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CLASS OF EXACT SOLUTION OF RELATIVISTIC GAS

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ABSTRACT

A method of obtaining exact solutions for the relativistic gas mixtures at high effective temperatures, subjected to a body force or no body force, is presented. The method yields the solution of the conservation equation by linear superposition of the energy momentum tensor of the electromagnetic field with or without electric charge and current. The nature of the flow field of high temperature gas is similar to that of the photon gas within the first two approximations.

I. INTRODUCTION

There has been a recent interest in the fluid dynamics of relativistic gas in the realm of astrophysics, particularly in those problems associated with high temperature gas in a stellar interior.

Various fundamental aspects of the relativistic gas have been exclusively treated by Synge[1], Taub[2-5], Lichnerowicz[6], McVitte[7], Guess[8], and most recently by Truitt[9]. Yet virtually no investigation has been made in obtaining the exact solutions of the ogverning equations, presumably because these equations are non linear. Undoubtedly obtaining exact solutions is of practical importance in view of the inadequacy of linearized theory prevailing at high velocity and high temperature.

It is the purpose of this paper to present the method of obtaining a class of exact solutions applicable for (1) photon gas, (11) material gas at high effective temperatures and (111) a mixture of the photon gas and the material gas at high effective temperatures.

The governing equations and the notations adopted here follow closely that of Synge[1]. The Greek affixes $\mu\nu\sigma$ ... takes the value 1,2,3,4, and the Latin affixes ab take values 1,2,3. c is the special relativity speed of light.

II. DEDUCTION OF THE ASYMPTOTIC PERTURBATION EQUATION OF THE RELATIVISTIC GAS

The equation of motion of the relativistic gases is the conservation of the energy momentum tensor $T_{\mu\nu}$

$$\partial_\mu T_{\nu\mu} = 0 \tag{1}$$

where ∂_μ stands for the operator $\frac{\partial}{ic\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$.

The energy tensor is given by

$$T_{\nu\mu} = (\epsilon + c^{-2} p)\, \lambda_\nu \lambda_\mu + \delta_{\nu\mu} c^{-2} p \tag{2}$$

ϵ is the mean density of the gas or gas mixture, p is the hydrostatic pressure, λ_μ is a 4-velocity vector satisfying the following relation

$$\lambda_\mu \lambda_\mu = -1 \tag{3}$$

The equations of continuity, and the equations of state of the photon gas, the gas composed material particles of proper mass m, and the gas mixture* of the photon and the material particles are assumed to be those given by Synge[(1)]

Photon gas $\quad \partial_\mu (\tilde{N}\lambda_\mu) = 0 \qquad$ (4a)

$$\epsilon + c^{-2} p = 4\tilde{N}/\xi \quad , \quad c^{-2} p = \tilde{N}/\xi \tag{4b}$$

Material gas $\quad \partial_\mu (N\lambda_\mu) = 0 \qquad$ (5a)

$$\epsilon + c^{-2} p = mNG(m\xi) \quad , \quad c^{-2} p = N/\xi \tag{5b}$$

Gas mixture $\quad \partial_\mu (\tilde{N}\lambda_\mu) = 0 \quad , \quad \partial_\mu (N\lambda_\mu) = 0 \qquad$ (6a)

$$\epsilon + c^{-2} p = mNG(m\xi) + 4\tilde{N}/\xi \quad , \quad c^{-2} p = (N + \tilde{N})/\xi \tag{6b}$$

where $\tilde{N}$, and N are the number densities of the photon, and the material particles respectively.

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* It will be assumed that the photons and the material particles are in dynamic and thermal equilibrium.

ξ is the reciprocal temperature related with the absolute temperature T as follows

$$kT = e^2/\xi \tag{7}$$

where k is the Baltzmann's constant. $m\xi$ is a non-dimensional number, and $1/m\xi$ is the effective temperature.

