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EFFECTS OF HEAT EXCHANGE ON RELATIVISTIC FLUID FLOW

L. A. SCHMID

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ABSTRACT

The established covariant formalism of relativistic fluid dynamics is used as a framework on which to construct a formalism for relativistic thermodynamics. A variational principle is first given for adiabatic flow of a single ideal fluid. For this case the problems of relativistic heat exchange are absent. In the variational approach, the scalar fluid rest-temperature is completely displaced by the thermasy, which enters the formalism as the Lagrange multiplier corresponding to the entropy-conservation constraint, but which turns out to have the property that its substantial time derivative equals the temperature.

The case of reversible heat exchange between two ideal fluids merely requires adding to the sum of the two separate single-fluid Lagrangian densities a constraint term that guarantees conservation of the total entropy of the two fluids, but not of each separately. The Lagrange multiplier for this constraint turns out to be the time-rate of entropy transfer per unit volume between the two fluids. This same constraint term requires the equality of the thermasies for the two interacting fluids. Thus the temperatures of both fluids are determined by a single thermasy. This is the mathematical statement of the intuitive idea that two fluids in reversible thermal contact should have the same temperature. In actual fact, however, the two scalar temperatures are not exactly equal if the two fluids have different velocities. The reason for this apparent discrepancy is explained by describing the heat exchange in terms of a relativistic Carnot cycle operating between two heat reservoirs having different velocities.

The fact that the two interacting fluids can be described by a single thermasy is given an intuitive explanation in terms of a heat reservoir which interacts with each of the fluids and mediates the heat exchange between them. The fact that the velocity and rest-temperature are described in terms of the gradient of a single scalar function, the thermasy, implies certain restrictions on their spatial variability. It is shown that these are exactly the restrictions that must be imposed if the concept of the relativistic Carnot cycle is to be extended so as to admit the possibility of spatial variation in the temperatures and velocities of the heat reservoirs.

All this indicates that relativistic thermodynamics finds its most natural description in terms of a 4-vector temperature, although it could also be described in terms of a scalar temperature (plus a velocity 4-vector), or a reciprocal temperature 4-vector (which would be especially well-adapted to the needs of relativistic statistical mechanics). The orthodox Planck formalism, although in principle a permissible alternative (if one ignores the universally accepted requirement of covariance), is in practice completely unworkable in any but the simplest of problems because of its non-covariance. In an appendix the early history of the subject is surveyed in order to make the point that the Planck formalism was developed before four-dimensional tensor analysis and the modern concept of covariance had evolved.

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EFFECTS OF HEAT EXCHANGE
ON RELATIVISTIC FLUID FLOW

I. INTRODUCTION

I-1. Summary of Various Approaches to Relativistic Thermodynamics

If the adiabatic constraint is removed from the description of relativistic fluid flow, an intimate involvement with the question of relativistic heat transfer between matter in different states of motion becomes unavoidable. Correspondingly, the formalism of relativistic fluid dynamics provides a convenient and straight-forward means of developing a formalism for relativistic thermodynamics. A beginning in this direction will be made in this paper.

Because such a fluid-dynamical approach to thermodynamics involves working with energy-momentum densities, rather than with the integrated quantities that characterize the more common box-of-gas approach, it is possible to avoid the delicate question as to exactly how such integrated quantities should be defined. It is well known[1] that if the total energy-momentum is defined to exclude the contribution resulting from the stress in the walls of the box, then the resulting total energy and momentum do not constitute a 4-vector, which results in a non-covariant formalism. Such non-covariance can be avoided in one of two ways: Either the total energy-momentum may be defined so as to include the effects of wall stress,[2] or the total energy-momentum vector may be defined[3] to be the rest-frame energy of the gas alone (a scalar) multiplied by the 4-vector velocity of the box in the observer's reference frame.

The first of these alternatives has the disadvantage of involving a nonhomogeneous system, part of which (the box) is incompletely specified. It has the advantage, however, that because all quantities are referred to the observer's frame, the description of the mutual interactions between the various parts of a complex system, such as two boxes having different velocities, presents no problem because all quantities have already been referred to a common reference frame.[4] The second alternative effectively takes the point of view that thermodynamic relations only have meaning in the rest-frame of the system,[5] so that the basic thermodynamic quantities are all scalars. Corresponding 4-vectors can be defined by multiplying these scalars by the 4-velocity of the system as seen by the observer. From this point of view, the 4-vector temperature, for example, would simply represent a convenient way to include two separate pieces of information—the scalar temperature in the rest-frame of the system and the 4-velocity of this system—in a single 4-vector package. As far as it goes, this point of view is unassailable. It avoids, however, the difficulties that must ultimately be faced in the problem of describing the interaction between two systems having sharply defined and very different velocities. It would still be possible to insist that everything be referred to the rest-frame of the total system. Aside from the computational difficulty that would often arise in implementing this approach, there is the conceptual difficulty that arises in the case of two very weakly interacting systems that have very different velocities, such as two boxes of gas having large relative velocity that are in weak thermal interaction through photon exchange. From the point of view of an observer in the interior of one of the boxes, who could hardly be aware that any interaction with the other box was taking place, referring the temperature of his box to the common rest-frame of the two boxes would appear to be a very strange and arbitrary procedure.

This brief review suffices to indicate that the history of the box-of-gas approach has been characterized by arbitrarily imposed definitions and postulates.[6] Moreover, the problem being considered (uniform gas in a rigid cylinder (one end of which is closed by a wall that may be chosen to be either fixed or movable) bears no resemblance to the physical situations in which there can be a reasonable expectation that relativistic thermodynamics will play a significant role. Such situations will involve either relativistic fluid dynamics, or relativistic statistical mechanics (as applied to either material or photon gases, i.e., blackbody radiation).

Thus it would appear that the best approach to the problem of developing a formalism for relativistic thermodynamics would be to admit the possibility of thermal energy exchange into the formalism of either relativistic fluid dynamics or relativistic statistical mechanics. The latter approach has received some attention in recent years.[7] The fluid-dynamical approach to relativistic thermodynamics has been sorely neglected, with the notable exception of van Dantzig's excellent work[8] which, however, until very recently, was completely overlooked.

The two fields, fluid dynamics and statistical mechanics, have different conceptual bases, and so correspondingly the thermodynamical formalisms that would grow out of each could be expected to have somewhat different forms. If, however, each of the two formalisms were covariant, it would be an easy matter to relate one to the other. In this way one would arrive at a deeper physical insight than could be afforded by either approach separately.

The present paper considers only the fluid-dynamical approach and can be regarded as only a beginning to the problem of developing a complete relativistic formalism for thermodynamics. It does, however, establish a more complete conceptual and formal basis for the recently introduced idea of a relativ[illegible] heat reservoir.[9]

A prime objective of this paper is to demonstrate the need for a th[illegible]y covariant thermodynamical formalism. The reasons why the earliest work in relativistic thermodynamics produced a non-covariant formalism become very obvious from a historical perspective. Moreover, the reasons why present work should no longer be fettered by this early work become equally obvious. For these reasons, a historical sketch of the development of relativistic thermodynamics, with special emphasis on the early work, is given in an appendix.

I-2. Outline of Approach Followed in This Paper

This analysis is carried out completely within the framework of Special Relativity, rather than General Relativity. Gravitational and electromagnetic effects are excluded, not because these are unimportant for those phenomena in which relativistic effects are important—quite the contrary—but rather because they can be taken into account very easily [10] without in any way altering the considerations that bear on thermodynamics.

The formalism is first developed for the case of adiabatic flow of an ideal (i.e., inviscid and thermally non-conducting) compressible fluid. The question of heat transfer does not enter into this problem, so it is possible to arrive at a variational formulation purely by mathematical manipulation, without the need for any physical postulates. The adiabatic condition on the flow enters into the variational formalism as a constraint that conserves the entropy flux of the fluid. It is then an easy matter to modify this constraint in such a way as to describe the case of two coexisting ideal fluids in reversible thermal contact with one another. The form of the Euler equation for each of the two fluids that results from the variational formalism makes it evident that a 4-vector description of temperature is best suited to the needs of relativistic fluid dynamics.

The physical aspects of the heat transfer between the two fluids, especially the reversibility condition, are explained in terms of a relativistic Carnot cycle between each of the two fluids and a heat reservoir whose 4-vector temperature is specified as the 4-gradient of a scalar function, the *temperature integral* or *thermasy*, to use van Dantzig's expression, which (in a non-relativistic context) was first introduced by Helmholtz.[11]

I-3. Notation

Everything will be referred to a Cartesian coordinate system in flat space-time using the diagonal metric characterized by the signature

$$(g_{00}, g_{11}, g_{22}, g_{33}) = (1, -1, -1\ -1). \tag{1.1}$$

The 4-gradient operator ∂_j is

$$\partial_j = \left(\partial_0, \partial_1, \partial_2, \partial_3\right) = \left(\partial^0, -\partial^1, -\partial^2, -\partial^3\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \tag{1.2}$$

where ∇ is the 3-gradient operator. The 4-velocity v^j is

$$v^j = \left(v^0, v^1, v^2, v^3\right) = \Gamma(c, \mathbf{v}) \tag{1.3a}$$

where

$$\Gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \text{ and } \beta \equiv v/c \tag{1.3b}$$

As indicated in (1.2), laboratory time is represented by t. The proper time for any fluid particle is τ.

$$\frac{d}{d\tau} = v^j \partial_j = \Gamma\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) \tag{1.4}$$

The invariant particle density of the fluid (i.e., the density in the fluid rest-frame) is n. The particle rest-mass is m. The fluid pressure and local rest-frame temperature (both invariants) are P and T respectively. The specific enthalpy (i.e., enthalpy per unit mass) is h, and the specific energy and entropy are u and s respectively. Thus mh may be considered to be the enthalpy per particle, and nmh is the enthalpy per unit volume. Further notation will be introduced as needed.

II. ADIABATIC FLOW OF A SINGLE IDEAL GAS

II-1. Euler Equation

The well-known relativistic equation of motion (Euler equation) for adiabatic flow of an ideal gas (no viscosity or thermal conduction) is

$$\frac{d(\mu v^j)}{d\tau} = \frac{1}{n} \partial^j P \tag{2.1}$$

where μ is an effective particle mass that includes the mass contribution (per particle) mh/c^2 that results from the thermal energy of the fluid:

$$\mu = m(1 + h/c^2) \ , \tag{2.2}$$

The formal justification for the use of h as the thermal potential of the fluid (rather than the internal energy u as one might expect) is that the form of equation of motion given in (2.1) automatically guarantees fulfillment of the adiabatic constraint placed on the fluid. This can be seen as follows: Contracting (2.1) with v_j and using the fact that

$$v_j \frac{dv^j}{d\tau} = \frac{1}{2} \frac{d(v_j \ v^j)}{d\tau} = \frac{1}{2} \frac{dc^2}{d\tau} = 0 \tag{2.3}$$

we find

$$m \frac{dh}{d\tau} = \frac{1}{n} \frac{dP}{d\tau} \ . \tag{2.4}$$

However, if we contract the thermodynamic identity

$$\frac{1}{n}\partial_j P = m\partial_j h - mT\partial_j s \tag{2.5}$$

with v^j we find

$$\frac{1}{n}\frac{dP}{d\tau} = m\frac{dh}{d\tau} - mT\frac{ds}{d\tau} . \tag{2.6}$$

Taken together, (2.4) and (2.6) imply that

$$\frac{ds}{d\tau} = 0 \tag{2.7}$$

which is the adiabatic constraint.

An intuitive justification for the use of h, rather than u, as a measure of the thermal energy per unit mass of the fluid results from imagining unit mass of the fluid to be encased in a rigid, thermally insulated container. By means of a suitable Carnot engine and an ideal heat reservoir at absolute zero we could extract the energy u in the container, at the end of which process the gas inside the container would have become a collection of motionless particles corresponding to zero gas pressure. We could then arrange to allow the container to collapse in such a way that the surrounding fluid would perform the work PV_1 where $V_1 = 1/nm$ is the specific volume of the fluid, and hence the initial volume of the container. (We have, of course, assumed the volume of the container to be so small compared with that of the surrounding fluid that no significant decrease in fluid pressure would be produced by collapsing the container). Thus the total energy to be associated with the region of space occupied by unit mass of the fluid is

$$h = u + PV_1 = u + P/nm \ . \tag{2.8}$$

It is evident that the energy PV_1 does not reside within the blob of fluid under consideration, but rather within the surrounding fluid, which could be regarded as a very thick-walled "container" for the blob under consideration (replacing the imaginary container introduced at the beginning of the argument). Thus PV_1 is the reactive stress energy, analogous to that of a coiled spring, that was induced by the pressure of the gas within the "container." This is an example of the statement made in the introduction, that the total energy-momentum of a box of gas will be a 4-vector only if the stress contribution of the container is included. In this case the 4-vector $(h/c^2)v^j$ represents the total thermal energy-momentum 4-vector, including the container stress contribution, of unit mass of fluid, and μv^j is the corresponding vector (per particle) if the particle rest energy mc^2 is taken into account.

