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COMPARISON OF BRAYTON AND RANKINE CYCLE MAGNETOGASDYNAMIC SPACE-POWER GENERATION SYSTEMS

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ABSTRACT

Idealized Brayton- and Rankine-MHD cycles are considered for use in space. The Brayton-MHD cycle uses neon as the working fluid; the Rankine-MHD uses lithium. Both are seeded with cesium. The systems are restricted to a specified generator length and specific radiator area. It is shown that generally an entrance Mach number of 1.0 provides maximum power output; for the same value of Hall parameter the Rankine-MHD system may be used at a lower temperature than the Brayton-MHD; but at the same maximum temperature the Brayton-MHD generates more power than the Rankine-MHD. Subsonic entrance Mach number is recommended for Brayton-MHD, and supersonic for Rankine-MHD.

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SUMMARY

Idealized Brayton- and Rankine-magnetogasdynamic (MHD) cycles are considered for use in space. These cycles are constrained to operate with a specified radiator area per unit power generated and a specific generator length. They are compared with a reference cycle. The reference cycle is a Rankine cycle boiling potassium at 1365 K using a turboalternator with an efficiency of 0.75. The Brayton-MHD cycle uses neon as a working fluid, the Rankine-MHD uses lithium. Both are seeded with cesium. The magnetic-field strength considered is limited to 20 tesla, and the maximum cycle temperature to 2500 K.

At the same value of Hall parameter and specific radiator area the Rankine-MHD cycles can be used at lower maximum cycle temperature than the Brayton-MHD; but at a maximum cycle temperature at which either cycle may be used the Rankine-MHD power output is a factor of six smaller than the Brayton-MHD. The necessary Hall parameter increases as the maximum cycle temperature of either system decreases.
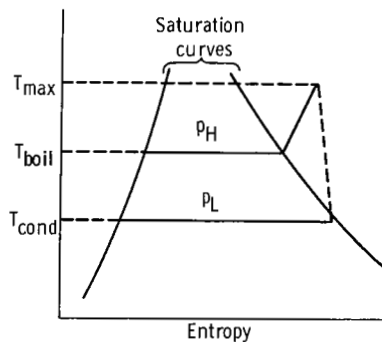
For the Rankine-MHD cycle there is an optimum seed fraction (0.01). For the Brayton-MHD, however, the seed fraction can be adjusted to minimize either the Hall parameter (s $\approx$ 0.020) or the magnetic-field strength (s $\approx$ 0.001) for a given total power generated.

A generator entrance Mach number of 1.0 provides the maximum power generated for a given specific radiator area, generator length, and maximum pressure for both Rankine and Brayton systems. Even so, subsonic operation for the Brayton-MHD system is preferable - primarily because the Hall parameter requirement is lower than for a Mach Number 1.0 at any maximum temperature. On the other hand, supersonic operation may be necessary for the Rankine-MHD system boiling at temperatures above 1645 K to avoid a prohibitively high magnetic-field strength requirement.
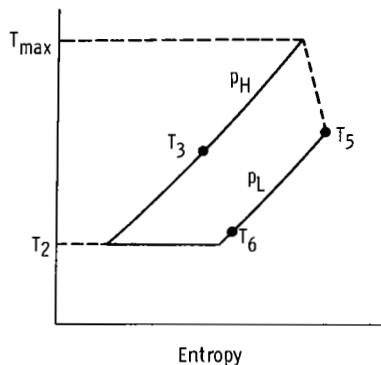
INTRODUCTION

The magnetogasdynamic (to be designated herein as MHD) power generator used in a closed thermodynamic cycle with a nuclear heat source is being considered as a means of providing electric power in space. The pressing requirement for a space system is light weight. Smaller system weights may be achieved by increasing the maximum cycle temperature. An MHD generator is one possible high-temperature generator. We are concerned with the problem of finding the lower limits of maximum cycle temperature (and associated design parameters) for which these MHD systems are acceptable for use in space. Acceptable, as used here, means that the specific weight (total weight divided by total power generated) is comparable to that of the competitive turboalternator system. As the maximum cycle temperature is increased above these lower limits, the competitive position of the MHD systems would improve.

We will consider two types of thermodynamic cycles: Brayton and Rankine. The temperature entropy diagrams for these two cycles are shown in figure 1. The Brayton cycle is modified to include isothermal compression, and the Rankine cycle includes

(a) Rankine cycle.

(b) Brayton cycle.

Figure 1. - Thermodynamic cycles.

superheat. The choice between the thermodynamic cycles depends not only on MHD generator performance characteristics but also on material properties and design difficulties. Is it easier to provide a reactor to boil lithium at 1650 K at a relatively low pressure (about 1 atm (10^5 N/m^2)) or to heat neon to 2000 K (or higher) at a higher pressure-for instance above 6 atmospheres? Is it easier to provide a condenser for lithium at 0.1 atmosphere or to compress neon efficiently and isothermally at approximately 1200 K? Is an efficient, light-weight regenerator easier to build than a superheater? The corrosion problems are likely to be more difficult in the Rankine system (with liquid alkali metal) than in the Brayton system (with a small fraction of alkali metal vapor in the working fluid). But the Brayton cycle performance is more sensitive than the Rankine cycle performance to system component efficiencies. We will attempt to compare the way in which the MHD generator characteristics affect the system specific weight and to determine the preferable values of system operating parameters for each cycle.

Some studies of the use of Brayton-MHD cycles in space have been made. Rosa (ref. 1) makes the case that a high-temperature (3500 K) system would be worth developing. MacNary and Jackson (ref. 2) chose a lower temperature (1500 K) and concluded that this temperature was too low for use in space. We will study an intermediate temperature region from 1700 to 2500 K.

Heighway and Nichols (ref. 3) considered only constant area generators at a maximum temperature of 2222 K where thermodynamic choking limited the cycle efficiency and generated power. Elliott (ref. 4) determined design values for a possible Brayton cycle but made no attempt to determine the range of design parameters which would provide an acceptable performance.

Little analysis of the effect of design parameters for Rankine-MHD space systems has been made. Emmrich and Voshall (ref. 5) studied only one set of generator parameters for lithium. Akers, et al., (ref. 6) investigated the effect of magnetic-field strength on system weight for one set of design conditions with zinc as working fluid. Nichols (ref. 7) considered lithium and studied the effects of Mach number and load parameter on generator performance as a function of maximum cycle temperature. A minimum volume generator was considered, but no study of the effect of the boiling temperature was made. In none of these references was an attempt made to determine the range of all variables where acceptable performance in space can be achieved.

A great many studies of closed MHD systems for ground-based use have also been made. (Refs. 8 to 17 are some of the most recent.) These studies are not entirely pertinent to space application. In most of these studies the main consideration is generator power density only, and the generator length required to achieve this pressure change (or total temperature change) is not considered as a serious design limitation. Also, in these ground-based application studies it has been assumed that the generator load pa-

rameter (or current-density ratio) is indepent of the entrance Mach number, the operating pressure ratio, or any other cycle conditions. In space this design freedom is not possible because the system weight is sensitive to the value of load parameter since it effects both the generator efficiency and power density.

CYCLE SPECIFICATIONS AND ASSUMPTIONS

System Weight

It will be assumed that most of the system weight is found in three major components: (1) the reactor and shielding, (2) the waste-heat radiator, and (3) the generator, magnet, and miscellaneous equipment. Zipkin (ref. 18) calculates the percentage of the total system weight which is contributed by each of these portions to be 60, 20, and 20 percent, respectively.

It will be assumed herein that the weight of each portion in one system is equal to the weight of the same portion in other systems. This ensures that the total weight of the systems will be equal. It also implies that there are no trade-offs made wherein the weight of one portion is reduced by increasing the weight of some other portion.

First, we will assume that the weight of the reactor and shielding is the same for all systems (including a turboalternator system which will be used as a reference for comparison) at a specified thermal power and maximum cycle temperature. Therefore, cycles with the same efficiency will be assumed to have the same reactor specific weight. This assumption is probably valid for comparing the MHD systems because the Rankine systems must be superheated. However, the MHD reactors are likely to be heavier than the turboelectric reactors, because the vapor is saturated.

In space the waste heat is rejected by radiation and must be done at a high temperature to keep the radiator area (and weight) small. Radiator weight is assumed herein to be proportional to the area which, in turn, depends on temperature. For a given maximum temperature there is a rejection temperature which minimizes the ratio of radiator area to total electrical power generated. This ratio will be called the specific radiator area ratio. The rejection temperature, which minimizes the specific radiator area ratio, also depends on generator efficiency. The rejection temperature of the space MHD cycles will be chosen to maintain a specific area ratio equal to that of the reference system. The reference cycle is assumed to have a fixed maximum temperature of 1365 K. As the maximum MHD cycle temperature changes, the generator efficiency must be adjusted to maintain an equal specific radiator area ratio. In this way, restrictions are placed on the MHD system so as to equal the reference system ratio. The restrictions placed on generator efficiency and maximum cycle temperature so as to improve the specific radiator area of the MHD system will be indicated.

The dominant component of the MHD generator weight will be the weight of the magnetic-field coils. These coils are assumed to be provided by superconducting magnets. Magnet systems of the size considered and providing about 9 tesla are presently feasible (ref. 19). Systems requiring magnetic-field strength greater than 20 tesla will not be considered.

The weights of the reactor and radiator are either assumed to be equal or set equal to those of the reference system. Consequently, if the power generating equipment weight equals that of the reference system, the total weight of the MHD system will equal the reference system. Therefore, the operating conditions where the magnet specific weight (total magnet weight divided by generator power output) is comparable to or less than the reference system's turboalternator specific weight will be determined. In this way the total specific weight will be comparable to the reference system total specific weight. It will be shown that it is possible to design MHD generators whose magnet specific weight is a rather small (a few percent) fraction of the total system specific weight.

Working Fluid Specification

The Brayton cycle working fluid is limited to the seeded noble gases. The choice among the noble gases requires consideration of reactor heat-transfer characteristics and compressor requirements so as to keep the reactor size small and working fluid density to keep small the number of compressor stages. Reactor heat-transfer rate is larger with low-molecular-weight fluids, whereas, high-molecular-weight fluids pose less severe compressor requirements. Glassman (ref. 20) concludes that neon is an acceptable compromise for Brayton cycles used in space. Heighway and Nichols (ref. 3) show that neon is an acceptable fluid for use in MHD generators. Therefore, neon was chosen for the present Brayton-MHD studies.

The proper working fluid for Rankine-MHD cycles has also been studied. Emmrich and Voshall (ref. 5) discovered that the critical parameter that determines the suitability of a Rankine-MHD working fluid for use in space is the electron mobility of its vapor. Because the boiling temperature and condensing temperature are fixed by Rankine cycle considerations (which, in turn, are fixed by materials and radiator weight considerations), the vapor pressure as well as the collision cross section determines the mobility. They conclude that lithium would be preferable, primarily because of its low vapor pressure and consequent high electron mobility. Akers, et al., (ref. 6) concluded that zinc would be acceptable. However, their conclusion is based on keeping the mobility high by boiling at a temperature of 1250 K. If the boiling temperature is increased to 1365 K, then zinc becomes unacceptable because of the reduced mobility. The same is true of the other alkali metals. Thus lithium will be used in the cycles studied in this analysis.

Basis of Cycle Comparisons

Assumptions. - Many assumptions have been made to arrive at the desired comparison as simply as possible but still with a degree of realism.

The thermodynamic cycles are assumed to be ideal as shown in figure 1 with the exception of generator efficiency. The generator efficiency is calculated based on an idealized model wherein the only nonisentropic effect considered is joule heating in the plasma. Hence the design conditions that are determined will be necessary, but not sufficient, to provide specific weight competitive with the reference system. In the generator, it is required that the value of jB and the value of j^2/σ are constant at every point from the entrance to the exit of the duct. These restrictions are chosen because they provide a uniform retarding force and uniform dissipation along the duct. The generators are designed to have a minimum volume. Also, the generators are compared on the basis of a fixed length and fixed aspect ratio.

An especially important assumption is that there are no "instabilities" present. Recent results summarized by Louis (ref. 21) have indicated that at large Hall parameters various forms of instability will appear which reduce the actual performance below the ideal. The reduction usually occurs at Hall parameters between 2.0 and 4.0 depending on the value of other parameters. Therefore, we will attempt to determine a region of acceptable performance at as low a value of Hall parameter as possible which is consistent with the requirements imposed by the space environment. The implication of the effects of these instabilities therefore remains unexplored in this study.

Calculation procedure. - We will determine for each cycle as a function of its maximum temperature the values of five design parameters: maximum pressure, generator entrance Mach number, entrance current density ratio, seed fraction, and entrance Hall parameter which provides a system with specific radiator weight equal to that of the reference cycle. The reference cycle is a Rankine cycle boiling potassium with no superheat at a maximum temperature of 1365 K and a turboalternator efficiency of 0.75. Values of the five-system design parameters will be selected that make the overall thermodynamic efficiency and specific radiator area equal to those of the reference cycle. The calculation procedure is as follows:

(1) Select a cycle and a maximum temperature. Calculate the radiator temperature which minimizes the specific radiator area with generator efficiency as parameter.

(2) Calculate specific radiator area at this rejection temperature and determine generator efficiency required to equal specific radiator area of reference cycle.

(3) Determine pressure ratio for cycle (equal to 0.5 for Brayton and 0.1 for Rankine).

(4) Design the generator to have minimum volume for that pressure ratio and a uniform value of both jB and j^2/σ. Calculate the generator efficiency and determine the value of entrance Mach number and entrance current-density ratio required to give the

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required generator efficiency (based on specific radiator area requirements described in step (2)).

(5) Calculate length required to provide the required pressure drop. Select values of conductivity and magnetic field strength to make this length equal to 0.5 meter.

This calculation procedure is followed in this report, and the results are analyzed. All symbols are defined in appendix A, and all quantities are expressed in International System of units.

EFFECT OF CYCLE PARAMETERS ON CYCLE PERFORMANCE CHARACTERISTICS

Radiator Temperature Which Minimizes Specific Radiator Area

Consider a thermodynamic cycle, with efficiency η, generating electrical power and radiating waste heat to a sink at absolute zero. The required radiator area $\mathscr{A}$ is

$$\mathscr{A} = \frac{1 - \eta}{\eta} \; \frac{wW}{\epsilon\sigma_{SB}T^4_{rad}} \tag{1}$$

If a radiator area parameter α is defined as

$$\alpha \equiv \frac{\mathscr{A}\epsilon\sigma_{SB}T^4_{max}}{wW} \tag{2}$$

then equation (1) becomes

$$\alpha = \frac{1 - \eta}{\eta}\left(\frac{T_{max}}{T_{rad}}\right)^4 \tag{3}$$

For a fixed maximum cycle temperature T_{max} there is a value of T_{rad} which minimizes α. This minimum provides a minimum value of the ratio of total radiator area to total power generated, $\mathscr{A}/wW$. We will call this ratio the specific radiator area. We assume that the raditor weight is proportional to its area, so that the minimum value of the specific radiator area corresponds to the minimum value of specific radiator weight.
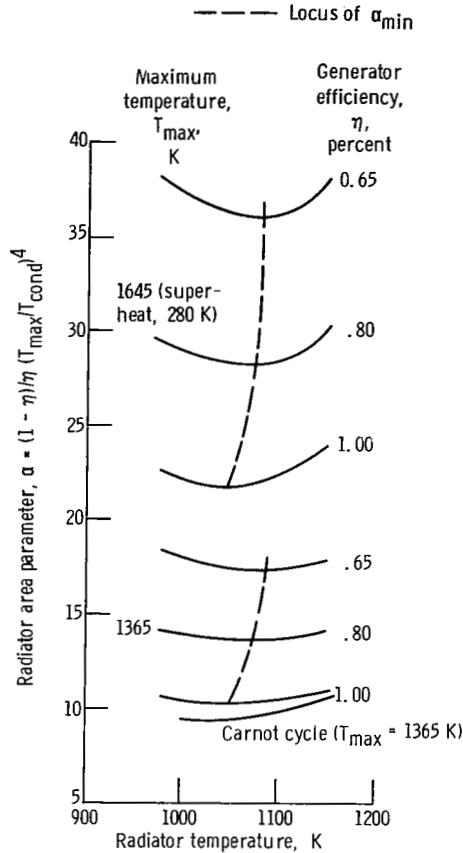
Figure 2. – Area parameter for two thermodynamic cycles as function of radiator (condenser) temperature with amount of superheat and generator efficiency as parameters. Cycle uses potassium boiling at 1365 K. Carnot maximum cycle temperature, 1365 K.

