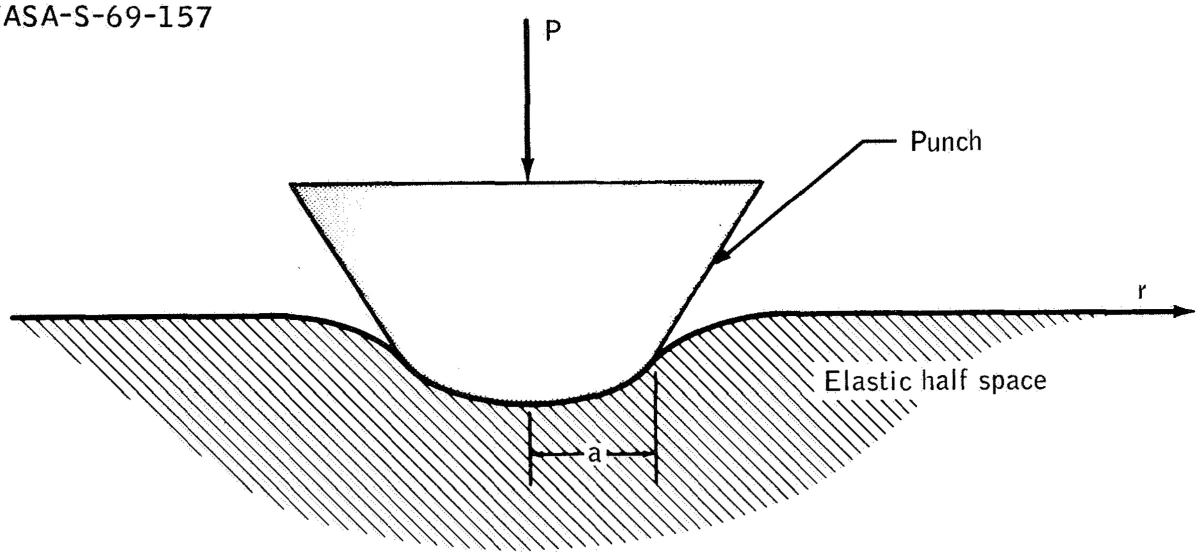


Figure 15. - Half space deformed by a punch.

NASA-S-69-149

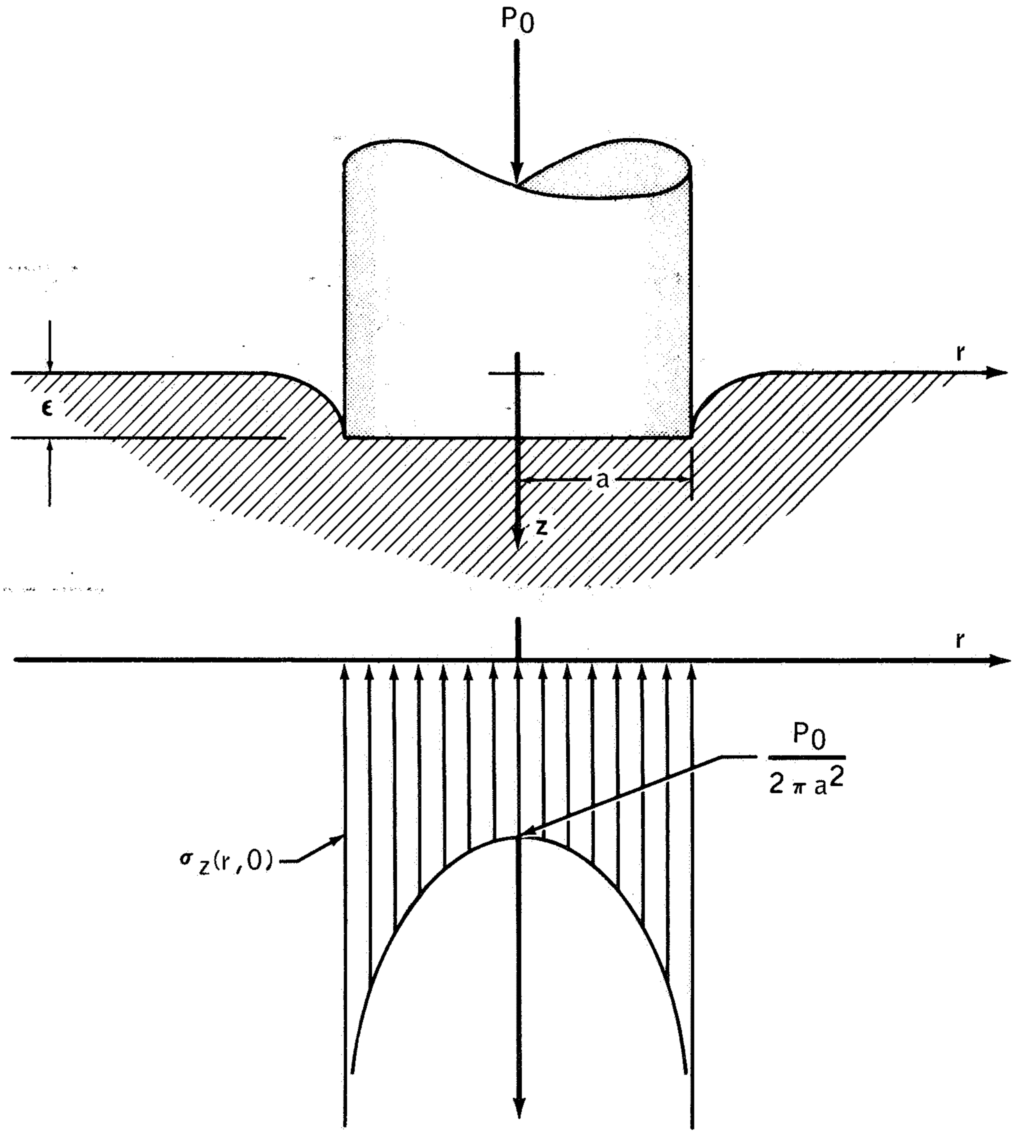


Figure 16. - Contact pressure between a half space and a flat-base cylindrical punch.