The thermodynamic identity (2.5) can be used to eliminate the particle density n from (2.1) yielding

$$\begin{aligned} \frac{d(\mu v^j)}{d\tau} &= m\partial^j h - mT\partial^j s \\ &= \partial^j(\mu c^2) - mT\partial^j s \ . \end{aligned} \tag{2.9}$$

Thus we see that the effective particle mass-energy serves a dual role: On the left side of (2.9) it describes the inertial properties of the particle, whereas on the right side it serves as a potential energy.

The second term on the right side of (2.9), which will be called the "entropy force," can be given a simple intuitive explanation by combining a virtual displacement argument with the Available Energy Theorem of thermodynamics which states that, if the coldest reservoir available to a system has temperature T, and if the system suffers an irreversible entropy increase Δs, then an amount of energy $T\Delta s$ becomes forever unavailable for conversion into mechanical form. Since the Euler equation is sensitive only to changes in available energy, if a virtual displacement carries a unit mass of gas into a new environment of higher entropy so that, in taking on the thermodynamic properties of its new environment (in the same way that the sample would assume the new value of any potential energy acting on the fluid), the sample suffers a virtual entropy increase Δs, then because the only heat reservoir available to the sample is the fluid itself, the virtual change in available energy (on a per-particle basis) is

$$\Delta E_{\text{available}} = - mT\Delta s \tag{2.10}$$

and the force acting in the direction of the displacement Δx is

$$\left[\left(-\Delta E_{\text{available}}\right)/\Delta x\right] = mT\left(\Delta s/\Delta x\right) .$$

Thus the total force is $mT\nabla s$ which is the space part of the entropy force in (2.9). (Recall from (1.2) that the space part of ∂^j is $-\nabla$, not $+\nabla$.) Thus the intuitive interpretation of (2.9) is that the total thermal energy of the fluid (per particle), which is equal to mh, serves as a potential function for the fluid except that in calculating force we must keep in mind that only available energy can produce momentum changes, and so that part of the total force $-m\nabla h$ that corresponds to gradients in unavailable energy must be subtracted from the total force.

II-2. Formal Integration of the Euler Equation

It is well known[12] that a formal integration of Euler's equation for adiabatic flow becomes possible if the fluid temperature T (a scalar because it refers to the local rest-frame of the fluid) is eliminated in favor of a scalar function[13] Θ defined by the relation

$$\frac{d\Theta}{d\tau} = v^i \partial_j \Theta = T \tag{2.11}$$

or

$$\Theta \equiv \int_{\tau_0}^{\tau} T d\tau \tag{2.12}$$

where the integration is carried out along a particle trajectory. Because of its definition as given in (2.12), Θ is often called the temperature integral. Van Dantzig, however, called it the thermasy,[14] and in recognition of his pioneering (albeit totally neglected) work, this usage will be followed here.

Applying (2.11) to eliminate T from (2.9), and making use of (2.7) and (2.3), we arrive at the following form of Euler's equation:

$$v_j\left\{\left[\partial^j\left(\mu v^k\right)-\partial^k\left(\mu v^j\right)\right]-m\left[\left(\partial^j s\right)\left(\partial^j \Theta\right)-\left(\partial^k s\right)\left(\partial^j \Theta\right)\right]\right\}=0\,, \tag{2.13}$$

which in turn implies that

$$\left[\partial^j\left(\mu v^k\right)-\partial^k\left(\mu v^j\right)\right]-m\left[\left(\partial^j s\right)\left(\partial^k \Theta\right)-\left(\partial^k s\right)\left(\partial^j \Theta\right)\right]+2\mu\omega^{jk}=0 \tag{2.14}$$

where ω^{jk} is an antisymmetric tensor that is completely unspecified[15] except for the requirement

$$v_j\,\omega^{jk}=-\omega^{kj}\,v_j=0\,. \tag{2.15}$$

For the case of constant h, s, and Θ, ω^{jk} becomes

$$\omega^{jk}=\Omega^{jk}\equiv-\frac{1}{2}\left(\partial^j v^k-\partial^k v^j\right) \tag{2.16}$$

where Ω^{jk} is the relativistic vorticity. Thus from (2.14) it is evident that ω^{jk} may be regarded as the contribution to the total fluid vorticity Ω^{jk} that is not produced by thermal effects. For this reason it is called the intrinsic vorticity, and may be regarded as that part of the total vorticity that is a retained residue of the initial boundary conditions of the fluid.[16]

Because

$$\left(\partial^j s\right)\left(\partial^k\Theta\right)-\left(\partial^k s\right)\left(\partial^j\Theta\right)=\partial^j\left(s\partial^k\Theta\right)-\partial^k\left(s\partial^j\Theta\right)\,, \tag{2.17}$$

it is possible to write (2.14) in the form

$$2\mu\omega^{jk} = -\left[\partial^j\left(\mu v^k - ms\partial^k\Theta\right) - \partial^k\left(\mu v^j - ms\partial^j\Theta\right)\right] \tag{2.18}$$

which states that the tensor $2\mu\omega^{jk}$ must be expressible as the curl of some 4-vector which, in fact, differs from $\left(\mu v^k - ms\partial^k\Theta\right)$ at most by the 4-gradient of a scalar function S. Thus there exists a 4-vector b^j such that

$$2\mu\omega^{jk} = \partial^j b^k - \partial^k b^j \tag{2.19}$$

where

$$-b^j = \mu v^j - ms\partial^j\Theta + \partial^j S \tag{2.20}$$

The fact that $2\mu\omega^{jk}$, and hence b^j, is actually a function of the initial boundary conditions of the fluid finds its most natural expression in the fact that it can be written in terms of two constants of the fluid motion, M and Φ:

$$b^j = M\partial^j\Phi \tag{2.21}$$

where

$$\frac{dM}{d\tau} = \frac{d\Phi}{d\tau} = 0 . \tag{2.22}$$

Using (2.21) in (2.19) we have

$$\begin{aligned} 2\mu\omega^{jk} &= \partial^j\left(M\partial^k\Phi\right) - \partial^k\left(M\partial^j\Phi\right) . \\ &= \left(\partial^j M\right)\left(\partial^k\Phi\right) - \left(\partial^k M\right)\left(\partial^j\Phi\right) . \end{aligned} \tag{2.23}$$

Because of (2.22), it is obvious that the requirement (2.15) is satisfied.

It might at first seem surprising that the 4-vector b^j can be expressed, without loss of generality, in terms of only <u>two</u> scalar functions. Firstly, since from (2.19), only the curl of b^j is observable, there is a gauge indeterminacy (just as with the electromagnetic 4-vector potential) which means that b^j has only three significant degrees of freedom. Secondly, we shall now show that condition (2.15) removes one more degree of freedom, which brings us down to two degrees of freedom, which means that writing b^j in the form (2.21) does not sacrifice any generality. The argument goes as follows: Condition (2.15) implies that there exists a reference frame at each point in space-time (the local fluid rest-frame in this case) in which the space-time components ω^{0j} of ω^{jk} vanish. This fact means that

$$\epsilon_{jkln}\,\omega^{jk}\omega^{ln} = 0 \qquad (2.24)$$

where ϵ_{jkln} is the Levi-Civita object. This quantity is one of the two characteristic invariants of the tensor (the other being $\omega^{jk}\omega_{jk}$), and if it vanishes in one frame (which is obvious for the frame in which $\omega^{0j} = 0$) it vanishes in all frames. This is a restrictive condition which sacrifices one degree of freedom. (Incidentally, in the case the electromagnetic tensor, the invariant corresponding to (2.24) is $\mathbf{E} \cdot \mathbf{B}$. Since this invariant must vanish for any <u>pure</u> electric or <u>pure</u> magnetic field, in the sense that there exists a frame at every point—not necessarily the same frame for each point—for which either $\mathbf{B}$ or $\mathbf{E}$ vanishes, it follows that the corresponding 4-vector potential can be expressed in the form (2.21), but (2.22) will in general not hold.)

From (2.20) and (2.21) we find

$$\mu v^j = -\partial^j S + ms\partial^j \Theta - M\partial^j \Phi \tag{2.25a}$$

which is the desired formal integration of Euler's equation (2.9). For completeness (2.25a) must be augmented by (2.22) and (2.7)

$$\frac{dM}{d\tau} = \frac{d\Phi}{d\tau} = 0 \ , \tag{2.25b}$$

$$\frac{ds}{d\tau} = 0 \tag{2.25c}$$

and the continuity and normalization conditions:

$$\partial_j \left(nv^j\right) = 0 \tag{2.25d}$$

$$v_j v^j = c^2 \ . \tag{2.25e}$$

Equation (2.25a) has been derived as a preliminary to exhibiting a variational formulation of compressible adiabatic flow. First two important points should be made concerning the constants of motion s, M, and Φ:

(1) Assuming that the three functions are independent, i.e., that the 3 × 3 matrix constructed from the elements of $\overset{\circ}{\nabla}M$, $\overset{\circ}{\nabla}\Phi$, $\overset{\circ}{\nabla}s$, where the 3-gradients are calculated in the local rest-frame, has a non-vanishing determinant, then these three scalars serve as unambiguous markers for each particle of the fluid, and may be regarded as the co-moving material coordinates of the fluid. Because we are dealing with identifiable classical particles (rather than indistinguishable

fermions or bosons) any trial variation carried out within the framework of a variational approach must not allow s, M, and Φ to vary on the surface of the action integral, and must guarantee the preservation of particle identity along every trajectory.[17] In other words, the variational formulation must automatically guarantee the fulfillment of (2.25b) and (2.25c).

(2) Inasmuch as s, M, and Φ may be regarded as properties that are attached to the particles, fluid turbulence can be expected to produce a diffusion of these properties with a resulting tendency for s, M, and Φ to become constant throughout the fluid, to the extent that boundary conditions and other requirements allow this to happen.[18] If s becomes constant, the entropy term in (2.25a) becomes the gradient of a scalar and can be absorbed into the term $-\partial^j S$. This is also true of the intrinsic vorticity term $-M\partial^j\Phi$ if either M or Φ becomes constant (or even if one simply becomes a function of the other). When such a state of affairs is achieved, we have the case of potential flow:[19]

$$\mu v^j = -\partial^j S . \qquad \text{(Potential Flow)} \tag{2.26}$$

Finally, it should be remarked that (2.25a) may be read as the definition of a generalized canonical momentum $\mathfrak{p}^j$:

$$\mathfrak{p}^j \equiv -\partial^j S = \mu v^j - ms\partial^j \Theta + M\partial^j \Phi \tag{2.27}$$

An interesting variation on this definition is

$$\tilde{\mathfrak{p}}^j = -\partial^j \tilde{S} = \mu v^j + m\Theta\partial^j s + M\partial^j \Phi \tag{2.28a}$$

where

$$\tilde{S} \equiv S - ms\Theta . \tag{2.28b}$$

Because of (2.25b) and (2.25c), the 4-vector $m\Theta\partial^j s + M\partial^j \Phi$ is space-like, which means that the negative of its norm

$$\epsilon^2 \equiv - \left(m\Theta\partial^j s + M\partial^j \Phi\right)\left(m\Theta\partial_j s + M\partial_j \Phi\right) \geq 0 \tag{2.29}$$

is positive-definite. Thus

$$\tilde{\mathfrak{p}}^j \tilde{\mathfrak{p}}_j + \epsilon^2 = (\mu c)^2 . \tag{2.30}$$

Using (2.28a), we arrive at an equation having the form of a generalization of the Hamilton-Jacobi equation[20] in particle dynamics:

$$\left(\partial^j \tilde{S}\right)\left(\partial_j \tilde{S}\right) + \epsilon^2 = (\mu c)^2 . \tag{2.31}$$

