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ANALYSIS OF THERMALLY INDUCED STRUCTURAL VIBRATIONS BY FINITE ELEMENT TECHNIQUES

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GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

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Test and Evaluation Division
Systems Reliability Directorate

August 1968

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

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THERMALLY INDUCED STRUCTURAL VIBRATIONS

BY

FINITE ELEMENT TECHNIQUES

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REPORT STATUS

This report describes procedures for treating thermally induced structural vibrations by finite element analyses techniques. It is intended for readers familiar with finite element methods but unfamiliar with thermoelasticity theory and procedure. The report demonstrates the applicability of finite element methods to the thermal vibration problem and, as such, should serve as a guide to users of various discrete element structural analysis programs when treating problems of this type.

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ANALYSIS OF

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SUMMARY

A brief review of the classical continuum approach to thermoelasticity problems is given. It is shown that an analogous technique can be used to reduce the study of thermally induced vibrations by finite element techniques to that of a forced response analysis of the idealized structure excited by time dependent equivalent mechanical loads.

Equivalent mechanical loads for beam and rectangular plate elements are presented. These time dependent loads are evaluated and employed to treat two sample problems for which exact solutions of the differential equations of motion exist. The samples considered are those of a simply supported beam and a simply supported plate exposed to a uniform step heat input over one face. Results of the finite element analyses are compared to the exact solutions for these two cases.

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ANALYSIS OF THERMALLY INDUCED STRUCTURAL VIBRATIONS BY FINITE ELEMENT TECHNIQUES

INTRODUCTION

It is the purpose of this report to indicate the general procedures to be used in treating thermally induced vibrations of complex structures by finite element techniques. Consideration of this problem is taking on new importance in both present and planned spacecraft structures because of their lightweight thin walled construction, their associated slender structural components (typified by long antennas and booms) and the unusually demanding performance requirements expected of many of these vehicles in their space environment.

The present investigation demonstrates that the thermal vibration problem can be treated by a forced response analysis of the idealized structure excited by time dependent equivalent mechanical loads. The problem is thus reduced to the determination of the equivalent mechanical loads associated with the particular finite elements used in the idealization of the real structure.

Time dependent mechanical loads are obtained in this report for a beam element of uniform cross section and a thin rectangular plate bending element. In order to demonstrate the analytical procedures, two sample problems for which exact solutions exist are analyzed using the beam and plate elements. The finite element computations were performed on the IBM 7094 computer using the Martin Company SB-038 computer program and the results of this analysis compared to the exact solutions.

ASSUMPTIONS AND BACKGROUND

In the following development all of the usual assumptions of linear elasticity for continuous, homogeneous, and isotropic materials are employed and Hooke's law is assumed valid. In addition, the conversion of mechanical energy into heat is neglected. With the latter assumption, the determination of the temperature distribution in the structure is uncoupled from the elasticity problem and may be obtained by the Fourier heat conduction equation.

A brief review of the method of restraint for treating thermoelasticity problems by the techniques of classical elasticity is given here. This is done to provide insight for the solution of these problems by the finite element approach.

Static thermoelasticity problems may be considered as initial strain problems and, in turn, the initial strains treated as additional mechanical loads acting on the structure. The field equations describing the heated structure in rectangular Cartesian coordinates may be expressed as:

Equilibrium Equation

$$\sigma_{ij,j} + X_i = 0 \tag{1}$$

Stress-Strain

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{\beta\beta}\,\delta_{ij} + \alpha T \delta_{ij} \tag{2}$$

Strain-Displacement

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

where the conventional summation convention is adapted, $(i, j, \beta = 1, 2, 3)$ and $(\ )_{,i}$ indicates $\frac{\partial (\ )}{\partial x_i}$, here

x_i ~ coordinate component

u_i ~ displacement component

X_i ~ body force component

σ_{ij} ~ stress component

ϵ_{ij} ~ strain component

δ_{ij} ~ Kronecker delta

α ~ coefficient of thermal expansion

T ~ temperature rise from a given base temperature

ν ~ Poisson's ratio

E ~ modulus of elasticity

subject to the boundary conditions:

Traction Boundary Conditions

$$\overset{n}{S}_i = n_j \, \sigma_{ji} \tag{4}$$

Displacement Boundary Conditions

$$u_i = f_i(p) \tag{5}$$

where

$\overset{n}{S}_i$ ~ surface traction component acting on the surface with outer normal $\vec{n}$

n_i ~ component of $\vec{n}$

$f_i(p)$ ~ prescribed displacement function component at points p of the boundary surface

Employing the method of restraint, we consider a body free of surface tractions to be subdivided into infinitely small cubical elements. Initially these small elements will fit together to form a continuum. If the temperature of the body is altered in a nonuniform manner, however, the cubical elements will expand an amount

$$\epsilon_{ij} = \alpha T \, \delta_{ij} \tag{6}$$

proportional to their own temperature change and no longer necessarily form a continuum. This expansion can be entirely suppressed by applying a uniform pressure

$$\sigma_{ij} = -\frac{\alpha E T}{1 - 2\nu} \delta_{ij} \tag{7}$$

to each of the individual elements. Under this stress system, referred to as "state A," the lack of fit between elements in the heated structure is removed and the body assumes its initial undeformed continuous configuration.

To this point we have determined the state A stress distribution, given by Equation (7), necessary to restrain the heated structure in its undeformed configuration. The requirement now is to obtain equivalent mechanical loads which, when applied to the heated structure, will produce this same restraining stress state. This technique of determining equivalent external loads is used in order that the well-known methods from classical elasticity for the analysis of structures under external loading can be utilized in the analysis of thermally deformed structures.

To obtain the equivalent mechanical loads, we note that the state A stress system ensures compatibility in that it is the stress system required to restrain the heated structure in its undeformed configuration. Substituting Equation (7) into the equilibrium Equation (1) results in the required body forces for restraint

$$X_i = \frac{\alpha E}{1 - 2\nu} T,_i \tag{8}$$

while substitution of Equation (7) into Equation (4) results in the required restraining surface tractions

$$\overset{n}{S}_i = -\frac{\alpha ET}{1 - 2\nu} n_i \tag{9}$$

Therefore, Equations (8) and (9) express the equivalent mechanical loads which produce the state A stress distribution and restrain the heated structure in its undeformed configuration.

Since in actuality these loads do not exist on the body, the undeformed structure is now loaded with the negative of the calculated body forces

$$X_i = -\frac{\alpha E}{1 - 2\nu} T,_i \tag{10}$$

and the negative of the calculated surface tractions

$$\overset{n}{S}_i = \frac{\alpha ET}{1 - 2\nu} n_i \tag{11}$$

in a manner which satisfies the boundary conditions existing on the structure. The solution of this problem is obtained by classical elasticity methods for a structure under the action of external loads given by Equations (10) and (11) and is referred to here as the "state B" solution.

