NASA TECHNICAL MEMORANDUM

NASA TM X-1761

NASA TM X-1761

CASE FILE COPY

ELECTROTHERMAL INSTABILITIES IN THE ENTRANCE REGION OF AN MHD GENERATOR

by J. Marlin Smith

Lewis Research Center
Cleveland, Ohio

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1969

NASA TM X-1761

ELECTROTHERMAL INSTABILITIES IN THE ENTRANCE REGION OF AN MHD GENERATOR

By J. Marlin Smith

Lewis Research Center
Cleveland, Ohio

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

ABSTRACT

In the entrance region of a nonequilibrium MHD generator one finds an abrupt rise in electron temperature followed by a four- or five-decade increase in electron density. We consider the stability of this region of abrupt electron density rise. A solution is found in the limit that the ambipolar diffusion field is small compared to the transverse applied electric field. The solution is found to be stable in the limit of zero magnetic field but to be unstable in the presence of a magnetic field.

ELECTROTHERMAL INSTABILITIES IN THE ENTRANCE REGION OF AN MHD GENERATOR

by J. Marlin Smith

Lewis Research Center

SUMMARY

Previous analyses of electrothermal instabilities in MHD generators have considered fluctuations from a steady state solution pertinent to the infinite homogeneous medium. In the entrance region of a nonequilibrium generator this condition is not valid since in this region one finds an abrupt rise in electron temperature followed by a four- or five-decade increase in electron density. We consider the stability of the subregion of abrupt electron density rise. The steady state solution is one of uniform gas dynamic properties and electron temperature but with an electron density gradient which is solely a function of the ionization rate. This rate is evaluated at an electron temperature obtained by equating the Joule heating to the elastic collisional and ionization energy loss. The governing equations upon which the perturbation is taken are those of reference 1 with the exception that recombination is neglected in the electron continuity equation. A solution is found in the limit that the ambipolar diffusion field is small compared to the transverse applied electric field. The solution is found to be stable in the limit of zero magnetic field but to be unstable in the presence of a magnetic field.

INTRODUCTION

In contrast to experiments performed with equilibrium MHD generators, nonequilibrium generators have conspicuously failed to operate at anything near the performance predicted by theory. These negative results have been attributed to such factors as Hall current leakages resulting from the higher Hall parameters required to operate in the nonequilibrium mode, insufficient residence time of the working fluid in the generator for the nonequilibrium ionization to buildup to its steady state value, instabilities arising from the nonequilibrium plasma state, etc. In this report another possible source of difficulty is considered. This is the possibility of the formation of instabilities in the ionization region of the entrance of the generator which would inhibit the growth of ionization to the fully nonequilibrium level which present theory now assumes.

Previous analyses of electrothermal instabilities in MHD generators have been concerned with fluctuations occurring in regions well removed from any large gradients of gas dynamic or electrical properties. In this regime the steady state solution is that pertinent to the infinite homogeneous medium or small deviations therefrom. In the entrance region of a nonequilibrium generator these solutions are only valid for extremely short wavelength component in the gradient direction since in this region one finds an abrupt rise in electron temperature followed by a four- or five-decade increase in electron density occurring within a few centimeters. It is therefore the stability of fluctuations with wavelength component the order of the ionization region which is of concern in this report.

The analysis is limited to the segmented electrode generator configuration operating in the Faraday mode. The ionizing electric field is restricted to that induced by the gas flow through the magnetic field or this field supplemented by an electric field applied solely in the Faraday current direction. All equations are in SI units and all symbols are defined in the appendix.

STATEMENT OF PROBLEM

Model

To facilitate the analysis a simplified model of the ionization region based upon the numerical results of Bertolini (ref. 2) is taken. In the model considered, it is assumed that at the entering electron temperature (gas temperature) the ionization rate is small compared to the electron heating rate so that the entrance region can be considered to consist of two subregions. In the first subregion the electron temperature is abruptly elevated with negligible change in electron density until the point is reached at which the Ohmic heating is balanced by elastic electron-neutral atom collisional and inelastic ionizing collisional energy losses. This is followed by a second subregion in which the electron density rapidly rises at constant electron temperature.

This model is realistic due to the very strong exponential dependence of the ionization rate on electron temperature so that the distance over which the electron density builds up and reaches equilibrium is controlled by the peak electron temperature. That the electron temperature remains constant in the second region is a consequence of the electron density dependence of the electron heating equation and will be demonstrated at a later point in the analysis.

Assumptions

On the basis of the previous model this report is limited to the study of the stability of fluctuations occurring in the second subregion. Furthermore, as in the case of homogeneous electrothermal instabilities, the following limitations on the analysis are made:

(1) Throughout the analysis the gas dynamic properties, that is, gas density, velocity, and temperature, are taken to be constant. In the steady state solution this is a valid assumption since these properties change on a scale which is the order of the generator length (typically the order of meters) while the ionization region is ideally the order of centimeters. Fluctuations in the gas dynamic properties are ignored relative to those of electron density and electron temperature due to the fact that the relatively lighter and more mobile electrons respond to disturbances of greatly different frequencies than do the heavier and less mobile neutral gas atoms which dominate the gas properties. Furthermore, due to the low degree of ionization in the region of interest, the gas dynamic properties are weakly coupled to the electrical properties.

(2) Only the propagation of magnetohydrodynamic type disturbances are considered. This restriction embodies the following limitations:

(a) The displacement current is neglected in Maxwell's equations. Under this restriction the relatively high frequency transverse electromagnetic disturbances are not considered.