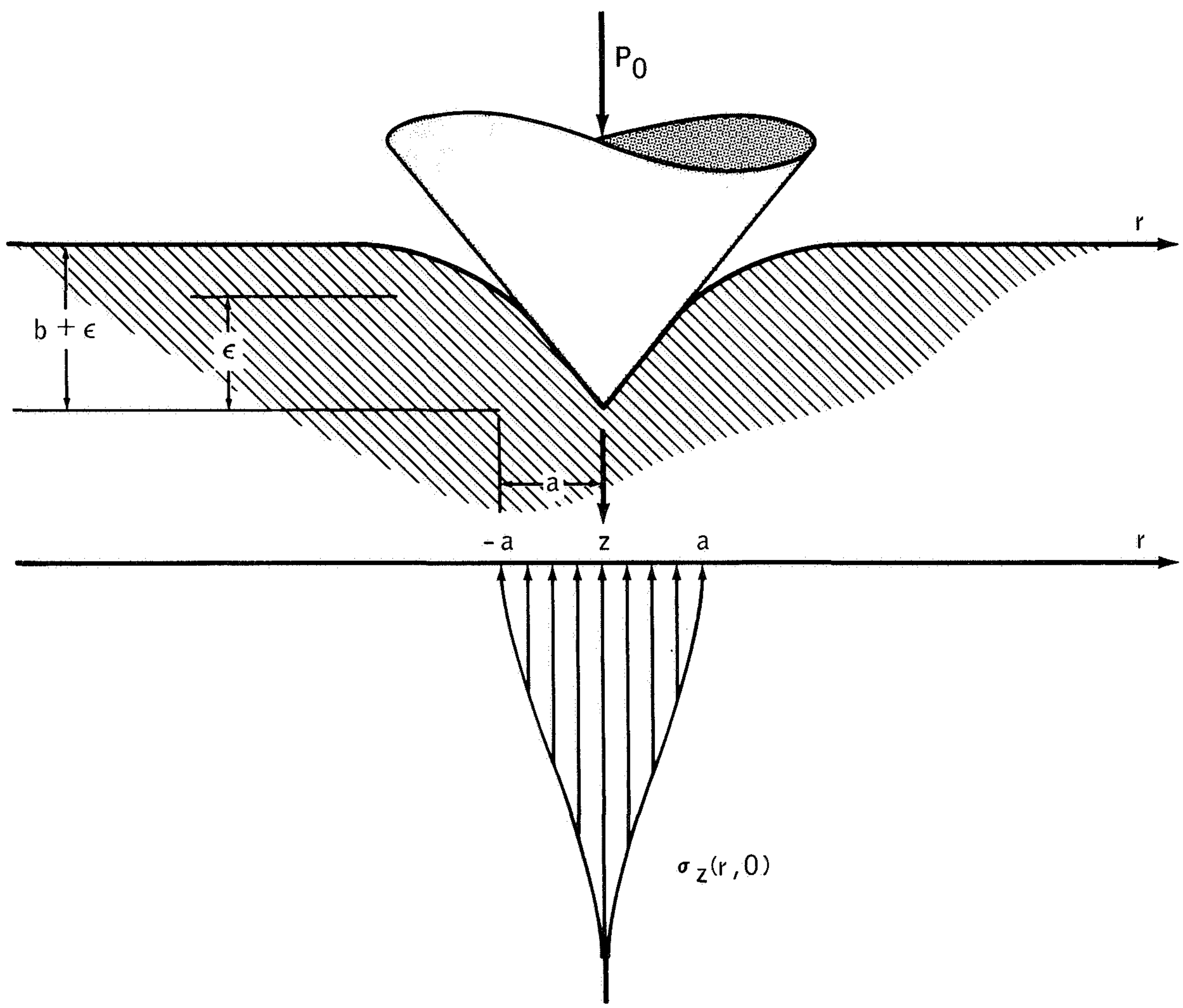


Figure 17. - Contact pressure between a half space and a conical-base punch.

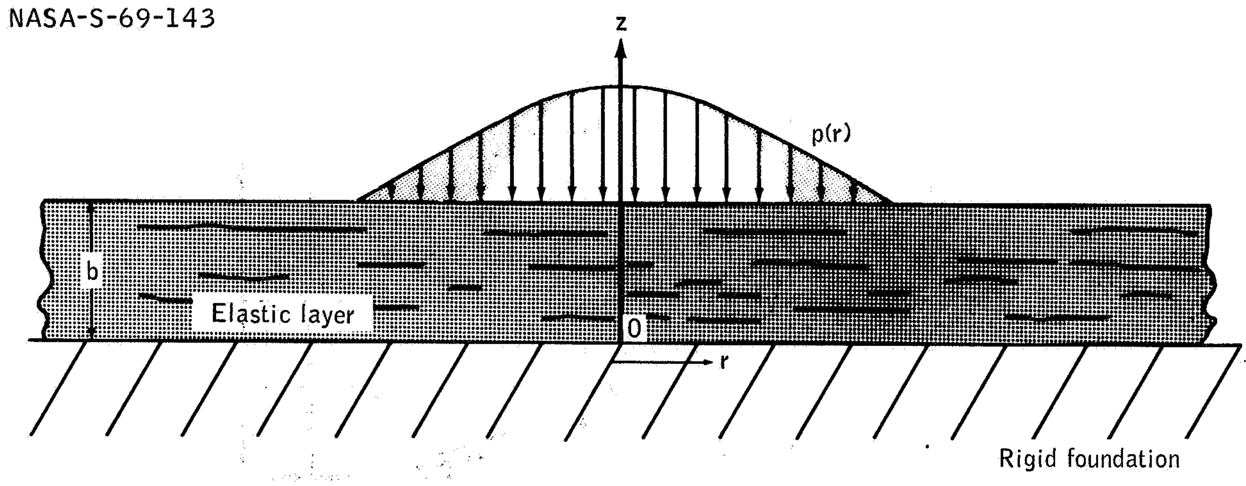


Figure 18. - Elastic layer resting on rigid foundation and subjected to a surface pressure.

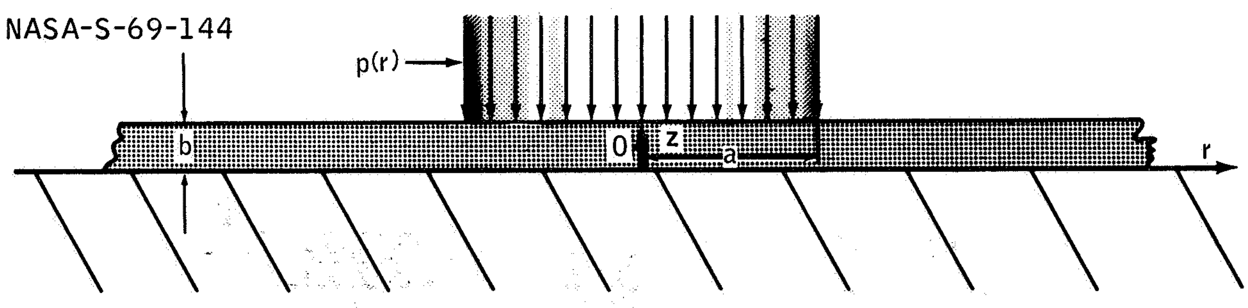


Figure 19. - Elastic layer resting on rigid foundation and subjected to a pressure uniformly distributed over a circle.

NASA-S-69-146

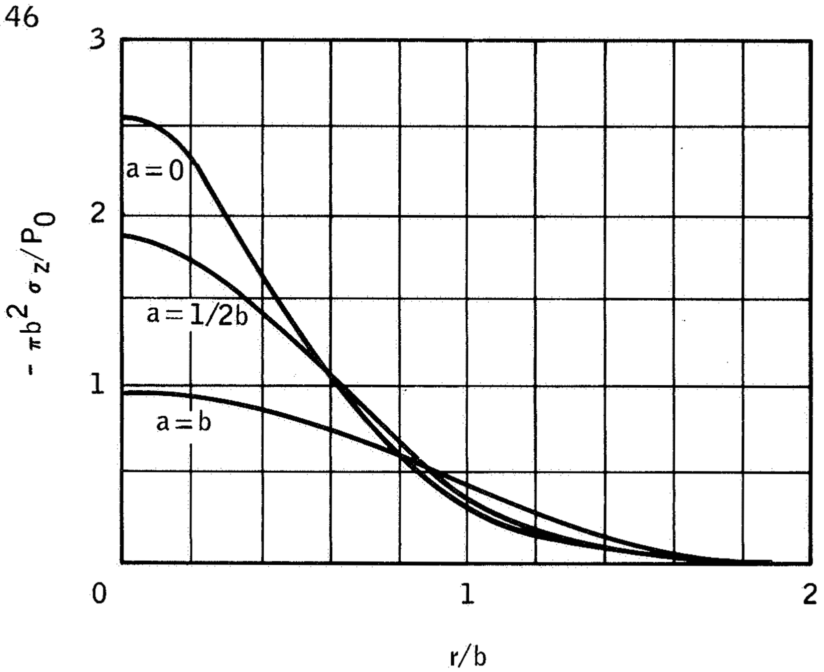


Figure 20. - Contact pressure between layer and rigid foundation (Sneddon's solution for a layer on a rigid foundation).