The final displacements are obtained from the state B analysis, since the state A displacements are zero, and the final stresses in the body are found by summing the state A and state B stress distributions.

It is seen that the effects of initial thermal strains have been treated by the introduction of equivalent external body and surface forces which were derived by restraining the deformations of the heated structure. The thermoelasticity problem can be treated by finite element methods in a manner completely analogous to the continuum approach described above. Conceptically, this is achieved by replacing the infinitesimal cubical elements and their associated restraining stress systems by finite elements and the necessary equivalent mechanical loads which must be applied at the element grid points to constrain the elements in their unheated shapes.

DISCRETE ELEMENT FORMULATION

The matrix-displacement formulation of the thermally induced vibration problem is used in the following development. The results, however, are applicable to other finite element formulations.

In the finite element approach to structural analysis problems, the body to be analyzed is idealized to a new structure composed of an assemblage of finite structural elements which are joined at grid points. The individual discrete elements are defined by a number of grid point degrees of freedom which are sufficient to adequately represent the stress and displacement behavior of the element. One may write the force-displacement relations for the thermally strained i^{th} element by the stiffness equation

$$\{X\}_i = [k]_i \{u\}_i + \{X_T\}_i \tag{12}$$

where

$\{X\}_i$ ~ forces acting at the grid points of element i

$[k]_i$ ~ stiffness matrix of element i

$\{u\}_i$ ~ grid point displacements of element i

$\{X_T\}_i$ ~ thermally equivalent mechanical loads acting at the grid points of element i

The equivalent mechanical loads $\{X_T\}_i$ are obtained as those required to remove the initial thermal strains in the discrete element. As such, the analogy with the restraining systems, given by Equations (8) and (9), is apparent.

Assembly of the element matrices, Equation (12), to form the idealized structure results in the following set of equations describing the static behavior of the structure

$$\{\underline{X}\} = [\underline{K}]\ \{U\} + \{\underline{X}_T\} \tag{13}$$

where

$\{\underline{X}\}$ ~ external applied loads acting at the grid points of the assembled structure

$[\underline{K}]$ ~ stiffness matrix of the assembled structure

$\{U\}$ ~ grid point displacements of the assembled structure

$\{\underline{X}_T\}$ ~ thermally equivalent mechanical loads acting at the grid points of the assembled structure

In Equation (13), the equivalent mechanical loads acting at a grid point of the assembled structure are obtained as the vectorial sum, at the grid point, of the loads $\{X_T\}_i$ of all elements joined at that grid point.

In many practical problems the temperature variations in the structure change rapidly with time and the effects of inertia can not be neglected. In these cases, the study of thermally induced vibrations by finite element techniques can be accomplished by modification of Equation (13) to include inertia effects. Using the well-known principle of d'Alembert, the negatives of the inertia forces

$$-\ [\underline{M}]\ \{\ddot{U}\} \tag{14}$$

where

$[\underline{M}]$ ~ mass matrix of the assembled structure

$\{\ddot{U}\}$ ~ grid point accelerations of the assembled structure

are treated as additional applied loads acting on the structure. In addition, the equivalent mechanical loads become time dependent forcing functions. Thus, from Equations (13) and (14) for the case of no externally applied loading, we can write

$$[M]\{\ddot{U}\} + [K]\{U\} = -\{X_T(t)\} \qquad (15)$$

Examination of Equation (15) shows that the problem of thermally induced vibrations in finite element structures is reduced to that of obtaining the dynamic response to a forced vibration in which the external forcing functions become the negatives of the equivalent mechanical loads. Upon the determination of $\{X_T\}$ as a function of time t and imposition of boundary conditions, Equation (15) is solved for the unknown grid point displacements $\{U(t)\}$ of the idealized structure.

The remainder of this report has been written to demonstrate and evaluate the above analysis procedures. Two sample problems, for which solutions of the differential equations of motion exist, have been chosen for finite element analysis. The determination of equivalent mechanical loads for the finite elements employed in these samples, i.e., uniform beams and rectangular plates, pose no particular difficulty. In general, however, the evaluation of mechanical loads for various finite elements is more involved since these loads are, in reality, functions of the assumed displacement or stress pattern for the element and are obtained by energy minimization procedures.

THERMALLY INDUCED BEAM VIBRATIONS

The problem to be considered in this section is that of a uniform simply supported rectangular beam of length L and thickness h exposed to a step heat input Q (constant in x) along edge $y = +\frac{h}{2}$ while edge $y = -\frac{h}{2}$ is insulated, see Figure 1. For this case the time varying temperature distribution in the beam, provided $T(y,o) = 0$, is[1]

$$T(y,\tau) = \frac{hQ}{\tilde{k}}\left\{\tau + \frac{1}{2}\left(\frac{y}{h}+\frac{1}{2}\right)^2 - \frac{1}{6} - \frac{2}{\pi^2}\sum_{j=1}^{\infty}\frac{(-1)^j e^{-j^2\pi^2\tau}}{j^2}\cos j\pi\left(\frac{y}{h}+\frac{1}{2}\right)\right\} \qquad (16)$$