For a Carnot cycle, for example, the efficiency is $1 - (T_{rad}/T_{max})$. Then the minimum value of α occurs for $T_{rad} = 3/4 \, T_{max}$, and is

$$\alpha_{min} = 4\left(\frac{4}{3}\right)^3 = \frac{256}{27}$$

The variation of α with T_{rad} is shown in figure 2 with $T_{max} = 1365$ K (2000° F).

Effect of Generator Efficiency on Cycle Pressure Ratio

Consider the cycles shown in figure 1. The radiator area required is again given by

equation (1). Actual generator effects can be included by introducing a generator efficiency η_{gen} which is the actual change in enthalpy divided by the isentropic change between the same pressure limits. With this definition, the cycle efficiency can be written as

$$\eta = \eta_{gen} \frac{(\Delta h)_{isen}}{Q_{in}} \tag{4}$$

and used to evaluate α in equation (3). Just as in the Carnot cycle, there is a lower temperature which minimizes the area of the radiator. This temperature depends on η_{gen}.

Rankine cycle. - Consider a Rankine cycle boiling potassium at 1365 K. The radiation area parameter α as a function of T_{rad} with generator efficiency and degree of superheat as parameters is calculated from equations (3) and (4) and is plotted in figure 2. The temperature at which the area is minimized can be seen to depend only slightly on generator efficiency and degree of superheat. The minimum is also rather flat. Since the temperature for minimum specific area is nearly constant for generator efficiencies greater than 0.60, it will be assumed that it is fixed at about 1070 K for potassium boiling at a temperature of 1365 K. A separate calculation shows that the condensing temperature is 1280 K for potassium boiling temperature of 1645 K. Since the boiling temperature and condensing temperatures are independent of generator efficiency, the high- to low-pressure ratio is independent of η_{gen} and has a value of about 0.1 for both potassium and lithium. This pressure ratio constraint will be placed on the MHD generator in the Rankine cycle system.

Brayton cycle. - The Brayton cycle to be considered is shown in figure 1(b). Ideal isothermal compression is assumed, rather than the usual nearly isentropic. It is advantageous to do this because the radiator area increases rapidly with a decrease in radiator temperature. Rosa (ref. 1) has shown that a three-stage intercooled compressor yields nearly the same radiator area as the isothermal compression, so the calculations will assume that isothermal compression can be sufficiently approximated. The thermodynamic efficiency is derived in appendix B and is

$$\eta = \frac{\eta_{gen} T_{max}(1 - y) - T_2 \ln\frac{1}{y}}{T_{max} - T_3} \tag{5}$$

where

$$y = \left(\frac{P_L}{P_H}\right)^{(\gamma-1)/\gamma}$$

Just as for the Rankine cycle, there is a radiator temperature which minimizes the radiator area for a given maximum temperature, generator efficiency, and (for the Brayton cycle) regenerator effectiveness. In appendix B the radiator temperature which minimized the radiator area for a regenerator effectiveness of 1.0 is found to be

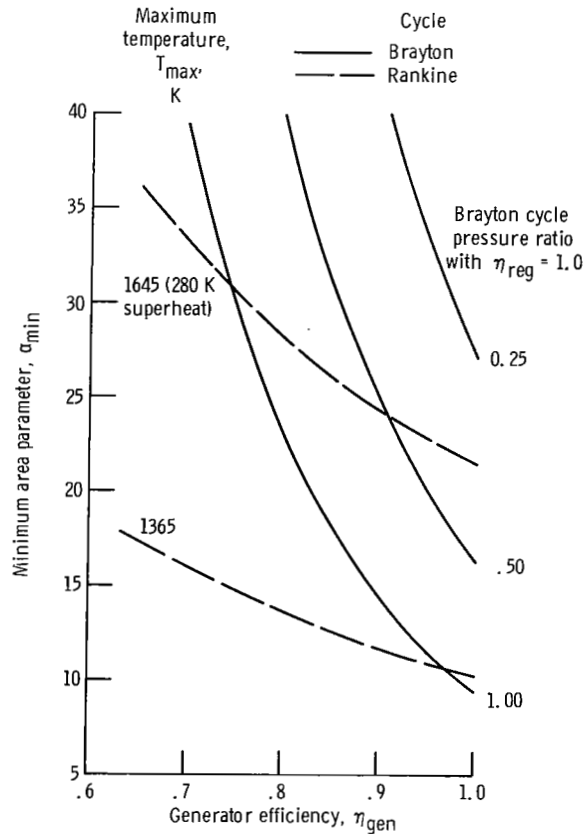$T_2 = 3/4 \ \eta_{gen}\left[(1 - y)/(\ln 1/y)\right]T_{max}$.

Figure 3. - Minimum value of radiator area parameter as function of generator efficiency with maximum temperature as parameter for potassium boiling at 1365 K in Rankine cycle and with cycle pressure ratio as parameter for Brayton cycle (determined from eq. (6)).

For this minimum area radiator temperature, the area parameter can be written as

$$\alpha_{min} = 4\left(\frac{4}{3}\right)^3 \left(\frac{\ln\frac{1}{y}}{1-y}\right)^4 \left(\frac{1}{\eta_{gen}}\right) \tag{6}$$

which differs from the Carnot value by the pressure ratio term and the generator efficiency. The minimum area parameter is shown in figure 3 for pressure ratios of 0.25, 0.50, and 1.00. It can be seen that the parameter increases with decreasing generator efficiency and less sensitively with decreasing pressure ratio. Rosa (ref. 1) showed that by considering frictional effects in the compressor, regenerator, and radiator, the optimum pressure ratio is 0.5. This pressure will be chosen as typical of the optimum for Brayton cycle. Another reason for selecting this ratio will be discussed in the generator design section.

Effect of Generator Efficiency on Minimum Specific Radiator Area

Reference cycle. - The minimum area parameter for a potassium Rankine cycle is plotted in figure 3 as a function of generator efficiency for two cases: no superheat and 280 K superheat. The value of α_{min} increases with increasing superheat and decreasing generator efficiency. However, the minimum specific radiator area $(\mathscr{A}/wW)_{min}$ is nearly independent of degree of superheat. This is true because α_{min} is found to be (eq. (3)) proportional to the fourth power of the maximum temperature. Hence the specific radiator area is constant (eq. (2)). The potassium cycle is typical of turboelectric cycles considered for use in space. A turboalternator efficiency of 0.75 may be reasonable for such a potassium cycle. Therefore, the specific area of MHD systems will be compared with the specific area of this cycle (called the reference cycle).

MHD Rankine cycle. - The specific radiator area ratio is $(\mathscr{A}/wW)/(\mathscr{A}/wW)_{ref}$ and by using equation (2) and the value of α_{min} for potassium boiling at 1365 K, can be written as

$$\frac{\left(\dfrac{\mathscr{A}}{wW}\right)}{\left(\dfrac{\mathscr{A}}{wW}\right)_{ref}} = \frac{\alpha_{min}}{14.8}\left(\frac{1365}{T_{max}}\right)^4 \tag{7}$$

This ratio is computed for lithium boiling at 1430 and 1645 K with generator efficiency

and degree of superheat as parameters. It can be seen in figure 4 that the ratio is rather sensitive to generator efficiency and boiling temperature, but relatively insensitive to the degree of superheat.

From figure 4 it is possible to determine what MHD generator efficiency is required for each T_{max} to provide a specific area which is the same as the potassium reference cycle. In figure 5 this MHD generator efficiency is plotted as a function T_{max} for the two boiling temperatures. Generally, efficiencies above 60 percent are required with a boiling temperature of 1430 K, but this requirement can be relaxed to 38 percent if the boiling temperature of the Rankine-MHD is increased to 1645 K.

Brayton cycle. - The specific area ratio is again given by equation (5) once α_{min} is known. In the case of perfect regeneration α_{min} is given in equation (7) and equation (5) becomes

$$\frac{\left(\dfrac{\mathscr{A}}{wW}\right)}{\left(\dfrac{\mathscr{A}}{wW}\right)_{ref}} = \frac{256}{27}\left(\frac{\ln\dfrac{1}{y}}{1-y}\right)^4 \frac{1}{\eta_{gen}^4}\left(\frac{1365}{T_{max}}\right)^4 \frac{1}{14.8} \tag{8}$$
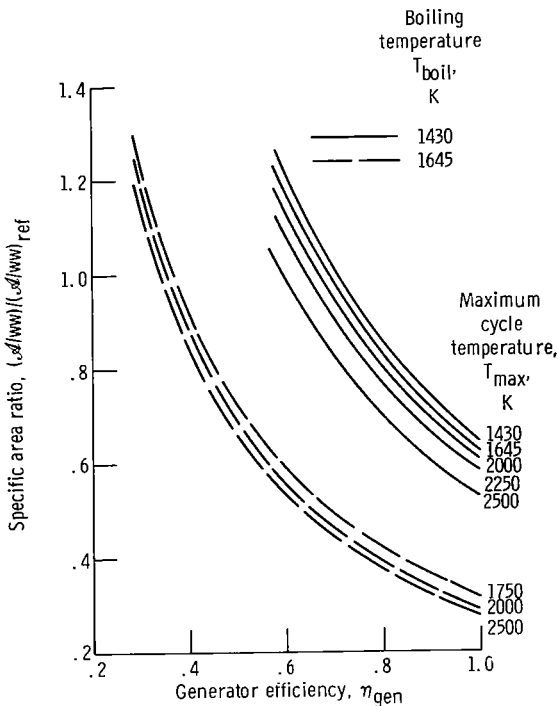

Figure 4. - Specific area ratio as function of generator efficiency with boiling temperature and maximum temperature (degree of superheat) as parameters. Working fluid, lithium.
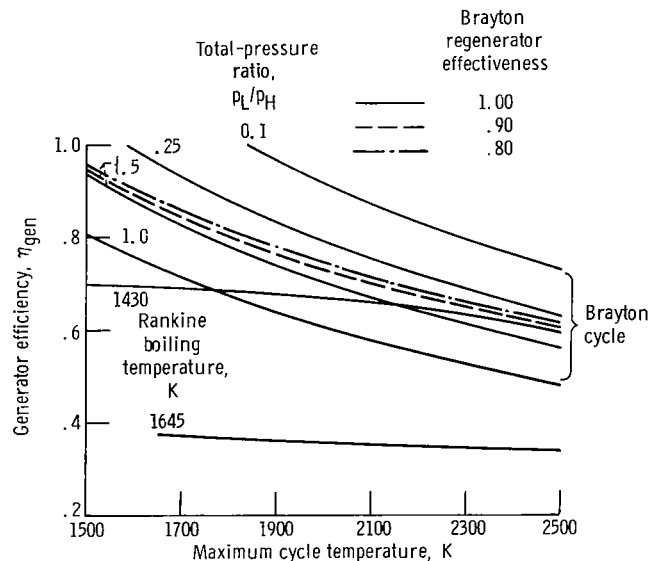

Figure 5. - Generator efficiency as function of maximum cycle temperature required to provide a specific area ratio of 1.0. Cycle pressure ratio and regenerator effectiveness are parameters for Brayton cycle. Brayton cycle curves determined using equation (8).
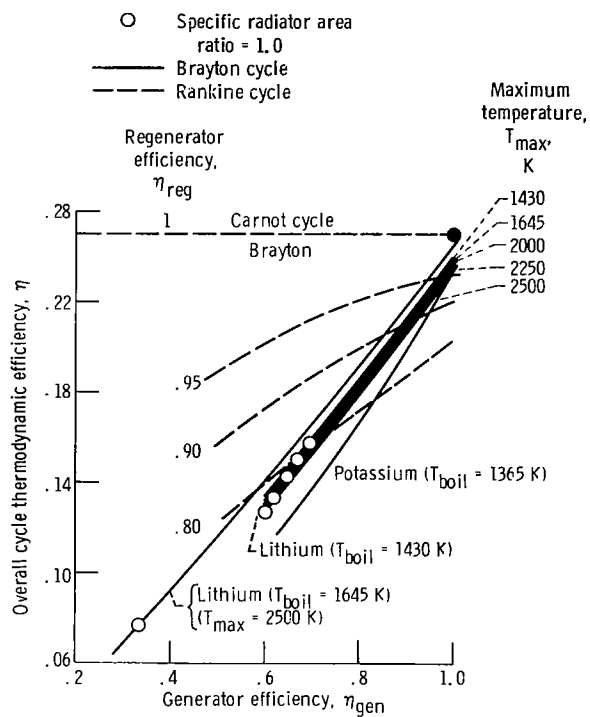
Figure 6. - Overall cycle thermodynamic efficiency as function of generator efficiency. Brayton cycle curves with various regenerator effectivenesses are calculated for pressure ratio of 0.5. All curves calculated for minimum radiator area parameter.

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The generator efficiency required to provide a specific area ratio of 1.0 as a function of maximum cycle temperature can be obtained from equation (8). It is also plotted in figure 5 as a function of maximum cycle temperature. Parameters are pressure ratio and regenerator effectiveness. The required generator efficiency increases with decreasing pressure ratio, and much less sensitively, with decreasing regenerator effectiveness. The calculations for the Brayton MHD cycle will be for a pressure ratio of 0.5 and a regenerator effectiveness of 1.0.

Overall Cycle Thermodynamic Efficiency

The overall cycle thermodynamic efficiency is shown in figure 6. The thermodynamic efficiency of the reference cycle is 15.2 percent. For the Rankine MHD cycle boiling at 1430 K with generator efficiency chosen to provide a specific radiator area ratio of 1.0 and for the Brayton cycle with pressure ratio of 0.5 with regenerator effectiveness greater than 0.8, the cycle efficiency is about the same as the reference cycle. However, for lithium boiling at 1645 K this is not the case. For this cycle with a specific radiator area ratio of 1.0 the efficiency is lower (about 7.8 percent) so that the specific weight of the reactor may be higher than for the other cycles. However, this boiling temperature is still considered because the generator power density will be increased over that at the 1430 K temperature. It appears that the Brayton cycle will have higher efficiency than the Rankine cycle if the regenerator effectiveness is greater than 0.80.

GENERATOR DESIGN

The generator will be designed to operate at the efficiency and at the specified pressure ratio as determined from the preceding analysis. An isentropic nozzle will be used to expand the working fluid to a specified velocity, the generator will convert some of the total enthalpy into electrical energy, and an isentropic diffuser will compress the fluid to the specified low pressure with negligible exit velocity.