III. VARIATIONAL FORMULATION

III-1. Adiabatic Flow of a Single Ideal Fluid

Our objective is to find a variational principle that will yield the system of equations (2.25a). Such a principle for the non-relativistic case has long been known.[21] Various forms for the corresponding relativistic principle have been proposed in recent years by several different authors. The form presented here is closest to that given by Tam.[22]

The desired Lagrangian density $\mathcal{L}$ can be obtained very easily simply by adding the necessary constraint contributions to the basic Lagrangian density

$\mathcal{L}_0$ that would apply to the case of a fluid of free, non-interacting particles, each of mass $\hat{\mu}$, where

$$\hat{\mu} \equiv m\left(1 + u/c^2\right) \tag{3.1}$$

includes the energy-mass (per particle) resulting from the internal energy u. Note that now it is u rather than h = u + PV_1 that plays the role of the thermal potential. The reason for this is that, as we saw in section II-1, the energy PV_1 does not reside within the blob of gas whose trajectory we are following, but rather in the surrounding fluid. We shall see that in the variational formulation this outside energy is taken into account by means of the constraint on the normalization of v^j. Thus, as a first approximation (i.e., neglect of constraints) we treat the blob as a free particle whose mass is determined only by the energy contained within the blob itself. The corresponding variational principle is just

$$\delta \int_{V_4} \mathcal{L}_0 \, dV_4 = 0 \tag{3.2}$$

where dV_4 is the volume element in 4-space and

$$\mathcal{L}_0 = - n\hat{\mu}c \left(v^j v_j\right)^{1/2} . \tag{3.3}$$

To see the similarity of (3.2) to the formalism for particle dynamics, imagine the integral in (3.2) to be carried out over a length of flux tube that contains the trajectories of N neighboring particles. We take the 4-volume element dV_4 to be $c\,d\tau\,d\mathring{V}_3$ where $d\mathring{V}_3$ is the 3-space volume element in the local fluid rest-frame.

Dropping the factor of c in this volume element, (3.2) can be written as

$$0 = \delta \int_{\tau_0}^{\tau} \mathring{L} d\tau = \delta \int_{t_0}^{t} L dt \tag{3.4}$$

where, since $d\tau = \sqrt{1 - \beta^2}\, dt$,

$$\begin{aligned} L &= \left(1 - \beta^2\right)^{1/2} \mathring{L} = \left(1 - \beta^2\right)^{1/2} \int_{\mathring{V}_3} \mathcal{L}_0 \, d\mathring{V}_3 \\ &= - N\hat{\mu} c^2 \left(1 - \beta^2\right)^{1/2} \\ &\approx \frac{1}{2} (N\hat{\mu})\, v^2 - N\hat{\mu} c^2 . \end{aligned} \tag{3.5}$$

Thus, as we would expect, in the non-relativistic limit L has the form of kinetic energy minus potential energy if we regard $N\hat{\mu}c^2$ as the potential energy of the mass moving through the flux tube.

To extend the variational principle from what is essentially particle dynamics to fluid dynamics, it is necessary to impose constraints that guarantee maintenance of the normalization of v^j, conservation of particles and entropy, and constancy of M and Φ along the trajectories:

$$0 = \delta \int P\left[1 - \left(v^j v_j\right)^{1/2}/c\right] dV_4 \; ; \tag{3.6a}$$

$$0 = \delta \int S\partial_j (nv^j)\, dV_4 = \delta \int \partial_j (Snv^j)\, dV_4 - \delta \int (nv^j \partial_j S)\, dV_4$$

$$= -\delta \int (nv^j \partial_j S)\, dV_4 \; ; \qquad (3.6b)$$

$$0 = \delta \int \left[-\Theta\partial_j (nv^j ms)\right] dV_4 = -\delta \int \partial_j (\Theta nv^j ms)\, dV_4 + \delta \int (nmsv^j \partial_j \Theta)\, dV_4$$

$$= \delta \int (nmsv^j \partial_j \Theta)\, dV_4 \; ; \qquad (3.6c)$$

$$0 = \delta \int \Phi\partial_j (nMv^j)\, dV_4 = \delta \int \partial_j (\Phi nMv^j)\, dV_4 - \delta \int (nMv^j \partial_j \Phi)\, dV_4$$

$$= -\delta \int (nMv^j \partial_j \Phi)\, dV_4 \; . \qquad (3.6d)$$

In (3.6b)-(3.6d) use has been made of the well-known fact that the variation of the integral of the divergence of a vector vanishes when the vector is held constant on the surface of integration (which the variational formalism requires). The arbitrary scalar functions P, S, and $(-\Theta)$ are Lagrange multipliers. They will subsequently be shown to be just the functions indicated by their symbols. In (3.6d) it is evident that we may regard Φ as the Lagrange multiplier corresponding to the condition $\partial_j(nMv^j) = 0$ (which is identical to $dM/d\tau = 0$ if the condition $\partial_j(nv^j) = 0$ is maintained), or M may be regarded as the multiplier corresponding to the condition $nv^j \partial_j \Phi = 0$. Thus the single constraint (3.6d) suffices to guarantee the fulfillment of both $dM/d\tau = 0$ and $d\phi/d\tau = 0$.

The normalization condition (3.6a) is not necessary in the particle case, because then the variables are the particle coordinates x^j and $(v^j v_j)^{1/2}$ in (3.3) actually represents $\left[(dx^j/d\tau)(dx_j/d\tau)\right]^{1/2}$. Using x^j as the variables in the formalism automatically takes care of the normalization $v^j v_j = c^2$. If, however, we take the Euler point of view in the fluid case, and so regard the components of v^j as the basic variables, then the fact that we have four variables with only three degrees of freedom requires the imposition of the constraint (3.6a).

In the usual way, we may add the constraint conditions (3.6) to the basic variation (3.2) to arrive at the desired variational principle for the fluid:

$$\delta \int \mathcal{L} dV_4 = 0 \tag{3.7a}$$

where

$$\mathcal{L} = - n\hat{\mu}c(v^j v_j)^{1/2} + P\left[1 - (v^j v_j)^{1/2}/c\right] - nv^j \partial_j S$$

$$+ nmsv^j \partial_j \Theta - nMv^j \partial_j \Phi \, . \tag{3.7b}$$

To simplify the analysis, we regard the surface of integration in (3.7a) as fixed, i.e., the walls that bound the fluid are rigid. We could dispense with walls entirely if we were to take the point of view that we are dealing with a gas that is contained by means of gravitational or electromagnetic forces which have not been included in $\mathcal{L}$ simply because they do not influence the description of thermodynamic effects, which is our primary concern, and could in any case be included without difficulty.

The Euler-Lagrange equation that results from variation of n, for example, is

$$\partial_j \left[\frac{\partial \mathcal{L}}{\partial(\partial_j n)} \right] - \frac{\partial \mathcal{L}}{\partial n} = 0 \, , \tag{3.8}$$

which yields

$$\delta n: \quad \hat{\mu} c \left(v^j v_j \right) + nc^2 \left(\frac{\partial \hat{\mu}}{\partial n} \right)_s - v^j \partial_j S + msv^j \partial_j \Theta - Mv^j \partial_j \Phi = 0 \, . \tag{3.9}$$

We note that the internal energy u which enters into $\hat{\mu}$ via (3.1) is to be regarded as a known function

$$u = u(n, s) \tag{3.10}$$

of the thermodynamic variables n and s. It is in this way that the equation of state of the fluid enters into the formalism.

The Euler-Lagrange equations for all the scalar variables have the same form as (3.8). Varying P, S, Θ, M, Φ, and s in turn, we find

$$\delta P: \quad v^j v_j = c^2 \tag{3.11a}$$

$$\delta S: \quad \partial_j \left(nv^j \right) = 0 \tag{3.11b}$$

$$\delta\Theta: \quad \frac{ds}{d\tau} = 0 \tag{3.11c}$$

$$\delta M: \quad \frac{d\Phi}{d\tau} = 0 \tag{3.11d}$$

$$\delta\Phi: \quad \frac{dM}{d\tau} = 0 \tag{3.11e}$$

$$\delta s: \quad \frac{d\Theta}{d\tau} = \left(\frac{\partial\hat{\mu}}{\partial s}\right)_n c^2 = \left(\frac{\partial u}{\partial s}\right)_n \equiv T \tag{3.12}$$

where in (3.12) we have used (3.1) and the thermodynamic definition of temperature $T \equiv (\partial u/\partial s)_n$.

The equations (3.11) are just the constraints that we built into the formalism, but (3.12) is a surprise. It says that Θ, which entered the formalism in (3.6c) as the Lagrange multiplier associated with conservation of entropy (i.e., the adiabatic condition), is in fact just the thermasy that was defined in (2.11).

The Euler-Lagrange equation that results from variation of the components of v^j is

$$\partial_k\left[\frac{\partial\mathcal{L}}{\partial(\partial_k v^j)}\right] - \frac{\partial\mathcal{L}}{\partial v^j} = 0 \tag{3.13}$$

which is simple to apply in this case because $\mathcal{L}$ does not depend on derivatives of v^j. Thus variation of v^j yields

$$\delta v^j: \quad -\partial_j S = \left(\hat{\mu} + P/nc^2\right)v_j - ms\partial_j\Theta + M\partial_j\Phi \tag{3.14}$$

where the equations (3.11) have been taken into account. Contracting (3.14) with v^j and using (3.11) and (3.12), we find

$$- v^j \partial_j S = - \frac{dS}{d\tau} = \hat{\mu} c^2 + P/n - msT . \qquad (3.15)$$

Using (3.11), (3.12) and (3.15) in (3.9), we have

$$P = mn^2 \left(\frac{\partial u}{\partial n}\right)_s = - \left(\frac{\partial u}{\partial V_1}\right)_s \equiv \text{Pressure} \qquad (3.16)$$

since

$$V_1 = \frac{1}{nm} = \text{specific volume} \qquad (3.17)$$

and $-\left(\partial u/\partial V_1\right)_s$ is the thermodynamic definition of pressure. Thus the Lagrange multiplier introduced in (3.6a) in connection with the normalization constraint is just the pressure. Using this fact, (3.14) may now be written

$$- \partial_j S = \mu v_j - ms\partial_j \Theta + M\partial_j \Phi \qquad (3.18)$$

since

$$\hat{\mu} + P/nc^2 = m + m(u + P/nm)/c^2 = m\left(1 + h/c^2\right) = \mu . \qquad (3.19)$$

Thus the variational principle (3.7) yields the system of equations (2.25), from which the Euler equation (2.1) follows.

III-2. Reversible Heat Transfer Between Two Ideal Fluids

If the individual Lagrangian densities for the two fluids are designated by $\mathcal{L}$ and $\mathcal{L}'$, each having the form given in (3.7b), it is easy to demonstrate that the appropriate Lagrangian density for the total system including reversible heat transfer is

$$\mathcal{L}_{\text{total}} = \mathcal{L} + \mathcal{L}' + \sigma(\Theta - \Theta') . \tag{3.20}$$

Only the Euler-Lagrange equations that result from variation of Θ, Θ', and σ will differ from the corresponding equations for isolated fluids. These three variations yield

$$\delta\Theta: \quad \partial_j\left(\text{nmsv}^j\right) = \sigma \tag{3.21a}$$

or

$$\frac{\text{ds}}{\text{d}\tau} = \sigma/\text{nm} = \sigma V_1 , \tag{3.21b}$$

where V_1 is the specific volume;

$$\delta\Theta': \quad \partial_j\left(n'\,m'\,s'\,v^{j\,\prime}\right) = -\sigma \tag{3.22a}$$

or

$$\frac{\text{ds}'}{\text{d}\tau'} = -\sigma/n'\,m' = -\sigma V_1' ; \tag{3.22b}$$

and

$$\delta\sigma: \quad \Theta = \Theta' . \tag{3.23}$$

Adding (3.21a) and (3.22a) we have

$$\partial_j\left(\text{nmsv}^j + n'\,m'\,s'\,v^{j\,\prime}\right) = 0 \tag{3.24}$$

which states that the total entropy flux of the total system is conserved. It is this fact that justifies the choice of $\mathcal{L}_{total}$ given in (3.20).