where

$\tau = \left(\frac{\kappa t}{h^2}\right)$ ~ nondimensional time parameter

$\tilde{k}$ ~ thermal conductivity

κ ~ thermal diffusivity

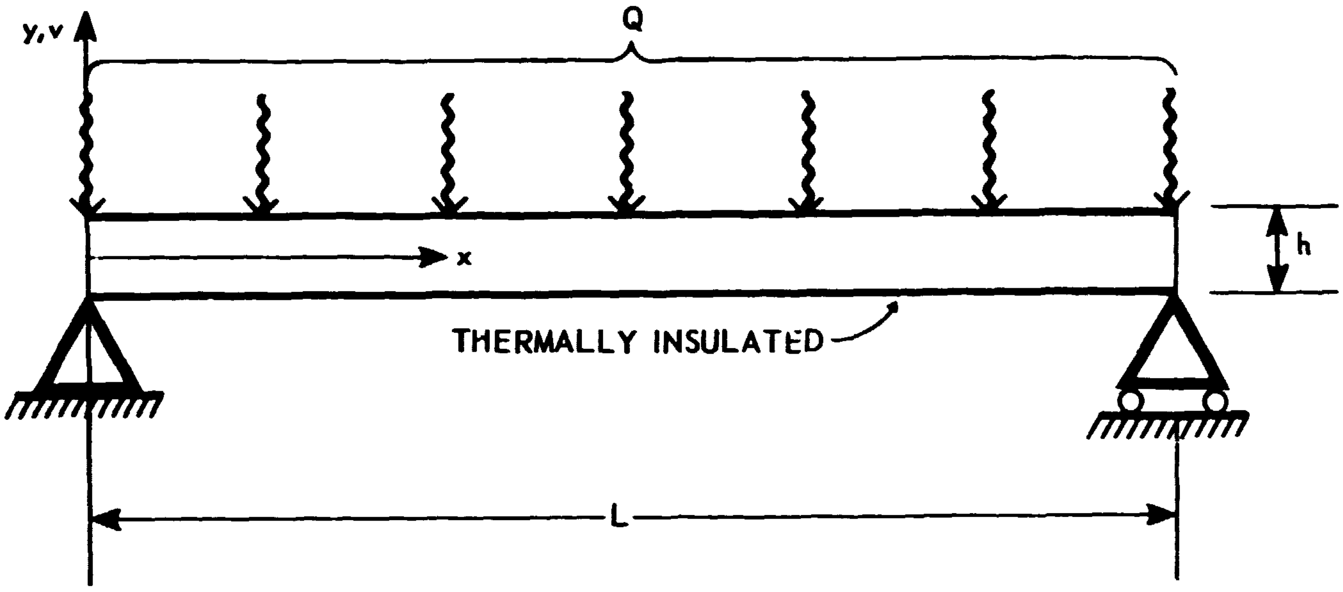

Figure 1. Simply Supported Beam Exposed to Step Heat Input

Elasticity Formulation[1]

The governing equation for the displacement of a uniform Bernoulli-Euler beam subjected to heating can be written as

$$\frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 v}{\partial x^2}\right) + \rho\, A\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 M_T}{\partial x^2} \tag{17}$$

where the forcing moment is

$$M_T(\tau) = \iint\limits_A E\,\alpha\, T\,(y,\tau)\; y\, dA \tag{18}$$

and

$x \sim$ longitudinal coordinate

$y \sim$ transverse coordinate

$v \sim$ transverse displacement

$I \sim$ area moment of inertia

$A \sim$ cross sectional area

$\rho \sim$ mass density

Substituting Equation (16) into Equation (18) and integrating yield the time dependent moment in terms of the basic input parameters as

$$M_T(\tau) = \frac{48EIQ\alpha}{\pi^4\tilde{k}}\left[\frac{\pi^4}{96} - \sum_{j=1,3,5}^{\infty} \frac{e^{-j^2\pi^2\tau}}{j^4}\right] \tag{19}$$

Defining the following dimensionless quantities

$$\xi = \frac{x}{L}$$

$$m_T(\tau) = \frac{\pi^4\tilde{k}M_T}{192EIQ\alpha} \tag{20}$$

$$V(\xi,\tau) = \frac{\pi^4\tilde{k}v}{192Q\alpha L^2}$$

$$B = \frac{h}{L\sqrt{\kappa}}\left(\frac{EI}{\rho A}\right)^{1/4}$$

and using Equation (19), noting that M_T does not depend on x, the governing equation becomes

$$B^4\frac{\partial^4 V}{\partial \xi^4} + \frac{\partial^2 V}{\partial t^2} = 0 \tag{21}$$

The boundary and initial conditions for the simply supported beam are

$$V(0,\tau) = V(1,\tau) = V(\xi,0) = \frac{\partial V}{\partial \tau}(\xi,0) = 0$$

$$\frac{\partial^2 V(0,\tau)}{\partial \xi^2} = \frac{\partial^2 V}{\partial \xi^2}(1,\tau) = -m_T(\tau) \tag{22}$$

Solution of the system given by Equations (21) and (22) yields the displacement function

$$V(\xi,\tau) = -\frac{m_T}{2}\left[\xi^2 - \xi\right] - \sum_{n=1,3,5}^{\infty} \frac{\sin n\pi\xi}{n^3\pi^3}\left[\frac{\pi^2}{8B^2n^2}\sin n^2\pi^2B^2\tau - \sum_{j=1,3,5}^{\infty} \frac{e^{-j^2\pi^2\tau} + \left(\frac{j}{nB}\right)^2 \sin n^2\pi^2B^2\tau - \cos n^2\pi^2B^2\tau}{j^4 + n^4B^4}\right] \tag{23}$$

Results are given for V versus τ for the case $B = 1$ and $\xi = ½$ in Figure 2. The results were obtained by a single precision program which summed to $n = j = 21$ in Equation (23).

Finite Element Formulation

It has been shown, see Equation 12, that the stiffness equation

$$\{X\}_i = [k]_i \{u\}_i + \{X_T\}_i \tag{24}$$

represents the force-displacement relations for a thermally strained finite element. The stiffness and mass matrices for beams are well-known, and the problem of thermally induced vibrations in finite element beam problems reduces to the determination of equivalent mechanical loads.

A typical beam element of uniform rectangular cross section is shown in Figure 3. Only temperature distributions T(y, t), constant along the axis of the element, need be considered if care is taken to model the structure such that stepwise variations in temperature closely approximate the axial temperature variation in the actual structure. Equivalent mechanical loads necessary to restrain the beam element from bending when subjected to a temperature variation through the depth of the beam element are required.

Since the temperature does not vary along the length of the element and equilibrium is considered in the axial direction only, no body forces are required for restraint and only the boundary stresses