The function G(x) is defined in terms of Bessel functions

$$G(x) = \frac{K_3(x)}{K_2(x)} = \frac{2}{x} - \frac{K_2'(x)}{K_2(x)} \tag{8}$$

For a gas composed of material particles at high effective temperature i.e. $m\xi \ll 1$, the equation of state is approximated(1) to be

$$\epsilon + c^{-2}p = mNG(m\xi) \simeq \frac{4N}{\xi} + \frac{m^2N\xi}{2} + \ldots \tag{9a}$$

$$c^{-2}p = N/\xi \ , \ \epsilon \simeq \frac{3N}{\xi} + \frac{m^2N\xi}{2} + \ldots \tag{9b}$$

Note that the left hand side of Eq. (9) reduced to that of the photon gas when m vanishes.

The asymptotic behavior of the equation of state suggests the plausibility of developing the asymptotic theory of the material gas which approaches that of the photon gas in the limit of vanishing rest mass.

To begin with, the reciprocal temperature is non-dimensionalized by some reference reciprocal temperature $\xi_{ref.}$

$$m\xi = (m\xi_{ref})\zeta = \varkappa\zeta \tag{10}$$

where $\varkappa$ will be regarded as a small parameter in the ensuing perturbative analysis.

All the fluid properties are expanded in terms of $\varkappa$

$$\epsilon = \epsilon^{(o)} + \varkappa\epsilon^{(1)} + \varkappa^2\epsilon^{(2)} +.. \quad (11a)$$

$$p = p^{(o)} + \varkappa p^{(1)} + \varkappa^2 p^{(1)} +.. \quad (11b)$$

$$N = N^{(o)} + \varkappa N^{(1)} + \varkappa^2 N^{(2)} +.. \quad (11c)$$

$$\widetilde{N} = \widetilde{N}^{(o)} + \varkappa\widetilde{N}^{(1)} + \varkappa^2\widetilde{N}^{(2)} +.. \quad (11d)$$

$$\lambda_\mu = \lambda_\mu^{(o)} + \varkappa\lambda_\mu^{(1)} + \varkappa^2\lambda_\mu^{(2)} +.. \quad (11e)$$

$$\zeta = \zeta^{(o)} + \varkappa\zeta^{(1)} + \varkappa^2\lambda^{(2)} +.. \quad (11f)$$

Substituting Eqs. (11a) to (11f) into (3), the energy tensor $T_{\mu\nu}$ is also expanded in ascending powers of $\varkappa$

$$T_{\mu\nu} = \frac{1}{\varkappa} T_{\mu\nu}^{(o)} + T_{\mu\nu}^{(1)} + \varkappa T_{\mu\nu}^{(2)} \quad (12a)$$

$T_{\mu\nu}^{(o)}$, $T_{\mu\nu}^{(1)}$, and $T_{\mu\nu}^{(2)}$ are given as

$$T_{\mu\nu}^{(o)} = 4\left(\psi^{(o)}+\chi^{(o)}\right)\lambda_\mu^{(o)}\lambda_\nu^{(o)} + \delta_{\mu\nu}\left(\psi^{(o)}+\chi^{(o)}\right) \quad (12b)$$

$$T_{\mu\nu}^{(1)} = 4\left[\psi^{(1)}+\chi^{(1)}-\left(\psi^{(o)}+\chi^{(o)}\right)\zeta^{(1)}/\zeta^{(o)}\right]\lambda_\mu^{(o)}\lambda_\nu^{(o)} + 4\left(\psi^{(o)}+\chi^{(o)}\right)\left(\lambda_\mu^{(1)}\lambda_\nu^{(o)}+\lambda_\mu^{(o)}\lambda_\nu^{(1)}\right)$$
$$+ \delta_{\mu\nu}\left[\psi^{(1)}+\chi^{(1)}-\left(\psi^{(o)}+\chi^{(o)}\right)\zeta^{(1)}/\zeta^{(o)}\right] \quad (12c)$$