Ideal Faraday Segmented Generator

The design of the generator will be based on the Faraday segmented concept (refs. 22

(pp. 479-480) and 23). The working fluid is assumed to be a perfect gas with zero viscosity and thermal conductivity. The generator is assumed to have infinitely fine segmentation so that there is zero current in the Hall direction. The flow can then be described by five one-dimensional equations with nine dependent variables. These equations are

Continuity:

$$\frac{d}{dx}(a\rho u) = 0 \tag{9}$$

Momentum:

$$\rho u \frac{du}{dx} + \frac{dp}{dx} + jB = 0 \tag{10}$$

Energy:

$$\rho u \frac{dh}{dx} + \rho u^2 \frac{du}{dx} - jE = 0 \tag{11}$$

where

$$h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \tag{12}$$

and Ohm's law

$$j = \sigma(uB + E) \tag{13}$$

These equations can be written in nondimensional terms by defining the following variables:

$$A = \frac{a}{a_{ent}} \qquad U = \frac{u}{u_{ent}}$$

$$\Sigma = \frac{\sigma}{\sigma_{ent}} \qquad C = \frac{B}{B_{ent}}$$

$$R = \frac{\rho}{\rho_{ent}} \qquad P = \frac{p}{\rho_{ent}u_{ent}^2}$$

$$K = \frac{-E}{u_{ent}B_{ent}} \qquad X = \frac{\sigma_{ent}B_{ent}^2 x}{\rho_{ent}u_{ent}}$$

$$J = \frac{j}{\sigma_{ent}u_{ent}B_{ent}}$$

With these substitutions, and combining equations (11) and (12) the equations become

$$ARU = 1 \tag{14}$$

$$\frac{dU}{dX} + A\frac{dP}{dX} + JAC = 0 \tag{15}$$

$$\frac{\gamma}{\gamma - 1}\frac{d}{dX}(AUP) + U\frac{dU}{dX} + JKA = 0 \tag{16}$$

$$\frac{J}{\Sigma} + K = UC \tag{17}$$

Equations (14) to (17) can be further reduced to

$$\frac{dU}{dX} + A\frac{dP}{dX} + JAC = 0 \tag{18}$$

$$\frac{\gamma}{\gamma - 1}\frac{d}{dX}(AUP) + U\frac{dU}{dX} + JAC\left(U - \frac{J}{\Sigma C}\right) = 0 \tag{19}$$

Generator Design Criteria

Further assumptions must be made since there are still six variables and only two equations. First, all solutions will be obtained in terms of J_{ent} which will then be specified. Next, two rather arbitrary assumptions will be made which will be shown later to yield generator performance that is quite acceptable. The assumptions are

$$JC = J_{ent} \tag{A}$$

and

$$\frac{J}{\Sigma C} = J_{ent} \qquad (B)$$

Equation (18) and (19) then become

$$\frac{dU}{dX} + A\frac{dP}{dX} + J_{ent}A = 0 \qquad (20)$$

and

$$\frac{\gamma}{\gamma - 1}\frac{d}{dX}(AUP) + U\frac{dU}{dX} + J_{ent}A(U - J_{ent}) = 0 \qquad (21)$$

The current-density ratio can be eliminated from assumptions (A) and (B). Then one has the requirement that

$$\Sigma C^2 = 1 \qquad (22)$$

which implies that

$$\sigma B^2 = \sigma_{ent} B^2_{ent}$$

This is a restriction on the electrical conductivity and magnetic-field strength. The electrical conductivity for the working fluids considered also depends on the Hall parameter which will be introduced and discussed in the section Electrical Conductivity. By adjusting this parameter a wide range of entrance values of σB^2 can be obtained, and the requirement that σB^2 be a constant can be satisfied. Thus the restriction on the conductivity is not as limiting for these working fluids as it may seem. This dependence of the conductivity on the Hall parameter will be examined critically in the section Electrical Conductivity.

Finally, one more assumption must be made before MHD generator performance can be evaluated. This assumption involves choosing the variation of generator cross-sectional area from the entrance to the exit of the generator. The change in total pressure is fixed from cycle considerations. The overall cycle efficiency is proportional to the generator efficiency at this pressure ratio. Also, the radiator area is sensitive to the generator efficiency. It is important, therefore, to maintain as high a generator efficiency as possible for the required pressure ratio.

Resistance heating of the working fluid is the only loss considered in the generator. It has been further assumed that the loss is uniform along the duct. Therefore, the total loss can be minimized by minimizing the volume of the generator. Also, a minimum-volume generator will tend to keep the volume and hence weight of the magnet system small. This may be an important factor especially if large magnetic-field strengths are required. For these reasons the generator will be designed to have a minimum volume for the specified pressure ratio. This minimizing procedure involves the use of the calculus of variations and is shown in appendix C.

The choice of minimizing the volume as opposed to maximizing efficiency is somewhat arbitrary. However, there is a further restriction on the parameters that sometimes appears when efficiency is maximized. Pleshanov (ref. 24) finds an optimum only for supersonic generators. It turns out that they are also constant Mach number generators. The restrictive values of the current (e.g., at $M_{ent} = 2.0$, $J_{ent} = 1/8$) severely limits the amount of nonequilibrium conductivity that one can achieve, which in turn limits the power density. This point will be discussed in the section Electrical Conductivity. Therefore a less restrictive assumption should be studied. It turns out that maximizing the efficiency rather than minimizing the volume under the same conditions provides nearly the same value of efficiency.

Solutions to Generator Equations

Typical solution. - In figure 7 the area ratio A and the volume V are shown as functions of pressure ratio. As the pressure ratio decreases, both A and V increase. There is some pressure ratio at which the area increases so rapidly that the one-dimensional flow equations are no longer valid (the flow probably would separate). However, it is assumed that this does not occur before a pressure ratio of 0.1, which is the pressure ratio chosen for the Rankine cycle.

Power output and generator efficiency. - Under the assumptions made, it is possible to calculate the generator efficiency as a function of M_{ent}, J_{ent}, and p_L/p_H. This derivation of the efficiency closely parallels that found in reference 3. First, the conversion effectiveness will be defined. The conversion effectiveness is the fraction of the stagnation enthalpy flux entering the generator which is converted into electrical energy. The stagnation enthalpy entering the generator is called the characteristic power $\mathscr{P}_{ch}$ and is defined as

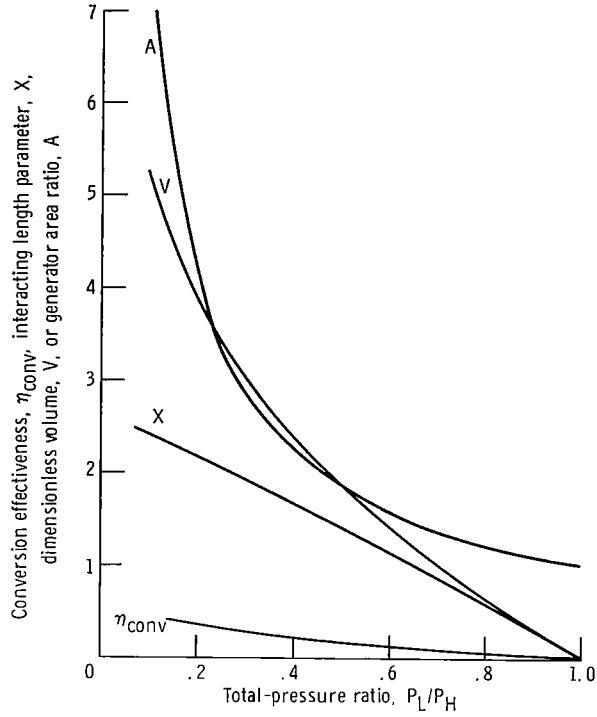$$\mathscr{P}_{ch} = u_{ent}\rho_{ent}h_{max} \tag{23}$$

Figure 7. - Variation of generator variables as function of total-pressure ratio. Entrance Mach number, 1.0; entrance current density ratio, 0.25; specific heat ratio, 5/3.

The characteristic power can be expressed in terms of the design parameters as

$$\mathscr{P}_{ch} = P_H \sqrt{\gamma \frac{k}{m} T_{max}} \ \frac{\frac{\gamma}{\gamma - 1} M_{ent}}{\left(1 + \frac{\gamma - 1}{2} M_{ent}^2\right)^{(\gamma+1)/2(\gamma-1)}} \tag{24}$$

Now, the conversion effectiveness can be defined as

$$\eta_{conv} \equiv \frac{1}{a_{ent} \mathscr{P}_{ch}} \int_0^L ja(-E) \, dx = 1 - \frac{T_5}{T_{max}} \tag{25}$$

The power output Π is

$$\Pi = \eta_{conv} a_{ent} \mathscr{P}_{ch} \tag{26}$$

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Introducing the nondimensional quantities so that

$$\eta_{conv} = \frac{\int_0^X (U - J)A \, dX}{\frac{1}{2} + \frac{1}{(\gamma - 1)M_{ent}^2}} \qquad (27)$$

The integral may be evaluated by using equation (21), and yields

$$\eta_{conv} = 1 - \frac{\frac{1}{2} + \frac{1}{(\gamma - 1)M^2}}{\frac{1}{2} + \frac{1}{(\gamma - 1)M_{ent}^2}} U^2 \qquad (28)$$

The conversion effectiveness is calculated from the generator solutions and is shown in figure 7. It can be seen that the variation of conversion effectiveness with pressure ratio is nearly linear. This is an interesting consequence of the variational procedure. A linear variation means that a constant amount of work is obtained for a given change in total pressure – whether this change is near the generator entrance (high pressure) or near the exit (low pressure). The conversion effectiveness is shown in figure 8 as a function of entrance current-density ratio with entrance Mach number, as parameter at the two pressure ratios of interest. Note that η_{conv} is higher at lower pressure ratio, low Mach number, and low entrance current-density ratio.

The generator efficiency is defined as the actual change in enthalpy divided by the

(a) Total-pressure ratio, 0.1.   (b) Total-pressure ratio, 0.5.
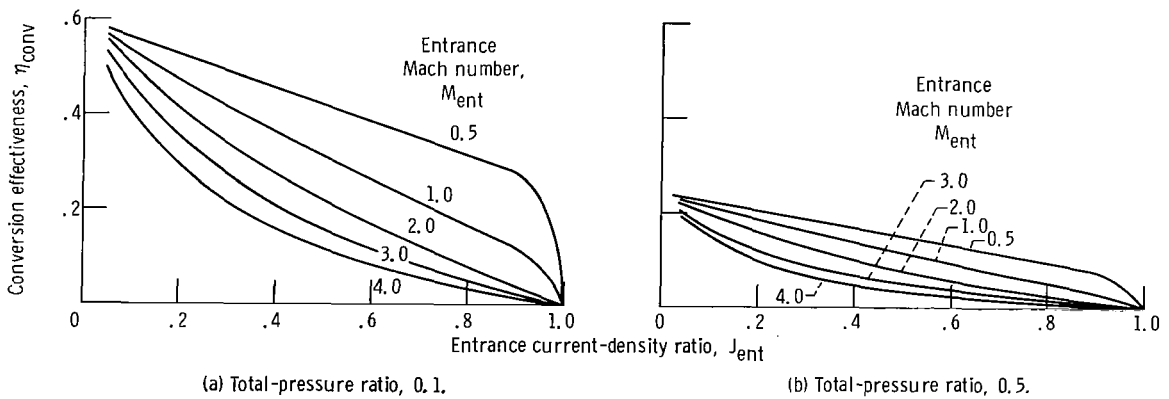
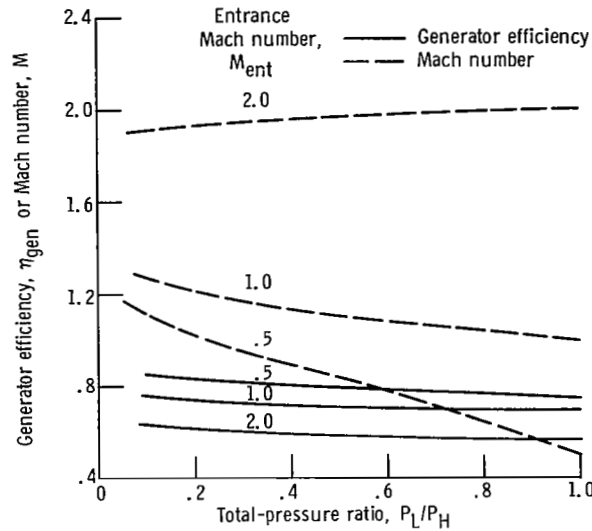Figure 8. - Generator effectiveness as function of current density ratio.

Figure 9. - Mach number and generator efficiency as function of total-pressure ratio. Entrance current density ratio, 0.25; specific heat ratio, 5/3.

isentropic change in enthalpy between the same pressure limits. The generator efficiency can be expressed in terms of the conversion effectiveness (refs. 3 and 7) as

$$\eta_{gen} = \frac{\eta_{conv}}{1 - \dfrac{\eta_{conv}}{AU\left(\gamma M_{ent}^2 P\right)^{1/\gamma}}} \tag{29}$$

The variation in both the Mach number calculated from the solution to equations (20) and (21) and the generator efficiency calculated from equation (28) is shown in figure 9 for three values of entrance Mach number.

The generator efficiency at a pressure ratio of 0.5 for the optimal conditions of Pleshanov (ref. 24) of $J_{ent} = 1/8$ and $M_{ent} = 2.0$ is 0.775, whereas from figure 9 the efficiency is 0.765. The minimum volume solution is very close to the maximum efficiency case as computed by Pleshonov. Notice that the generator efficiency is rather insensitive to pressure ratio. The Mach number varies much more for subsonic than sonic or supersonic entrance velocities.

In figure 10 the generator efficiency is shown as a function of entrance current-density ratio with entrance Mach number as parameter at the two pressure ratios of interest. The efficiency approaches 1.0 as the entrance current-density ratio approaches zero. The efficiency decreases with increasing current-density ratio much faster for the

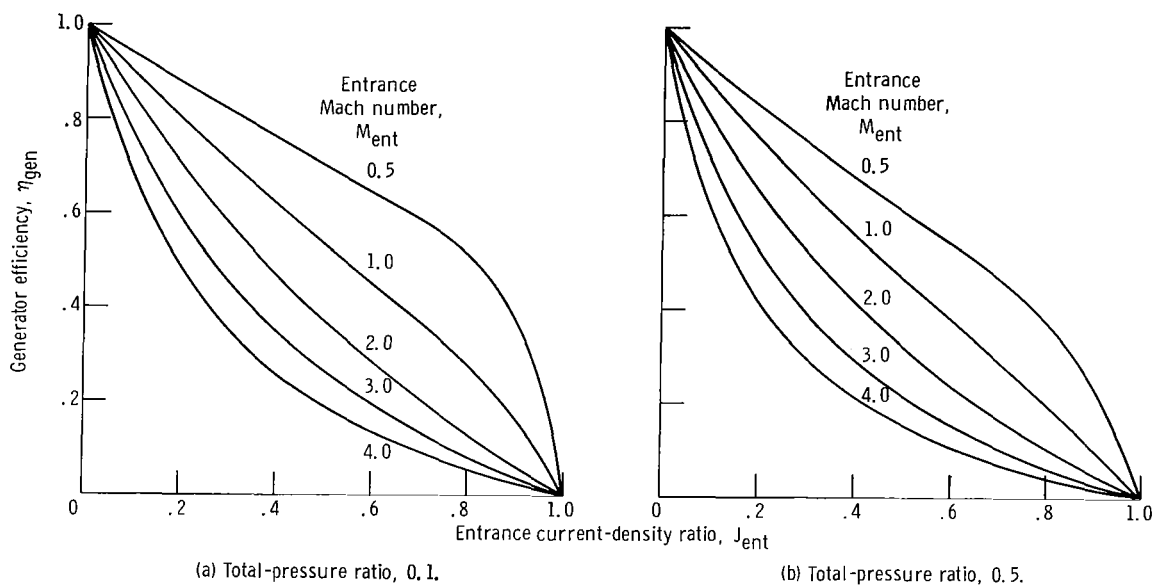(a) Total-pressure ratio, 0.1. (b) Total-pressure ratio, 0.5.