(b) Induced magnetic fields are ignored, that is, the analysis is limited to conditions of low magnetic Reynolds number which is, in general, satisfied in MHD generator applications.

(c) The plasma is assumed to be quasi-charge neutral, that is, to zero order the electron and ion number densities are equal.

(3) Terms of the order of the electron to heavy particle mass ratio are neglected.

(4) Ion slip is neglected. This is justified since in cases of interest the Hall parameter is less than 10.

(5) Consider only propagation in the plane perpendicular to the applied magnetic field.

Equations

For the previous model and assumptions the analysis is completely determined in terms of the electron density and electron temperature by the following set of equations.

Maxwell equations. - Under the previously discussed conditions of charge neutrality and neglect of the induced magnetic field these equations reduce to

$$\overline{\nabla} \times \underline{E} = 0 \tag{1}$$

$$\overline{\nabla} \cdot \underline{j} = 0 \tag{2}$$

Electron continuity equation. - In the region of interest the dominant process is that of ionization and the analysis is restricted to the case in which the ionization process is dominated by electron-neutral atom ionizing collisions. Then

$$\frac{\partial}{\partial t} n_e + \underline{v}_0 \cdot \overline{\nabla} n_e = n_e n_a \nu_i \tag{3}$$

It is noted that the equation depends upon the gas flow velocity $\underline{v}_0$ rather than the electron velocity $\underline{v}_e$. This arises as a result of the neglect of ion slip and the assumption of quasi-charge neutrality for which the definition of $\underline{j}$ is $n_e e(\underline{v}_0 - \underline{v}_e)$ and $\overline{\nabla} \cdot \underline{j} = 0$. Therefore

$$\overline{\nabla} \cdot n_e \underline{v}_e = \overline{\nabla} \cdot n_e \left(\underline{v}_0 - \frac{1}{n_e e} \underline{j} \right) = \overline{\nabla} \cdot n_e \underline{v}_0 = \underline{v}_0 \cdot \overline{\nabla} n_e \tag{4}$$

that is, gas flow velocity is constant.

Generalized Ohm's law. -

$$\underline{j} = \sigma \underline{E}^* - \frac{e}{m_e \nu_0} \underline{j} \times \underline{B}_0 + \frac{e}{m_e \nu_0} \overline{\nabla} P_e \tag{5}$$

where $\underline{E}^* = \underline{E} + \underline{v}_0 \times \underline{B}_0$ is the electric field in the frame of reference moving with the velocity $\underline{v}_0$ and $\underline{B}_0$ is the applied magnetic field.

In the above form of the generalized Ohm's law the contribution from the heat conduction is neglected. Its effect is to contribute terms of the form $\overline{\nabla} n_e$ and $\overline{\nabla} T_e$ which lead to a slight modification of the $\overline{\nabla} P_e$ term already appearing in the Ohm's law equation. However, the slightness of the correction does not seem to justify the additional complexity resulting from its inclusion and therefore is ignored.

The electrical conductivity σ is defined in the usual manner as

$$\sigma = \frac{n_e e^2}{m_e \nu_0} \tag{6}$$

where, as in reference 1, the total electron momentum collision frequency ν_0 is taken to be constant throughout the analysis. This condition is justified, particularly in the region considered here, because the electron density is small so that the collision frequency is dominated by electron-neutral atom collisions. In this case the frequency depends upon the neutral atom density ($\simeq$ gas density) which by prior restriction is constant. We need then only require that the energy dependence of the cross section be such that the frequency have a negligibly small dependence upon electron temperature (nearly Maxwellian molecule interaction). This is assumed to be the case in this analysis.

Electron energy equation. -

$$\frac{3}{2} k\left(n_e \frac{\partial}{\partial t} T_e + n_e \underline{v}_0 \cdot \overline{\nabla} T_e\right) - \frac{5}{2} \underline{j} \cdot \overline{\nabla} \frac{kT_e}{e}$$

$$= -2 \frac{m_e}{m_a} \nu_0 n_e \frac{3}{2} k(T_e - T_0) + j \cdot \underline{E}^* - \left(kT_i + \frac{3}{2} kT_e\right)\left(\frac{\partial}{\partial t} n_e + \underline{v}_0 \cdot \overline{\nabla} n_e\right) \qquad (7)$$

where the $\overline{\nabla} T_e$ contribution to the heat conduction term and radiation losses are neglected. These terms contribute wavelength dependent damping terms to the dispersion relation which decrease with increasing wavelength relative to the wavelength independent elastic collision damping. The analysis is therefore restricted to wavelengths of sufficient magnitude so as to make these terms ignorable. For a further discussion of this point see reference 3.

ANALYSIS

The analysis is carried out for a segmented electrode MHD generator operating in the Faraday mode. The coordinates are oriented such that the x-axis is in the fluid flow direction and the z-axis is in the direction of the applied magnetic field.

Steady State Solution

Under the previously discussed assumptions the zeroth order or steady state solution is depicted by a region of constant fluid dynamic quantities and a constant electron temperature which is given by equating the Joule heating to the elastic and ionization collisional energy losses. From equation (7) the steady state electron temperature is then given by

$$\underline{j}_0 \cdot \underline{E}_0^* = 2\frac{m_e}{m_a}\nu_0 n_{e0}\frac{3}{2}k(T_{e0} - T_0) + \left(kT_i + \frac{3}{2}kT_{e0}\right)\underline{v}_0 \cdot \overline{\nabla} n_{e0} \tag{8}$$

where the subscript "0" refers to the zeroth order or steady state solution.