$$T_{\mu\nu}^{(2)} = 4\left\{\psi^{(2)}+\chi^{(2)}-\left(\psi^{(1)}+\chi^{(1)}\right)\zeta^{(1)}/\zeta^{(o)}-\psi^{(o)}\left[\zeta^{(2)}/\zeta^{(o)}-\left(\zeta^{(1)}/\zeta^{(o)}\right)^2\right] + \frac{1}{8}\psi^{(o)}\zeta^{(o)^2}\right\}$$
$$\lambda_\mu^{(o)}\lambda_\nu^{(o)} + 4\left\{\left(\psi^{(1)}+\chi^{(1)}\right)-\left(\psi^{(o)}+\chi^{(o)}\right)\zeta^{(1)}/\zeta^{(o)}\right\}\left\{\lambda_\mu^{(o)}\lambda_\nu^{(1)}+\lambda_\mu^{(1)}\lambda_\nu^{(o)}\right\}$$
$$+ 4\left(\psi^{(o)}+\chi^{(o)}\right)\left(\lambda_\mu^{(2)}\lambda_\nu^{(o)}+\lambda_\mu^{(o)}\lambda_\nu^{(2)}+\lambda_\mu^{(1)}\lambda_\nu^{(1)}\right)$$
$$+ \delta_{\mu\nu}\left\{\psi^{(2)}+\chi^{(2)}-\left(\psi^{(1)}+\chi^{(1)}\right)\left(\zeta^{(1)}/\zeta^{(o)}\right)-\left(\psi^{(o)}+\chi^{(o)}\right)\left[\zeta^{(2)}/\zeta^{(o)}-\left(\zeta^{(1)}/\zeta^{(o)}\right)^2\right]\right\}$$
$$(12d)$$

where $\psi^{(n)} = \dfrac{4mN^{(n)}}{\zeta^{(o)}} \qquad \chi^{(n)} = \dfrac{4m\widetilde{N}^{(n)}}{\zeta^{(o)}}$ (12e)

The perturbative conservation equations are

$$\partial_\nu \, T^{(o)}_{\mu\nu} = 0 \tag{13a}$$

$$\partial_\nu \, T^{(1)}_{\mu\nu} = 0 \tag{13b}$$

$$\partial_\nu \, T^{(2)}_{\mu\nu} = 0 \tag{13c}$$

The continuity equations are

$$\partial_\nu\left(N^{(o)}\lambda^{(o)}_\nu\right) = 0 \qquad \partial_\nu\left(\tilde{N}^{(o)}\lambda^{(o)}_\nu\right) = 0 \tag{14a}$$

$$\partial_\nu\left(N^{(1)}\lambda^{(o)}_\nu + N^{(o)}\lambda^{(1)}_\nu\right) = 0 \qquad \partial_\nu\left(\tilde{N}^{(1)}\lambda^{(o)}_\nu + \tilde{N}^{(o)}\lambda^{(1)}_\nu\right) = 0 \tag{14b}$$

$$\partial_\nu\left(N^{(2)}\lambda^{(o)}_\nu + N^{(1)}\lambda^{(1)}_\nu + N^{(o)}\lambda^{(2)}_\nu\right) = 0 \qquad \partial_\nu\left(\tilde{N}^{(2)}\lambda^{(o)}_\nu + \tilde{N}^{(1)}\lambda^{(1)}_\nu + \tilde{N}^{(o)}\lambda^{(2)}_\nu = 0 \tag{14c}$$

The four velocity vectors $\lambda^{(n)}_\mu$ satisfy the following relations

$$\lambda^{(o)}_\mu \lambda^{(o)}_\mu = -1 \tag{15a}$$

$$\lambda^{(o)}_\mu \lambda^{(1)}_\mu = 0 \tag{15b}$$

$$\lambda^{(o)}_\mu \lambda^{(2)}_\mu + \frac{1}{2}\lambda^{(1)}_\mu \lambda^{(1)}_\mu = 0 \tag{15c}$$