From (3.21b) and (3.22b) we see that σ (assuming it is positive) is the time rate at which entropy is added, per unit volume, to the unprimed fluid and removed from the primed fluid. From (3.20) it is evident that σ is the Lagrange multiplier corresponding to the constraint (3.23). In the same way that Θ was introduced as the Lagrange multiplier associated with conservation of entropy, but was subsequently found to play the role of a temperature potential, so also has σ been introduced in connection with the constraint (3.23) that the two thermasies be equal, and then subsequently found to describe the entropy exchange between the two fluids.

Because the relation (3.12) connecting thermasy and rest-temperature resulted from the variation of s, which is unaffected by the addition of the term $\sigma(\Theta - \Theta')$ in (3.20), this relation is unaltered:

$$\frac{d\Theta}{d\tau} = v^j \partial_j \Theta = T \equiv \left(\frac{\partial u}{\partial s}\right)_n \tag{3.25a}$$

$$\frac{d\Theta'}{d\tau'} = v^{j\prime} \partial_j \Theta' = T' \equiv \left(\frac{\partial u'}{\partial s'}\right)_{n'} . \tag{3.25b}$$

From (3.23), (3.25), and the fact that in general $v^j \neq v^{j\prime}$, it follows that in general

$$T \neq T' . \tag{3.26}$$

The fact that reversible heat transfer, as evidenced by the entropy conservation condition (3.24), can occur between two fluids having unequal proper temperatures, is one of the most striking apparent paradoxes of relativistic thermodynamics. It has, in fact, a very simple and reasonable explanation that is most easily visualized by imagining that the local heat transfer between the two fluids is carried out by means of a relativistic Carnot cycle. This will be discussed in Section V.

It is an easy matter to verify that the Euler equations for each of the two fluids that result from the variational formalism have the following forms:

$$n \frac{d(\mu v^j)}{d\tau} = \partial^j P + \sigma \partial^j \Theta \tag{3.27a}$$

$$n' \frac{d(\mu' v^{j\prime})}{d\tau'} = \partial^j P' - \sigma \partial^j \Theta \tag{3.27b}$$

where P and P' are the partial pressures of each of the two fluids. Contraction of these equations with v_j and $v_j{}'$ respectively yields (using (3.21b) and (3.22b))

$$T \frac{ds}{d\tau} = \frac{dh}{d\tau} - V_1 \frac{dP}{d\tau} \tag{3.28}$$

and exactly the same equation for the primed quantities. This is just a statement of the combined First and Second Laws of Thermodynamics in the case of reversible heat transfer.

Note that so far the only thermodynamics that has entered into the formalism has been relations that refer to the fluid rest frame. No assumption whatever has been made about the transformation properties of the various thermodynamic quantities. Even without any such assumptions we find the role of temperature most naturally taken over by a 4-vector $\partial_j \Theta$. This fact would appear to make a 4-vector description of temperature in fluid dynamics inevitable. An alternative would be to drop temperature as a fundamental quantity, and let this role be taken over by the scalar Θ. After all, from (3.23) and (3.26) we note that what had been perhaps the most fundamental property of non-relativistic thermodynamics—its equality for two systems in reversible thermal contact—has been taken over by the thermasy in the relativistic case.

IV. RELATIVISTIC DESCRIPTION OF HEAT TRANSFER

IV-1. Transformation Properties of Heat and Temperature

In recent years there has occurred a rather lively controversy over the question of the correct transformation laws for heat and temperature, the leading contenders being the non-covariant formalism set forth by Planck in 1907 and the covariant formalism, first advocated by Ott in 1963 and independently by Arzeliès in 1965, that is characterized by 4-vector representations of heat and temperature. Fundamentally the question is one of definition rather than substance, a point that was made by von Laue[23] in his 1911 textbook on relativity (but subsequently ignored), and re-emphasized by Ott. Ironically, in all probability this dispute never would have occurred if the 1939 work of van Dantzig had not been so completely ignored. Without in any way attacking the Planck formalism, he demonstrated the existence of a covariant alternative. He appears to have taken the position that Planck's non-covariant temperature, although not a convenient quantity on which to build a theory, nevertheless corresponded to the operational temperature that one would actually observe. He attempted to elucidate the rather exotic-seeming Planck transformation law for temperature, which has temperature transforming like volume, by identifying physical temperature with the reading (a volume) of a moving constant-pressure gas thermometer.[24] The fact that we now know that such volume contractions are not actually observable by optical means[25] illustrates the hazards of letting what might appear to be hard-headed operational considerations out-weigh profound theoretical necessities in determining the structure of a theory (a mistake that van Dantzig did not make!).

A brief historical sketch of the development of relativistic thermodynamics, given in an appendix, affords a very natural explanation why the first attempts, which were carried out before 4-space tensor analysis and the modern statement of the Relativity Principle in terms of covariance requirements had been developed, should have fallen into what we now recognize as a non-covariant mold.

The physical differences between the Planck and the Ott formalisms were clearly set forth by Ott, and are illustrated in Figure 1 for the simple case of heat absorption by a homogeneous body, such as a block of metal, that is rigid enough so that the work of expansion against the surrounding atmosphere is negligible. In the common rest-frame of the reservoir and the absorbing body,

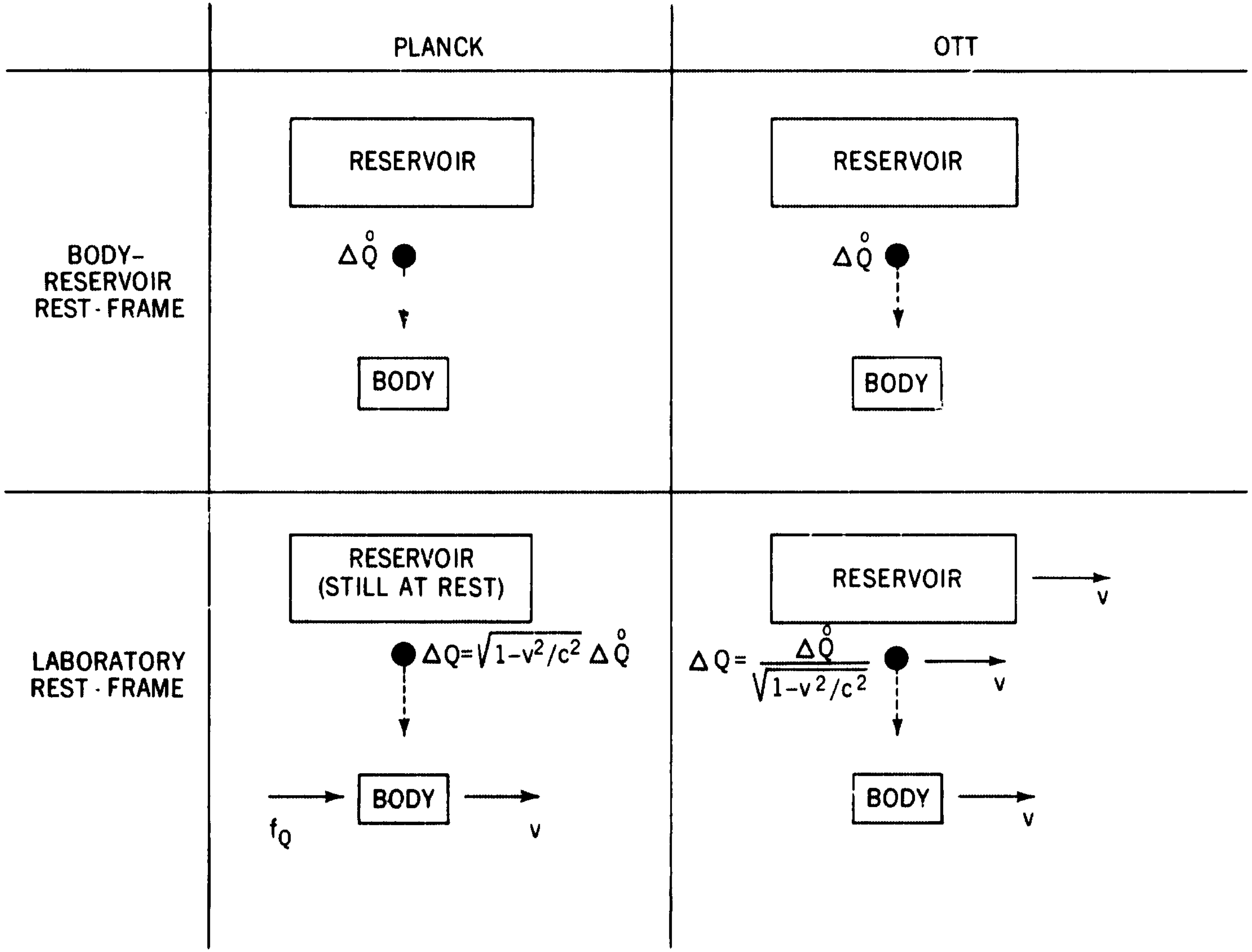

Figure 1. Schematic Comparison of Planck and Ott Formalisms

the description is the same in both formalisms: The body increases its total rest-energy $\mathring{E}$ by the amount

$$\text{(both)} \qquad \Delta\mathring{E} = \Delta\mathring{Q} \tag{4.1}$$

which corresponds to an increase in the body rest-mass $\mathring{M}$ of amount

$$\text{(both)} \qquad \Delta\mathring{M} = \Delta\mathring{E}/c^2 = \Delta\mathring{Q}/c^2 \ . \tag{4.2}$$

In both formalisms the energy of the body is regarded as the time-like component of a 4-vector with the result that, in the laboratory frame in which the body has velocity $\mathbf{v}$, the change in energy is

$$\text{(both)} \qquad \Delta E = \Gamma\Delta\mathring{E} = \Gamma\Delta\mathring{Q} \ . \tag{4.3}$$

The difference between the two approaches centers on whether or not heat is considered to carry momentum. In the Ott approach, the heat coming from a moving reservoir carries momentum with it of amount $(\Gamma\Delta\mathring{Q}/c^2)\mathbf{v}$. Thus, if the reservoir and body have the same velocity, the heat will have just the right momentum so that when it is absorbed by the body, there will be no tendency for the body to speed up or slow down, hence no need for an applied force to maintain the constancy of velocity in the face of the increase of momentum caused by the increase of mass of the body. In such a case, the laboratory observer would write the First Law in the form

$$\text{(Ott)} \qquad \Delta Q = \Delta E \tag{4.4}$$

since the work $(-\Delta W)$ done on the body is zero. From (4.3) and (4.4) it follows that

(Ott) $$\Delta Q = \Gamma\Delta\mathring{Q} \ . \tag{4.5}$$

In the Planck picture, heat is a pure energy, and it makes no sense to talk about the velocity of pure heat (although one may talk about the velocity of hot bodies). Thus, in the laboratory frame, as well as in the common rest-frame of the body and reservoir, the reservoir and the transferred heat have no momentum, and so must be regarded as effectively still at rest. We are then confronted with an apparent paradox: On the one hand the velocity of the body must remain constant when it absorbs heat, because it is constant in its own rest-frame, and both frames are inertial frames related by a constant velocity difference. On the other hand, the laboratory observer sees the absorbed heat as increasing the body mass, but not its momentum, which would seem to imply a slow-down. The paradox is resolved by invoking an ad-hoc force $\mathbf{f}_Q$, the so-called "translation force" (Führungskraft), that is defined as

(Planck) $$\mathbf{f}_Q \equiv \mathbf{v}\,\frac{dM}{dt} \ . \tag{4.6}$$

Because the relation $M = E/c^2$ is regarded as having universal validity in the Planck as well as the Ott approach, we have from (4.3) and (4.6)

(Planck) $$\mathbf{f}_Q \equiv \mathbf{v}\left(\frac{dE}{dt}\right)\Big/c^2 = \left(\mathbf{v}\Gamma/c^2\right)\frac{d\mathring{Q}}{dt} \ . \tag{4.7}$$

The force $\mathbf{f}_Q$, once having been introduced, must be regarded as capable of producing work. Thus the work $(-\Delta W)$ performed on the body in the interval Δt

is

$$\text{(Planck)} \qquad -\Delta W = \mathbf{v} \cdot \mathbf{f}_Q \Delta t = \beta^2 \Gamma \Delta \mathring{Q} \,, \tag{4.8}$$

and the First Law in the Planck formalism is

$$\begin{aligned} \text{(Planck)} \qquad \Delta Q &= \Delta E + \Delta W \\ &= \Gamma \Delta \mathring{Q} - \beta^2 \Gamma \Delta \mathring{Q} = \Gamma\left(1 - \beta^2\right) \Delta \mathring{Q} \\ &= \Delta \mathring{Q} / \Gamma \,, \end{aligned} \tag{4.9}$$

which may be written

$$\text{(Planck)} \qquad \Delta Q = \left(1 - v^2/c^2\right)^{1/2} \Delta \mathring{Q} \,. \tag{4.10}$$

This implies that the heat energy of a body decreases with increasing velocity and vanishes entirely as we approach the speed of light. This is perhaps not the total offense to intuition it might at first seem if one takes the position that heat is a measure of the randomness in molecular velocities. The Einstein Addition Law for velocities has the effect of suppressing such randomness at high speeds, and this, one might suggest, is the physical meaning of (4.10).