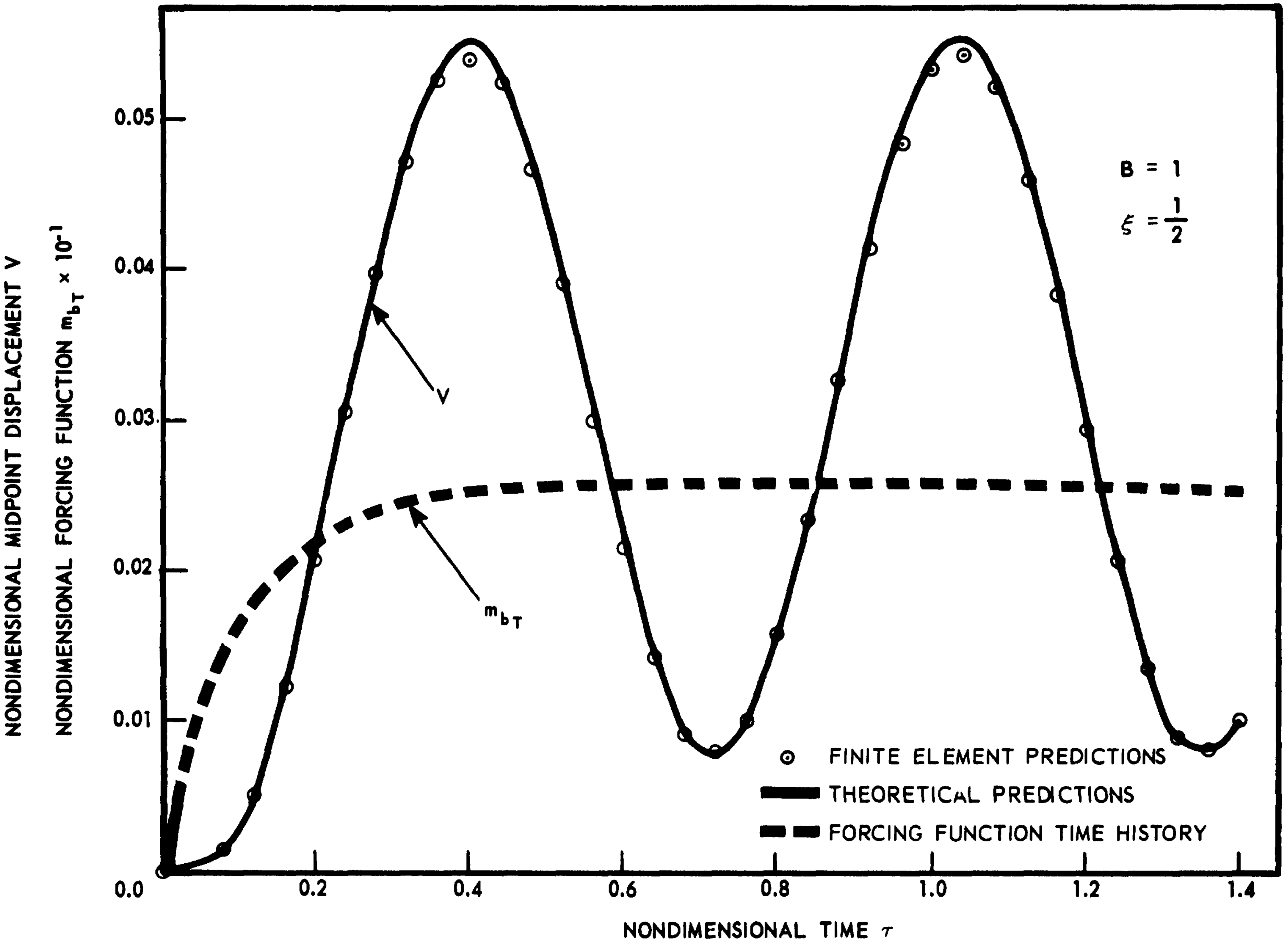

Figure 2. Thermally Induced Beam Vibrations (Theoretical and Finite Element Midpoint Displacement Response Predictions for Beam Forcing Function m_{b_T})

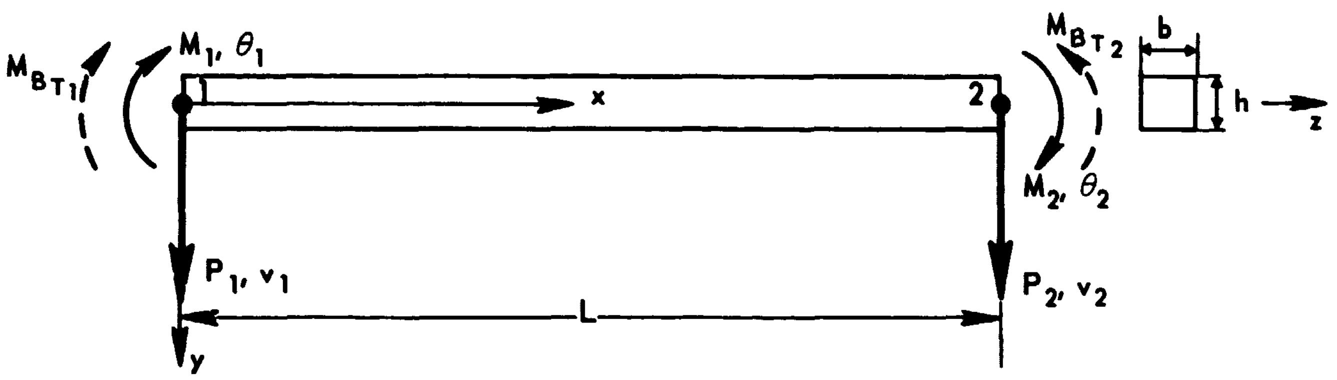

Figure 3. Grid Point Forces, Displacements and Thermal Loads for Finite Beam Element

$$\sigma_{xx}\Big|_{x=o} = \sigma_{xx}\Big|_{x=L} = -E\alpha T(y,t) \tag{25}$$

need be applied. This boundary stress distribution can be replaced by the resultant equivalent mechanical moments

$$M_{b_T}(\tau) = \int_{-h/2}^{h/2} b\alpha ET(y,\tau)y\,dy \tag{26}$$

acting at the grid points of the element, see Figure 3.

Referring to Figure 3 and Equation (26), we see that the stiffness equation for a typical heated beam element can be written as:

$$\begin{Bmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \\ K_{Beam} \\ \\ \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ M_{b_{T_1}} \\ 0 \\ -M_{b_{T_2}} \end{Bmatrix} \tag{27}$$

Having found the equivalent mechanical loads for the beam element, we can determine the required forcing functions for the complete model. The assembled finite element model of the simply supported beam of Figure 1 consisted of 11 grid points connected by 10 beam elements of length L/10, see Figure 4. For

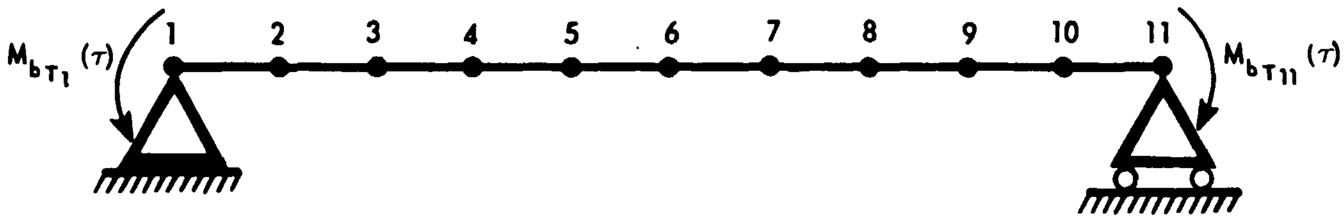

Figure 4. Finite Element Model of Simply Supported Beam

the temperature distribution given by Equation (16), the equivalent mechanical moments acting on each beam element are obtained from Equation (26), assuming E and α constant, as

$$M_{b_T}(\tau) = \frac{48EIQ\alpha}{\pi^4\tilde{k}}\left[\frac{\pi^4}{96} - \sum_{j=1,3,5}^{\infty}\frac{e^{-j^2\pi^2\tau}}{j^4}\right] \tag{28}$$

Since the temperature distribution does not vary along the length in this sample problem, the vectorial sum of the thermal moments at the internal grid points (2 through 10) of the beam are zero, and the problem reduces, according to Equation (15), to that shown in Figure 4.