The traces of $T^{(o)}_{\mu\nu}$, $T^{(1)}_{\mu\nu}$ and $T^{(2)}_{\mu\nu}$ are found to be

$$T^{(o)}_{\mu\mu} = 0 \tag{16a}$$

$$T^{(1)}_{\mu\mu} = 0 \tag{16b}$$

$$T^{(2)}_{\mu\mu} = -\frac{1}{2}\,\psi^{(o)}\zeta^{(o)^2} \tag{16c}$$

III. SOLUTION OF THE PHOTON GAS

The conservation equations of the photon gas, and the zeroth order equation of motion of the material gas are given by Eqs. (13a) and (14a). The energy tensor is given by

$$\frac{1}{\varkappa} T_{\mu\nu}^{(o)} = \frac{1}{\varkappa}\left\{\frac{4\chi^{(o)}}{\zeta^{(o)}} \lambda_{\mu}^{(o)}\lambda_{\nu}^{(o)} + \delta_{\mu\nu}\frac{\chi^{(o)}}{\zeta^{(o)}}\right\}$$

$$= \frac{4\tilde{N}^{(o)}}{\xi^{(o)}} \lambda_{\mu}^{(o)}\lambda_{\nu}^{(o)} + \delta_{\mu\nu}\frac{\tilde{N}^{(o)}}{\xi^{(o)}} \tag{17}$$

The energy tensor $T_{\mu\nu}^{(o)}$ given above is an exact expression for the photon gas, and is a zeroth order approximation for the material particle. It is therefore expected that the behavior of the gas composed of the material particles is similar to that of the photon gas within this approximation.

The equations governing the motion of the photons are non-linear, and despite the fact that Eq. (13a) admits the "First Integral"(1) representing the adiabatic law, the general solution is difficult to obtain.

The approach adopted in this paper is to seek a solution of the flow field in terms of electromagnetic fields which are properly superposed so that the flow variables constructed from such electromagnetic fields satisfy the physical boundary condition imposed on the flow field. The plausibility of the present approach is suggested by the facts that the energy momentum tensors of the photon like gas and the electromagnetic field are both traceless, and that they satisfy the same conservation equation.

According to the above, one writes

$$T_{\mu\nu} \text{ (photon gas)} = \sum_n A_n \left(\Pi_{\mu\nu}\right)_n \tag{18}$$

where A_n is an arbitrary constant, $\left(\Pi_{\mu\nu}\right)_n$ is an energy momentum tensor of an electromagnetic field. The constants A_n and the elementary tensor $\left(\Pi_{\mu\nu}\right)_n$ are to be properly selected. The elementary tensor $\left(\Pi_{\mu\nu}\right)_n$ is given in terms of field tensors $F_{\mu\nu}$ and $F^*_{\mu\nu}$ as follow

$$\left(\Pi_{\mu\nu}\right)_n = c^{-2}\left(F_{\mu\tau}F_{\nu\tau} - \frac{1}{4}\,\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta}\right)_n \tag{19a}$$

or

$$\left(\Pi_{\mu\nu}\right)_n = \frac{1}{2}\,c^{-2}\left(F_{\mu\tau}F_{\nu\tau} + F^*_{\mu\tau}F^*_{\nu\tau}\right)_n \tag{19b}$$

$F^*_{\mu\nu}$ is the tensor dual to $F_{\mu\nu}$

$$F^*_{\mu\nu} = \frac{i}{2}\,\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{20}$$

$$\epsilon_{\mu\nu\alpha\beta} = \begin{cases} 1 & \text{if the set of number } \mu\nu\alpha\beta \text{ is the set } (1,2,3,4) \\ & \text{in order or in an even permutation of that order} \\ -1 & \text{if } \mu\nu\alpha\beta \text{ is an odd permutation of } (1,2,3,4) \\ 0 & \text{otherwise} \end{cases}$$