Ott showed[26] that in formal terms the difference between the two approaches is that in the Ott formalism heat transfer is described by a 4-force which, being a 4-vector, must have space-like components. In fluid dynamics this is just the 4-vector $\sigma \partial^j \Theta$ that appears in (3.27). This heat-transfer 4-vector is completely distinct from any other 4-force acting on the body (e.g., the pressure force $\partial^j P$ in (3.27)).

In the Planck formalism, on the other hand, heat transfer is not described by an independent 4-vector. Rather, it is described by a contribution to the time-like component—and only this component—of the total 4-force acting on the body. Moreover, this is the case in every frame of reference. (Ordinarily, a contribution to the time-like component in one frame of reference will give rise to contributions to the space-like components in other frames.) Thus Q in the Planck formalism is neither scalar nor vector. It falls completely outside the framework of tensor algebra. But tensors (in the general sense, including scalars and vectors and spinors) are the irreducible representations of the Lorentz group, and as such are intimately related to the most fundamental symmetries, and associated conservation laws, that are encountered in physics. Thus plane waves and spherical harmonics induce irreducible representations of the group of translations and rotations respectively and for this reason are characteristic of states of definite linear or angular momentum. This is the explanation behind the calculational convenience that makes them of such universal importance in physics. One could, of course, solve problems without ever using sinusoids or spherical harmonics, working instead with some set of intricately defined linear combinations of the members of one of these sets. In the same way, it would be quite possible to solve difficult problems in relativistic thermodynamics using a non-covariant formalism. Such an undertaking should, however, be considered a private penance, not to be inflicted on others.

Once the transformation properties of heat are decided, those of temperature are automatically determined by the fact that entropy[27] S is a scalar, which follows either from its probabilistic significance or from the fact that it is possible to change the velocity of a system by a reversible, adiabatic acceleration that produces no change in the entropy. (This was Planck's argument.)[28] Thus in the Ott formalism

$$\text{(Ott)} \qquad \Delta Q^j = T^j \, \Delta S \qquad (4.11)$$

and so temperature is necessarily a 4-vector.

Although in relativistic fluid dynamics it is most convenient to work with T^j, in statistical mechanics a formalism built on the reciprocal-temperature 4-vector β^j is more convenient. β^j and T^j are related as follows:

$$\kappa \beta^j = T^j / (T^n T_n) \qquad (4.12)$$

where κ is the Boltzmann constant. This, in fact, was the vector on which van Dantzig based both his statistical mechanics and his fluid dynamics. (He called it the "temperature vector" which he designated as Θ^j.) Note that if one <u>defines</u> T in the laboratory frame as

$$\text{(non-covariant)} \qquad \beta^0 \equiv \frac{1}{\kappa T} \qquad (4.13)$$

and $\overset{\circ}{T}$ in the frame for which $\beta^1 = \beta^2 = \beta^3 = 0$ as

$$\overset{\circ}{\beta}{}^0 \equiv \frac{1}{\kappa\overset{\circ}{T}} \tag{4.14}$$

then from the fact that

$$\beta^0 = \Gamma\overset{\circ}{\beta}{}^0 \tag{4.15}$$

it follows that

(Planck) $$T = \overset{\circ}{T}/\Gamma \tag{4.16}$$

which is the Planck transformation law for temperature. This is what Mosengeil (1907) implicitly did when, by a direct application of electromagnetic theory, he worked out the form of the Planck Radiation Law for a moving radiation cavity.

The definition (4.13) is non-covariant because it defines T in terms of <u>only</u> β^0 (and none of the space-like components) in <u>every</u> frame of reference. Any definition that treats the time-like component of a 4-vector on a different basis from the space-like ones is non-covariant. Van Dantzig defined temperature by means of (4.13), but had the good sense not to make any further use of it, letting the 4-vector β^j completely displace T. His use of β^j in place of T^j did, however, make the fluid-dynamical part of his work more intricate than would have been the case if he had used T^j.

Even if one wishes to treat the 4-vector β^j as the fundamental quantity, the non-covariant definition of temperature given in (4.13) is by no means necessary. One could either invert (4.12) to arrive at a 4-vector definition of temperature

$$T^j \equiv \beta^j / \kappa(\beta^n \beta_n) , \qquad (4.13)$$

or one could define a scalar temperature as

$$T \equiv 1/\kappa(\beta^j \beta_j)^{1/2} . \qquad (4.14)$$

This latter definition is the one used in Israel's (1963) paper,[29] which incidentally provides an excellent demonstration of the need for a completely covariant formalism if one is to carry through meaningful calculations in relativistic statistical mechanics.

IV-2. Relativistic Carnot Cycle

The relativistic Carnot cycle can be described in either the Planck or the Ott formalisms. The analysis in terms of the Planck formalism was given already by von Laue in his 1911 relativity textbook.[30] As might be expected, the need to take the translation force into account not only complicates the analysis, but also obscures the basic physical simplicity of the process. The Ott formalism, on the other hand, presents the relativistic cycle as the direct and obvious extension of the non-relativistic one. The details of this analysis have been given elsewhere,[31] and only the results will be summarized here.

The relativistic Carnot cycle is illustrated in Figure 2. It consists of the same basic four parts as the non-relativistic cycle: isothermal, reversible heat exchanges with hot and cold reservoirs separated by adiabatic transitions

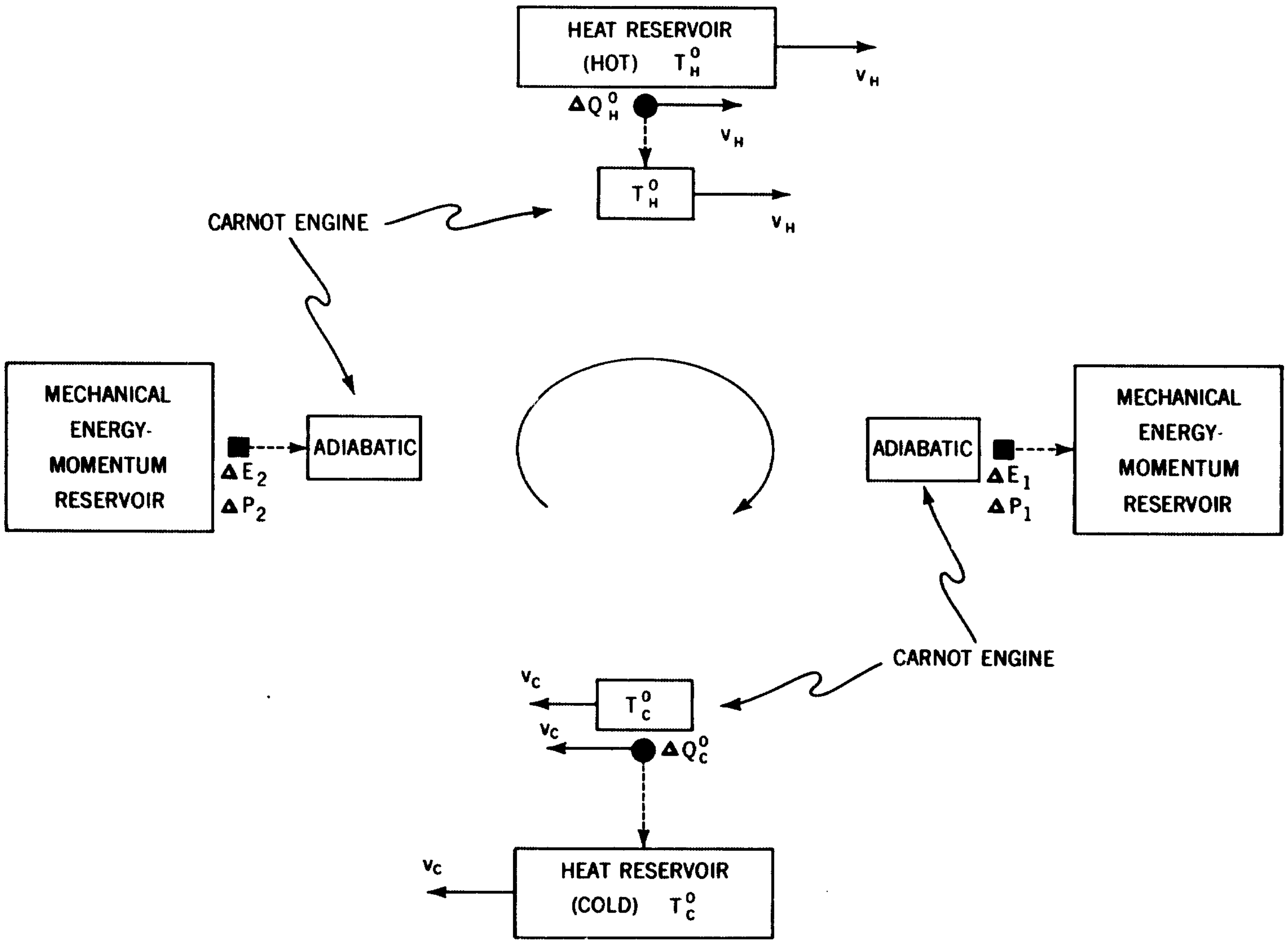

Figure 2. Schematic Representation of Relativistic Carnot Cycle

during which the engine delivers or receives mechanical work energy. The difference is that now momentum must be treated on an equal footing with energy. Thus, during the adiabatic transitions, the engine exchanges not only work energy, but also momentum. These transitions may be regarded as consisting of two stages: First the engine (which may be thought of as a gas-containing cylinder with a movable piston) either delivers or receives momentum. This involves an adiabatic acceleration which in general not only changes the momentum of the gas, but also its temperature. At the end of this process the engine has the velocity of the reservoir with which it will next interact, but in general its temperature will not match that of the reservoir. This temperature match is then accomplished by means of an appropriate adiabatic compression or expansion.

During the interactions with either reservoir, the Carnot engine must not have any relative velocity with respect to the reservoir. This is a matter of definition of what constitutes a reversible heat exchange. There is, however, physical necessity behind this definition. If, for example, we imagine a lake to be the reservoir, then if the Carnot engine, which must have finite dimensions, has a relative velocity with respect to the lake water, turbulence, and hence entropy, will be generated. It is to avoid such entropy generation that the two velocities, as well as the two temperatures must be equal. In the Ott formalism these two requirements are combined in the single statement that the two temperature 4-vectors must be equal. By definition, a temperature 4-vector of a body is

$$T^j \equiv Tv^j/c \tag{4.14}$$

where T is the scalar rest-temperature of the body and v^j is its 4-velocity. If v and $T^0 = \Gamma T$ are the 3-velocity and temperature respectively as seen in the laboratory frame, then (4.14) may be written

$$T^j = T^0 (1, v/c) . \tag{4.15}$$

Note that besides the physical need to have the 4-vector temperatures of engine and reservoir equal during heat exchange, the requirement of covariance makes the same demand. If the two laboratory temperatures were equal, but not the velocities, then in a different frame even the two temperatures would not be equal.