NASA-S-69-148

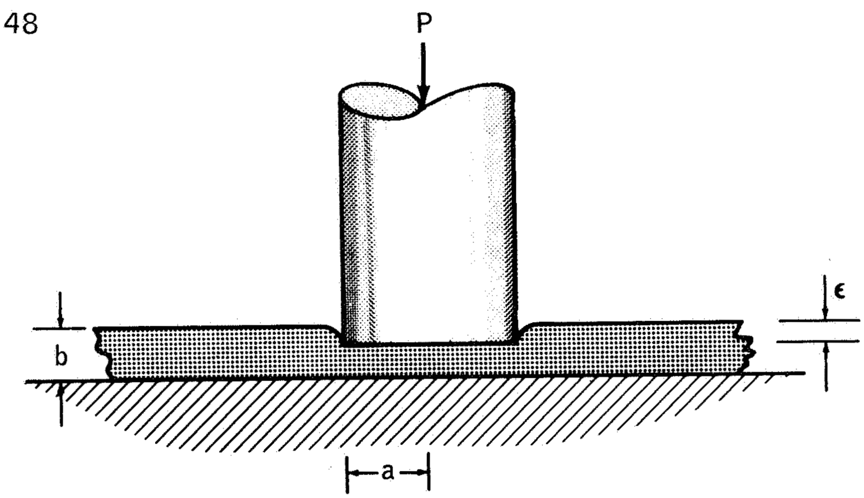
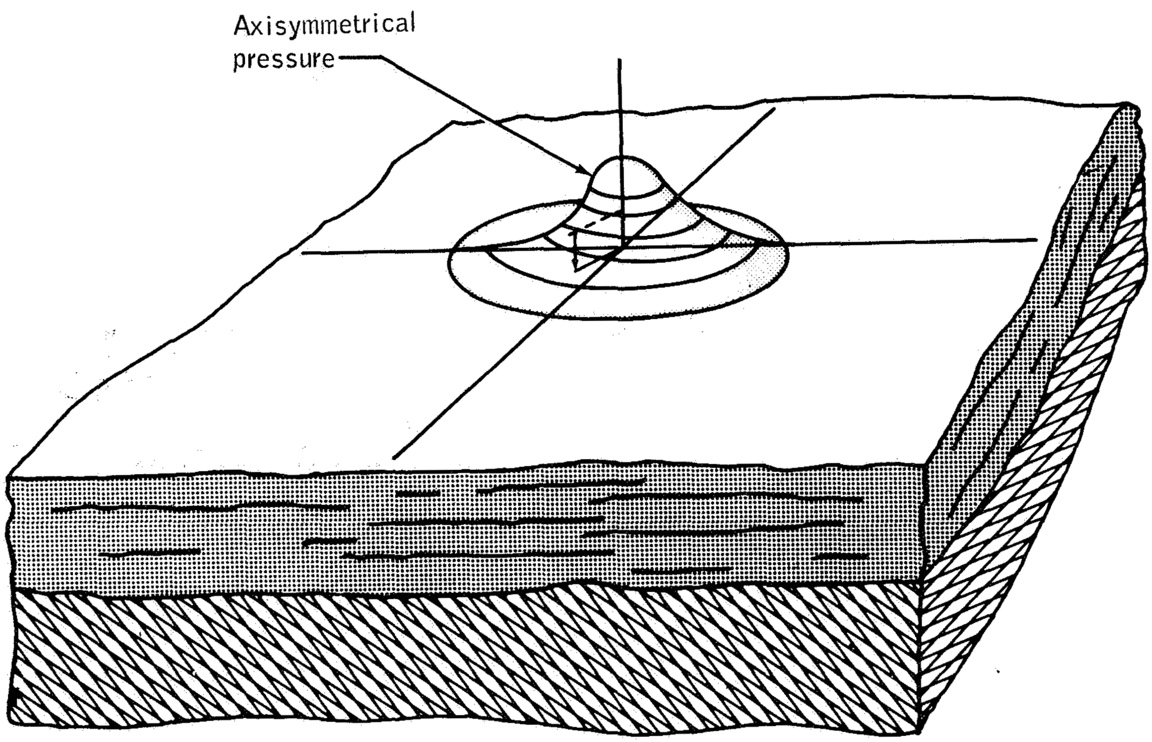


Figure 21. - Elastic layer resting on rigid foundation deformed by a flat-ended cylindrical punch.

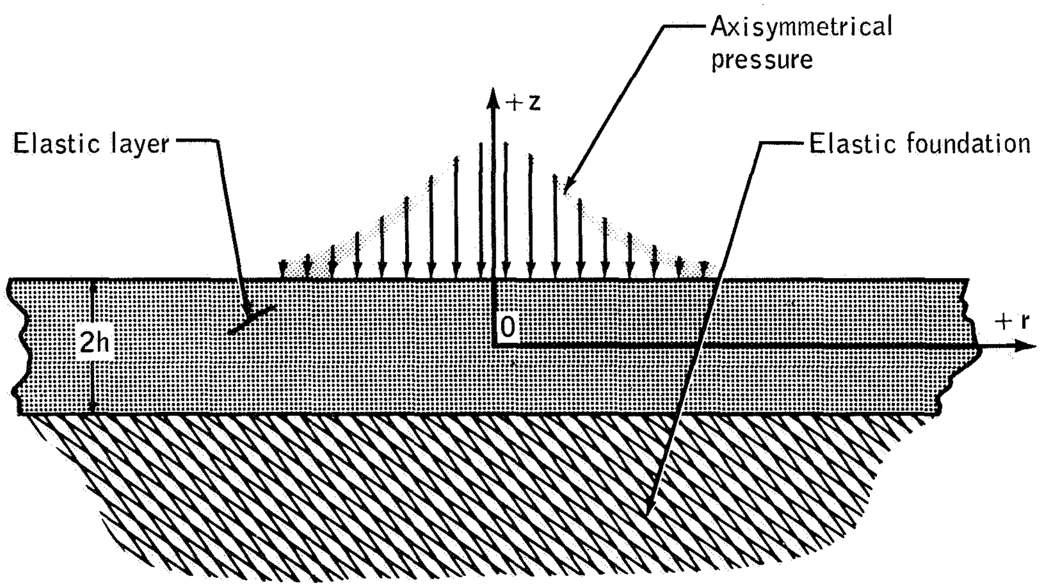
NASA-S-69-159



(a) Layer on elastic foundation subject to an axially symmetrical pressure.

Figure 22. - Thick layer on elastic foundation.