From equation (3) the steady state electron density is given by

$$\underline{v}_0 \cdot \overline{\nabla} n_{e0} = n_{e0} n_{a0} \nu_{i0} \tag{9}$$

Since the fluid properties are constant $n_{a0} \simeq$ constant and since ν_{i0} only depends upon the constant electron temperature it is constant. Therefore, with $\underline{v}_0$ a constant and orientated along the x-axis

$$\frac{1}{n_{e0}}\underline{v}_0 \cdot \overline{\nabla} n_{e0} = v_0 \frac{d}{dx} \ln n_{e0} = n_{a0}\nu_{i0} = \text{constant} \tag{10}$$

Obviously the space dependence of the steady state electron density is exponential with an e-folding length λ_i defined by

$$\frac{1}{\lambda_i} = \frac{d}{dx} \ln n_{e0} = \frac{n_{a0}\nu_{i0}}{v_0} \tag{11}$$

Under conditions typical of nonequilibrium generator operation the incoming electron density which is in Saha equilibrium with the electron temperature equal to the gas temperature is of order 10^{16} electrons per cubic meter. In the ionization region the electron temperature is elevated by the preferential Joule heating of the electrons to a level at which the electron density in Saha equilibrium with the electron temperature is in the range of 10^{20} to 10^{21} electrons per cubic meter. Furthermore, if this increased ionization is to be effectively used in the generator then the ionization region must be the order of a few centimeters at most. Therefore, the e-folding length must be on the order of a tenth of a centimeter.

The steady state Ohm's law relation as obtained from equation (5) is

$$\underline{j}_0 = \sigma_0 \underline{E}_0^* - \frac{e}{m_e \nu_0}\underline{j}_0 \times \underline{B}_0 + \sigma_0 \frac{kT_{e0}}{e}\overline{\nabla} \ln n_{e0} \tag{12}$$

With the definition of the segmented Faraday generator,

$$j_{0x} = 0 \tag{13}$$

and with the usual insulating walls in the magnetic field direction so that

$$j_{0z} = E_{0z}^* = 0 \tag{14}$$

the other components of equation (12) are

$$E_{0x}^* = E_{0x} = \beta_0(E_{0y} - v_0 B_0) - \frac{kT_{e0}}{e} \frac{d}{dx} \ln n_{e0} \tag{15}$$

$$j_{0y} = \sigma_0(E_{0y} - v_0 B_0) \tag{16}$$

where

$$\beta_0 = \frac{eB_0}{m_e \nu_0} \tag{17}$$

is the Hall parameter.

Equations (8), (10), and (13) to (17) give a complete description of the steady state solution in the region of interest. Before proceeding to answer the question as to whether or not this solution is stable to small perturbations we will show that the assumption of constant electron temperature is consistent with this set of equations.

Since there is no charge separation and v_0 and B_0 are constant

$$E_{0y}^* = E_{0y} - v_0 B_0 = \text{constant} \tag{18}$$

Then dividing equation (8) by σ_0 and using equation (16)

$$E_{0y}^{*2} = 2 \frac{m_e}{m_a} \nu_0 \left(\frac{m_e \nu_0}{e^2}\right) \frac{3}{2} k(T_{e0} - T_0) + \left(kT_i + \frac{3}{2} kT_{e0}\right) \left(\frac{m_e \nu_0}{e^2}\right) \underline{v}_0 \cdot \overline{\nabla} \ln n_{e0} \tag{19}$$

Since by prior argument ν_0 and T_0 are constant, it is obvious from equations (11) and (18) that equation (19) requires T_{e0} to be constant.

It is also convenient to note from equations (10), (17), and (18) that equation (15) requires that

$$E_{0x} = \text{constant} \tag{20}$$

and, therefore, that the steady state electric field

$$\underline{E}_0^* = \text{constant} \tag{21}$$

Furthermore, since

$$\sigma_0 = \frac{e^2}{m_e \nu_0} n_{e0}$$

the conductivity has the same exponential x-direction dependence as n_{e0} so that by equations (18) and (16) j_{0y} must have this same dependence. The steady state solution can now be summarized as

$$n_{e0} \sim e^{(1/\lambda_i)x} \tag{22}$$

$$\left.\begin{aligned} E_{0x}^*, E_{0y}^* &= \text{constant} \\ E_{0z}^* &= 0 \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} j_{0x} = j_{0z} &= 0 \\ j_{0y} &\sim e^{(1/\lambda_i)x} \end{aligned}\right\} \tag{24}$$

$$T_{e0} = \text{constant} \tag{25}$$

Perturbation Equations

The essence of this analysis is to determine whether an infinitesimal perturbation of the static system will initially grow or decay. To this end we therefore consider solutions of the governing equations of the form

$$f = f_0 + f'(\underline{r}, t) \tag{26}$$

where f_0 is the steady state solution and f' is an infinitesimal perturbation to this solution such that products of the f' can be neglected. Substituting solutions of this form into the governing equations, neglecting perturbations in the gas dynamic quantities, and utilizing the steady state solutions the equations for the perturbation quantities become

Maxwell's equations

$$\overline{\nabla} \times \underline{E}' = 0 \tag{27}$$

$$\overline{\nabla} \cdot \underline{j}' = 0 \tag{28}$$

Electron density

$$\frac{\partial}{\partial t} \frac{n_e'}{n_{e0}} + \underline{v}_0 \cdot \overline{\nabla} \frac{n_e'}{n_{e0}} = \frac{\nu_i'}{\nu_{i0}} \underline{v}_0 \cdot \overline{\nabla} \ln n_{e0} \tag{29}$$