Figure 10. - Generator efficiency as function of current-density ratio.

high Mach numbers. This fact will limit the use of supersonic MHD generators to applications where generator efficiency is not critical (e.g., in a high-temperature cycle).

Brayton cycle pressure ratio. - At this point it is possible to make a further comment about the pressure ratio for the Brayton cycle. Figure 5 allows the determination of the required generator efficiency as a function of pressure ratio for a fixed temperature and fixed regenerator effectiveness. From curves similar to those in figure 10 it is possible to calculate the current-density ratio required to provide that generator efficiency as a function of pressure for a fixed entrance Mach number. The variation in entrance current-density ratio with pressure ratio for T_{max} = 2000 K, η_{reg} = 1.0, and M_{ent} = 0.5 is shown in figure 11. A significant change in the current-density ratio occurs at a pressure ratio of about 0.5. It will be shown later that operating at a lower current-density ratio will have an adverse effect on power density and is to be avoided if possible. This is another argument in favor of a pressure ratio of about 0.5 for the Brayton cycle.

Specification of entrance current-density ratio. - Now it is possible to relate the generator entrance current density to the maximum cycle temperature. This is accomplished by requiring the specific radiator area ratio to be 1.0. Then one can take the generator efficiency requirements from figure 5 and deduce the required entrance current-density ratio from figure 10. The results are shown in figure 12 for the Brayton cycle and the Rankine cycle boiling lithium at 1430 K. Note that the current density increases with decreasing Mach number at any fixed maximum cycle temperature.
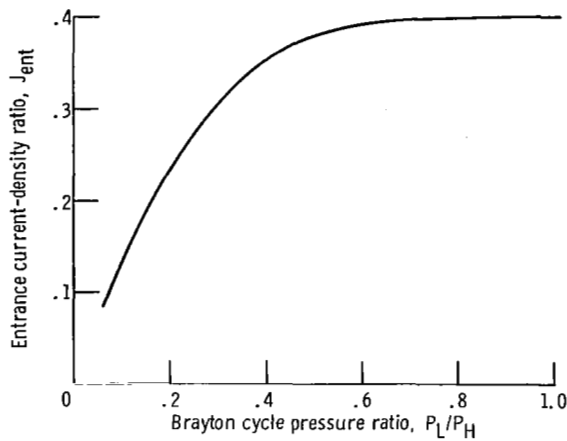
Figure 11. - Entrance current density ratio required to provide specific area ratio of 1.0 for Brayton cycle as function of cycle pressure ratio. Entrance Mach number, 0.5; maximum cycle temperature, 2000 K.

(a) Brayton cycle; regenerator efficiency, 1.0; total-pressure ratio, 0.5.

(b) Rankine cycle; boiling temperature, 1430 K; total-pressure ratio, 0.1.

Figure 12. - Entrance current-density ratio as function of maximum cycle temperature. Specific radiator area ratio, 1.0.

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Generator Interaction Parameter

The generator pressure drop is specified from overall cycle considerations. It is important to calculate the nondimensional interaction parameter X required to achieve this pressure drop. This parameter can be calculated by rewriting equation (20) as

$$J_{ent} \, dX = - \, dP - \frac{1}{A} \, dU \qquad\qquad (30)$$

and integrating it from the entrance to the exit of the generator, such that

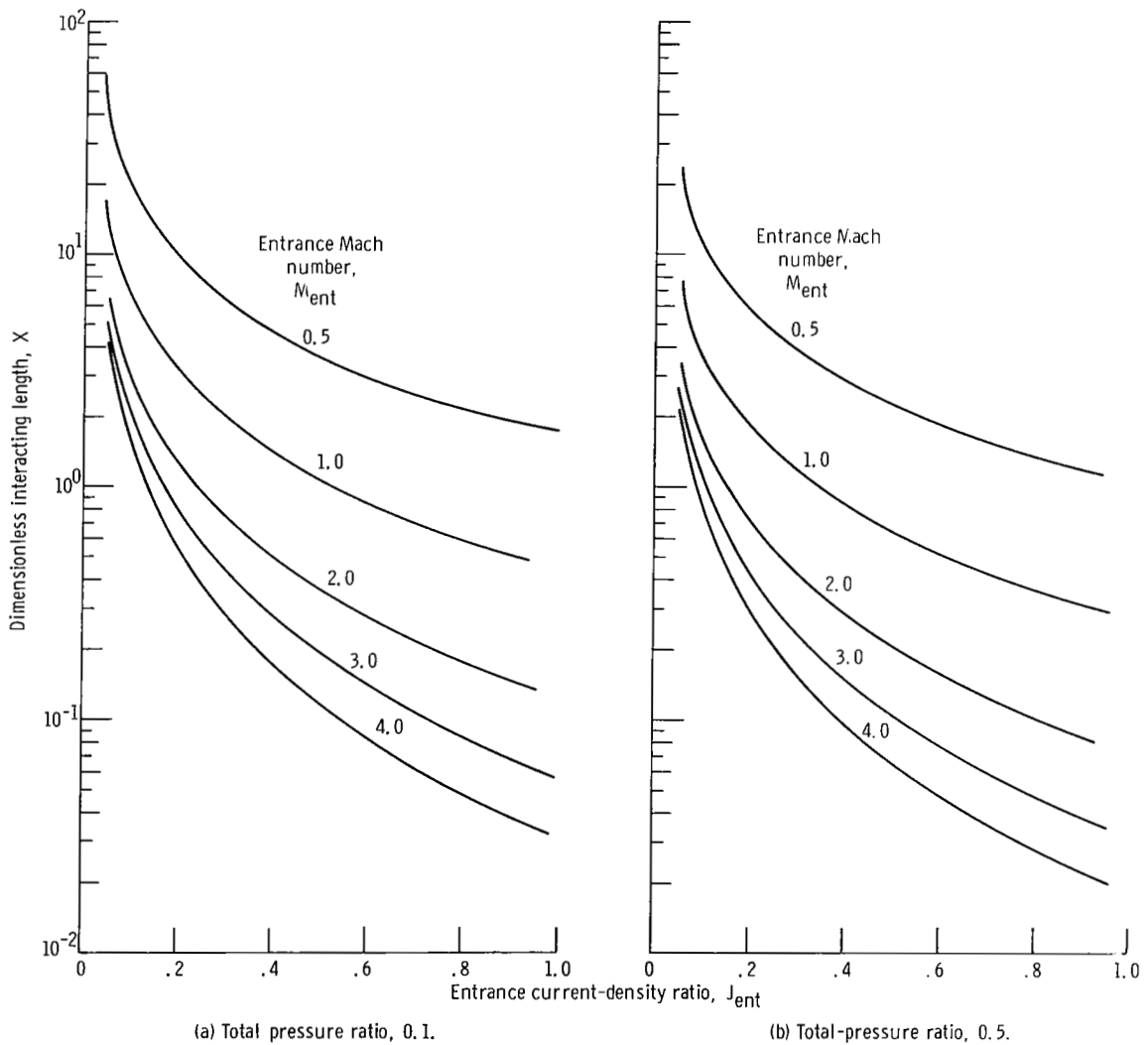(a) Total pressure ratio, 0.1.

(b) Total-pressure ratio, 0.5.

Figure 13. – Dimensionless interacting length as function of current-density ratio. Curves determined from equation (30).

$$J_{ent}X = \frac{1}{\gamma M_{ent}^2} - P - \int_1^U \frac{dU}{A} \qquad (31)$$

The variation of the area with the velocity is known from the minimum volume solution. The variation of the interacting length with total-pressure ratio is shown in figure 7 for a typical case $M_{ent} = 1.0$, $J_{ent} = 1/4$, and $\gamma = 5/3$. The nearly linear variation of X with pressure ratio, and hence conversion effectiveness (see fig. 7), means that a uniform amount of power per unit length is generated from one end of the generator to the other. This is an acceptable criterion for designing a generator, and it provides a rationale for the choice of $JC = J_{ent} = J/\Sigma C$. Two generator pressure ratios are of interest: 0.5 for the Brayton cycle and 0.1 for the Rankine. Figure 13 shows the interacting length required to achieve these pressure ratios with the entrance Mach number as parameter. Notice that the interacting length increases with decreasing current at a fixed Mach number and with decreasing Mach number at a fixed current. Also, the length required is longer for the lower pressure ratio.

Specification of Generator Length, Area, and Power Output

Generator length specification. - The generator length is a very important consideration, especially in space. Thermodynamic considerations dictate the total-pressure drop that must be achieved from the entrance to the exit of the generator. Momentum conservation relates this pressure drop to the generator length. The length of the magnet (and hence the generator) must be kept small to keep the weight small. The generators considered in this study will therefore be designed to have a specified length of 0.5 meter. This generator length with an entrance area of 0.01 square meter will generate between 0.1 and 1.0 megawatt depending on the values of other parameters. If the characteristic length L_{ch} is defined as

$$L_{ch} \equiv \frac{\rho_{ent} u_{ent}}{\sigma_{ent} B_{ent}^2} \qquad (32)$$

and the actual generator length is L, then, from the definition of the dimensionless length X, one has $L_{ch} = L/X$ where $L = 0.5$ meter and X is given in figure 13.

Generator entrance area. - The generator length to provide the required pressure drop has been specified. The generator entrance area can be varied independently to determine the amount of power generated for that pressure drop. However, in actual

generators, the entrance area must be neither too large nor too small relative to the length of the generator. If the area is too large, then there will be generator end effects and field fringing, whereas if the area is too small, then friction and heat-transfer effects become important. Therefore, in this analysis the entrance area will be expressed in terms of the generator length and an aspect ratio. The square root of the entrance area is chosen as a characteristic dimension to define the aspect ratio, $\mathcal{R}$:

$$\mathcal{R} = \frac{L}{\sqrt{a_{ent}}} \tag{33}$$

Reasonable values of $\mathcal{R}$ would be 5 to 10.

Power output. - Now the power output of the generator Π from equation (26) can be written in terms of $\mathcal{R}$

$$\Pi = \frac{L^2}{\mathcal{R}^2} \eta_{conv} \mathcal{P}_{ch} \tag{34}$$

The function η_{conv} can be determined from the minimum volume solution and is shown in figure 13 for pressure ratios of 0.5 and 0.1. In the comparisons made in this analysis, it will be assumed that all generators have the same value of the aspect ratio. Therefore, the comparison can be made on the basis of $\mathcal{R}^2\Pi$.

Specification of generator length and aspect ratio sets the power level of the generators, but the appropriate values of the other design parameters are insensitive to the level of power output. For example, for a typical set of design conditions, increasing the length by a factor of 10 increases the power level by a factor of 100, but the required Hall parameter and magnetic-field strength is lowered by only about 25 percent.

Specifications of Working Fluid Properties and Magnetic-Field Strength

The specification of L and calculation of X lead to a specification of the characteristic length L_{ch} where

$$L_{ch} = \frac{\rho_{ent} u_{ent}}{\sigma_{ent} B_{ent}^2}$$

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The characteristic length can be expressed in terms of the high pressure and the maximum temperature by using the isentropic relations for a perfect gas:

$$L_{ch} = \frac{p_H}{\sqrt{\gamma \frac{k}{m} T_{max}}} \frac{1}{\sigma_{ent} B_{ent}^2} \frac{\gamma M_{ent}}{\left(1 + \frac{\gamma - 1}{2} M_{ent}^2\right)^{(\gamma+1)/2(\gamma-1)}} = \frac{L}{X} \tag{35}$$

which can also be rewritten as

$$\sigma_{ent} B_{ent}^2 = \frac{p_H}{\sqrt{\gamma \frac{k}{m} T_{max}}} \frac{X}{L} \frac{\gamma M_{ent}}{\left(1 + \frac{\gamma - 1}{2} M_{ent}^2\right)^{(\gamma+1)/2(\gamma-1)}} \tag{36}$$

This equation specifies $\sigma_{ent} B_{ent}$ in terms of entrance Mach number, high pressure, generator length, and maximum cycle temperature (because X is known at a given M_{ent} and T_{max}). In order to evaluate the B_{ent} and σ_{ent} values separately, one must express the conductivity σ_{ent} in terms of the same parameters that appear on the right side of equation (36).

ELECTRICAL CONDUCTIVITY

Two-Temperature Conductivity for Faraday Segmented Generator

For a Faraday segmented generator the proper electrical conductivity, when ion-slip is negligible, is (ref. 3):

$$\sigma = \frac{e^2 N_e}{m_e \gamma_e} \tag{37}$$

The terms appearing in this expression can be evaluated for the conditions pertinent in this analysis by using the two-temperature plasma (refs. 3 and 22 (pp. 490-495)). In this calculation the electrons may have a temperature different from the heavy species because of the joule heating. This temperature is determined by the requirement that the energy added to the electrons by joule heating must (in the steady state) be balanced by the heat lost from the electrons by collisions with the heavy species; that is,

$$\frac{j^2}{\sigma} = 3 \frac{m_e}{\overline{m}} \frac{\nu_e N_e}{\Delta_{incl}} k(T_e - T) \tag{38}$$

One further assumes that the electron number density can be determined by using the Saha equation at the electron temperature (refs. 25 and 26). Introducing the nondimensional variables and the Hall parameter β_e (where $\beta_e = eB/m_e \nu_e$) into equation (38) results in

$$\frac{T_e}{T} = 1 + \frac{\gamma}{3} \frac{J^2}{\Delta_{inel}} \frac{M^2}{U^2} \frac{N_{e,ent}}{N_e} \frac{\nu_e}{\nu_{e,ent}} \frac{\beta_e^2}{C^2} \tag{39}$$

But it has been assumed that

$$\sigma B^2 = \sigma_{ent} B_{ent}^2$$

which from equation (37) implies that

$$\frac{N_e}{N_{e,ent}} \frac{\nu_{e,ent}}{\nu_e} C^2 = 1$$

Since $J^2/\Sigma = J_{ent}^2$ equation (39) can be written as

$$\frac{T_e}{T} = 1 + \frac{\gamma}{3} \frac{J_{ent}^2}{\Delta_{inel}} \frac{M^2}{U^2} \beta_e^2 \tag{40}$$

At the generator entrance, the equation is

$$\frac{T_e}{T_{max}} = \frac{1 + \dfrac{\gamma}{3} \dfrac{J_{ent}^2}{\Delta_{inel}} M_{ent}^2 \beta_{e,ent}^2}{1 + \dfrac{\gamma - 1}{2} M_{ent}^2} \tag{41}$$

where Δ_{inel} and ν_e are given in reference 3. The conductivity is calculated from the equations given in reference 3 at the entrance condition. As in reference 3, it is assumed

that there may be a seed material required to increase the electron concentration. The product σB^2 is assumed to be a constant equal to its value at the entrance of the generator. It is now apparent that σ_{ent} can be expressed in terms of the working fluid, the seed material, and six independent parameters: p_H, T_{max}, M_{ent}, J_{ent}, $\beta_{e,ent}$, and s (the amount of seed material).