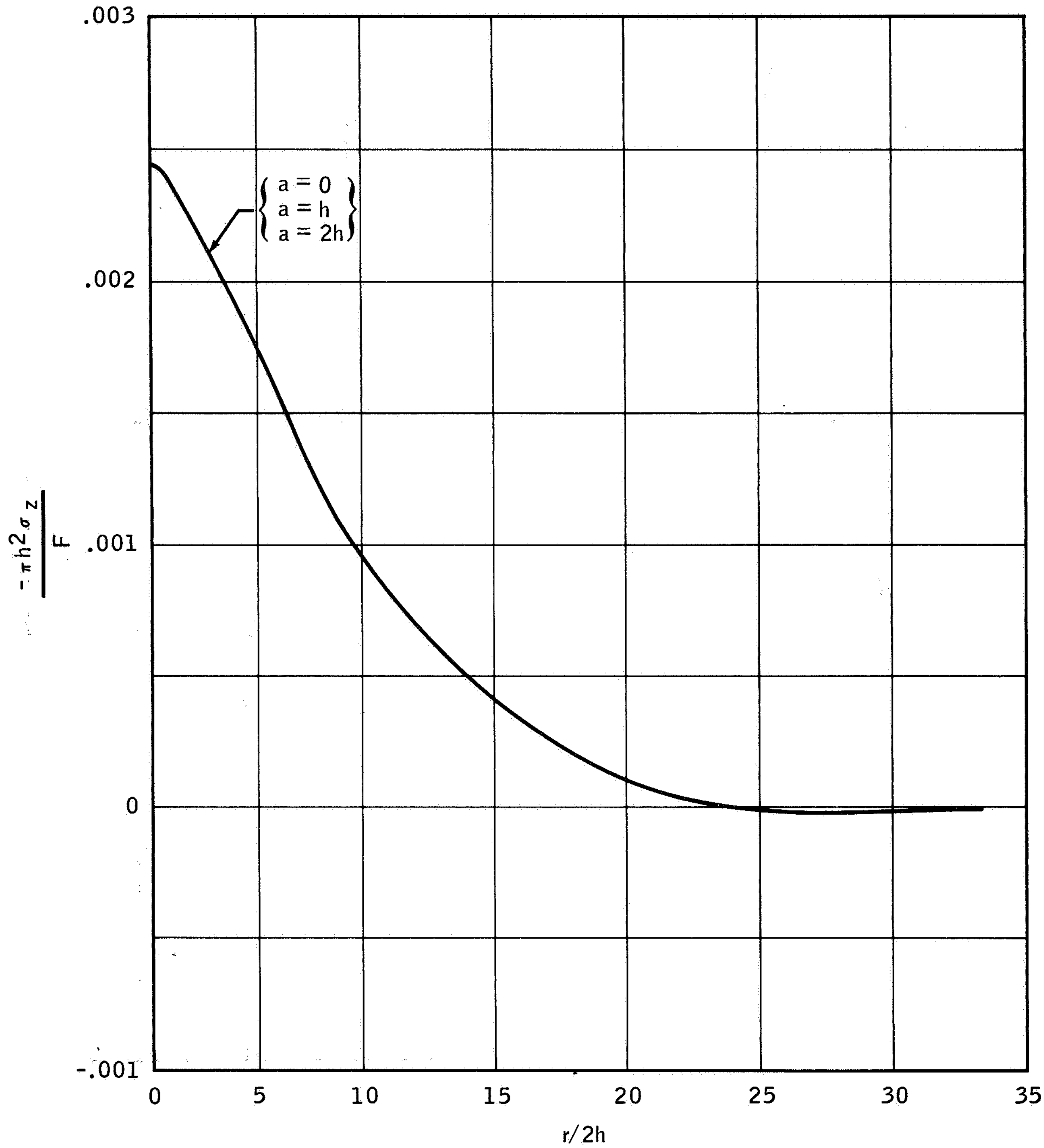
NASA-S-69-158



(b) Cross section through axis of symmetry showing coordinate system.

Figure 22. - Concluded.

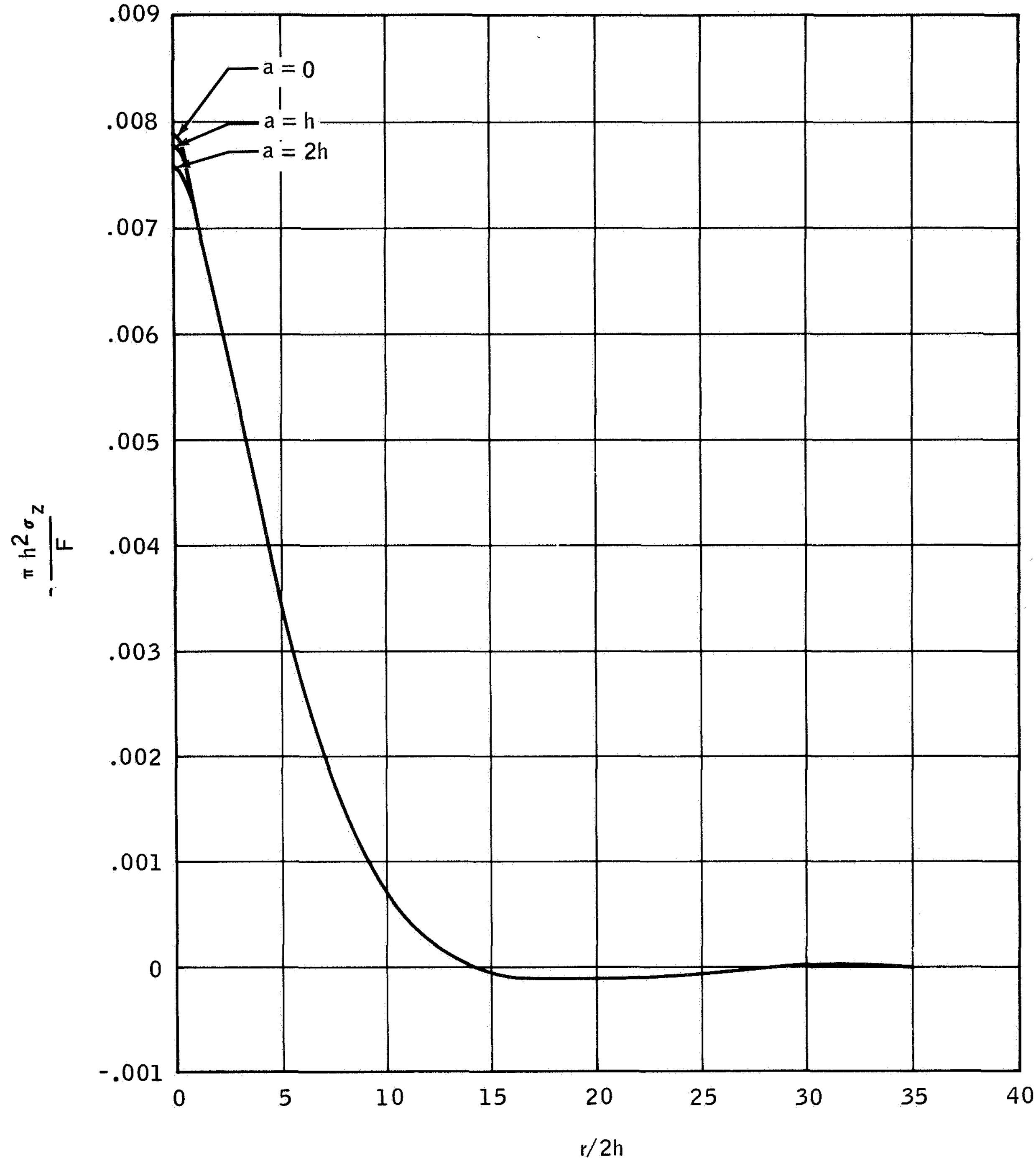
NASA-S-69-213



(a) $K = \frac{0.0001\mu}{2h(1 - \nu)}$.

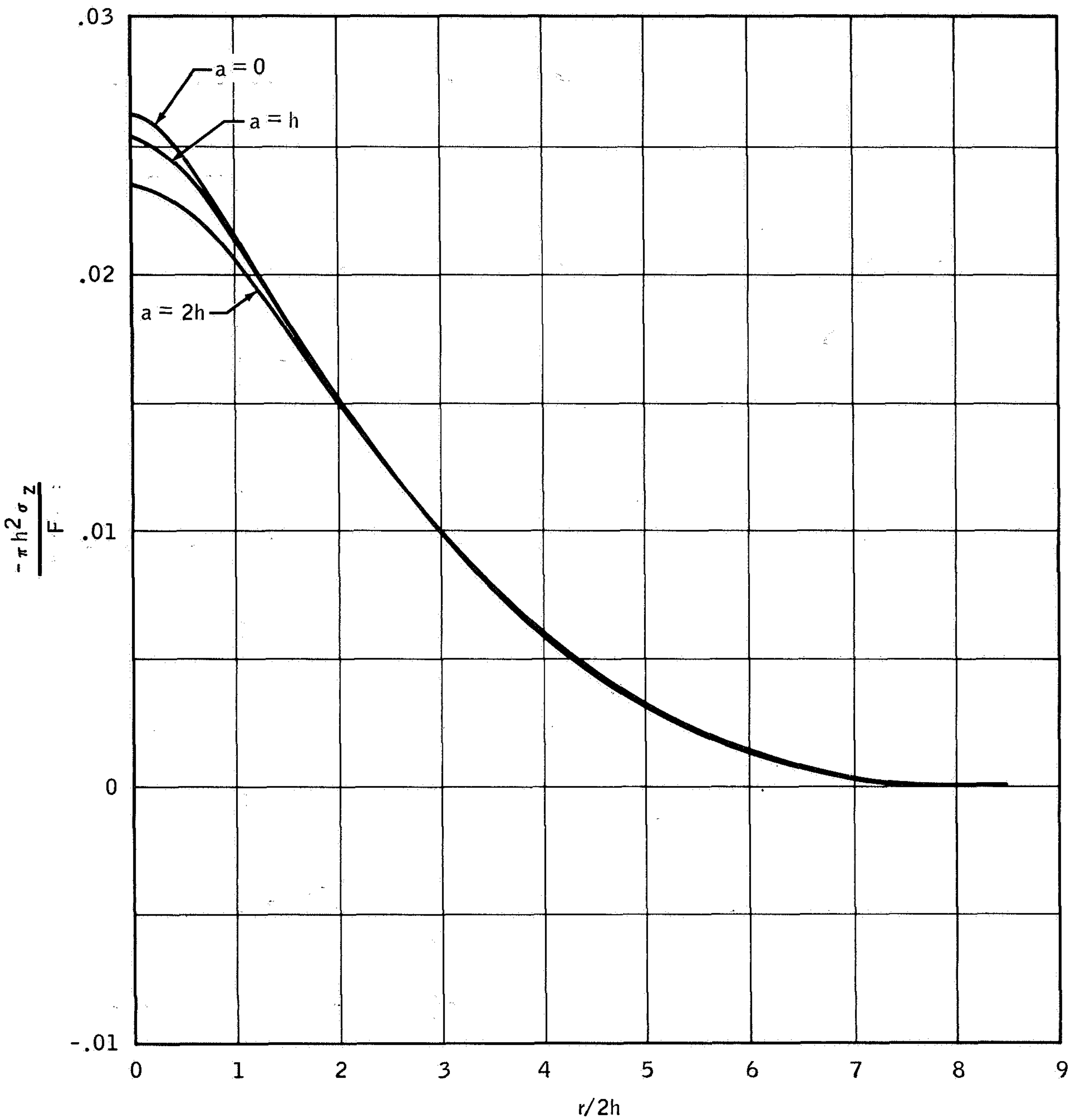
Figure 23. - Contact stress between a layer and its elastic foundation K; the surface loading is a uniform pressure distributed over a circle of radius a.