Solution of the finite element model subjected to the time varying end moments given by Equation (28) and shown in Figure 4 was obtained on the IBM 7094 computer using the Martin Company SB-038 force method program.[2] Lumped mass and modal displacement techniques were utilized for this analysis. Results are given for V versus τ for the case B = 1 and ξ = ½ in Figure 2. A plot of nondimensional forcing function for this case

$$m_{b_T}(\tau) = \frac{\pi^4\tilde{k}}{192EIQ\alpha} M_{b_T}(\tau) \tag{29}$$

is also shown in this figure.

THERMALLY INDUCED PLATE VIBRATIONS

The problem considered in this section is that of a simply supported square plate subjected to a step heat input Q (constant in x and y) on face

$z = +\frac{h}{2}$ while face $z = -\frac{h}{2}$ is insulated, see Figure 5. The temperature distribution is identical to that of Equation (16) with y replaced by z.

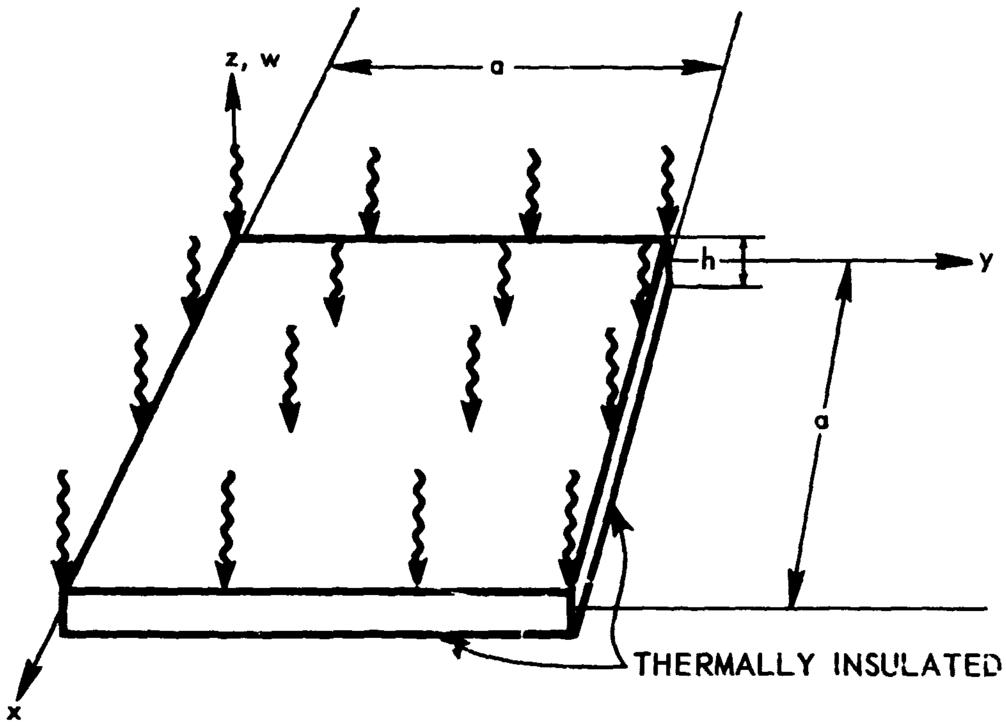

Figure 5. Simply Supported Square Plate Exposed to Step Heat Input

Elasticity Formulation

The governing differential equation for the square plate is:

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = -\frac{1}{1-\nu} \nabla^2 M_T \tag{30}$$

where the time dependent forcing moment is

$$M_T(t) = \int_{-h/2}^{h/2} \alpha E T\,(z,t)\, z\, dz \tag{31}$$

and

$x, y \sim$ middle surface coordinates

$z \sim$ transverse coordinate

$h \sim$ thickness of plate

$w \sim$ transverse displacement

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$T(z,\tau) = \frac{hQ}{\tilde{k}}\left\{\tau + \frac{1}{2}\left(\frac{z}{h}+\frac{1}{2}\right)^2 - \frac{1}{6} - \frac{2}{\pi^2}\sum_{j=1}^{\infty}\frac{(-1)^j\, e^{-j^2\pi^2\tau}}{j^2}\cos j\pi\left(\frac{z}{h}+\frac{1}{2}\right)\right\}$$

Solution of Equation (30) for the simply supported plate yields

$$w = \frac{16\,M_T}{(1-\nu)D\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{mn\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{a}\right)^2\right]}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$-\frac{768Q\alpha a^2(1+\nu)}{\tilde{k}\pi^8}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}{mn(m^2+n^2)} \tag{32}$$

$$\cdot\left\{\frac{\pi^2}{8\,B_1^2(m^2+n^2)}\sin\left[B_1^2\pi^2(m^2+n^2)\tau\right] - \sum_{j=1,3,5}^{\infty}\left[\frac{e^{-j^2\pi^2\tau} - \cos\left[B_1^2\pi^2(m^2+n^2)\tau\right]}{j^4+B_1^4(m^2+n^2)^2}\right.\right.$$

$$\left.\left.+\frac{j^2\sin\left[B_1^2\pi^2(m^2+n^2)\tau\right]}{\left[j^4+B_1^4(m^2+n^2)^2\right]\left[B_1^2(m^2+n^2)\right]}\right]\right\}$$

where

$$B_1 = \frac{h}{a\sqrt{\kappa}}\left(\frac{D}{h\rho}\right)^{1/4} \tag{33}$$

and

a ~ square plate dimension

Results for the following hypothetical problem

$$E = 1.44 \times 10^9 \ \frac{\text{lb}}{\text{ft}^3}$$

$$\nu = 1/3$$

$$\rho = 5.34 \ \frac{\text{lb-sec}^2}{\text{ft}^4}$$

$$\tilde{k} = 1.18 \ \frac{\text{BTU}}{\text{ft-sec-°F}}$$

$$\alpha = 12.0 \times 10^{-6} \ \frac{\text{ft}}{\text{ft-°F}}$$

$$c_p = 6.90 \ \frac{\text{BTU-ft}}{\text{lb-°F-sec}^2}$$

$$a = 8.0 \text{ ft}$$

$$h = 0.1 \text{ ft}$$

$$Q = 1440 \ \frac{\text{BTU}}{\text{ft}^2\text{-sec}}$$

were obtained from a single precision computer program which summed to $m = n = j = 25$ in Equation (32). Results of this calculation for midpoint displacement w versus time t are shown in Figure 6. It should be mentioned here that the numerical values were chosen to take advantage of an existing finite element plate model, see discussion of this below, and are not intended to be realistic.