The analysis shows,[31] that if W^j is the energy-momentum delivered to the energy-momentum reservoir, then (referring to Figure 2 for the notation)

$$\Delta W^j \equiv (\Delta E, c\Delta P) = \left(T_H^j - T_C^j\right) \Delta S \tag{4.16}$$

where ΔS is the entropy transfer that characterizes the cycle, ΔE is the delivered work energy, and ΔP the delivered 3-momentum. From (4.15) and (4.16), it follows that

$$\Delta E = \left(T_H^0 - T_C^0\right) \Delta S , \tag{4.17a}$$

$$\Delta\mathbf{P} = \left(T_H^0 \Delta S/c^2\right) \mathbf{v}_H - \left(T_C^0 \Delta S/c^2\right) \mathbf{v}_C \; . \tag{4.17b}$$

IV-3. Reversibility Condition

So long as no restrictions are placed on the energy ΔE that is delivered to the energy-momentum reservoir, the Carnot cycle is completely reversible. Such a restriction does result, however, if we take the point of view that no such separate energy-momentum reservoir exists, but that a real fluid combines the functions of energy-momentum reservoir and either hot or cold heat reservoir. (For definiteness we shall identify the fluid with the cold reservoir, but this is not necessary.) Henceforth the C subscript will be dropped, the corresponding quantities being identified with the fluid, and the H subscript will be replaced by R (for "reservoir") to indicate that these quantities may correspond to either a hot or a cold reservoir, the important thing being that the reservoir designated by subscript R only supplies (or absorbs) heat, but is not called upon to absorb the ordered energy ΔE and momentum $\Delta\mathbf{P}$.

The delivery of the momentum $d\mathbf{P}$ to the fluid, as observed in the laboratory frame, entails simultaneous delivery of an amount of work energy given by

$$\mathbf{v} \cdot d\mathbf{P} = \left(\frac{d\mathbf{r}}{dt}\right) \cdot d\mathbf{P} = d\mathbf{r} \cdot \left(\frac{d\mathbf{P}}{dt}\right) \tag{4.13}$$

where $d\mathbf{P}/dt$ is the force exerted on the fluid and $d\mathbf{r}$ is the displacement of the fluid during the time interval dt during which the momentum transfer takes place.

If this momentum transfer is viewed in the rest-frame of the fluid, however, no corresponding energy transfer takes place. The reason for this is that before the transfer $v = 0$ in this frame, and even after the impulse ΔP is delivered, the kinetic energy acquired by the fluid is of order $(\Delta P)^2$ and so vanishes in first order. Thus in the fluid rest-frame the total delivered mechanical energy is

$$\Delta \mathring{E} = \left(\Delta W^j\right) v_j / c \,. \tag{4.19}$$

The fluid, lacking a mechanism for absorbing this energy in ordered form without creating net momentum, has no alternative but to degrade it into heat, which constitutes an irreversible process. Thus the reversibility condition is simply

$$\text{(Reversibility Condition):} \quad 0 = \Delta \mathring{E} = \Delta W^j v_j / c$$

$$= \Delta S \left(T_R^{\ j} - T^j\right) v_j / c \,, \tag{4.20}$$

where use has been made of (4.16). From (4.14) we see that

$$T^j v_j / c = T \,. \tag{4.21}$$

Thus the reversibility condition may be written in either of two ways:

$$\text{(Reversibility Condition)} \qquad \left(T_R^{\ j} - T^j\right) v_j = 0 \tag{4.22a}$$

or

$$\text{(Reversibility Condition)} \qquad T = T_R^{\ j} v_j / c \,. \tag{4.22b}$$

The first of these statements in words is: Reversibility of the local heat transfer occurring between two fluids requires the vanishing of the projection of the difference of the two temperature 4-vectors of the fluids upon the velocity 4-vector of the fluid that absorbs the ordered energy-momentum that would be delivered by the equivalent Carnot cycle for the heat transfer.

The total energy-momentum per cycle absorbed by the fluid is equal to that given up by the reservoir:

$$\begin{aligned}\Delta Q_R^{\,j} &= T_R^{\,j}\,\Delta S = T^j\,\Delta S + \Delta W^j \\ &= \left[\left(T^0\,\Delta S + \Delta E\right),\ c\left(T^0\,\Delta S/c^2\right)\mathbf{v} + c\Delta\mathbf{P}\right] \qquad (4.23)\end{aligned}$$

where use has been made of (4.15) and (4.16). If (4.23) is divided by $c\Delta\tau$, where $\Delta\tau$ is the duration of the cycle as measured in the rest-frame of the fluid, we arrive at the 4-force

$$F^j = \frac{T_R^{\,j}}{c}\,\frac{dS}{d\tau} = \left[T^j\,\frac{dS}{d\tau} + \frac{dW^j}{d\tau}\right]\Big/c \qquad (4.24)$$

of the heat transfer on the fluid. We can put this into a form more applicable to fluid dynamics if we imagine that a separate Carnot cycle is carried out in each unit volume (as measured in the fluid rest-frame). $dS/d\tau$ would then be the entropy increase per unit rest-volume of the fluid:

$$\frac{dS}{d\tau} \equiv nm\,\frac{ds}{d\tau} = \sigma \quad . \qquad (4.25)$$

where we have made use of (3.21b). Let f^j be the 4-force of heat transfer per unit rest-volume[32] (i.e., a force density), and $dw^j/d\tau$ be the 4-force per unit rest-volume caused by delivery of the ordered energy-momentum to the fluid. Then (4.24) becomes

$$f^j = \frac{\sigma}{c} T_R^j = \frac{\sigma}{c} T^j + \frac{dw^j}{c d\tau} \tag{4.26}$$

which may also be read as a definition of the 4-force $dw^j/d\tau$.

$$\frac{dw^j}{d\tau} = \sigma\left(T_R^j - T^j\right) = \frac{ds}{d\tau}\left(T_R^j - T^j\right)\Big/nm \ . \tag{4.27}$$

Using (4.22a) we see that

$$v_j \frac{dw^j}{d\tau} = 0 \ . \tag{4.28}$$

Thus the force density $dw^i/d\tau$ produces no change in the fluid rest-mass per unit volume, which is just another statement of the reversibility condition.

Equation (4.22a) tells us that the time-like components of T_R^j and T^j, as viewed in the fluid rest-frame, must be equal, but not necessarily the space-like components. Equation (4.27) gives a physical explanation of this difference in space-like components in terms of the delivered energy-momentum of a Carnot cycle. Further discussion of the intuitive significance of $dw^j/d\tau$ has been given elsewhere.[33]

V. THERMASY AND THE FLUID HEAT RESERVOIR CONCEPT

V-1. Thermasy as Temperature Potential

In (4.26) the heat-exchange 4-force density f^j was found to be $\sigma T_R^j/c$. In (3.27a) it was found to be $\sigma \partial^j \Theta$. From this we make the important identification

$$T_R^j \equiv c\partial^j \Theta \tag{5.1}$$

which shows the thermasy to be the potential function for the reservoir temperature 4-vector. The physical significance of the thermasy gradient becomes clearer if we use (4.15) to write (5.1) in the form

$$\partial^j \Theta = T_R^j/c = \left[T_R^0/c,\ \left(T_R^0/c^2\right) \mathbf{v}_R\right] . \tag{5.2}$$

If we interpret T_R^0 as the average thermal energy per particle of the reservoir as seen in the laboratory frame, then $\left(T_R^0/c^2\right)$ is the corresponding mass and so $\left(T_R^0/c^2\right) \mathbf{v}_R$ would be the thermal momentum per particle of the reservoir. Thus $\partial^j \Theta$ has the standard form, $[E/c, \mathbf{P}]$, of a momentum 4-vector.

The Euler equations for two reversibly interacting ideal fluids were given in (3.27), and can now be written in the form

$$n \frac{d\left(\mu v^j\right)}{d\tau} = \partial^j P + f^j \tag{5.3a}$$

$$n' \frac{d\left(\mu' v^{j\prime}\right)}{d\tau'} = \partial^j P' - f^j \tag{5.3b}$$

where

$$f^j \doteq \sigma\partial^j\Theta = \sigma T_R^j/c = \left(\frac{nm}{c}\right)\frac{ds}{d\tau}T_R^j = \left(\frac{nm}{c}\right)\frac{ds}{d\tau}T_R^0\left(1,\ v_R/c\right)$$

$$= \left[(nm\Gamma)\frac{ds}{dt}T_R^0\right]\left(1/c,\ v_R/c^2\right) . \tag{5.4}$$

Because of (3.22b), the expression for $-f^j$ that appears in (5.3b) would have exactly the same form as (5.4) with primes appended to all fluid quantities. The interpretation to be given to (5.4) is similar to that for (5.2) except that now the mass in question is that associated with heat injected per unit time per unit volume.

It is interesting to note that the heat reservoir concept can be used to give intuitive meaning to the entropy term in the expression for canonical particle momentum that was defined in (2.27). First we note that from (5.1) and (4.14) $\partial^j\Theta$ may be written in the following form:

$$\partial^j\Theta = T_R^j\ c = T_R\ v_R^j/c^2 . \tag{5.5}$$

Using this in (2.27) we may write the canonical particle momentum $\mathfrak{p}^j$ as follows:

$$\mathfrak{p}^j \equiv -\partial^j S = mv^j + M\partial^j\Phi + \left[\left(mh/c^2\right)v^j - \left(msT_R/c^2\right)v_R^j\right] . \tag{5.6}$$

It is the expression in brackets that is of interest. Since $\left(mh/c^2\right)$ is the mass (per particle) of the <u>total</u> thermal energy h, the term $\left(mh/c^2\right)v^j$ is just the corresponding energy-momentum 4-vector. But not all of this energy-momentum is available for conversion into mechanical form. $\left(msT_R/c^2\right) v_R^j$ represents the energy-momentum that, on the basis of the Unavailable Energy Theorem, has become unavailable for conversion into mechanical form, and so should be regarded as residing in the reservoir rather than in the fluid. In other words, $\left(h/c^2\right) v^j$ is the total thermal energy-momentum that resides in the *region of space* occupied by unit mass of the fluid. But the reservoir also occupies this same region of space, and a part of this total energy-momentum, namely $\left(sT_R/c^2\right) v_R^j$, should be assigned to the reservoir rather than to the fluid. It is only the difference, that energy-momentum which may properly be associated with the fluid rather than with the reservoir, that appears in the definition (5.6) of the canonical momentum.

V-2. Equation of Motion of Heat Reservoir

The physical interpretation of equation (5.3) is that the two fluids are best regarded as interacting, not directly with each other, but rather through the mediation of a heat reservoir described by $\partial^j \Theta = T_R^j/c$, which may be regarded as a third "fluid" having a 4-velocity v_R^j different from that of the other two fluids. In fact, it is an easy matter to find the velocity and the "equation of motion" of this third "fluid." First we note that, from (4.14) and (5.2), the 4-velocity v_R^j of the reservoir is given by

$$\begin{aligned} v_R^j &= cT_R^j/T_R = cT_R^j/\left(T_R^j\, T_{Rj}\right)^{1/2} \\ &= c\partial^j \Theta\Big/\left[\left(\partial^j \Theta\right)\left(\partial_j \Theta\right)\right]^{1/2} \end{aligned} \qquad (5.7)$$

Because, from (5.2), T_R^j is the 4-gradient of a scalar, its 4-curl vanishes:

$$\partial^j T_R^k - \partial^k T_R^j = 0 . \tag{5.8}$$

Contracting this with v_{Rj}, and using the fact that

$$\begin{aligned} v_{Rj}\, \partial^k T_R^j &= v_{Rj}\, \partial^k \left(T_R\, v_R^j/c\right) \\ &= c\, \partial^k T_R + \frac{1}{2} \left(T_R/c\right) \partial^k \left(v_{Rj}\, v_R^j\right) \\ &= c\, \partial^k T_R , \end{aligned} \tag{5.9}$$

and the definition

$$v_{Rj}\, \partial^j \equiv \frac{d}{d\tau_R} , \tag{5.10}$$

we arrive at the desired equation of motion of the fluid:

$$\frac{d\left(T_R^k/c\right)}{d\tau_R} = \partial^k T_R . \tag{5.11}$$

This is the analog of the Euler equation in the form (2.9) for the case of isentropic flow. It is as close as we could expect the equation of motion of a heat reservoir to come to the equation of motion of a material fluid because, whereas the latter must involve properties, other than temperature, that are specific to that particular fluid (e.g., particle mass and enthalpy, which is a function of the equation of state), a heat reservoir—by definition—must be characterized by nothing except its temperature 4-vector. Even a photon gas (i.e., blackbody radiation), which might at first glance seem to depend on nothing but the temperature 4-vector (or its reciprocal), actually depends very critically on its statistics (bosons). Thus, despite the great temptation, it would not be strictly valid to identify the heat reservoir with a photon gas that coexists with and exchanges heat energy with the two material fluids that we have been considering.