NASA-S-69-140



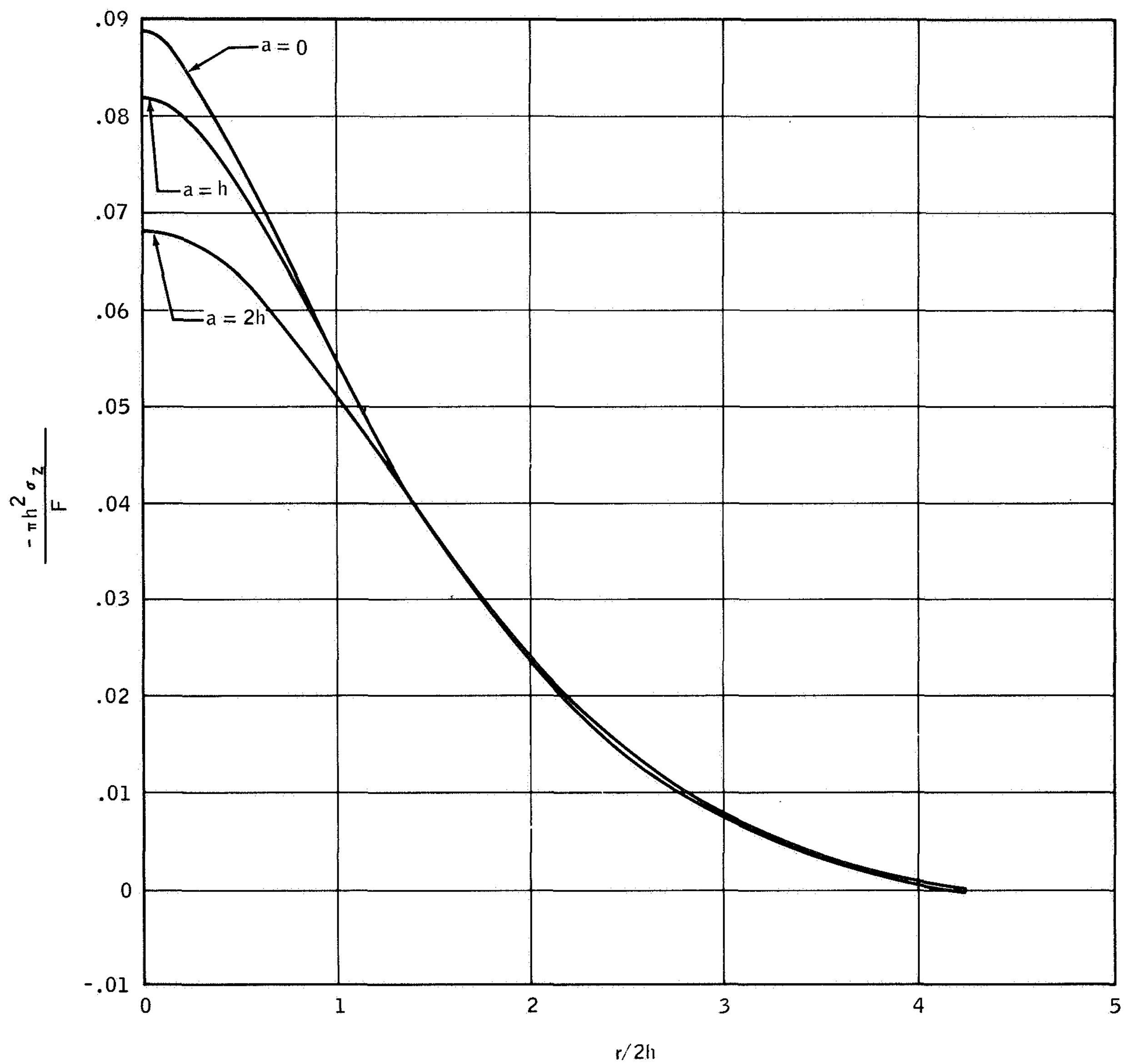
(b) $K = \frac{0.001\mu}{2h(1 - \nu)}$.

Figure 23. - Continued.



(c) $K = \frac{0.01\mu}{2h(1 - \nu)}$.

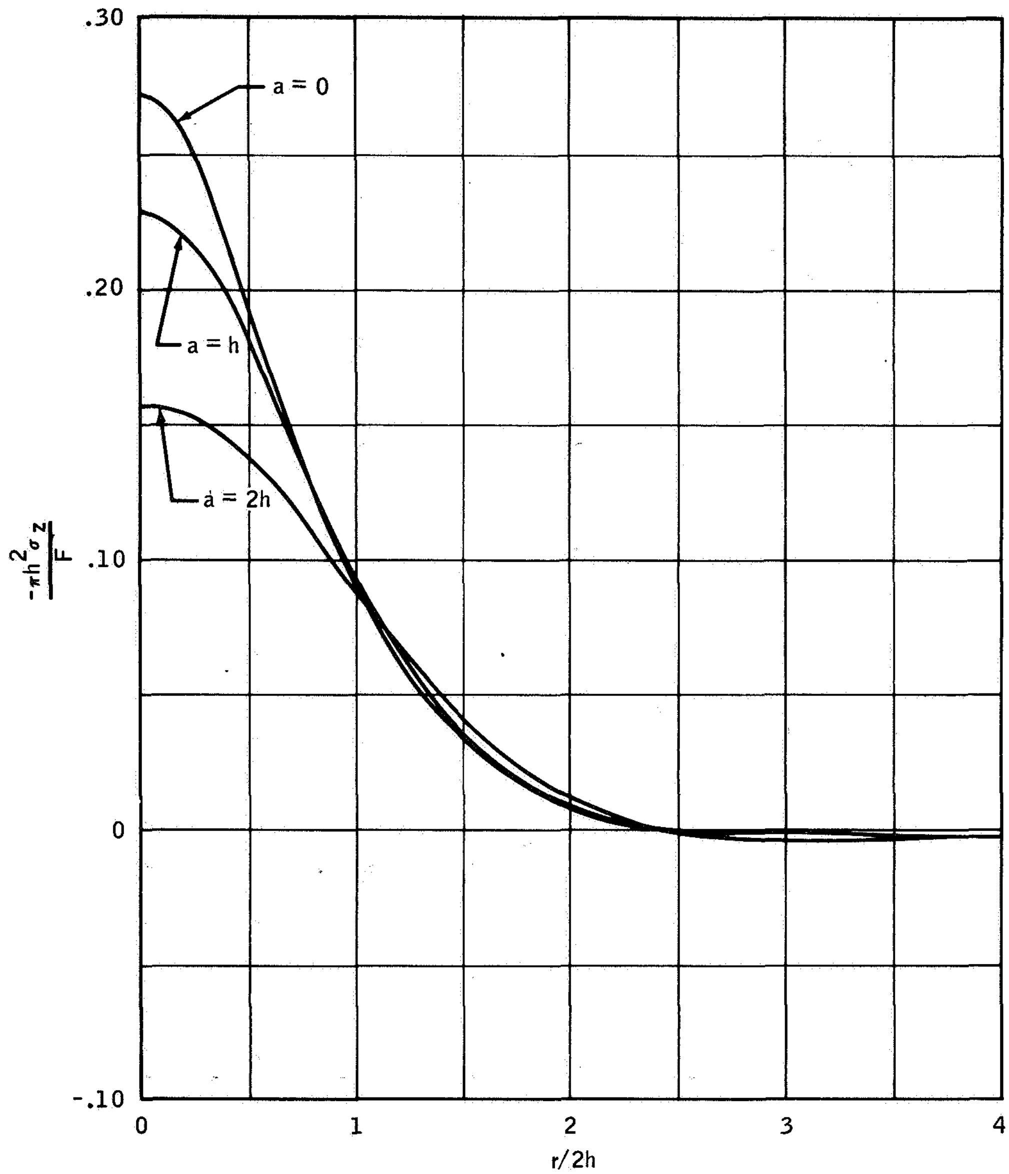
Figure 23. - Continued.