Finite Element Formulation

An existing quarter symmetry model of an 8.0 ft × 8.0 ft × 0.1 ft simply supported plate was used in this study. The model consisted of a four by four network of 1.0 ft × 1.0 ft × 0.1 ft plate elements, see Figure 7. Unlike the beam, however, various stiffness matrices have been suggested to describe the behavior of plate elements. Hrennikoff plate elements,[3] obtained as an equivalent network of beams connected at the four corner points of the square element, were used. Each grid point of the model has three-degrees-of-freedom, one translation (w) and two rotations (θ, ϕ). The equivalent mechanical loads acting at the grid points of the element were obtained by prorating the distributed thermal restraining loads of a continuous square plate to the grid points of the finite element.

A rectangular plate element with an assumed temperature distribution T(z, t) constant over the planform of the plate is considered. As with the beam, this element will be satisfactory if care is taken to model the structure so that stepwise variations in temperature closely approximate the actual planform temperature. Since the temperature does not vary over the middle surface of the plate, body forces are not required for constraint. We may regard the plate as free to expand in the direction normal to the middle surface and constrain the x, y expansion by providing the boundary stress system

$$\sigma_{xx} = \sigma_{yy} = -\frac{\alpha ET}{1-\nu}; \ \sigma_{xy} = 0 \tag{34}$$

Thus, the plate bending may be constrained by replacing the boundary stress system of Equation (34) by the resultant thermal moments

$$\bar{M}_{PT}(t) = \int_{-h/2}^{h/2} \frac{\alpha E}{1-\nu} T(z,t)\, z\, dz \tag{35}$$

distributed along the edges of the element.

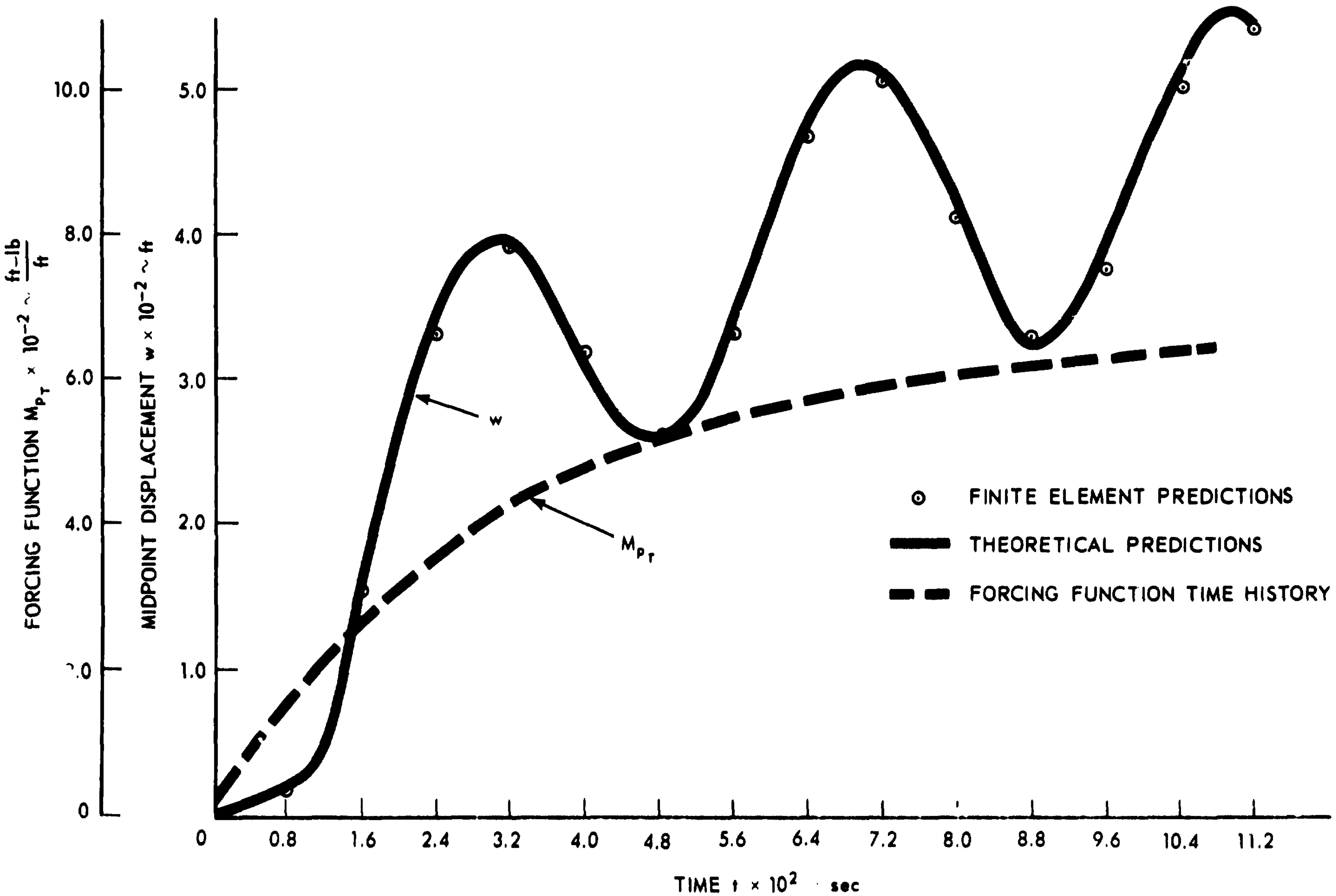

Figure 6. Thermally Induced Plate Vibrations (Theoretical And Finite Element Midpoint Displacement Response Predictions For Plate Forcing Function M_{P_T})