ν_i is determined as a function of T_e by approximating the ionization cross section by a linear curve fit to the low energy portion of the curve and averaging over a Maxwellian electron distribution so that

$$\nu_i \sim T_e^{1/2} \left(1 + 2 \frac{T_e}{T_i}\right) e^{-T_i/T_e} \tag{30}$$

where T_i is the temperature below which the ionization cross section goes to zero. At low electron densities where multistage ionization processes are negligible, T_i is the temperature corresponding to the ionization potential. This is assumed to be approximately true in the region being considered here. The perturbed value of equation (30) is then

$$\frac{\nu_i'}{\nu_{i0}} \simeq \left(\frac{1}{2} + \frac{T_i}{T_{e0}}\right) \frac{T_e'}{T_{e0}} \tag{31}$$

where we have neglected terms the order of T_{e0}/T_i since in the regime of interest $T_{e0}/T_i << 1$. Equation (30) then becomes

$$\frac{\partial}{\partial t} \frac{n_e'}{n_{e0}} + \underline{v}_0 \cdot \overline{\nabla} \frac{n_e'}{n_{e0}} = \left(\frac{1}{2} + \frac{T_i}{T_{e0}}\right) (\underline{v}_0 \cdot \overline{\nabla} \ln n_{e0}) \frac{T_e'}{T_{e0}} \tag{32}$$

Generalized Ohm's law

$$\left(\frac{1}{\sigma_0}\underline{j}'\right) = \frac{n_e'}{n_{e0}}\underline{E}_0^* + \underline{E}' - \frac{e}{m_e\nu_0}\left(\frac{1}{\sigma_0}\underline{j}'\right)\times\underline{B}_0 + \frac{kT_{e0}}{e}\overline{\nabla}\left(\frac{n_e'}{n_{e0}} + \frac{T_e'}{T_{e0}}\right)$$

$$+\frac{kT_{e0}}{e}\left(\frac{n_e'}{n_{e0}} + \frac{T_e'}{T_{e0}}\right)\overline{\nabla}\ln n_{e0} \tag{33}$$

Electron energy equation

$$\frac{\partial}{\partial t}\frac{T_e'}{T_{e0}} + \underline{v}_0\cdot\overline{\nabla}\frac{T_e'}{T_{e0}} - \frac{5}{3}\frac{e}{m_e\nu_0}\left(\frac{1}{\sigma_0}\underline{j}_0\right)\cdot\overline{\nabla}\frac{T_e'}{T_{e0}}$$

$$= -2\frac{m_e}{m_a}\nu_0\left(1 - \frac{T_0}{T_{e0}}\right)\left(\frac{n_e'}{n_{e0}} + \frac{T_e'}{T_{e0} - T_0}\right) + \frac{2}{3kT_{e0}}\frac{e^2}{m_e\nu_0}\left[\left(\frac{1}{\sigma_0}\underline{j}'\right)\cdot\underline{E}_0^* + \left(\frac{1}{\sigma_0}\underline{j}_0\right)\cdot\underline{E}'\right]$$

$$-\frac{2}{3}\left(\frac{3}{2} + \frac{T_i}{T_{e0}}\right)\left(\frac{\partial}{\partial t}\frac{n_e'}{n_{e0}} + \underline{v}_0\cdot\overline{\nabla}\frac{n_e'}{n_{e0}} + \frac{n_e'}{n_{e0}}\underline{v}_0\cdot\overline{\nabla}\ln n_{e0}\right) \tag{34}$$

Solution of the Perturbation Equations

We first write equation (28) in the form

$$\overline{\nabla}\cdot\left(\frac{1}{\sigma_0}\underline{j}'\right) = -\left(\frac{1}{\sigma_0}\underline{j}'\right)\cdot\overline{\nabla}\ln n_{e0} \tag{35}$$

It is now noted that equations (27) and (32) to (35) are simply a linear set of coupled equations with <u>constant</u> coefficients for the variables

$$(\underline{E}'),\ \left(\frac{n_e'}{n_{e0}}\right),\ \left(\frac{1}{\sigma_0}\underline{j}'\right),\ \text{and}\left(\frac{T_e'}{T_{e0}}\right)$$

Therefore, the solution can be taken in the form

$$\left(\underline{E}', \frac{n'_e}{n_{e0}}, \frac{1}{\sigma_0}\underline{j}', \frac{T'_e}{T_{e0}}\right) = \left(\underline{\tilde{E}}, \frac{\tilde{n}_e}{n_{e0}}, \frac{1}{\sigma_0}\underline{\tilde{j}}, \frac{\tilde{T}_e}{T_{e0}}\right) e^{i(\underline{l}\cdot\underline{r}+\omega t)} \tag{36}$$

where $\underline{\tilde{E}}$, $\tilde{n}_e/n_{e0}$, $(1/\sigma_0)\underline{\tilde{j}}$, and $\tilde{T}_e/T_{e0}$ are all constant and considering only propagation in the plane perpendicular to the applied magnetic field $(l_x, l_y, l_z) = (l_x, l_y, 0)$.