(d) $K = \frac{0.1\mu}{2h(1 - \nu)}$.

Figure 23. - Continued.

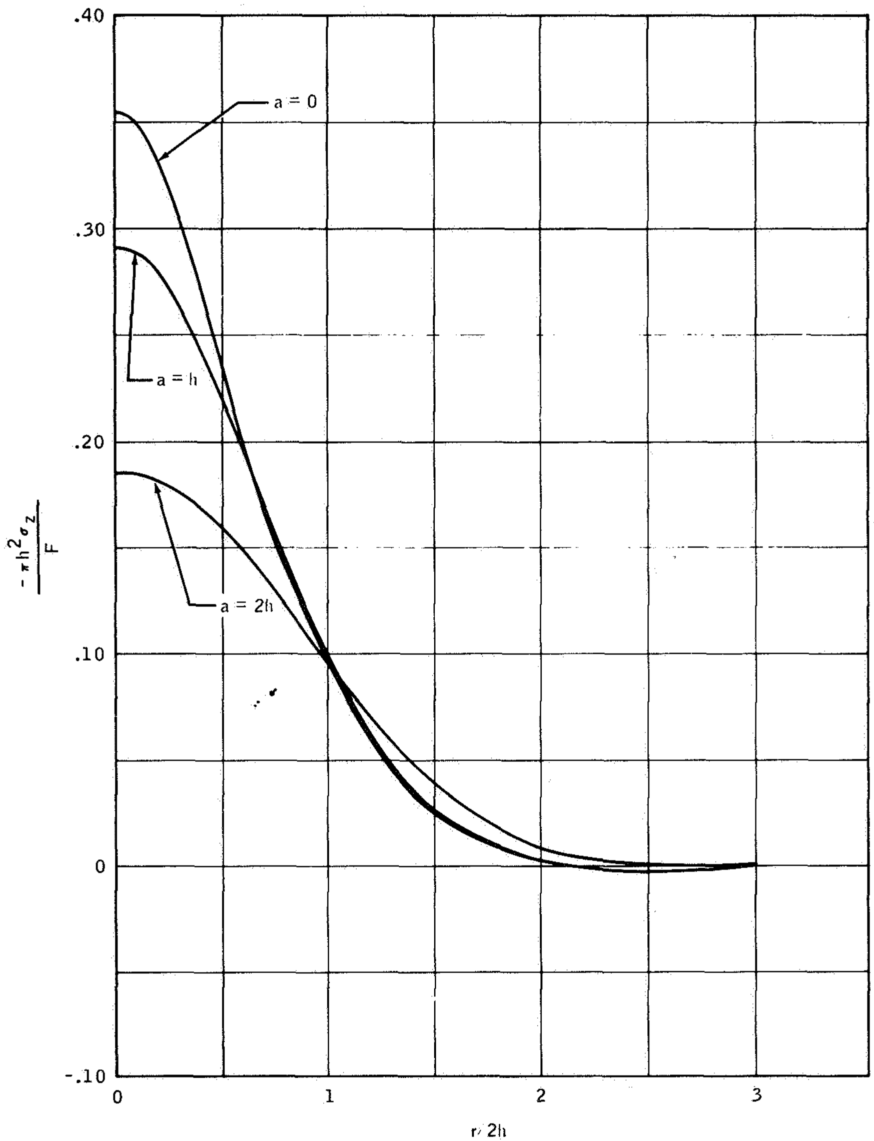
NASA-S-69-132



(e) $K = \frac{\mu}{2h(1 - \nu)}$.

Figure 23. - Continued.

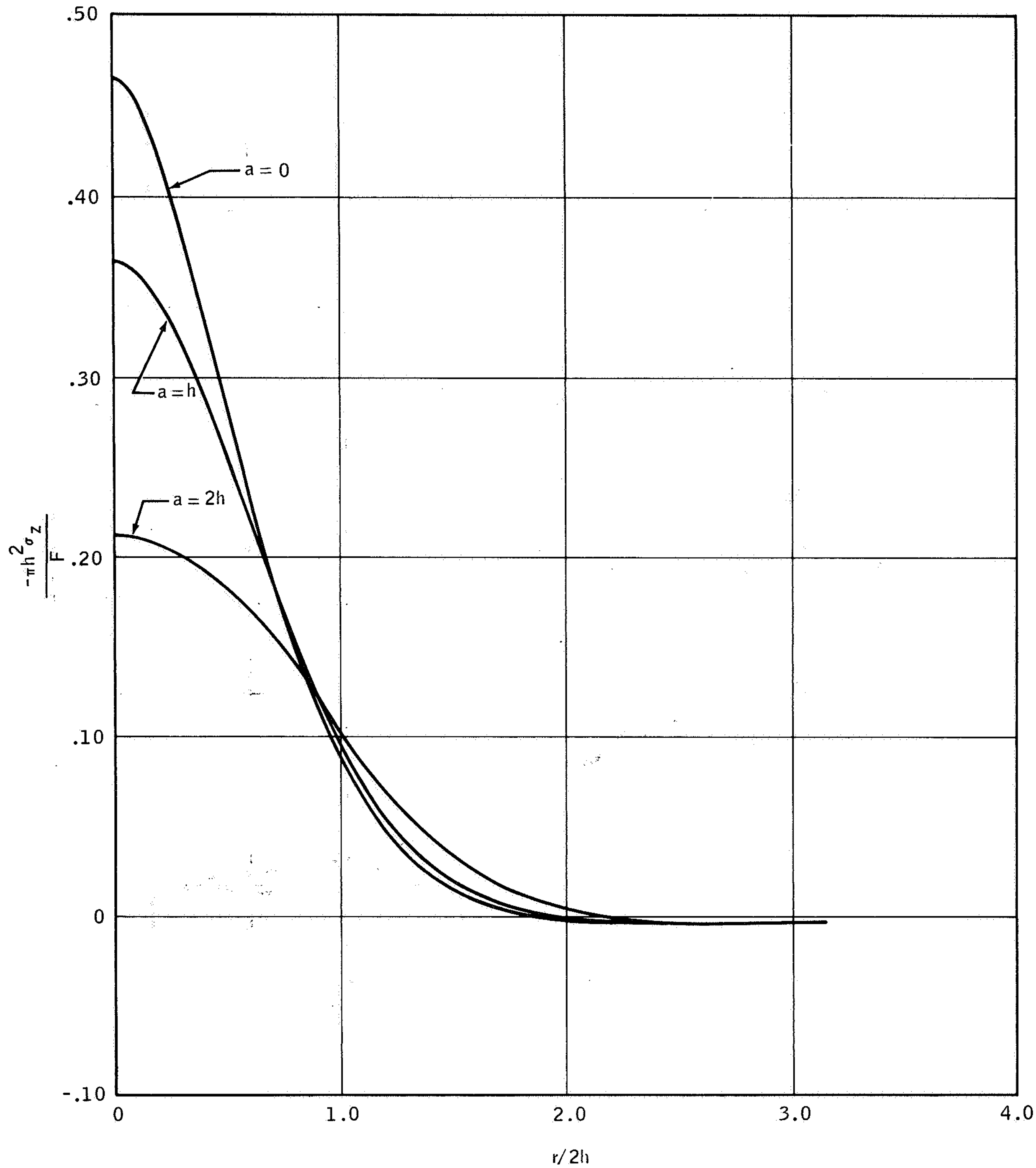
NASA-S-69-133



(f) $K = \frac{2\mu}{2h(1 - \nu)}$.