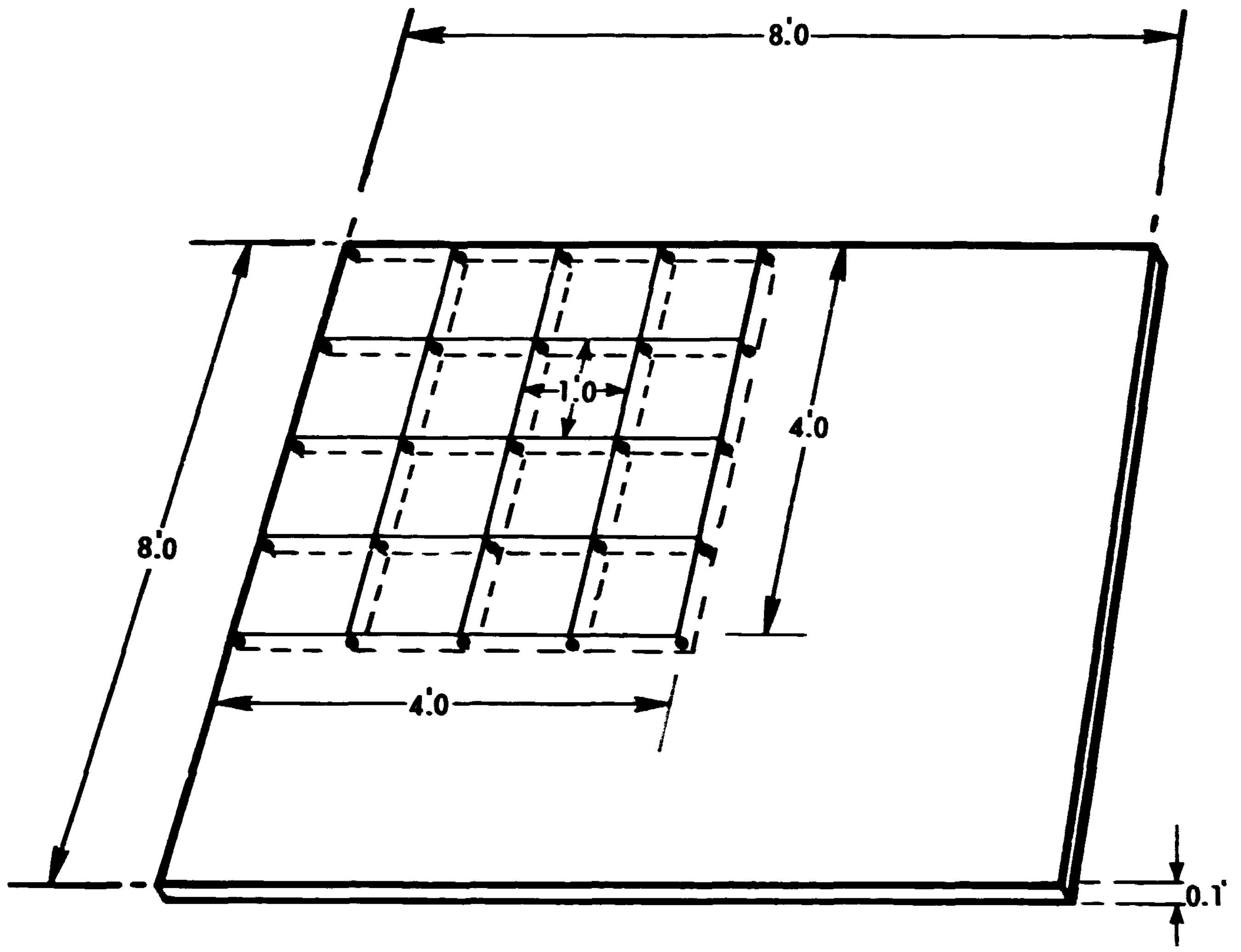

Figure 7. Quarter Symmetry Model of Simply Supported Square Plate

Prorating the distributed moments, Equation (35), to the grid points of the finite plate element results in a system of equivalent mechanical moments required for restraint. For the square plate element of the sample problem these thermal moments become

$$M_{PT}(t) = \frac{\bar{M}_{PT}(t)\,\bar{s}}{2} \tag{36}$$

where $\bar{s}$ is the planform dimension of the square plate element, see Figure 8.

The stiffness equation for a typical square plate element, referring to Figure 8, can be written as

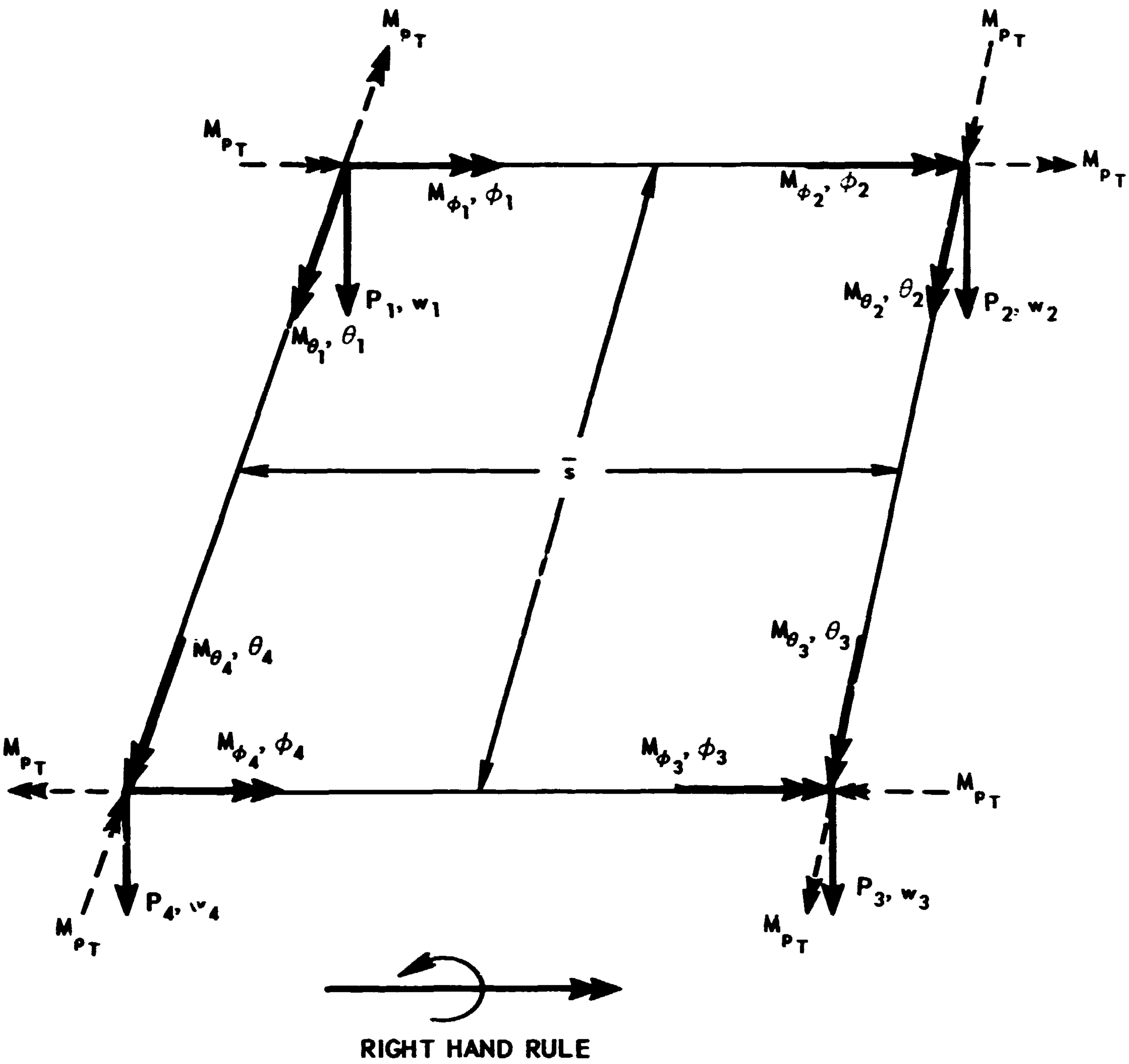

Figure 8. Grid Point Forces, Displacements and Thermal Loads for Finite Plate Element