It should be noted at this point that the normalization of the perturbation equations has resulted in solutions in the form of superimposed waves. However, this differs from the homogeneous case in that not all of the perturbed quantities have a wave-like form. Consider, for example, the perturbation in the electron density which from equations (22) and (36) is of the form

$$n'_e \sim e^{(1/\lambda_i)x} e^{i(\underline{l}\cdot\underline{r}+\omega t)}$$

This solution exhibits growth regardless of the eventual solution of the dispersion relation. However, it is noted that this growth is at the same rate as the zeroth order electron density so that if the perturbation is small to begin with it will be small to the same ratio throughout the region. This is not unstable behavior since the infinitesimal perturbation remains infinitesimal relative to the zeroth order. Therefore, just as in the case of the infinite homogeneous medium, when the imaginary part of ω is negative the perturbation grows without bound and the system is termed unstable; a negative value of the imaginary part of ω in the case considered here indicates that the perturbation will grow <u>faster than</u> the zeroth order and hence that the system is unstable.

Introducing the solutions represented in equation (36) into equations (27) and (32) to (35) we obtain

$$\underline{l} \times \underline{\tilde{E}} = 0 \tag{37}$$

$$\omega^* \frac{\tilde{n}_e}{n_{e0}} = -i\left(\frac{1}{2} + \frac{T_i}{T_{e0}}\right)(\underline{v}_0 \cdot \overline{\nabla} \ln n_{e0}) \frac{\tilde{T}_e}{T_{e0}} \tag{38}$$

$$\left(\frac{1}{\sigma_0}\underline{\tilde{j}}\right) = \frac{\tilde{n}_e}{n_{e0}}\underline{E}_0 + \underline{\tilde{E}} - \frac{e}{m_e \nu_0}\left(\frac{1}{\sigma_0}\underline{\tilde{j}}\right) \times \underline{B}_0 + i\frac{kT_{e0}}{e}\underline{l}\left(\frac{\tilde{n}_e}{n_{e0}} + \frac{\tilde{T}_e}{T_{e0}}\right)$$

$$+ \frac{kT_{e0}}{e}\left(\frac{\tilde{n}_e}{n_{e0}} + \frac{\tilde{T}_e}{T_{e0}}\right)\overline{\nabla} \ln n_{e0} \tag{39}$$

$$\omega^* \frac{\tilde{T}_e}{T_{e0}} - \frac{5}{3}\underline{l}\cdot\left(\frac{1}{n_{e0}e}\underline{j}_0\right)\frac{\tilde{T}_e}{T_{e0}} = i2\frac{m_e}{m_a}\nu_0\left(1 - \frac{T_0}{T_{e0}}\right)\left(\frac{\tilde{n}_e}{n_{e0}} + \frac{\tilde{T}_e}{T_{e0} - T_0}\right)$$

$$- i\frac{2}{3kT_{e0}}\frac{e^2}{m_e\nu_0}\left[\left(\frac{1}{\sigma_0}\tilde{\underline{j}}\right)\cdot\underline{E}_0^* + \left(\frac{1}{\sigma_0}\underline{j}_0\right)\cdot\tilde{\underline{E}}\right]$$

$$- \frac{2}{3}\left(\frac{3}{2} + \frac{T_i}{T_{e0}}\right)\left(\omega^*\frac{\tilde{n}_e}{n_{e0}} - i\frac{\tilde{n}_e}{n_{e0}}\underline{v}_0\cdot\overline{\nabla}\ln n_{e0}\right) \tag{40}$$

$$\left(\underline{l}\cdot\frac{1}{\sigma_0}\tilde{\underline{j}}\right) = i\left(\frac{1}{\sigma_0}\tilde{\underline{j}}\right)\cdot\overline{\nabla}\ln n_{e0} \tag{41}$$

where $\omega^* = \omega + \underline{l}\cdot\underline{v}_0$ is the frequency in the frame of reference moving with the gas.

The dispersion relation can then be determined from the condition for the existence of a solution, namely, that the determinant of coefficients be zero. However, it is somewhat more illustrative to derive the relation by the process of eliminating the various variables in the electron energy equation (eq. (40)). Therefore, the procedure to be used in this report is to use equations (37), (39), and (41) to eliminate the

$$\left[\left(\frac{1}{\sigma_0}\tilde{\underline{j}}\right)\cdot\underline{E}_0^* + \left(\frac{1}{\sigma_0}\underline{j}_0\right)\cdot\tilde{\underline{E}}\right]$$

term in terms of $\tilde{T}_e/T_{e0}$ and $\tilde{n}_e/n_{e0}$, then eliminate $\tilde{n}_e/n_{e0}$ by equation (38).

The simplest way to do this is to observe from equation (5) that $\underline{j}\cdot\underline{E}^*$ can be written as

$$\frac{1}{\sigma_0}\underline{j}\cdot\underline{E}^* = \frac{1}{\sigma_0}\underline{j}\cdot\left[\frac{1}{\sigma}\underline{j} + \frac{e}{m_e\nu_0}\left(\frac{1}{\sigma}\underline{j}\right)\times\underline{B}_0 - \frac{1}{n_e e}\overline{\nabla}P_e\right]$$

$$= \frac{1}{\sigma_0}\frac{1}{\sigma}\underline{j}\cdot\underline{j} - \frac{1}{n_e e}\left(\frac{1}{\sigma_0}\underline{j}\right)\cdot\overline{\nabla}P_e$$