Figure 23. - Continued.

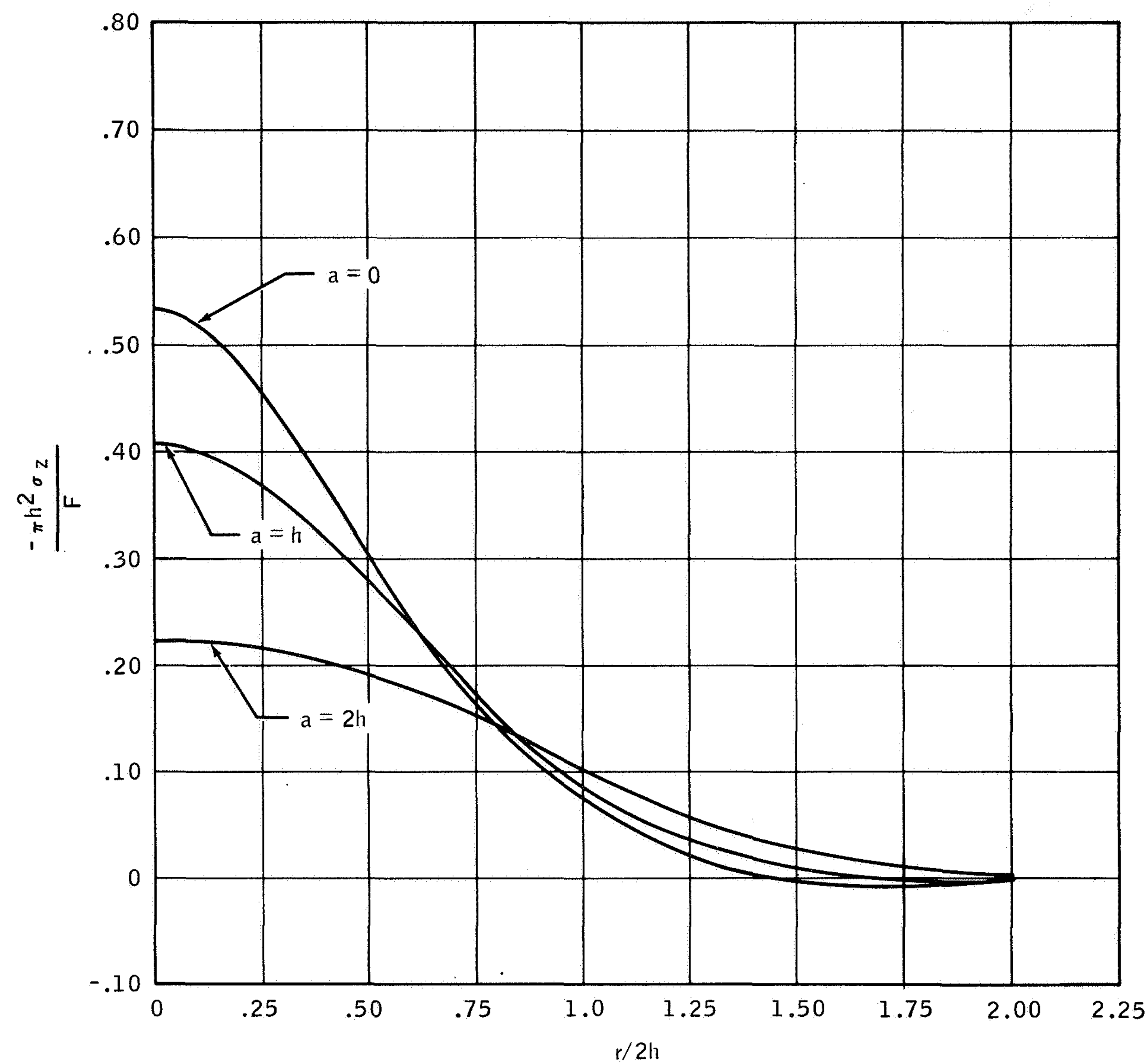
NASA-S-69-162



(g) $K = \frac{5\mu}{2h(1 - \nu)}$.

Figure 23. - Continued.

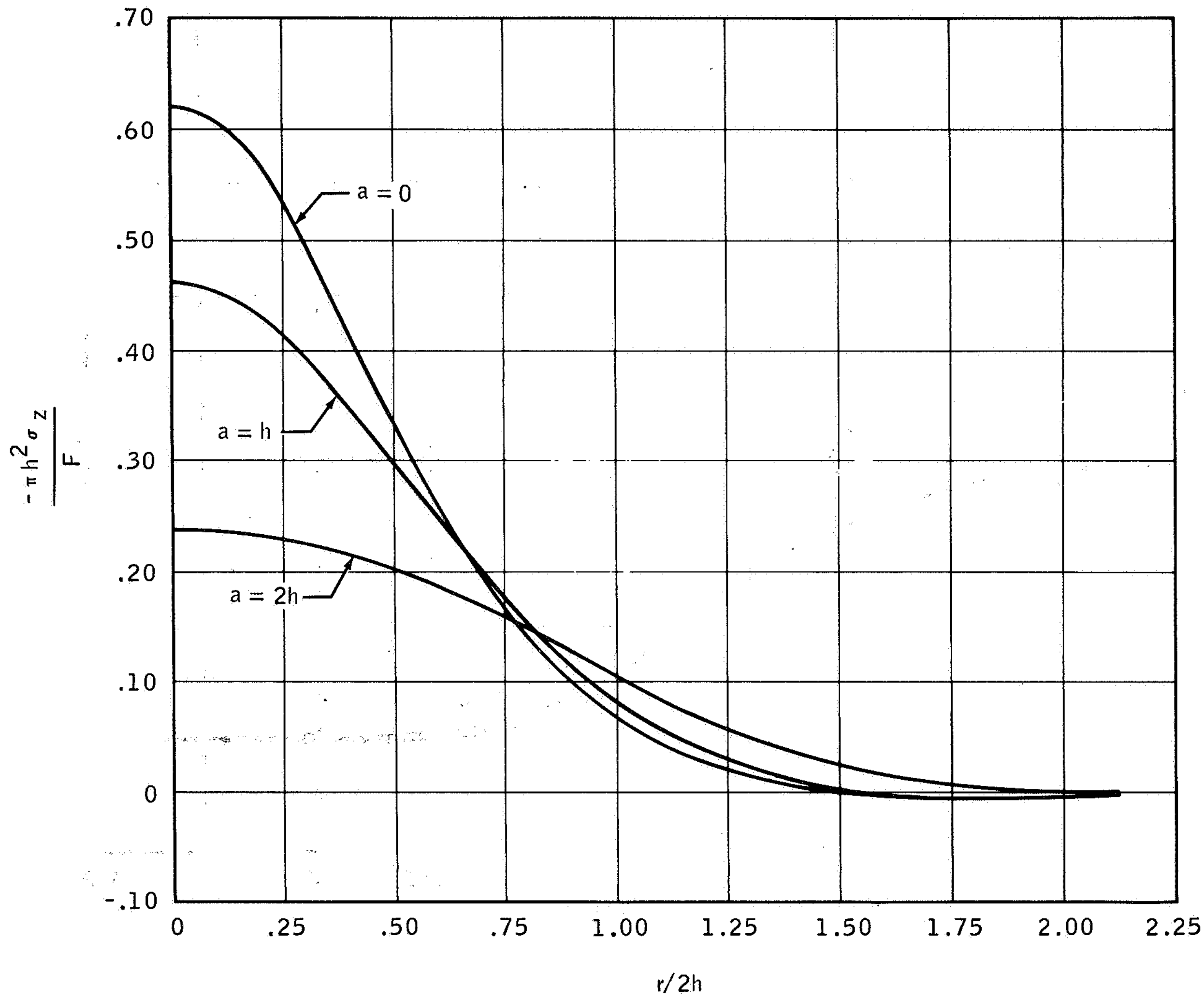
NASA-S-69-136



(h) $K = \frac{10\mu}{2h(1 - \nu)}$.