The important thing about (5.11), aside from its appealing resemblance to the Euler equation for isentropic flow, is that it demonstrates that the behavior of the reservoir depends only on its own properties. There are no terms in this equation that involve the properties or the behavior of the fluids with which it interacts. There is, of course, the reversibility condition on the temperature, but this could be interpreted as a condition on the fluids rather than on the reservoir. Thus the reservoir motion influences the fluid motion (witness the appearance of $f^j = \omega \partial^j \Theta$ in (5.3)), but not vice versa.

V-3. Physical Restrictions on an Ideal Heat Reservoir

Classically, a heat reservoir is regarded as completely uniform in its properties throughout its entire extent. In fluid dynamics (even non-relativistic fluid dynamics), however, we are forced to face the fact that this is an unrealistic idealisation. We must now confront the following problem: Granting the need to relax the definition of a heat reservoir so as to admit the possibility of its having different temperatures and velocities at different points in space-time, what restrictions must be placed on such variation in its properties so that the heat reservoir concept will still be preserved?

We shall answer this question by taking the position that the concept of heat reservoir is defined in terms of the Carnot cycle, and has meaning only if such a cycle (granting the usual idealisations) is a possibility, at least in the sense of some limit. In particular, we shall concentrate on the fact that any real Carnot engine must have finite dimensions. We admit the limiting process of using ever smaller engines, but to be realistic about such a process, we must concede that the capacity of the engine will decrease in proportion to its size, so that in order to effect a given entropy transfer between two given reservoirs, the number N of necessary cycles will increase in inverse proportion to the size of the engine. Thus if, with given reservoirs, an engine of given size unavoidably produces irreversible entropy generation, if this entropy generation decreases only in proportion to the engine size, the total entropy generated after the N cycles will be independent of the engine size. If, however, the entropy generated decreases more rapidly than the engine size, then the total entropy generated after the necessary number of cycles N will approach zero as we approach the limit of zero engine size.

The result of all this is to put the following restriction on the heat reservoir: In the co-moving frame of reference, in which at the point in question both the local velocity and acceleration vanish, the spatial gradient of the proper temperature must vanish (but not necessarily the second derivatives). The reason for this is that if this condition is not satisfied, a co-moving Carnot engine of finite size having a constant temperature throughout its extent (which is necessary to avoid internal entropy generation) cannot match the reservoir temperature at every point of its interface with the reservoir. The temperature differences and resulting entropy generation would be proportional to the size of the engine, and so, as remarked above, going over to the limit of an engine of vanishing size would not help. If, however, the temperature gradient in the co-moving frame does vanish, then in the limit it will be possible to carry out the assigned entropy transfer without entropy generation.

The second essential feature of the classical Carnot cycle that must be preserved is the complete lack of dependence of the cycle on the physical properties of the engine. From this we conclude that the rotation of the reservoir (i.e., the 3-curl of the reservoir velocity) must vanish in the co-moving frame. The reason for this is that, if this condition were not satisfied, the cycle would depend on the radius of gyration of the engine, contrary to our insistence that it be independent of the properties of the engine. This dependence would follow from the fact that, to avoid velocity differences at the interface of engine and reservoir, it would be necessary that the engine co-rotate with the reservoir during the heat exchange. Thus, as viewed from the fixed laboratory frame, there would be a corresponding exchange of angular momentum. In making the transition between reservoirs the engine would have to exchange angular momentum with the energy-momentum reservoir, and on completion of the cycle, a net angular momentum would have been transferred that was dependent not only on the properties of the two reservoirs and the amount of entropy transferred, but also on the radius of gyration of the engine. The same type of argument used above suffices to show that this difficulty cannot be avoided by going over to the limit of vanishingly small engine. It is necessary instead to require that the rotation of the reservoir vanish in its co-moving frame.

A reservoir satisfying these two requirements will be called an <u>ideal</u> heat reservoir. It is easy to show that these requirements together guarantee that the 4-curl of the temperature 4-vector (which we write without the subscript, because everything refers to the reservoir) must vanish in the co-moving frame. Since $T^k = Tv^k/c$, we have (using a bar over a subscript to indicate the range 1, 2, 3 but not 0).

$$\begin{aligned}\partial^0 cT^{\bar{k}} - \partial^{\bar{k}} cT^0 &= \left(\partial^0 T\right) v^{\bar{k}} + T\partial^0 v^{\bar{k}} - \left(\partial^{\bar{k}} T\right) v^0 - T\partial^{\bar{k}} v^0 \\ &= -\left(\partial^{\bar{k}} T\right) c \qquad (5.12a)\end{aligned}$$

$$\begin{aligned}\partial^{\bar{j}} cT^{\bar{k}} - \partial^{\bar{k}} cT^{\bar{j}} &= \left(\partial^{\bar{j}} T\right) v^{\bar{k}} + T\partial^{\bar{j}} v^{\bar{k}} - \left(\partial^{\bar{k}} T\right) v^{\bar{j}} - T\partial^{\bar{k}} v^{\bar{j}} \\ &= T\left(\partial^{\bar{j}} v^{\bar{k}} - \partial^{\bar{k}} v^{\bar{j}}\right) \qquad (5.12b)\end{aligned}$$

where we have used the fact that by definition of the co-moving frame $v^{\bar{k}} = 0$, $\partial^0 v^{\bar{k}} = 0$, $v^0 = c$, and $\partial^k v^0 = 0$. The condition that the spatial gradient vanish makes (5.12a) vanish, and the condition that the rotation vanish makes (5.12b) vanish. Thus in the co-moving frame

$$\partial^j T^k - \partial^k T^j = 0 \, . \qquad (5.13)$$

It is well known that the 4-curl of a 4-vector in general curvilinear coordinates transforms like a second-order tensor[34] (i.e., there is no need to include terms involving the Christoffel symbol when making the transition from one reference to another). Thus (5.13) is valid in the laboratory frame as well as in the co-moving frame. But this means that T_R^j must be the gradient of a scalar, which of course is just the way it is defined in (5.1). Thus a reservoir temperature 4-vector of the form $T_R^j = c\partial^j \Theta$ describes the most general ideal heat reservoir, so, assuming the necessity of the two requirements that went into the definition of an ideal reservoir, no generality is lost by using the thermasy to specify the reservoir.

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APPENDIX

HISTORICAL SKETCH OF THE DEVELOPMENT OF RELATIVISTIC THERMODYNAMICS

A-1. Development of the Covariance Concept

The history of relativistic thermodynamics can be understood only in terms of its relative timing with respect to the development of the covariant tensor formalism, and the resultant evolution in the covariance concept.

Maxwell's electrodynamic theory was implicitly harmonious with Special Relativity, so no modifications or extensions were necessary. This was not true, however, of both mechanics and three-dimensional vector and tensor analysis, and a conscious effort was needed to make them consistent with Relativity Theory. In the case of vector and tensor analysis, the first steps were made by Minkowski[35] in 1908. This work was taken up and continued by others after Minkowski's death in 1909 and culminated in the 1910 papers of Sommerfeld which summarized the tensor formalism as it is used in Special Relativity. These papers may be regarded as marking the advent of the four-dimensional tensor formalism. The full implications of this formalism began really to be understood only after Einstein's 1916 paper that introduced General Relativity, and Emmy Noether's 1918 paper that proved the intimate relation between the fundamental conservation laws of physics and tensor analysis. Group theory and the immensely important concept of irreducibility—familiar matters to many mathematicians—became widely known among physicists only after the advent of quantum mechanics. The fact that tensors (in the general sense including scalars, vectors, and spinors) induce the irreducible representations of the Lorentz group, threw tensor analysis into a new and deeper perspective. The outcome of this long development is the modern formulation of the Relativity Principle which requires that every fundamental equation of physics must belong to an irreducible representation of the Lorentz group, which in turn requires that all operators and variables entering into the equation must be combined in such a way that each term of the equation belongs to the same irreducible representation of the Lorentz group. (More general groups that include the Lorentz group as a subgroup are often considered, but the same kind of irreducibility requirement is always imposed on the equations.) Such an equation is covariant in the sense that it satisfies the same irreducibility requirement in every frame of reference. Note that this irreducibility or covariance requirement would eliminate any equation involving a quantity, such as heat in the Planck formulation of relativistic thermodynamics, that appeared <u>only</u> in the time-like component of a 4-vector (the force 4-vector) in <u>every frame of reference</u>.

Before the Relativity Principle had evolved into its present form, a much vaguer statement of it was used which simply required that the form of a physical equation remain the same under a Lorentz transformation of the coordinates.

The development of relativistic mechanics clearly shows the evolution toward a more exact statement of the Relativity Principle. (Incidentally, this historical evolution is still mirrored in the presentation of mechanics given in most modern relativity textbooks.) At the end of his 1905 paper, Einstein considered the relativistic dynamics of a particle, arguing directly from Newton's Second Law. In 1906 Planck gave a more elegant and general treatment of the subject based on the variational principle

$$\delta \int_{t_1}^{t_2} L dt = 0 \tag{A.1a}$$

where for a free particle

$$L = - mc^2 \left[1 - \left(\frac{dx}{dt}\right) \cdot \left(\frac{dx}{dt}\right) \Big/ c^2\right]^{1/2} \tag{A.1b}$$

This formulation is not covariant because it is built on a Lagrangian L that is not invariant. Because Ldt must be invariant, the transformation law for L is the inverse of that for time. Thus L is very similar to Planck's Q and T in that it has the same transformation law that they do, and like them is neither a scalar nor a 4-vector component. The covariant formulation of the same problem results, of course, by starting with

$$\delta \int_{\tau_1}^{\tau_2} \overset{o}{L} d\tau = 0 \tag{A.2a}$$

where for a free particle

$$\overset{o}{L} = - mc \left(\frac{dx^j}{d\tau} \frac{dx_j}{d\tau}\right)^{1/2} \tag{A.2b}$$

This covariant formulation had not yet been developed in the pre-1910 period and, as we shall see, this fact played a critical role in the history of relativistic thermodynamics.

A-2. Early Development of Relativistic Thermodynamics

The papers of Mosengeil (1907), Einstein (1907), and Planck[36] (1907 and 1908) are often cited in such a way as to indicate that these represented three independent approaches that all arrived at the same result, thereby suggesting the inevitability, if not the uniqueness of the Planck formulation of relativistic thermodynamics. This is misleading. Kurd von Mosengeil was Planck's student, and his paper was based on his dissertation. (He died shortly after finishing this dissertation, and it was Planck who edited the work for publication.) Einstein's 1907 paper was very wide-ranging, and only a small part[37] of it dealt specifically with thermodynamics. He was aware of Planck's work, and cited his 1907 paper. His work on thermodynamics should more properly be regarded as a presentation of Planck's approach from a different point of view. Specifically, he merely wrote down, without discussion, the form of the First Law that includes the work contributed by the translation force and, as pointed out in Section V-1, this already decides the issue in favor of the Planck formulation. Einstein's principal contribution to the thermodynamical problem was to emphasize[38] that whether or not the total energy and momentum of a system are the components of a 4-vector depends on whether or not the stress energy of the container is included. From this it is evident that the papers of Mosengeil and of Einstein did not represent independent approaches to relativistic thermodynamics, but rather were contributory to the main work, which was Planck's.

Mosengeil's work was undertaken in response to a calculation by Hasenöhrl (1904 and 1905) that indicated that a specific new postulate was necessary in order to accommodate the Planck Radiation Law for blackbody radiation to Relativity Theory. It was Mosengeil's purpose, in which he succeeded, to show that the Maxwell electromagnetic theory alone, without the need for new postulates, sufficed to arrive at the Planck Radiation Law for moving blackbody cavities. Mosengeil was not directly concerned with thermodynamics, but of necessity he was obliged to include temperature. He implicitly defined it in the fashion of equation (4.13), and so he naturally arrived at the Planck transformation law for temperature given in (4.16). (In fact, he derived several different transformation laws, depending on the conditions that he imposed on the volume of the cavity container during the acceleration from one velocity to another, including the rather exotic-seeming condition that the moving volume (not the rest-volume) of the container be kept constant during the acceleration).