The field tensor $F_{\mu\nu}$ and its dual $F^*_{\mu\nu}$ satisfy Maxwell equation.

$$\partial_\nu F_{\mu\nu} = 0 \qquad \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \tag{21a}$$

$$\partial_\nu F^*_{\mu\nu} = 0 \qquad \partial_\sigma F^*_{\mu\nu} + \partial_\mu F^*_{\nu\sigma} + \partial_\nu F^*_{\sigma\mu} = 0 \tag{21b}$$

Substituting (17) and (19a) into Eq. (18), we have

$$3\left(N^{(o)}/\xi^{(o)}\right)\lambda_\mu^{(o)}\lambda_\nu^{(o)} + N^{(o)}/\xi^{(o)}\delta_{\mu\nu} = \sum_n c^{-2}A_n\left(F_{\mu\sigma}F_{\nu\sigma} - \frac{1}{4}\,\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta}\right)_n \tag{22}$$

The above tensor equations contain 5 unknowns i.e. $N^{(o)}/\xi^{(o)}$, and

$\lambda_\mu^{(o)}$, which are presumably to be solved in terms of the components of the energy momentum tensor.

By multiplying both sides of Eq. (22) by $\lambda_\nu^{(o)}$, and noticing that $\lambda_\nu^{(o)}\lambda_\nu^{(o)} = -1$, the following equation is obtained

$$-\epsilon^{(o)}\lambda_\mu^{(o)} = -3\tilde{N}^{(o)}/\xi^{(o)}\lambda_\mu^{(o)} = \sum_n c^{-2} A_n\left(F_{\mu\sigma}F_{\nu\sigma} - \frac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta}\right)_n \lambda_\nu^{(o)} \tag{23}$$

we rewrite the above equation in the following eigenvalue problem

$$\left\{\epsilon^{(o)}\delta_{\mu\nu} + \sum_n c^{-2}A_n\left(F_{\mu\sigma}F_{\nu\sigma} - \frac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta}\right)_n\right\}\lambda_\mu^{(o)} = 0 \tag{24}$$

The energy density of the photon gas is thereby given by the eigenvalue of the energy momentum tensor $\sum_n \left(\Pi_{\mu\nu}\right)_n$ of the electromagnetic field, whereas the four velocity vectors are the corresponding eigenvector.

For n = 1, the following four eigenvalues are obtained

$$\epsilon_{1,2}^{(o)} = \epsilon_{3,4}^{(o)} = \pm\frac{1}{4}c^{-2}\left[\left(F_{\mu\nu}F_{\mu\nu}\right)^2 + \left(F_{\mu\nu}F_{\mu\nu}^*\right)^2\right]^{\frac{1}{2}} \tag{25a}$$

$$= \pm\frac{1}{2}c^{-2}\left[\left(E^2 - H^2\right)^2 + 4\left(\underline{E}\cdot\underline{H}\right)^2\right]^{\frac{1}{2}} \tag{25b}$$

Since $\epsilon^{(o)}$ is greater than or equal to zero, the negative value should be discarded.

The pressure of the photon gas is thus given by

$$p^{(o)} = \frac{1}{12}c^{-2}\left\{\left(F_{\mu\nu}F_{\mu\nu}\right)^2 + \left(F_{\mu\nu}F_{\mu\nu}^*\right)^2\right\}^{\frac{1}{2}} \tag{26}$$

Of the four algebraic equations contained in (24), only three of them are independent by virtue of the relation $\lambda_\mu^{(o)}\lambda_\mu^{(o)} = -1$.