Figure 23. - Continued.

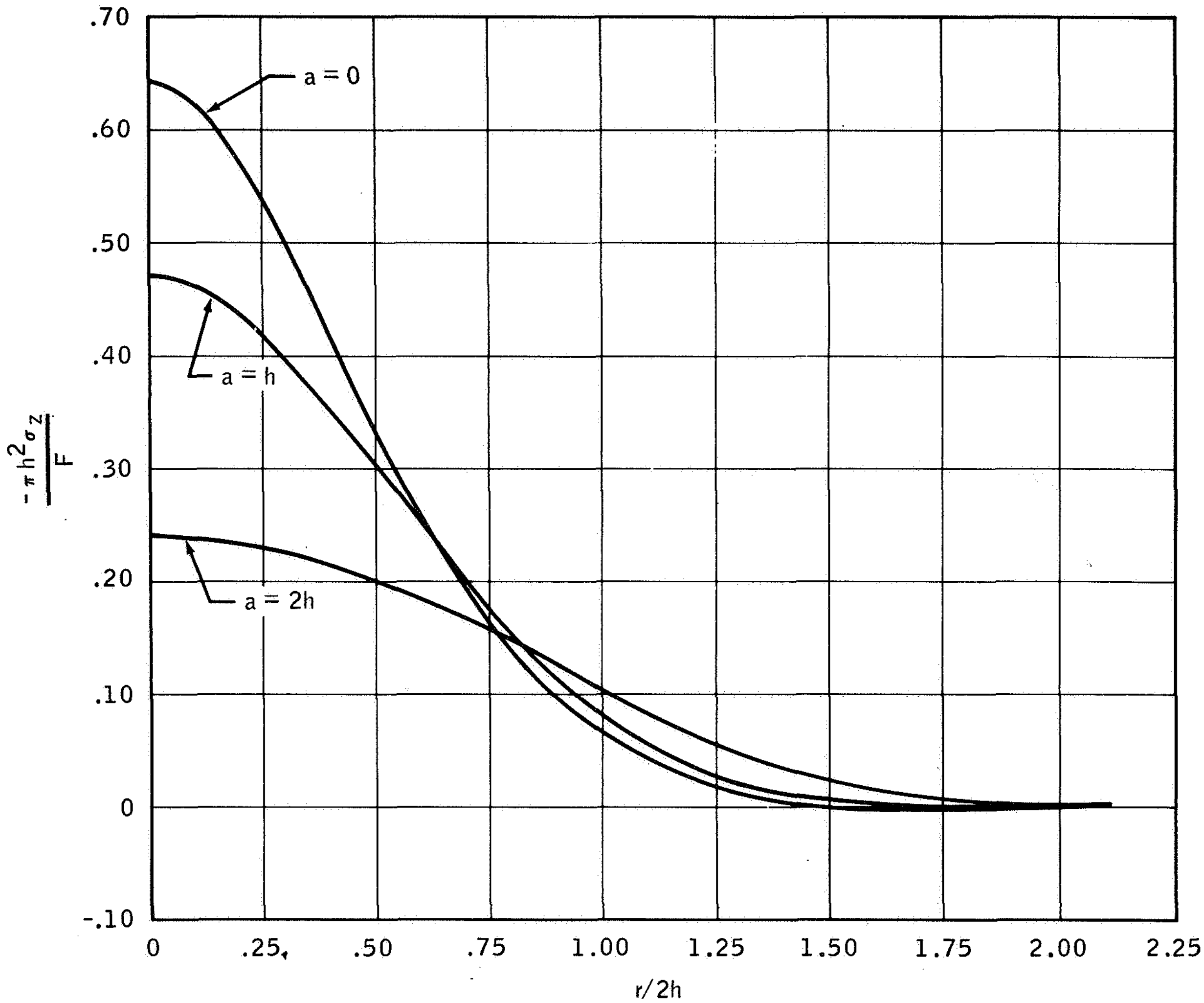
NASA-S-69-139



(i) $K = \frac{100\mu}{2h(1 - \nu)}$.

Figure 23. - Continued.

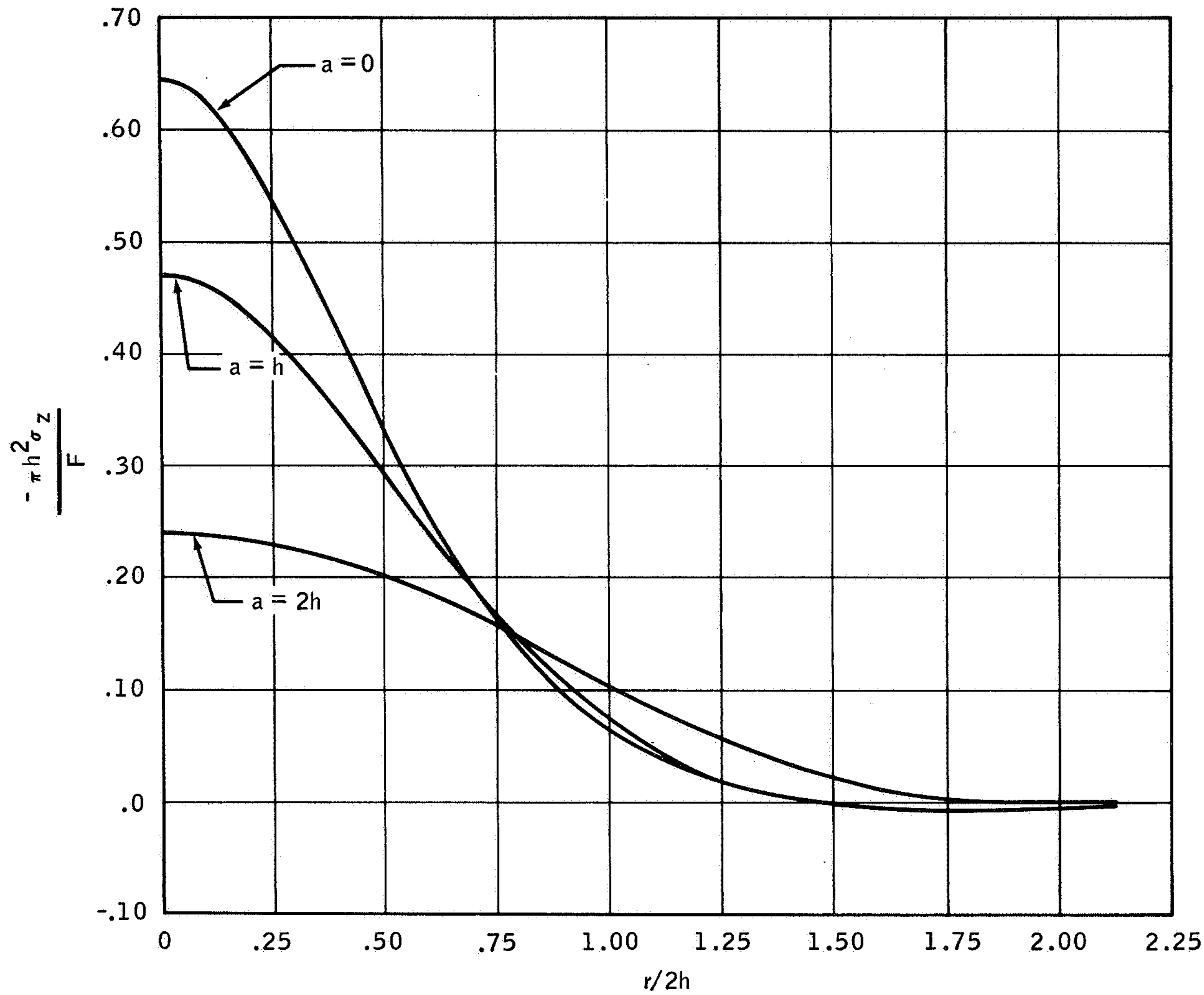
NASA-S-69-135



(j) $K = \frac{1000\mu}{2h(1 - \nu)}$.

Figure 23. - Continued.

NASA-S-69-137



(k) K = ∞.

Figure 23. - Concluded.