Planck began his 1907 paper (or the 1908 one—as noted in footnote 36, these are identical) by summarizing Mosengeil's results and their implications for relativistic thermodynamics. His purpose was to incorporate these results into a more general formalism that would combine both mechanics and thermodynamics. The approach that he chose was both natural and elegant, namely a relativistic generalization of Helmholtz's (1886) variational principle that combined mechanics and thermodynamics. This approach was an especially natural one for Planck because the year before (1906) he had published a variational principle for relativistic mechanics. It was only necessary then to generalize this along the lines indicated by Helmholtz's work, in order to include thermodynamics.

Helmholtz's variational principle was

$$\delta \int_{t_1}^{t_2} L dt = 0 \qquad \text{(A.3a)}$$

where (for one-dimensional motion)

$$L = \frac{1}{2} M \left(\frac{dx}{dt}\right)^2 - F \qquad \text{(A.3b)}$$

and

$$F = U - TS = F(V,T) . \qquad \text{(A.3c)}$$

F is what we now call the free-energy of the body or its Helmholtz function. The Euler-Lagrange equation for the variable x is

$$\frac{d\mathfrak{p}}{dt} \equiv \frac{d}{dt}\left[\frac{\partial L}{\partial\left(\frac{dx}{dt}\right)}\right] = \frac{\partial L}{\partial x} \qquad \text{(A.4a)}$$

or

$$\frac{d\mathfrak{p}}{dt} = - \frac{\partial F}{\partial x} \qquad \text{(A.4b)}$$

which is in accord with the well-known fact[39] that, for a body which absorbs or rejects heat reversibly and isothermally, F is a potential function for the work done on the body. Helmholtz developed a canonical formalism treating V and T, as well as x, as variables. Naturally the derivatives

$$\left(\frac{\partial L}{\partial T}\right)_{x,V} = -\left(\frac{\partial F}{\partial T}\right)_V = S \tag{A.5a}$$

and

$$\left(\frac{\partial L}{\partial V}\right)_{x,T} = -\left(\frac{\partial F}{\partial V}\right)_T = P \tag{A.5b}$$

played an important role in this formalism.

Planck generalized Helmholtz's work by replacing (A.3b) with

$$\begin{aligned} L &= -\left(Mc^2 + \overset{\circ}{F}\right)\left[1 - \left(\frac{dx}{dt}\right)^2 \Big/ c^2\right]^{1/2} \\ &\approx \frac{1}{2}\left(M + \overset{\circ}{F}/c^2\right)\left(\frac{dx}{dt}\right)^2 - \left(Mc^2 + \overset{\circ}{F}\right), \end{aligned} \tag{A.6}$$

where $\overset{\circ}{F}$ is the rest-frame Helmholtz function. One can arrive at a transformation law for temperature very simply: If $\overset{\circ}{T}$ is the rest-frame temperature and T the laboratory temperature, then let

$$T = \alpha\overset{\circ}{T} \tag{A.7}$$

where α is an as yet undetermined function of dx/dt. Then

$$\frac{\partial L}{\partial T} = -\left(1-\beta^2\right)^{1/2} \frac{\partial \mathring{F}}{\partial T} = -\frac{\left(1-\beta^2\right)^{1/2}}{\alpha} \frac{\partial \mathring{F}}{\partial \mathring{T}} = \frac{\left(1-\beta^2\right)^{1/2}}{\alpha} S \qquad \text{(A.8)}$$

In the rest-frame this equation has the form

$$\frac{\partial \mathring{L}}{\partial \mathring{T}} = S \; . \qquad \text{(A.9)}$$

(As noted in Section IV, Planck had given a general argument for the invariance of S.) One now invokes the Relativity Principle in its early form to insist that the form of the equation $\partial L/\partial T = S$ be Lorentz invariant. From (A.8) it is evident that this requires that $\alpha = \left(1-\beta^2\right)^{1/2}$ with the result that

$$T = \left(1-\beta^2\right)^{1/2} \mathring{T} \qquad \text{(A.10)}$$

which is the Planck transformation law for temperature. In a similar way, from (A.5b) and the known fact that pressure is invariant, one recovers the familiar transformation law for volume, which has the same form as (A.10).

It is obvious that the lack of covariance in Planck's formalism followed from his use of the variational principle in the form (A.1a), rather than in the form (A.2a). Note also that this non-covariance was in no way inconsistent with the Relativity Principle in its early, vague form.

With Planck's work, the die was cast. In 1911 Jüttner, working directly under Planck's influence (which he acknowledges with thanks at the beginning of his first paper), adapted the Planck formalism to the kinetic theory of gases.

Tolman (1914) extended Jüttner's work to systems containing molecules of different masses in order to derive a relativistic equipartition law. Both these men did subsequent work in relativistic thermodynamics [Jüttner (1928), Tolman (1934)] well after the Relativity Principle in its present form demanding covariance had become well established, but it is not surprising that their work remained in its original mold. In a footnote[40] in his 1911 textbook von Laue made the point that the Planck formulation was not unique. His motive in this footnote was to concede the formal possibility of a relativistic thermodynamics built on a scalar temperature, and then to dismiss such a possibility as intuitively unacceptable. (He defended the Planck formalism.) In an earlier footnote[41] he was even more explicit in pointing out that an infinity of formulations was conceivable, i.e., that it was fundamentally a question of definition. These two footnotes were completely ignored until Ott (1963) made their implications very explicit.

A-3. Later Development of Relativistic Thermodynamics

Mechanics, unlike relativistic thermodynamics, did not long remain in its early non-covariant mold. It lay at the center of research in first general relativity and then later relativistic quantum mechanics. The non-covariant formalism, adequate enough for simple problems, would have been utterly unworkable for the complicated problems encountered in these two fields.

The contrast with the case of relativistic thermodynamics is striking. There was certainly no experimental interest in it, and almost no theoretical interest. The research in relativistic fluid dynamics concentrated exclusively on those special types of flow for which it is possible to avoid any real thermodynamical considerations, namely adiabatic flow, or the even more special case of barotropic flow. Relativistic thermodynamics was of peripheral interest in cosmology. Tolman was already committed to the Planck formalism. Eddington[42] effectively opted for a scalar thermodynamics with the reasonable remark that constitutive equations had physical significance only in the local rest-frame of the fluid.

It is significant that the first person, van Dantzig (1939-1940), who attempted to do anything with relativistic fluid dynamics going beyond the adiabatic restriction, immediately replaced the Planck formulation with a covariant alternative. (He actually could not have done otherwise, since any non-covariant formalism is unworkable.) Unfortunately, however, van Dantzig's work was completely ignored until very recently (cf. footnote 8).

Whereas van Dantzig in no way attacked the Planck formulation, but rather contented himself with demonstrating the existence of a covariant alternative, Ott, Arzeliès, and most of the other recent proponents of a covariant formalism did attack the Planck formulation, characterizing it as being simply wrong (rather than just unworkable). In view of the counter-attack these papers subsequently provoked (itself an interesting phenomenon!), it appears van Dantzig's approach would have been the more effective tactic. One can only regret the absence of van Dantzig's contribution in the debate, and hope that this strange episode in the history of physics has run its course.

FOOTNOTES

1. First emphasized by Einstein (1907), Sec. 12.
2. Einstein (1907), Sec. 14. For references to recent advocates of this and other points of view see Landsberg and Johns (1967).
3. Gamba (1966), Rohrlich (1966).
4. Kibble (1966).
5. Eddington (1923), cf. p. 34 of 2nd edition; Gamba (1966).
6. See survey of recent literature given by Landsberg and Johns (1967), esp. chart on p. 42.
7. Israel (1963).
8. van Dantzig (1939-1940). I learned of van Dantzig's work only very recently from Prof. Peter Bergmann who told me that one of his students, Mr. Victor Hamity, discovered these papers in the course of preparing a literature survey for his dissertation.
9. Schmid (1967a).
10. Schmid (1967 b, c).
11. von Laue (1911), p. 181 of 7th edition (1961), credits Helmholtz with having first introduced the temperature integral (in connection with bodies rather than fluids) and cites Helmholtz's 1886 paper. This is a long paper, of which only pp. 225-227 and pp. 234-235 (of Helmholtz's Wissenschaftliche Abhandlungen) deal directly with thermodynamics. If Helmholtz did indeed introduce the temperature integral in this paper, it is so indirect as to be

virtually invisible. It seems most probable that Helmholtz introduced it in another paper, and von Laue erroneously cited this one (which is one of Helmholtz's most famous papers) from memory.

12. See Schmid (1967c), p. 326 for references and further details.
13. Θ is the same function that was designated by $\mathfrak{Z}$ in Schmid (1967c), and is not to be confused with the entropy force tensor Θ^{jk} of Schmid (1967a). Θ is the designation used by van Dantzig. In recognition of his pioneering (albeit neglected) work, his designation and nomenclature ("Thermasy") will be adopted in this paper.
14. Presumably from the Greek Θέρμανσις (heat generator) with elusion of the ν and final ς in anglicization, the latter of which is omitted in modern Greek, in any case. My thanks to my colleague E. G. Stassinopoulos for this suggestion as to the etymological origin of the term "thermasy," which was in no way indicated by van Dantzig. Indeed, he gave no specific indication that the term originated with him, but I have never seen it used elsewhere.
15. The factor 2μ has been inserted in front of ω^{jk} in (2.14) in order that ω^{jk} will satisfy (2.16) for the case of a cold fluid, and hence correspond to a vorticity.
16. For further details, see Schmid (1967 b and c).
17. Serrin (1959), p. 148, footnote 2, credits C. C. Lin (unpublished) with having first made this point explicit, although the correct variational principle for

(non-relativistic) barotropic flow (which includes M and Φ, but not s and Θ) has long been known. (See Bateman (1932), p. 164-165).

18. Schmid (1966). See pp. 106-107.

19. For a more detailed discussion of potential flow (in the presence of gravitational and electromagnetic fields) see Schmid (1967 b and c).

20. For more details see Schmid (1966 and 1967c).

21. For a brief review see Serrin (1959), pp. 144-150.

22. Tam (1966).

23. von Laue (1911), see the footnote on p. 138 and the footnote running from p. 177 to p. 178 of the 7th edition (1961). See also discussion at the end of Section A.2 of the appendix of this paper.

24. van Dantzig (1939a), footnote on p. 700.

25. Terrell (1959).

26. Ott (1963). See also Schmid (1967a).

27. Only in this section and in the appendix is entropy designated by capital S. Elsewhere we always deal with the specific entropy s. Thus there is no danger of confusing entropy with the scalar S introduced in (2.20), which in Section III was shown to be the Lagrange multiplier associated with the conservation of particles.

28. Planck (1907) p. 552, or (1908) p. 13.

29. Israel (1963). See his Equations (6.2) and (6.18).

30. von Laue (1911). Sec. 23c, pp. 175-178 of the 7th edition (1961).

31. Schmid (1967a) Sec. 4.

32. In Schmid (1967a), this force was expressed on a per-particle basis and denoted by π^j, i.e., $\pi^j = f^j/n$. (Particle density was designated by ρ rather than n).

33. Schmid (1967a) pp. 17-19. In this reference the 4-vector $dw^j/d\tau$ was expressed on a per-particle (rather than per unit volume) basis, and designated as ϕ^j, i.e., $\phi^j = (dw^j/d\tau)/n$.

34. See, for example, Sedov (1966) p. 56, Equation (8.9). The treatment in this reference is for three-dimensional curvilinear coordinates, but the formalism is the same as for the four-dimensional case.

35. Minkowski gave lectures on the subject "Space and Time" before several audiences. For a listing of the various places where these talks were published see p. 74 of Part II of Felix Klein, "Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert," (Chelsea Publ. Co., New York 1967) (originally published Berlin 1926-27). The most accessible reprinting of Minkowski's lecture "Raum und Zeit," (besides his collected works cited with the references) is to be found (in English) in "The Principle of Relativity" (Dover Publications, New York), pp. 75-91 followed by notes by A. Sommerfeld, pp. 92-96.

36. The 1908 paper is a verbatim reprint of the 1907 paper.

37. Einstein (1907), sec. 15, pp. 451-453.

38. Einstein (1907), sections 12 and 14.

39. See, for example, Sommerfeld (1956), p. 53.

40. von Laue (1911). See footnote running from p. 177 to p. 178 of 7th edition (1961).

41. von Laue (1911). See footnote on p. 138 of 7th edition (1961).

42. Eddington (1923). See p. 34 of 2nd edition (1924).