which upon linearization and substitution of equations (12) and (36) yields

$$\left(\frac{1}{\sigma_0}\underline{\tilde{j}}\right)\cdot\underline{E}_0^* + \left(\frac{1}{\sigma_0}\underline{j}_0\right)\cdot\underline{\tilde{E}} = -\frac{\tilde{n}_e}{n_{e0}}\left(\frac{1}{\sigma_0}\underline{j}_0\right)\cdot\underline{E}_0^* + 2\left(\frac{1}{\sigma_0}\underline{j}_0\right)\cdot\left(\frac{1}{\sigma_0}\underline{\tilde{j}}\right)$$

$$-\frac{kT_{e0}}{e}\left(\frac{1}{\sigma_0}\underline{\tilde{j}}\right)\cdot\overline{\nabla}\ln n_{e0} - i\frac{kT_{e0}}{e}\underline{l}\cdot\left(\frac{1}{\sigma_0}\underline{j}_0\right)\left(\frac{\tilde{n}_e}{n_{e0}}+\frac{\tilde{T}_e}{T_{e0}}\right)$$

$$-\frac{kT_{e0}}{e}\left(\frac{\tilde{n}_e}{n_{e0}}+\frac{\tilde{T}_e}{T_{e0}}\right)\left(\frac{1}{\sigma_0}\underline{j}_0\right)\cdot\overline{\nabla}\ln n_{e0} \qquad (42)$$

To determine $(1/\sigma_0)\underline{\tilde{j}}$, the cross product $\underline{l}\times\left[\underline{l}\times(1/\sigma_0)\underline{\tilde{j}}\right]$ is formed from equation (39), which, with the use of equations (37) and (41) yields

$$l^2\left(\frac{1}{\sigma_0}\underline{\tilde{j}}\right) = i\left(\underline{l} - \frac{e}{m_e\nu_0}\underline{l}\times\underline{B}_0\right)\left(\frac{1}{\sigma_0}\underline{\tilde{j}}\right)\cdot\overline{\nabla}\ln n_{e0} - \frac{\tilde{n}_e}{n_{e0}}\left(\underline{l}\,\underline{l}\cdot\underline{E}_0^* - l^2\underline{E}_0^*\right)$$

$$-\frac{kT_{e0}}{e}\left(\frac{\tilde{n}_e}{n_{e0}}+\frac{\tilde{T}_e}{T_{e0}}\right)\left(\underline{l}\,\underline{l}\cdot\overline{\nabla}\ln n_{e0} - l^2\overline{\nabla}\ln n_{e0}\right) \qquad (43)$$

The quantity

$$\left(\frac{1}{\sigma_0}\underline{\tilde{j}}\right)\cdot\overline{\nabla}\ln n_{e0}$$

is then determined from equation (43) by dotting $\overline{\nabla}\ln n_{e0}$ into the equation to obtain

$$\left(\frac{1}{\sigma_0}\underline{\tilde{j}}\right)\cdot\overline{\nabla}\ln n_{e0} = -\frac{1}{l^4 + (l_x - \beta_0 l_y)^2\frac{1}{\lambda_i^2}}\left[l^2 + i(l_x - \beta_0 l_y)\frac{1}{\lambda_i}\right]$$

$$\times\left[\frac{\tilde{n}_e}{n_{e0}}(l_x - \beta_0 l_y)l_y E_{0y}^* - \frac{\tilde{T}_e}{T_{e0}}l_y^2\left(\frac{kT_{e0}}{e}\frac{d\ln n_{e0}}{dx}\right)\right]\frac{1}{\lambda_i} \qquad (44)$$

Substitution of equations (43) and (44) into equation (42) after considerable algebraic manipulation yields

$$\left(\frac{1}{\sigma_0}\underline{\tilde{j}}\right)\cdot\underline{E}_0^* + \left(\frac{1}{\sigma_0}\underline{j}_0\right)\cdot\underline{\tilde{E}} \equiv \left(\frac{1}{\sigma_0}\underline{j}_0\right)\cdot\underline{E}_0^*\frac{\tilde{n}_e}{n_{e0}} - i\frac{kT_{e0}}{e}\underline{l}\cdot\left(\frac{1}{\sigma_0}\underline{j}_0\right)\frac{\tilde{T}_e}{T_{e0}}$$

$$-\frac{a - iA}{l^4 + \dfrac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}}\frac{\tilde{n}_e}{n_{e0}} - \frac{g - iG}{l^4 + \dfrac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}}\frac{\tilde{T}_e}{T_{e0}} \tag{45}$$

where using equations (15) and (16) to eliminate E_{0x} and j_{0y} in terms of E_{0y}^*

$$a \equiv 2l^2 l_y(l_y + \beta_0 l_x)E_{0y}^{*2} - l^2 l_y(l_x - \beta_0 l_y)E_{0y}^*\left(\frac{kT_{e0}}{e}\frac{d\ \ln n_{e0}}{dx}\right) \tag{46}$$

$$A \equiv 2l_y(l_y + \beta_0 l_x)(l_x - \beta_0 l_y)\frac{1}{\lambda_i}E_{0y}^{*2} + l_y l^4\frac{1}{\lambda_i}E_{0y}^*\left(\frac{kT_{e0}}{e}\frac{d\ \ln n_{e0}}{dx}\right) \tag{47}$$

$$g \equiv 2\left[l_x l_y l^2 + l_y(l_x - \beta_0 l_y)\frac{1}{\lambda_i^2}\right]E_{0y}^*\left(\frac{kT_{e0}}{e}\frac{d\ \ln n_{e0}}{dx}\right) + l_y^2 l^2\left(\frac{kT_{e0}}{e}\frac{d\ \ln n_{e0}}{dx}\right)^2 \tag{48}$$

$$G \equiv 2l_y^2(l_y + \beta_0 l_x)\frac{1}{\lambda_i}E_{0y}^*\left(\frac{kT_{e0}}{e}\frac{d\ \ln n_{e0}}{dx}\right) + l_y^2(l_x - \beta_0 l_y)\frac{1}{\lambda_i}\left(\frac{kT_{e0}}{e}\frac{d\ \ln n_{e0}}{dx}\right) \tag{49}$$