Seed Fraction Specification

<u>Rankine cycle</u>. – The author has determined by a separate calculation that there is a value of s which minimizes L_{ch} for Rankine cycle application. A few typical curves from this calculation are shown in figure 14. The curves illustrate that the optimum value

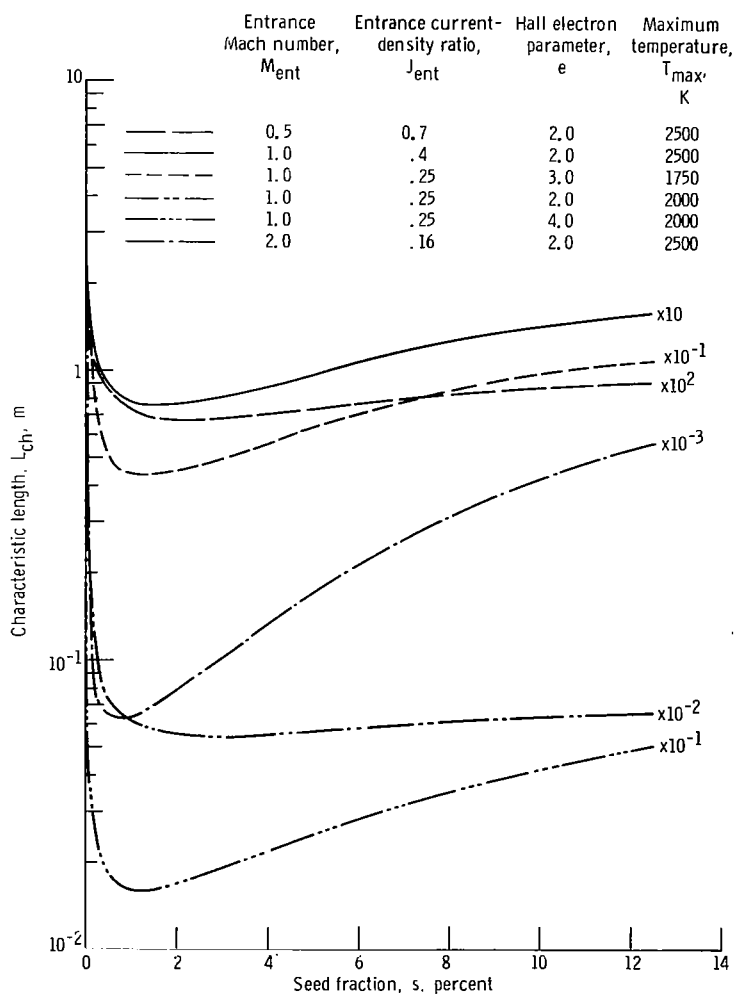| Entrance Mach number, M_{ent}	Entrance current-density ratio, J_{ent}	Hall electron parameter, β_e	Maximum temperature, T_{max}, K
0.5	0.7	2.0	2500
1.0	.4	2.0	2500
1.0	.25	3.0	1750
1.0	.25	2.0	2000
1.0	.25	4.0	2000
2.0	.16	2.0	2500

Figure 14. – Characteristic length as function of seed fraction for lithium boiling at 1430 K.

of s is only slightly dependent on the other five parameters for the range of interest and equals about 0.01. A value for s of 0.01 will therefore be used for all Rankine calculations made in this report.

Brayton cycle. - The seed fraction specification for the Brayton cycle is quite different. No seed fraction minimizes the characteristic length. In figure 15 the magnetic-field strength and Hall parameter necessary to provide the required pressure ratio in a 0.5-meter generator length at fixed current-density ratio, Mach number, and maximum
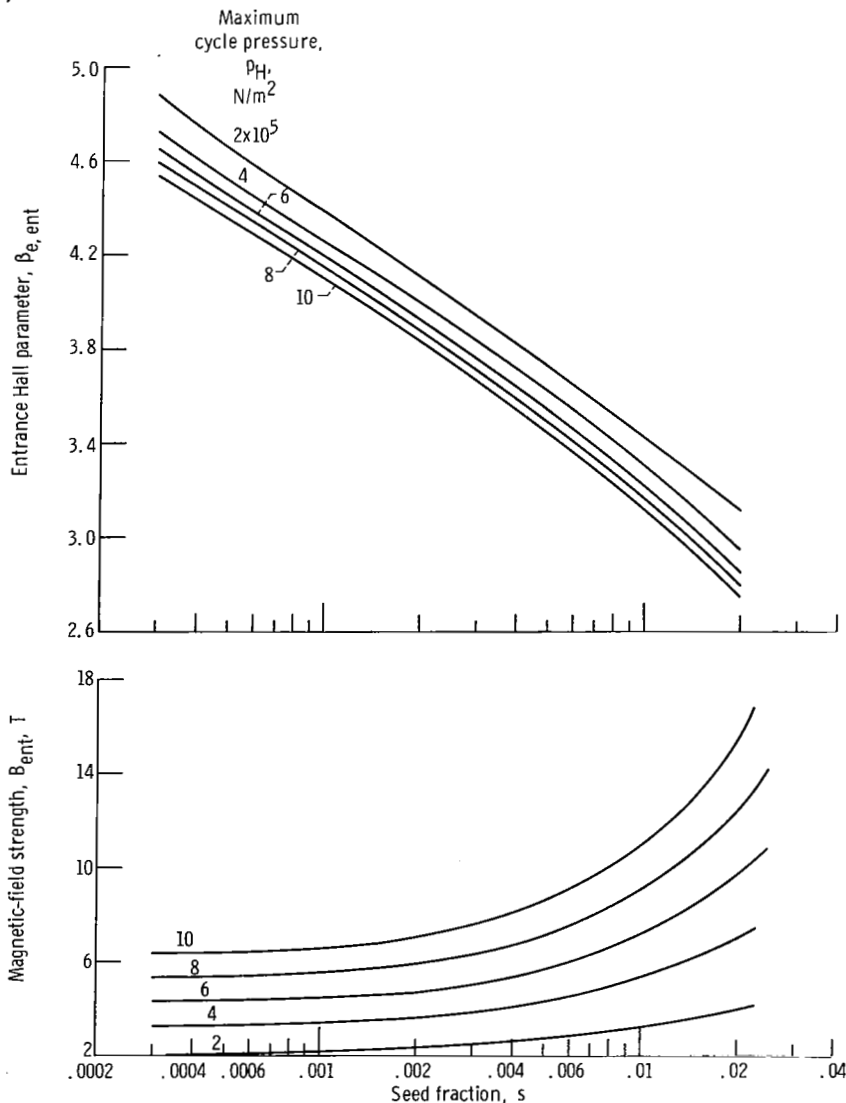

Figure 15. - Entrance Hall parameter and magnetic-field strength required to provide a generator length of 0.5 meter as function of seed fraction. Entrance Mach number, 0.5; entrance current-density ratio, 0.418; maximum cycle temperature, 2100 K; generator pressure ratio, 0.5; working fluid neon; seeded with cesium. Each curve corresponds to constant total electrical power generated.

temperature (this implies an almost constant total-power generated) is shown as a function of seed fraction. The magnetic-field required increases with increasing seed, while the Hall parameter decreases with increasing seed. If the seed fraction gets greater than 0.02, the magnetic-field strength begins to increase rapidly. As a result, the amount of power generated begins to decrease at the same value of s. At seed fractions below 0.001, the magnetic-field strength is approxiamtely at its minimum. There is a minimum because of ion-slip. A variation in s from 0.001 to 0.020 can therefore be considered as acceptable. It is possible to select the value of seed fraction based on either a low Hall parameter or low magnetic field strength. Both 0.020 and 0.001 will be considered in the calculations.

RESULTS AND DISCUSSION

There are six design parameters (s, p_{max}, T_{max}, M_{ent}, J_{ent}, and β_e) that are considered. The parameters J_{ent} and β_e will be chosen so as to provide a specific radiator area ratio of 1.0 (as shown in fig. 12) and a generator length of 0.5 meter (using eq. (41) to specify T_e, which, in turn, provides σ_{ent} for use in eq. (32)). This reduces to four the number of design parameters that are free to be chosen. The effect of these remaining variables on the generator performance was parametrically examined for the space MHD cycles.

Rankine Cycle

In the Rankine cycle the seed fraction is fixed at 0.01 since this minimizes the generator characteristic length. The value of p_H is fixed when the boiling temperature is picked. Two boiling temperatures are considered: 1430 and 1645 K.

<u>Boiling at 1430 K.</u> - For the 1430 K (p_H = 2.7 N/cm^2) generator, figure 16 shows the magnetic-field strength required to provide a generator length of 0.5 meter as a function of the maximum cycle temperature for entrance Mach numbers of 0.5, 1.0, and 2.0. Hall parameter and generator efficiency are parameters. Generally, at the lower temperatures a high Hall parameter but relatively low magnetic-field strength is required for a supersonic Mach number, whereas a high magnetic-field strength but lower Hall parameter is needed for a subsonic Mach number. Also, high Hall parameter and magnetic-field strength are both required at high generator efficiencies. The dotted curves in figure 16 show that the efficiencies necessary to achieve a specific radiator area
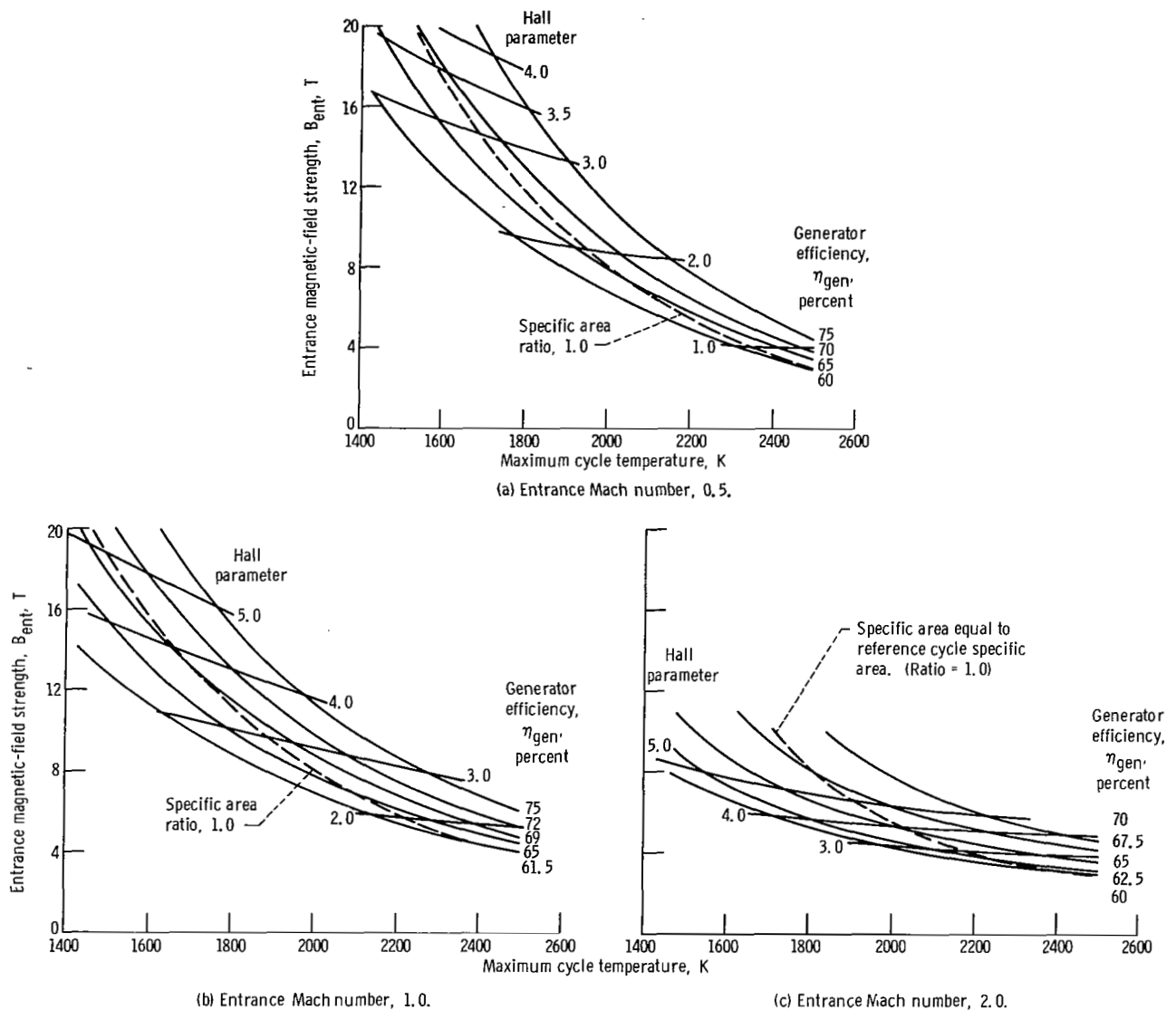
Figure 16. - Entrance magnetic-field strength required to provide generator length of 0.5 meter as function of maximum cycle temperature with generator efficiency as parameter. Lithium boiling at 1430 K with a seed fraction of 0.01.

ratio of 1.0. The power generated is shown in figure 17 for Mach number 0.5, 1, and 2, with generator efficiency as parameter. It can be seen that increasing generator efficiency (e.g., 61.5 to 75 percent) will increase the power output. Remember that this will also increase the required magnetic-field strength and the Hall parameter. The dotted curves are calculated to provide a specific area ratio of 1.0. It can be seen that the $M_{ent} = 1.0$ provides the maximum power output.