By taking the first three algebraic equations, $\lambda_\ell^{(o)}$ for $\ell = 1,2,3$ are obtained as follows

$$\lambda_\ell^{(o)} = -\left(\Pi_{k\ell} + \delta_{k\ell}\epsilon^{(o)}\right)^{-1} \Pi_{k4}\lambda_4^{(o)} \tag{27}$$

where $\left(\Pi_{k\ell} + \delta_{k\ell}\epsilon^{(o)}\right)^{-1}$ is the inverse of the matrix $\Pi_{k\ell} + \delta_{k\ell}\epsilon_\ell^{(o)}$. Note that the results (25a,b), (26) and (27) are only applicable for $n = 1$. When $n > 2$, the solutions need be properly modified. After the energy density, pressure and the four velocity vectors are found, and the constants A_n's properly chosen, the numerical density $\widetilde{N}^{(o)}$ is calculated from the continuity equation (14a), which becomes effectively a linear partial differential equation by virtue that $\lambda_\nu^{(o)}$ is a known vector. The reciprocal temperature $m\xi$ is subsequently obtained from the equation of state.

The present analysis can be extended to treat the case where the flow field is subject to the four force f_i. Let the body force acting on the fluid be f_i, the conservation equation reads

$$\partial_\mu T_{\nu\mu}^{(o)} = f_\mu \tag{28}$$

Let f_μ be expressed by

$$f_\mu = \sum_n B_n \left(f_\mu\right)_n \tag{29}$$

where B_n is an arbitrary constant.

Assuming further that $\left(f_\mu\right)_n$ is expressed by

$$\left(f_\mu\right)_n = - c^{-2}\left\{F_{\mu\sigma} J_\sigma\right\}_n \tag{30}$$

where J_σ is the electric four current, i.e. $\left(J_a, i\rho_e\right)$. In analogy to the case previously considered we write the energy momentum tensor of the gas as

$$T_{\mu\nu} \text{ (photon gas)} = \sum_n D_n \left(\Pi_{\mu\nu}\right)_n \tag{31}$$

where $\left(\Pi_{\mu\nu}\right)_n$ satisfies the conservation equation

$$\partial_\nu\left(\Pi_{\nu\mu}\right)_n = - c^{-2}\left\{F_{\mu\sigma}J_\sigma\right\}_n \tag{32}$$

The tensor $\left(\Pi_{\mu\nu}\right)_n$ is traceless and symmetric.

The electromagnetic field with charge and current is obtained by solving the system of linear equations, i.e.

$$F_{\mu\nu,\nu} = J_\mu \quad , \quad \overset{*}{F}_{\mu\nu,\nu} = 0 \tag{33}$$

The flow variables associated with the photon like gas can therefore be expressed by superposition of the electromagnetic fields.

For the gas mixture of photon and the material particles at high temperature, the zeroth order energy tensor is given by

$$T^{(o)}_{\mu\nu} = 4\left\{\left(N^{(o)}+\widetilde{N}^{(o)}\right)/\xi^{(o)}\right\} \lambda_\mu^{(o)}\lambda_\nu^{(o)} + \delta_{\mu\nu}\left(N^{(o)}+\widetilde{N}^{(o)}\right)/\xi^{(o)} \tag{34}$$

The results obtained previously for the energy density (25a), pressure (26), and the mean 4-velocity (27) are also applicable for the present case.

IV. EFFECT OF FINITE MASS

Since trace of the first order energy tnesor (Eq. 12c) is null, and the divergence of the energy tensor is zero, the solution of the first order approximation is also photon gas like. Hence without loss of generality the first order solution may be taken to be zero.

The effect of the finite mass appears in the second order approximation. Consequently the flow field of the high energy gas deviates from that of the photon gas in the second and the higher order theory, which will not be treated in the present analysis.

V. APPLICATION OF THE EQUIVALENCE PRINCIPLE

In this section, the flow pattern of a gas at high effective temperature will be examined under the equivalence principle.

For simplicity, the case of one-dimensional flow will be assumed.