Upon substitution of equation (45) into equation (40) and using equation (8) to eliminate

$$\left(\frac{1}{\sigma_0}\underline{j}_0\right)\cdot\underline{E}_0^*$$

we obtain

$$\omega^* \frac{\tilde{T}_e}{T_{e0}} - \frac{\underline{l} \cdot \underline{j}_0}{n_{e0} e} \frac{\tilde{T}_e}{T_{e0}} = i2 \frac{m_e}{m_a} \nu_0 \frac{\tilde{T}_e}{T_{e0}} + i \frac{2}{3kT_{e0}} \frac{e^2}{m_e \nu_0} \left[\frac{a + iA}{l^4 + \frac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}} \frac{\tilde{n}_e}{n_{e0}} + \frac{g - iG}{l^4 + \frac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}} \frac{\tilde{T}_e}{T_{e0}} \right]$$

$$+ i \frac{\tilde{T}_e}{T_{e0}} \underline{v}_0 \cdot \overline{\nabla} \ln n_{e0} - \frac{2}{3} \left(\frac{3}{2} + \frac{T_i}{T_{e0}} \right) \omega^* \frac{\tilde{n}_e}{n_{e0}} \tag{50}$$

$\tilde{n}_e/n_{e0}$ is eliminated from equation (50) by multiplying through by ω^* and using equation (38) to give an equation of the form

$$X \frac{\tilde{T}_e}{T_{e0}} = 0$$

so that the condition for the existence of a nontrivial solution requires that X be zero. The result of setting $X = 0$ yields

$$\omega^{*2} - \left(\left[\frac{\underline{l} \cdot \underline{j}_0}{n_{e0} e} + \frac{2}{3kT_{e0}} \frac{e^2}{m_e \nu_0} \frac{G}{l^4 + \frac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}} \right] \right.$$

$$\left. - i \left\{ 2 \frac{m_e}{m_a} \nu_0 + \left[1 + \frac{2}{3} \left(\frac{3}{2} + \frac{T_i}{T_{e0}} \right) \left(\frac{1}{2} + \frac{T_i}{T_{e0}} \right) \right] \underline{v}_0 \cdot \overline{\nabla} \ln n_{e0} + \frac{2}{3kT_{e0}} \frac{e^2}{m_e \nu_0} \frac{g}{l^4 + \frac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}} \right\} \right) \omega^*$$

$$= \frac{2}{3kT_{e0}} \frac{e^2}{m_e \nu_0} \frac{a + iA}{l^4 + \frac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}} \left(\frac{1}{2} + \frac{T_i}{T_{e0}} \right) \underline{v}_0 \cdot \overline{\nabla} \ln n_{e0} \tag{51}$$

which is the general dispersion relation for the problem under consideration. Unfortunately it is too complicated to consider for other than numerical evaluation. Fortunately, however, in the regime of parameters of interest in most nonequilibrium MHD experiments the relation can be greatly simplified.

It is noted that the coefficients a, A, g, and G are functions of the Faraday electric field E_{0y}^* and the ambipolar electric field

$$\frac{kT_{e0}}{e}\frac{d\ln n_{e0}}{dx}$$

and by virtue of equations (16) and (19) so are the coefficients of equation (51). Under typical generator operating conditions $E_{0y}^* \simeq 2\times10^3$ volts per meter and $T_{e0} \simeq 3000$ K. At this temperature the ambipolar field $\simeq 3\times10^2$ volts per meter ($d \ln n_{e0}/dx \simeq 10^3$ 1/M, see discussion following eq. (11)). Therefore

$$\frac{\dfrac{kT_{e0}}{e}\dfrac{d\ln n_{e0}}{dx}}{E_{0y}^*} << 1$$

The coefficients of equation (51) are now simplified by retaining only the dominant power of the above ratio in each of the coefficients to yield

$$\omega^{*2} - \left(\left[1 + \frac{4}{3}\frac{l_y(l_y + \beta_0 l_x)\dfrac{1}{\lambda_i^2}}{l^4 + \dfrac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}}\right]\frac{\underline{l}\cdot\underline{j}_0}{n_{e0}e} + i\left\{2\frac{m_e}{m_a}\nu_0 + \left[1 + \frac{2}{3}\left(\frac{3}{2} + \frac{T_i}{T_{e0}}\right)\left(\frac{1}{2} + \frac{T_i}{T_{e0}}\right)\right]\underline{v}_0\cdot\overline{\nabla}\ln n_{e0}\right\}\right)\omega^*$$

$$= 2\left[\frac{l^2 l_y(l_y + \beta_0 l_x)}{l^4 + \dfrac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}} + i\frac{l_y(l_y + \beta_0 l_x)(l_x - \beta_0 l_y)\dfrac{1}{\lambda_i}}{l^4 + \dfrac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}}\right]$$

$$\times\left[2\frac{m_e}{m_a}\nu_0 + \frac{2}{3}\left(\frac{3}{2} + \frac{T_i}{T_{e0}}\right)\underline{v}_0\cdot\overline{\nabla}\ln n_{e0}\right]\left(\frac{1}{2} + \frac{T_i}{T_{e0}}\right)\underline{v}_0\cdot\overline{\nabla}\ln n_{e0} \qquad (52)$$