Boiling at 1645 K. - In figure 18 the magnetic-field strength required to provide a generator length of 0.5 meter with an entrance Mach number of 2.0 is plotted as a function of maximum temperature for a system with boiling temperature at 1645 K. Only an
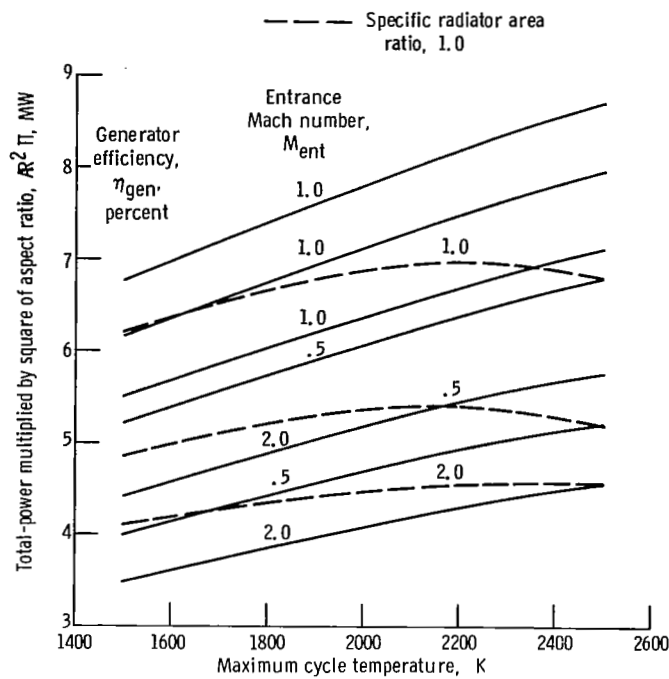
Figure 17. – Total output power multiplied by square of aspect ratio as function of maximum cycle temperature with entrance Mach number and generator efficiency as parameters. Rankine cycle; lithium boiling temperature, 1430 K; seed fraction, 0.01; generator length, 0.5 meter.
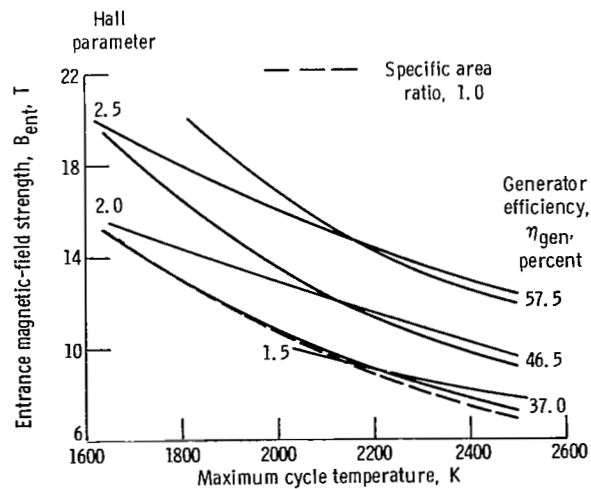

Figure 18. – Entrance magnetic-field strength required to provide generator length of 0.5 meter as function of maximum cycle temperature. Lithium boiling temperature, 1645 K; seed fraction, 0.01; entrance Mach number, 2.0.

entrance Mach number of 2.0 is considered because the magnetic-field strength required to yield a generator length of 0.5 meter is excessive for Mach numbers much less than 2.0. Generator efficiency is used as a parameter in figure 18. Generator efficiency of 37 percent corresponds approximately to a specific radiator area ratio of 1.0, and generator efficiency of 57.5 percent corresponds approximately to overall cycle efficiencies comparable to the reference cycle. At the 57.5-percent efficiency, the magnetic-field strength becomes greater than 20 tesla at temperatures below 1800 K. The choice of generator efficiency depends on whether the weight of the reactor is diminished sufficiently at the higher generator efficiency to compensate for the increased magnet weight. It appears that the specific weight of such magnet systems may be excessive - so we will confine our discussion to the former system. In this case the overall cycle efficiency of the system with lithium boiling at 1645 K will be low. This is not consistent with the assumption that all systems have the same overall cycle efficiency. Nevertheless it is included as an illustration of one method by which generator power output density may be improved. It may still be possible to maintain the same reactor specific weight if the thermal power of the reactor can be increased sufficiently (nearly doubled) for the same weight. This possibility may exist as a result of increasing the system pressure by a factor of 6. Thus, the overall system weight may not suffer as a result of the reduction in generator efficiency. Whether or not this is possible depends on the reactor design and the system requirements.

In figure 19 the power output is plotted for the generator with efficiency of 37 percent (i.e., a specific radiator area ratio of 1.0). The power output is increased by a factor of 4 over the 1430 K boiling temperature for the same magnetic-field strength. Also shown in figure 19 are the generator entrance and exit values of the Hall parameter and the magnetic-field strength. Remember that the design criteria for the generator was based on constant σB^2, which, in turn, provided us with nearly uniform power generation along the duct. The calculations so far have been for the entrance condition, where the working fluid pressure is the highest. The pressure decreases along the generator (to 0.1 of its value in the case of the Rankine cycle). The conductivity will increase as a result. Hence, the magnetic field must decrease to keep σB^2 constant. The change is shown in figure 19. It also turns out that the Hall parameter will increase but not nearly as much as may be indicated by the decrease in pressure. At any rate, the maximum Hall parameter in the generator will occur at the exit. This must be considered in the design.

The dependence of the pressure on the temperature of the working fluid (because of the saturation curve) is a severe limitation in the Rankine-MHD system. The electron mobility is proportional to the fluid density. The density can therefore be changed only by changing the temperature. Even though radiator specific weight decreases with increasing boiling temperature, the magnet specific weight increases because of the
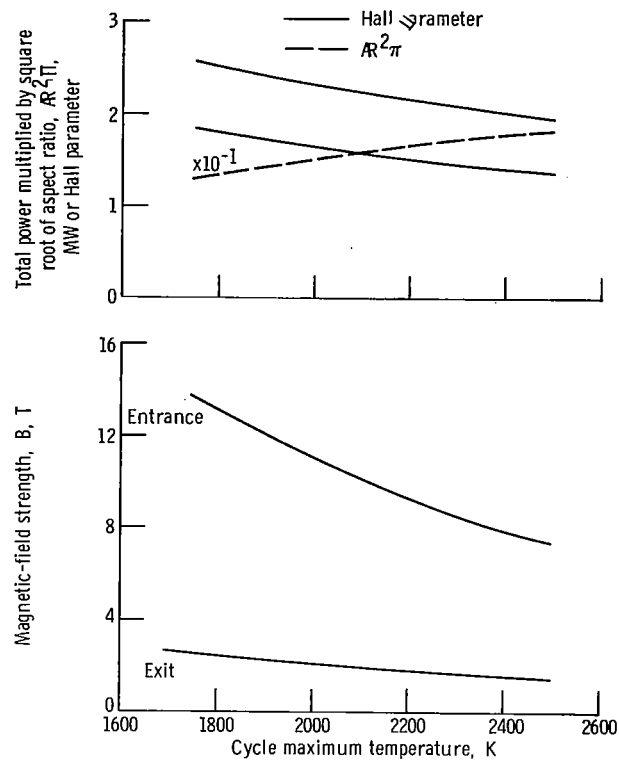
Figure 19. – Magnetic-field strength and Hall parameter required to provide given amount of total power (times aspect ratio squared) as function of maximum cycle temperature. Rankine cycle; lithium boiling temperature, 1645 K; seed fraction, 0.01; generator efficiency, 38 percent; length, 0.5 meter; entrance Mach number, 2.0; specific area ratio, 1.0.

decrease in mobility. In a Brayton cycle the pressure, and hence the electron mobility, can be changed independently of the temperature because there is no change of phase of the working fluid.

Brayton Cycle

Calculations have been made for a neon-cesium Brayton cycle. Again a generator length of 0.5 meter is selected, and the efficiency picked so as to maintain a specific area equal to that of the reference cycle. The effect of system pressure is determined for entrance Mach numbers of 0.5, 1.0, and 2.0 at both entrance and exit conditions. For a seed fraction of 0.02 the entrance curves are shown in figure 20, and the exit con-
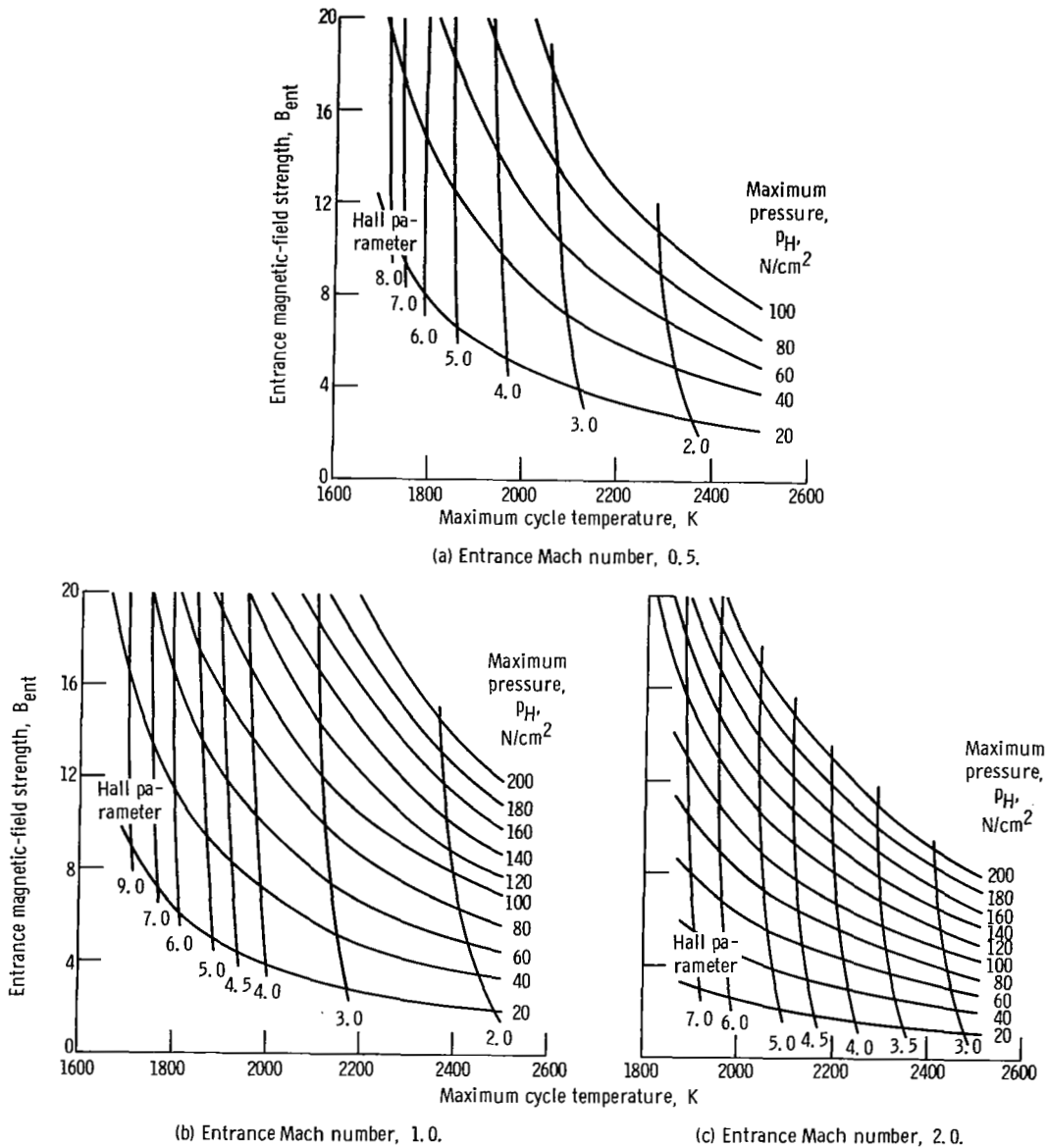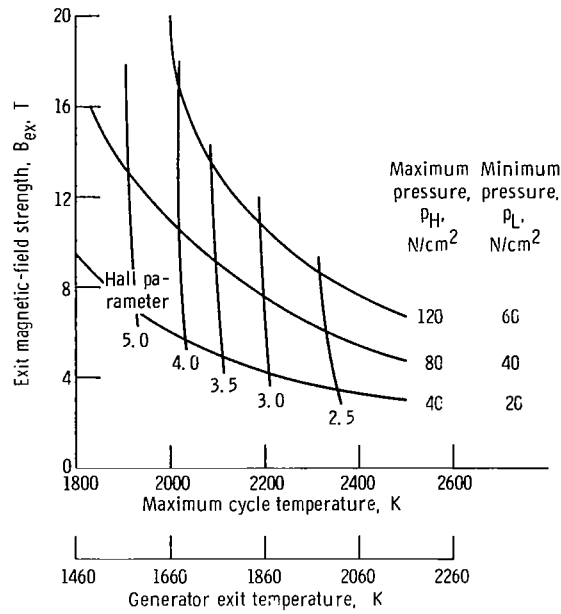
Figure 20. - Entrance magnetic-field strength required for Brayton cycle as function of maximum temperature. Seed fraction, 0.02; specific area ratio, 1.0; generator length, 0.5 meter.

ditions in figure 21. This seed rate is the condition that maintains as low a value of Hall parameter as possible without reducing the power output significantly. However, the magnetic-field strength required is increased.

Effect of pressure, temperature, and Mach number. - In figure 20 it is apparent that the magnetic-field strength increases with decreasing maximum temperature and increasing pressure. The resulting Hall parameter, however, is relatively insensitive to pressure, but also increases with decreasing temperature. For a fixed maximum

(a) Entrance Mach number, 0.5.

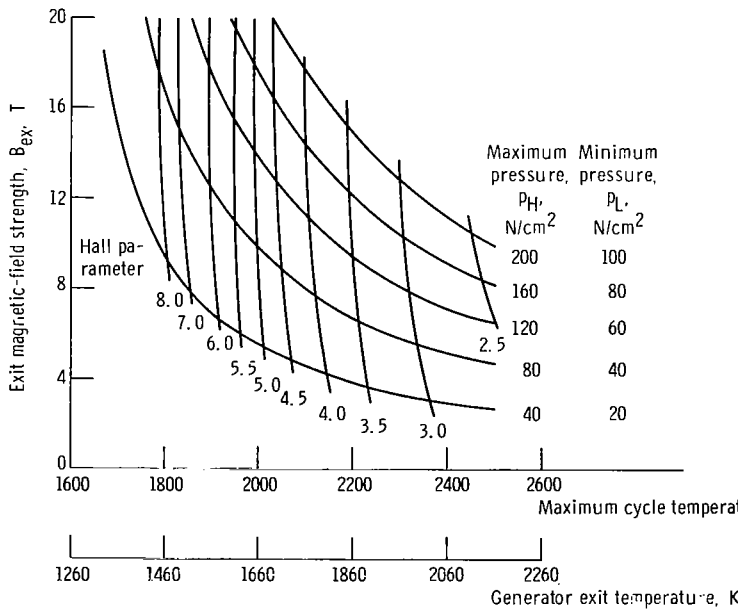(b) Entrance Mach number, 1.0.

(c) Entrance Mach number, 2.0.

Figure 21. - Exit magnetic-field strength required for Brayton cycle as function of maximum cycle temperature. Seed fraction, 0.02; specific radiator area ratio, 1.0; generator length, 0.5 meter.

temperature the Hall parameter increases with Mach number. Also, at the same temperature and pressure the required magnetic-field strength decreases with increasing Mach number. Or, put another way, for a fixed magnetic-field strength and temperature a higher pressure can be used with the higher entrance Mach number. This may be advantageous in the design of the reactor.

Figure 21 shows that the Hall parameter is larger at the exit than at the entrance and that the magnetic-field strength is smaller at the exit than the entrance. The difference is less for an entrance Mach number of 0.5 than for 1.0 and 2.0.