Let $\Pi_{\mu\nu}$ consist of three individual fields $S_{\mu\nu}$, $R_{\mu\nu}$ and $V_{\mu\nu}$. Each energy momentum tensor will be assumed to be associated with plane electromagnetic waves given by

$$S_{\mu\nu}: \quad E_1 = f(x,t), \quad E_2 = 0, \quad E_3 = 0$$
$$H_1 = 0, \quad H_2 = f(x,t), \quad H_3 = 0$$

$$R_{\mu\nu}: \quad E_1 = 0, \quad E_2 = g(x,t), \quad E_3 = 0$$
$$H_1 = g(x,t), \quad H_2 = 0, \quad H_3 = 0$$

$$V_{\mu\nu}: \quad E_1 = 0, \quad E_2 = 0, \quad E_3 = a(t)$$
$$H_1 = 0, \quad H_2 = 0, \quad H_3 = b(t)$$

It can be shown that the components of each energy momentum tensor are given by

$$S_{11} = S_{22} = S_{23} = S_{31} = S_{12} = S_{14} = S_{24} = 0$$
$$S_{33} = c^{-2} f^2, \quad S_{44} = -c^{-2} f^2, \quad S_{34} = ic^{-2} f^2$$

$$R_{11} = R_{22} = R_{23} = R_{31} = R_{12} = R_{14} = R_{24} = 0$$
$$R_{33} = c^{-2} g^2, \quad R_{44} = -c^{-2} g^2, \quad R_{34} = -ic^{-2} g^2$$

$$V_{11} = \frac{1}{2} c^{-2} (a^2+b^2), \quad V_{22} = \frac{1}{2} c^{-2} (a^2+b^2)$$

$$V_{23} = V_{31} = V_{12} = V_{14} = V_{24} = 0$$

$$V_{33} = -\frac{1}{2} c^{-2} (a^2+b^2), \quad V_{44} = -\frac{1}{2} c^{-2} (a^2+b^2)$$

By equality $T_{\mu\nu} = \Pi_{\mu\nu}$, and solving for the 2 components λ_1 and λ_4 of the 4-velocity in terms of f, g, a and b, yields

$$\lambda_3 = \frac{1}{\sqrt{2}} \frac{2f^2 - (a^2+b^2)}{2f\sqrt{a^2+b^2}}$$

$$\lambda_4 = \frac{i}{\sqrt{2}} \frac{2f^2 + (a^2+b^2)}{2f\sqrt{a^2+b^2}}$$

$$\frac{N}{\xi} = \frac{1}{2} c^{-2} (a^2+b^2)$$

From the continuity equation, one obtains the following equation governing N

$$\frac{\partial\psi}{\partial t} + c \frac{2f^2 - (a^2+b^2)}{2f^2 + (a^2+b^2)} \frac{\partial\psi}{\partial x} = - \frac{\partial}{\partial t} \left\{ \ell n \frac{2f^2 + a^2 + b^2}{2f\sqrt{a^2+b^2}} \right\}$$

$$- \frac{2cf\sqrt{a^2+b^2}}{2f^2 + a^2 + b^2} \frac{\partial}{\partial x} \left[\frac{2f^2 - (a^2+b^2)}{2f\sqrt{a^2+b^2}} \right]$$

where $\psi = \ell n N$.

The above equation is a linear partial differential equation of the first order and can be solved in principle, by the standard mathematical technique.

VI. CONCLUDING REMARKS

Since the four divergence of the electric four current J_a appears in the body force expression (30) vanishes, the type of body force occurring is subjected to the following scalar equation

$$\partial_\sigma \left\{ \left(F^{-1}_{\mu\sigma} \right)_n \left(f_\mu \right)_n \right\} = -c^2 \partial_\sigma J_a = 0$$

where $F^{-1}_{\mu\sigma}$ is the tensor element of the inverse of matrix F. Despite the above limitation, the case of flow without body force is subjected to no constraining condition, and appears to have general applicability in treating relativistic flow with various boundary and initial conditions.

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