It is further noted that typically $\nu_0 \simeq 10^{10}$ collisions per second and $m_e/m_a \simeq 10^{-5}$ so that

$$2 \frac{m_e}{m_a} \nu_0 \simeq 10^5 \frac{1}{\text{sec}} \tag{53}$$

Also

$$v_0 \simeq 10^3 \frac{\text{M}}{\text{sec}}$$

and

$$\left(\frac{1}{2} + \frac{T_i}{T_{e0}}\right) \simeq 20$$

so that

$$\left(\frac{1}{2} + \frac{T_i}{T_{e0}}\right) v_0 \frac{d \ln n_{e0}}{dx} \simeq 10^7 \tag{54}$$

Furthermore

$$\frac{\underline{l} \cdot \underline{j}_0}{n_{e0} e} = \frac{e}{m_e \nu_0} l_y E^*_{0y} \leq \beta_0 l_y v_0 = \beta_0 \frac{2\pi}{\lambda_y} v_0$$

where λ_y is the wavelength in the y-direction. In general, $\beta_0 \lesssim 1$ so that for wavelengths the order of centimeters in the y-direction

$$\frac{\underline{l} \cdot \underline{j}_0}{n_{e0} e} \simeq 10^5 \tag{55}$$

From relations (53) to (55) we conclude that the factor

$$\left(\frac{1}{2} + \frac{T_i}{T_{e0}}\right) v_0 \frac{d \ln n_{e0}}{dx}$$

is by far the dominant one in equation (52) and hence we seek a solution in the limit as this factor approaches infinity. For this limit, the real and imaginary parts of ω become

$$\omega_r^* = -2 \frac{l_y(l_y + \beta_0 l_x)(l_x - \beta_0 l_y) \frac{1}{\lambda_i}}{l^4 + \frac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}} \underline{v}_0 \cdot \overline{\nabla} \ln n_{e0} \tag{56}$$

$$\omega_i = 2 \frac{l^2 l_y(l_y + \beta_0 l_x)}{l^4 + \frac{(l_x - \beta_0 l_y)^2}{\lambda_i^2}} \underline{v}_0 \cdot \overline{\nabla} \ln n_{e0} \tag{57}$$

That this is the correct solution to equation (52) to the order of the ratio of relations (53) and (55) to relation (54) can be shown by direct substitution of the solution back into equation (52).

It is noted from equation (57) that for zero magnetic field, $\beta_0 = 0$, ω_i is positive and hence the system is stable at least up to terms the order of the ratio of ambipolar to Faraday electric fields. However, with $\beta_0 > 0$, ω_i becomes negative for orienta- of the wave vector for which

$$\beta_0 \frac{l_x}{l_y} < -1 \tag{58}$$

Since all orientations are possible, the result indicates that the system is unstable for all nonzero values of magnetic field. However, again the result is only correct up to terms the order of the ratio of ambipolar to Faraday electric fields. If these higher order terms are positive then there will be a critical value of β_0 for which the system goes unstable.

CONCLUSIONS

It is concluded that the ionization region of a nonequilibrium, segmented electrode MHD generator operating in the Faraday mode is unstable in the presence of a magnetic field. To the present order of solution, which is the dominant order in the ratio of the ambipolar to Faraday electric fields, the system is unstable to any nonzero magnetic field. It therefore appears that some auxiliary means of ionizing the gas in this region may be required or the region should be shielded from the magnetic field. Under investigation is the possibility of ionizing by applied electric fields within the magnetic field region but in the magnetic field direction or in the Hall field direction.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, November 7, 1968,
129-02-08-05-22.

APPENDIX - SYMBOLS

A	defined by eq. (47)
a	defined by eq. (46)
$\underline{B}$	magnetic field vector
$\underline{E}$	electric field vector
e	unit electric charge
G	defined by eq. (49)
g	defined by eq. (48)
$\underline{j}$	electric current density
k	Boltzmann constant
$\underline{l}$	wave vector
m	particle mass
n	particle number density
P	pressure
T	temperature
$\underline{v}$	velocity
β_0	Hall parameter
λ	wavelength or e-folding length
ν	collision frequency
σ	electrical conductivity
ω	wave frequency

Subscripts:

a	gas atoms
e	electrons
i	ionization value
x, y, z	components of x, y, z coordinate system

Superscripts:

*	evaluated in frame of reference moving with gas
'	perturbed quantities
~	defined by eq. (36)

REFERENCES

1. Kerrebrock, J. L.: Nonequilibrium Ionization Due to Electron Heating. 1. Theory. AIAA J., vol. 2, no. 6, June 1964, pp. 1072-1080.

2. Bertolini, E.; Toschi, R.; and McNab, I. R.: Relaxation Phenomena in MPD Generators. Vol. 1 of Magnetohydrodynamic Electrical Power Generation; Proc. Int. Symposium; Salzburg, Austria, July 5-8, 1966. European Nuclear Energy Agency (Vienna), 1966, pp. 533-545.

3. Lutz, M. H.: Radiation and Its Effect on the Nonequilibrium Properties of a Seeded Plasma. AIAA J., vol. 5, no. 8, Aug. 1967, pp. 1416-1423.

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

— NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546