Effect of seed fraction. - It does not seem prudent to consider lowering the seed fraction for an entrance Mach number greater than 1.0 because the magnetic-field strength requirements are not limiting the pressure severely. Also, the required Hall parameter is already rather large and the situation would only be aggravated. However, it may be reasonable to consider lower seed fraction for the subsonic case, because the Hall parameter is not excessive. In figure 22 the conditions for entrance Mach 0.5 are shown for a seed fraction of 0.001. Remember from figure 15 that there is no advantage in considering lower values of seed fraction. Now an entrance Hall parameter has increased for a given temperature, but the pressure that can be used for a given magnetic-field strength has also increased. Now the exit and entrance Hall parameters do not differ.
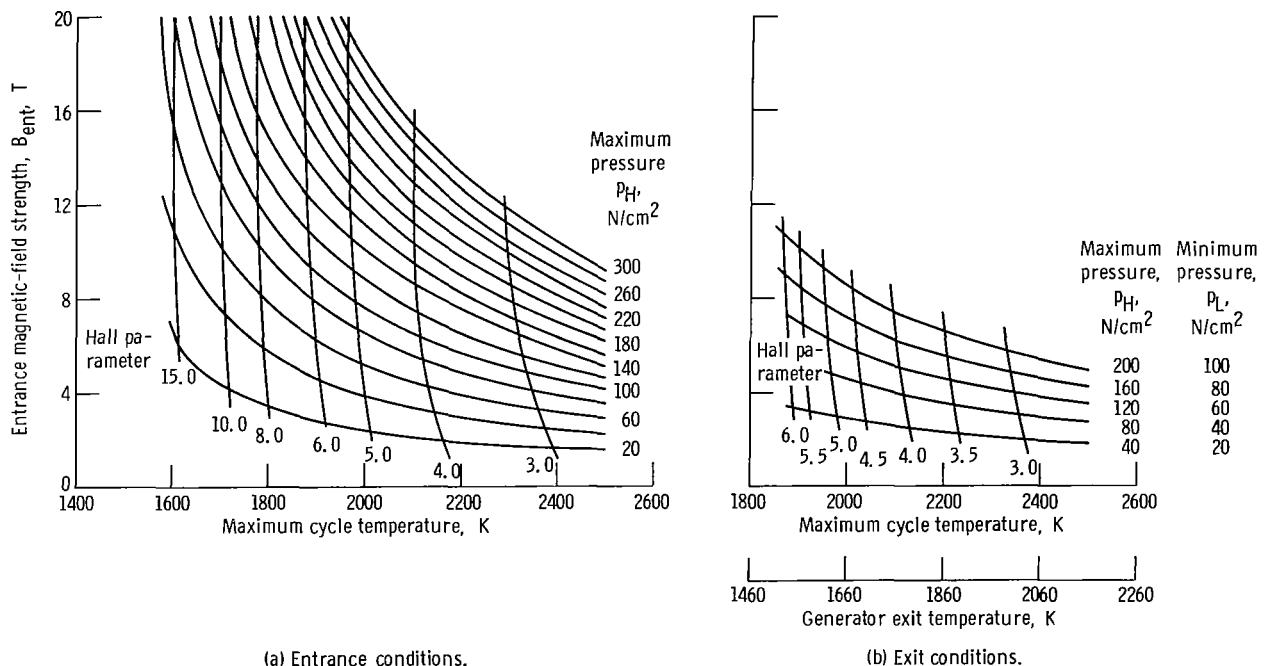

(a) Entrance conditions.

(b) Exit conditions.

Figure 22. - Magnetic-field strength required for Brayton cycle as function of maximum cycle temperature. Seed fraction, 0.001; specific radiator area ratio, 1.0; generator length, 0.5 meter; entrance Mach number, 0.5.

Comparison of Cycles

Power output. - In figure 23 the power generated multiplied by the square of the aspect ratio is plotted for the various cases considered. The curve is plotted for a fixed entrance magnetic-field strength of 10 tesla. (The exit field is about 6 T for the Brayton and about 2 T for the Rankine.) This value of magnetic-field strength is not unreasonable considering the present state of superconducting magnet technology. The first result to notice is nearly a factor of 6 improvement in power output for the Brayton over the Rankine for a seed of 0.001 and a factor between 3 and 4 for a seed fraction of 0.02. The 0.02 seed fraction reduces the Hall parameter by about 10 percent. The second trend to notice is that power output decreases with decreasing maximum cycle temperature.

Hall parameter. - The Hall parameter requirements can be seen in figure 24. The lower limit of cycle operation for the MHD system depends on whether the low power output (fig. 23) and also low Hall parameter (fig. 24) of the low-pressure Rankine are desired, or whether one relies on obtaining the ideal Brayton cycle performance at high Hall parameter. If maximum Brayton cycle temperatures are below 2000 K, then good
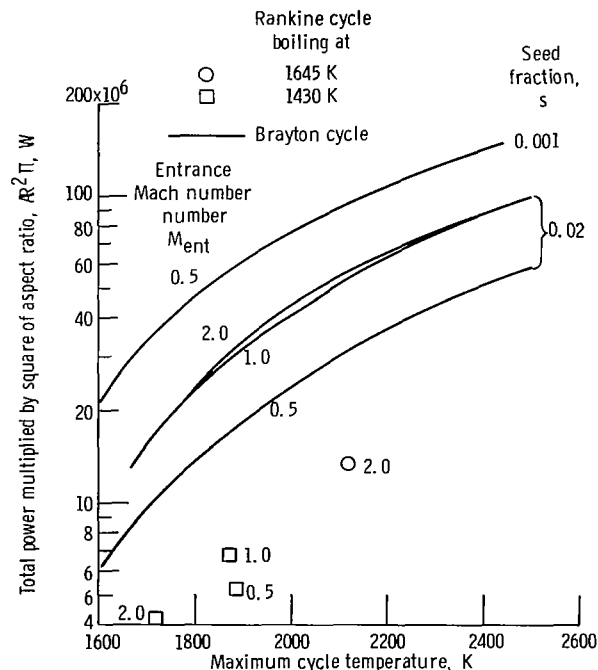

Figure 23. - Total power multiplied by square of aspect ratio as function of maximum cycle temperature. Specific radiator area equal to that of reference cycle; generator length, 0.5 meter; entrance magnetic-field strength, 10 tesla; Rankine cycle seed fraction, 0.01.

Conditions | Cycle | Boiling lithium, K, at–

△   Exit   Rankine   1645
□   Entrance   Rankine   1430
○   1645
——   Exit   Brayton   ----
– – –   Entrance   Brayton   ----

Seed fraction, s

Hall parameter, β_e

Maximum cycle temperature, K
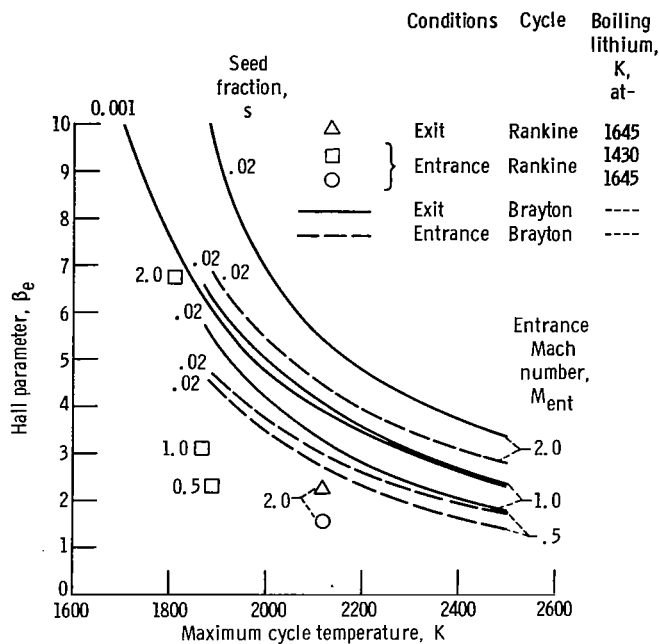
Entrance Mach number, M_{ent}

Figure 24. - Hall parameter at entrance and exit of generator required to provide specific radiator area equal to that of reference cycle and generator length of 0.5 meter as function of maximum temperature. Magnetic-field strength, 10 tesla; Rankine cycle seed fraction, 0.01.

Weight ratio, kg/kW

Magnet weight, Mg

Power output, MW

Maximum cycle pressure, P_H, N/cm^2
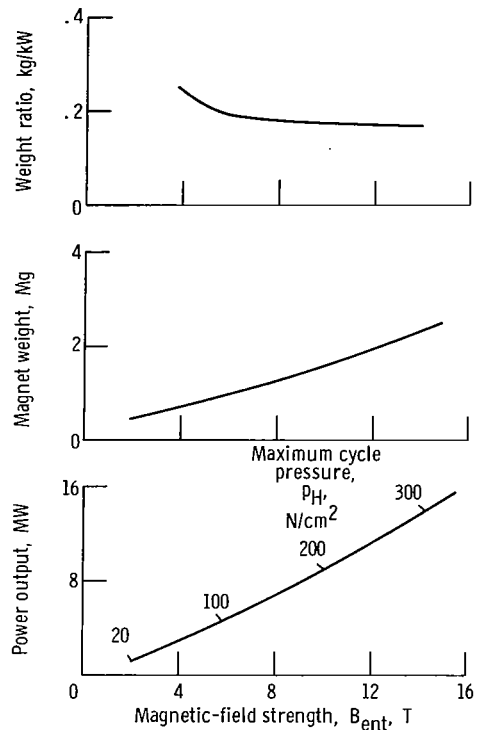
Magnetic-field strength, B_{ent}, T

Figure 25. - Generator power output, magnet weight, and weight ratio as functions of magnetic-field intensity. Generator length, 1.5 meters; aspect ratio, 10; entrance Mach number, 0.5; specific area ratio, 1.0; seed fraction, 0.001; maximum cycle temperature, 2100 K.

performance must be achieved at Hall parameters above 5.0 (and at least 4.5 for the reduced power output case).

Magnet weight. - The weight of a magnet suitable for a generator 1.5 meters long with an aspect ratio of 10 was estimated in reference 15. The weight of this magnet will be used even though a generator length of 0.5 meter was used in this analysis. The values of Hall parameter and magnetic-field strength determined for the 0.5-meter generator are relatively insensitive to generator length and should therefore also be applicable to the 1.5-meter generator. The weight variation for the 1.5-meter generator is shown in figure 25. Also shown in figure 25 is the power output for a 1.5-meter-long generator operation in a Brayton cycle. The system has a maximum temperature of 2100 K, entrance Mach 0.5, specific radiator area ratio of 1.0, and a seed fraction of 0.001. The results are plotted as a function of magnetic-field strength (varied by changing system pressure). The ratio of these curves is also shown in figure 25. Note that specific magnet weight is relatively insensitive to magnetic-field strength, which means that it is relatively insensitive to system pressure. It is therefore possible to adjust system pres-

sure to improve heat-transfer characteristics in the reactor and not suffer a specific weight penalty in the magnet system for a Brayton cycle.

The specific weight of the magnet plus support equipment for the Brayton cycle is of the order of 0.2 kilogram per kilowatt, which compares favorably with estimated turbogenerator weights (~0.5 kg/kW in ref. 18). However, the specific magnet weight for the Rankine system is about 0.6 kilogram per kilowatt, which takes into account both the decreased power output and the increased average magnetic field strength. The Rankine systems seem to be heavier than the Brayton systems for the same maximum cycle temperature. However, this difference is not significant since the magnet weight is only a small fraction of the total power-system weight.

CONCLUDING REMARKS

The performance regions outlined herein are certainly necessary - but not sufficient - conditions; that is, it will not be possible to achieve this performance at temperatures lower than indicated. This is the result of the idealized generator and cycle model considered. When actual system effects are considered (e.g., realistic component efficiencies, friction and heat transfer in the generator, pressure drops in the piping and heat source, etc.) the lower temperature limits may require raising. Also, the case against using supersonic Mach numbers (except for inherently inefficient systems) is based on the ideal generator model. This case may be modified if the actual effect of the Hall parameter on the electron temperature is markedly different from that chosen (this may be the case when instabilities are present), if electrode voltage drops are significant and a higher voltage is required to overcome them, or if the effects of heat transfer and friction are less detrimental to the supersonic Mach number generator than the subsonic Mach number.

An improved appraisal of the system power level and weight can be made only when a parametric study of the weights of the reactor, radiator, and magnet systems is available. Until then, it is difficult to evaluate the power level at which these systems will have minimum weight.

Finally, how high a temperature is required to produce competitive system weights depends on how high a Hall parameter can be used and still maintain approximately the performance of the ideal generator. This question appears to be the most important one to answer in order to evaluate the feasibility of MHD systems for use in space.

CONCLUSIONS

The effect of varying certain design parameters on the performance of both Rankine and Brayton MHD systems has been studied. The results of the analysis are as follows:

1. At a fixed value of Hall parameter, the Rankine-MHD cycles can operate with specific radiator area equal to the reference specific radiator area at lower maximum cycle temperature than Brayton-MHD.

2. The Brayton cycle has a factor of 6 greater power generated than the Rankine for the same entrance magnet-field strength and maximum cycle temperature.

3. The maximum cycle temperature at which MHD systems can be used in space decreases as the allowable Hall parameter increases. In order to use a maximum temperature lower than 2000 K for the Brayton cycle, the required Hall parameter at entrance Mach 1.0 would be greater than 5.0. To achieve temperatures below 2000 K for the Rankine cycle with overall efficiency equal to the reference cycle, either the Hall parameter must be greater than 5.0 (M_{ent} = 2.0) or the magnetic-field strength must be greater than 8.0 tesla (M_{ent} = 0.5 or 1.0). At the high magnetic-field strengths the specific weight of the magnet is higher than the specific weight of the energy conversion equipment in the reference cycle. The specific magnet weight can be reduced by raising the boiling temperature. When the magnet specific weight is about the same as for the turbo-alternator and associated equipment of the reference cycle (i. e. , about 1645 K), the overall cycle efficiency is cut to nearly 60 percent of its former value, and the entrance magnetic-field strength must be above 11 tesla. But the greater magnet weight is still a small fraction of the total system weight, and it may not be prudent to operate at a low generator efficiency at the higher boiling temperature.

4. The seed fraction can be specified to minimize the characteristic length for the Rankine cycle. However, this is not the case for the Brayton cycle. The seed fraction for the Brayton cycle can be selected at a low value, in which case the magnetic-field strength is low but the Hall parameter is high; or the seed fraction can be selected at a high value, in which case the magnetic field strength is higher and the Hall parameter lower.

5. Generator entrance Mach number of 1.0 for both cycles provides the maximum power generated for a specified generator efficiency, length, and maximum pressure. Even so, subsonic operation for the Brayton-MHD system is preferable; primarily because the Hall parameter required to provide a generator length of 0.5 meter is lower at any maximum temperature than for an entrance Mach number of 1.0. On the other hand, supersonic operation may be preferable for the Rankine-MHD system boiling at temperatures above 1645 K because the magnetic-field strength requirement is lower than for Mach 1.0. It appears that in the Brayton cycle, operating with a high Mach number is not a good method of providing both a high-pressure working fluid in the reactor

for good heat transfer and a low pressure in the generator for good electrical conductivity. The reason involves the adverse effect of increasing Mach number on the generator efficiency (see fig. 10) coupled with the sensitive dependence of system weight on generator efficiency. On the other hand, in the Rankine cycle this lowered generator efficiency can be partially compensated by raising the boiling temperature and thereby increasing the thermodynamic cycle efficiency. Here the increase in radiator temperature (which is proportional to the boiling temperature) is so great that the same specific radiator weight can be attained in spite of the reduction in generator efficiency. But the overall cycle efficiency drops, and this will probably lead to a heavier reactor for the same power output. A further increase in temperature (and pressure) may make it possible to achieve acceptable generator performance at even higher Mach numbers. The limit to this would most likely appear because of material difficulties in the heat source and the lowered overall cycle efficiency.