$$
\begin{Bmatrix} P_1 \\ M_{\theta_1} \\ M_{\phi_1} \\ P_2 \\ M_{\theta_2} \\ M_{\phi_2} \\ P_3 \\ M_{\theta_3} \\ M_{\phi_3} \\ P_4 \\ M_{\theta_4} \\ M_{\phi_4} \end{Bmatrix} = \begin{bmatrix} K_{Plate} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ \phi_1 \\ w_2 \\ \theta_2 \\ \phi_2 \\ w_3 \\ \theta_3 \\ \phi_3 \\ w_4 \\ \theta_4 \\ \phi_4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -M_{PT} \\ M_{PT} \\ 0 \\ M_{PT} \\ M_{PT} \\ 0 \\ M_{PT} \\ -M_{PT} \\ 0 \\ -M_{PT} \\ -M_{PT} \end{Bmatrix} \tag{37}
$$

The equivalent mechanical loads for the 1 ft × 1 ft × 0.1 ft plate element with temperature distribution Equation (16) are found using Equations (35), and (36) to be

$$M_{P_T}(t) = \frac{2\,\alpha\, QEh^3}{(1-\nu)\,\pi^4\tilde{k}}\left[\frac{\pi^4}{96} - \sum_{j=1,3,5}^{\infty} \frac{e^{-j^2\pi^2\tau}}{j^4}\right] \tag{38}$$

when E and α are taken as constants. Since the temperature distribution does not vary over the planform of the plate in the sample problem, the vectorial sum of the thermal moments at the internal grid points of the assembled plate model are zero and the problem, according to Equation (15), reduces to the quarter symmetry model loaded as shown in Figure 9.

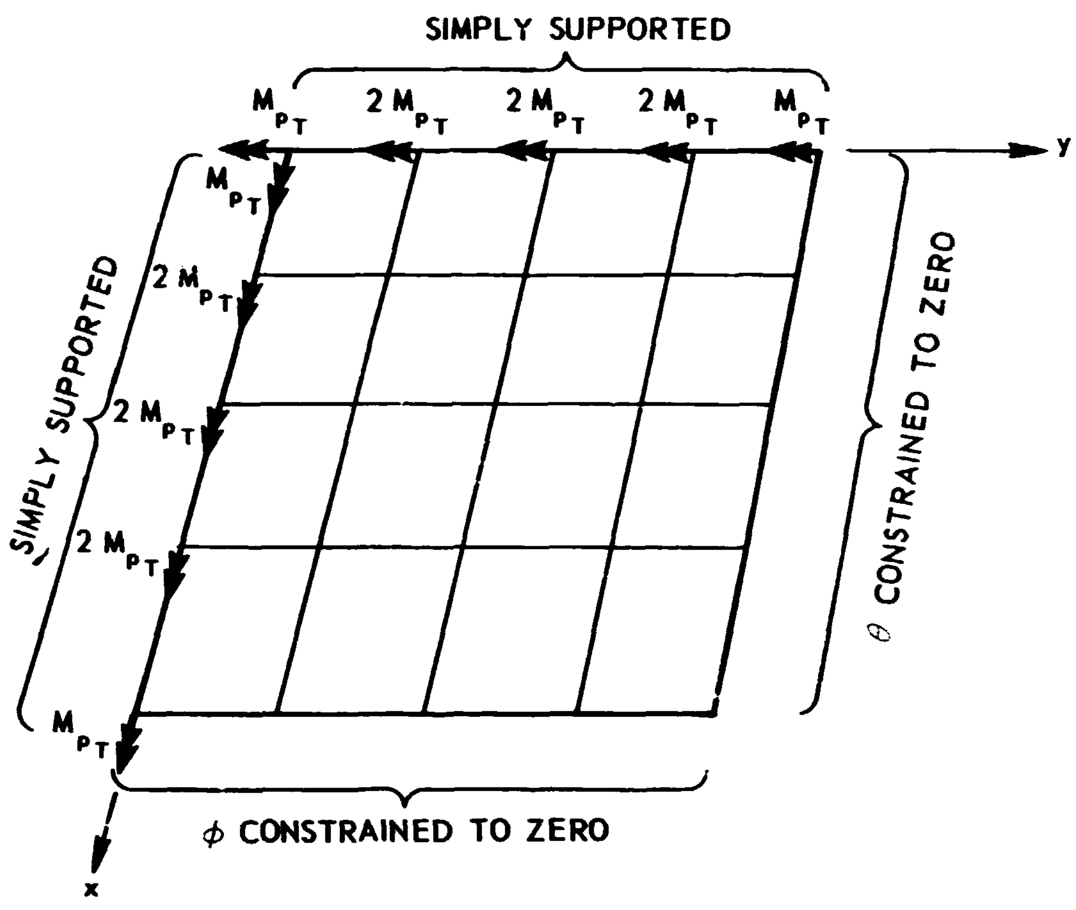

Figure 9. Quarter Symmetry Hrennikoff Model of Simply Supported Square Plate

Solution of the finite element model shown in Figure 9 was obtained by the Martin Company's SB-038 program. Lumped mass and modal acceleration techniques were utilized and the results are given for midpoint displacement w versus time t in Figure 6. A plot of forcing functions M_{P_T} versus time is also shown.

CONCLUSIONS

It has been shown that structural vibrations induced by uncoupled transient temperature distributions may be treated by a forced response analysis of the idealized structure excited by time dependent mechanical loads applied at the grid points of the model. Time dependent mechanical loads for a uniform rectangular beam element and a thin rectangular plate bending element have been presented.

Two thermal vibration problems have been solved by the finite element method:

1. Simply supported beam subjected to a step heat input over one face: The finite element results compare favorably with those of the elasticity solution. Differences between the finite element results and the elasticity solution can be accounted for by the coarseness of partition in the beam model and numerical inaccuracies inherent in obtaining both solutions.

2. Simply supported plate subjected to a step heat input over one face: The results obtained from the finite element analysis compare well with those from the elasticity solution. Differences can be attributed to the coarseness of partition in the plate model, inadequacies inherent in the Hrennikoff plate bending element and inaccuracies in numerical evaluation of both solutions.

At the option of the analyst, improvements in the finite element results for both of the sample problems could be obtained by use of a finer element partition.

The results of this study demonstrate that the finite element method offers a powerful tool for use in the analysis of thermally induced vibrations. Future work in this area will concentrate on the automatic calculation of equivalent mechanical loads for the various finite elements incorporated in the NASTRAN computer program.

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