6. The output power for the Brayton MHD increases with increasing pressure. The magnetic-field strength must also increase to maintain the required Hall parameter. But the ratio of magnet weight to generator power output remains constant at a maximum temperature of 2100 K from a pressure of 10^6 newtons per square meter (a magnetic-field strength of 6 T) to 3×10^6 newtons per square meter (a magnetic-field strength of 14.0 T). Thus it may be possible to match the required generator pressure to the required nuclear reactor pressure in the Brayton cycle by increasing the magnetic-field strength and system pressure without suffering a specific weight penalty. This may imply very large amounts of generated power, or very short generators.

7. An estimate of magnet weight indicates that this weight is a very small fraction (a few percent) of the total system weight. Therefore, the assumption of not reducing the magnet weight at the expense of other component weights is justified. However, the low magnet weight may make the minimum generator volume condition used in designing the generator less significant. Since the reactor (and its shielding) is the heaviest portion, it may be better to specify the generator area variation so as to maximize the generator efficiency and hence overall thermodynamic efficiency. It turns out, however, that the minimum volume design is very close to the maximum efficiency design.

Lewis Research Center,
 National Aeronautics and Space Administration,
  Cleveland, Ohio, December 11, 1968,
   129-02-08-05-22.

APPENDIX A

SYMBOLS

A   dimensionless generator area ratio

$\mathscr{A}$ radiator area

$\mathcal{R}$ aspect ratio

a   generator area

B   magnetic-field strength specific heat

C   dimensionless magnetic-field strength ratio

c_p specific heat

e   electron change

F   variable defined in eq. (C11)

G   variable defined in eq. (C17)

H   function defined in eq. (C15)

h   enthalpy

I   integral defined in eq. (C12)

J   dimensionless current density ratio

j   current density

K   load parameter

k   Boltzmann constant

L   generator length

M   Mach number

m   particle mass

N   particle number density

P   dimensionless pressure

$\mathscr{P}$ stagnation enthalpy flux defined in eq. (23)

p   pressure

Q   heat transferred by heat source

q   variables defined in eq. (C5)

R   dimensionless density ratio

s   seed fraction

T   temperature

U   dimensionless fluid velocity ratio

u   fluid velocity

V   dimensionless volume ratio

v   variable defined in equation (C13)

W   work done by working fluid per unit mass

w   working fluid mass flow rate

X   dimensionless interacting length parameter

x   axial distance

y   function of pressure ratio defined in eq. (B4)

z   temperature ratio defined in eq. (B7)

α radiator area parameter defined in eq. (2)

β Hall parameter

γ ratio of specific heats

Δ	loss factor	comp	compressor
δ	variational symbol	cond	condenser
ϵ	radiator emissivity	conv	conversion
η	overall thermodynamic cycle efficiency	e	electron
		ent	entrance
λ	Lagrange multiplier	exit	exit of generator
$\mu(\gamma)$	function of specific heat ratio	gen	generator
ν	collision frequency	H	high
Π	power generated	isen	isentropic
ρ	working fluid density	L	low
Σ	dimensionless electrical conductivity ratio	max	maximum value
		min	minimum value
σ	electrical conductivity	reg	regenerator
σ_{SB}	Stefan-Boltzmann constant	T	total property
τ	generator volume	2, 3, 4, 5	points in cycle
Φ	constraint given by eq. (C14)		

Subscripts:

ch   characteristic

APPENDIX B

SPECIFICATION OF RADIATOR TEMPERATURE FOR MINIMUM AREA

Consider the Brayton cycle shown in figure 1(b). The work output from the generator is

$$W_{gen} = \eta_{gen} w c_p T_{max} \left[1 - \left(\frac{p_L}{p_H} \right)^{(\gamma-1)/\gamma} \right] \tag{B1}$$

and the work of isothermal compression is

$$W_{comp} = w c_p T_2 \ln \left(\frac{p_H}{p_L} \right) \left(\frac{\gamma - 1}{\gamma} \right) \tag{B2}$$

The heat added to the cycle is

$$Q_{in} = w c_p (T_{max} - T_3) \tag{B3}$$

so that the cycle efficiency can be written as

$$\eta = \frac{W_{gen} - W_{comp}}{Q_{in}} = \frac{\eta_{gen} T_{max} (1 - y) - T_2 \ln \frac{1}{y}}{T_{max} - T_3}$$

where

$$y = \left(\frac{p_L}{p_H} \right)^{(\gamma-1)/\gamma} \tag{B4}$$

If the regenerator effectiveness and generator efficiency are defined as

$$\left. \begin{array}{l} (T_5 - T_6) = (T_3 - T_2) = \eta_{reg}(T_5 - T_2) \\[2ex] T_{max} \eta_{gen}(1 - y) = T_{max} - T_5 \end{array} \right\} \tag{B5}$$

then the overall cycle efficiency can be written as (ref. 12)

$$\eta = \frac{\eta_{gen}(1-y) - z \ln \frac{1}{y}}{(1-z)(1-\eta_{reg}) + \eta_{reg}\eta_{gen}(1-y)} \tag{B6}$$

where

$$z = \frac{T_2}{T_{max}} \tag{B7}$$

If the radiator temperature equals the fluid temperature at any point in the radiator, and the radiation sink temperature is zero, then the incremental heat balance in the radiator requires that

$$wc_p \, dT = \epsilon\sigma_{SB}T^4 \, d\mathscr{A} \tag{B8}$$

The radiator area required to reduce the working fluid temperature from T_6 to T_2 can be obtained by integration of this equation to read

$$\mathscr{A}_{6,2} = \frac{wc_p}{3\epsilon\sigma_{SB}}\left(\frac{1}{T_2^3} - \frac{1}{T_6^3}\right) \tag{B9}$$

The radiator area required to maintain isothermal compression is

$$\mathscr{A}_2 = \frac{wc_p}{\epsilon\sigma_{SB}T_2^3} \ln\left(\frac{p_H}{p_L}\right)\left(\frac{\gamma - 1}{\gamma}\right) \tag{B10}$$

The total radiator area is

$$\mathscr{A} = \mathscr{A}_{6,2} + \mathscr{A}_2 = \frac{wc_p}{\epsilon\sigma_{SB}T_2^3}\left[\ln\frac{1}{y} + \frac{1}{3}\left(1 - \frac{T_2^3}{T_6^3}\right)\right] \tag{B11}$$

The radiator area per unit power generated is thus obtained by dividing equation (B11) by the net power output such that

$$\frac{\mathscr{A}}{wW} = \frac{\left[\ln\frac{1}{y} + \frac{1}{3}\left(1 - \frac{T_2^3}{T_6^3}\right)\right]}{\epsilon\sigma_{SB}T_2^3\left[\eta_g T_{max}(1 - y) - T_2 \ln\frac{1}{y}\right]} \qquad (B12)$$

T_2/T_6 can be related to the regenerator effectiveness and generator efficiency by elimination of T_5 from equation (B5):

$$\frac{T_2}{T_6} = \frac{z}{z\eta_{reg} + (1 - \eta_{reg})\left[1 - (1 - y)\eta_{gen}\right]} \qquad (B13)$$

Equation (B12) can be written as

$$\frac{\mathscr{A}}{wW} = \frac{1}{\epsilon\sigma_{SB}T_{max}^4} \frac{\ln\frac{1}{y} + \frac{1}{3}\left(1 - \frac{T_2^3}{T_6^3}\right)}{z^3\left[(1 - y)\eta_{gen} - z\ln\frac{1}{y}\right]} \qquad (B14)$$

There is a Z which minimizes the area parameter

$$\alpha = \frac{\epsilon\sigma_{SB}T_{max}^4\mathscr{A}}{wW} \qquad (B15)$$

for specified values of y, η_{gen}, and η_{reg}. This value of $\eta_{reg} = 1.0$ is

$$4z \ln\frac{1}{y} = 3\eta_{gen}(1 - y) \qquad (B16)$$

The efficiency is 0.25 for this case, and the minimum area parameter is given as equation (6).

APPENDIX C

DETERMINATION OF GENERATOR AREA VARIATION
WHICH MINIMIZES GENERATOR VOLUME

The generator volume τ is the integral of the cross-sectional area:

$$\tau = \int_0^L a \, dx \tag{C1}$$

The integral over the axial distance can be transformed into an integral over the pressure by substituting for the area from equation (20). If the nondimensional volume V is defined as

$$V \equiv \frac{\sigma_{ent} B_{ent}^2 \tau}{a_{ent} \rho_{ent} u_{ent}}$$

equation (C1) can be written as

$$V = \frac{1}{J_{ent}} \left(1 - U - \int_{P_{ent}}^{P_{exit}} A \, dP \right) \tag{C2}$$

Since the volume is to be minimized for a given ratio of total pressure, the total pressure will be introduced into the equations. The total pressure is

$$P_T = P \left(1 + \frac{\mu}{2} \frac{U}{AP} \right)^{1/\mu} \tag{C3}$$

where

$$\mu = \frac{\gamma - 1}{\gamma}$$

Solve equation (C3) for the area A such that

$$A = \frac{\mu}{2} \; \frac{UP^{\mu-1}}{P_T^\mu - P^\mu} \tag{C4}$$

Introduce variables that are functions of the total and static pressure

$$q_T = P_T^\mu \qquad q = P^\mu \tag{C5}$$

So that

$$A \, dP = \frac{U}{2} \; \frac{dq}{q_T - q}$$

and

$$d(AUP) = \frac{\mu}{2} \; \frac{U}{q_T - q} \left(2q \, dU + \frac{Uq_T}{q_T - q} \, dq - \frac{Uq}{q_T - q} \, dq_T \right) \tag{C6}$$

Equations (C5) and (C6) can be combined to eliminate the nondimensional interacting length X. In this way the effects of the conductivity can be removed from the problem. Since the specification for the generator is going to involve a prescribed total-pressure ratio, the generator will be designed to meet this specification first. The total-pressure function q_T will be taken as the independent variable. The combined equations become

$$\left(J_{ent} + \frac{Uq}{q_T - q} \right) \left(\frac{du}{dq_T} + \frac{U}{2(q_T - q)} \; \frac{dq}{dq_T} \right) = \frac{U^2 q}{2(q_T - q)^2} \tag{C7}$$

The interacting volume then becomes

$$V = \frac{1 - U}{J_{ent}} - \frac{1}{2J_{ent}} \int_{q_{T\,ent}}^{q_{T\,exit}} \frac{Uq'}{q_T - q} \, dq_T \tag{C8}$$

where the prime now denotes differentiation with respect to q_T. Because of the method of nondimensionalizing, both U and q are known at the entrance:

$$q\left(q_{T, ent}\right) = \frac{1}{\left(\gamma M_{ent}^2\right)^{\mu}} \qquad U\left(q_{T, ent}\right) = 1.0 \qquad (C9)$$

The constraint (eq. (C7)) plus the requirement that V be minimized for a specified pressure ratio are sufficient to determine the generator area variation. Then all variables can be determined.

The problem can be stated in mathematical terms as the minimization of the volume V given by equation (C8) subject to the constraint given in equation (C7). This problem is not the type wherein at the endpoints of the independent variable the dependent variables are specified. However, it can be easily shown that the conditions for the minimum are the same. At the extremum, the variation in V must vanish (ref. 27, p. 74):

$$J_{ent}\delta V = -\delta U - \frac{1}{2}\delta\left(\int_{q_{ent}}^{q_{exit}} F\, dq_T\right)$$

$$= -\left(1 + \frac{1}{2}\frac{\partial F}{\partial U'}\right)\delta U - \frac{1}{2}\frac{\partial F}{\partial q'}\delta q - \frac{1}{2}F\,\delta q_T - \frac{1}{2}\int_{q_{ent}}^{q_{exit}}\left[\left(\frac{\partial F}{\partial U} - \frac{d}{dq_T}\frac{\partial F}{\partial U'}\right)\delta u\right.$$

$$\left. + \left(\frac{\partial F}{\partial q} - \frac{d}{dq_T}\frac{\partial F}{\partial q_T'}\right)\delta q\right]dq_T \qquad (C10)$$

where

$$F = \frac{Uq'}{q_T - q} \qquad (C11)$$

and the prime denotes differentiation with respect to q_T. The third term on the right side vanishes because the exit-total pressure is fixed, so that a variation of q_T must also vanish. The first two terms vanish because the constraint must hold (eq. (C7)) and $\delta q_T = 0$. The fourth and last term therefore must vanish. This is equivalent to minimizing the integral I, where

$$I = \int_{q_{ent}}^{q_{exit}} F\, dq_T \qquad (C12)$$

subject to the constraint of equation (C7). This problem can be handled as a fixed end-point problem with a constraint. Introduce

$$v = \frac{U}{q_T - q} \qquad (C13)$$

So that equation (C7) becomes

$$\Phi \equiv (q_T - q)v' - \frac{v}{2}q' + \frac{v}{2}\left(1 + \frac{J_{ent}}{J_{ent} + qv}\right) = 0 \qquad (C14)$$

Using the technique of LaGrange multipliers (ref. 27, p. 129) introduce

$$H = vq' + 2(\lambda - 1)\Phi \qquad (C15)$$

where λ is the undertemined LaGrange multiplier. Then I will be minimized if the following equations

$$\left.\begin{array}{c} \dfrac{\partial H}{\partial q} - \dfrac{d}{dq_T}\dfrac{\partial H}{\partial q'} = 0 \\[2em] \dfrac{\partial H}{\partial v} - \dfrac{d}{dq_T}\dfrac{\partial H}{\partial v'} = 0 \\[2em] \Phi = 0 \end{array}\right\} \qquad (C16)$$

are satisfied. These three equations for q, v, and λ can be written as

$$\begin{pmatrix} \lambda & 0 & -v \\ 0 & -\lambda & 2(q_T - q) \\ 2(q_T - q) & -v & 0 \end{pmatrix} \begin{pmatrix} v' \\ q' \\ \lambda' \end{pmatrix} = \begin{pmatrix} -\dfrac{(\lambda - 1)J_{ent}v^2}{(J_{ent} + qv)^2} \\[1.5em] -(\lambda - 1)\left[1 - \left(\dfrac{J_{ent}}{J_{ent} + qv}\right)^2\right] \\[1.5em] -v\left(1 + \dfrac{J_{ent}}{J_{ent} + qv}\right) \end{pmatrix}$$

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This matrix is singular and unless

$$G \equiv (J_{ent} + qv)J_{ent}\left(J_{ent} + \frac{qv}{2}\right) - (\lambda - 1)\left[(q_T - q)v - \left(J_{ent} + \frac{qv}{2}\right)\right] = 0 \qquad (C17)$$

there are no solutions. The requirement that the matrix have a solution defines the LaGrange multiplier λ. With this condition (that G and $dG = 0$) the matrix can be rewritten as

$$
\begin{pmatrix}
\lambda & 0 & -v \\
\\
0 & -\lambda & 2(q_T - q) \\
\\
G_v & G_q & G_\lambda
\end{pmatrix}
\begin{pmatrix}
\dfrac{v'}{\lambda - 1} \\
\\
\dfrac{q'}{\lambda - 1} \\
\\
\dfrac{\lambda'}{\lambda - 1}
\end{pmatrix}
=
\begin{pmatrix}
-\dfrac{J_{ent}v^2}{(J_{ent} + qv)^2} \\
\\
-\left[1 - \left(\dfrac{J_{ent}}{J_{ent} + qv}\right)^2\right] \\
\\
J_{ent}v
\end{pmatrix}
$$

The solution to this system determines the proper variation of v, q, and λ which can be used in equation (C4) to give the proper area variation. A solution is shown in figure 7 for $M_{ent} = 1.0$, $J_{ent} = 0.25$, and $\gamma = 5/3$.

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