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EULER POTENTIALS AND THEIR APPLICATION IN THE HAMILTONIAN FORMULATION OF GUIDING CENTER MOTION

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Hamiltonian Formulation of Guiding Center Motion

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Abstract

The representation of a magnetic field by the cross-product of the gradients of two scalars has recently seen wide use in plasma physics and in the study of energetic particles in space. The properties of such a representation are reviewed, and as an example of its application, the first-order guiding center motion of a charged particle in a time-independent magnetic field is derived in canonical form.

"From my earliest experiments on the relation of electricity and magnetism, I have had to think and speak of lines of magnetic force as representations of the magnetic power; not merely in the points of quality and direction, but also in quantity"

Michael Faraday, Experimental Researches in Electricity, § 34.

"As I proceeded with the study of Faraday, I perceived that his method of conceiving the phenomena was also a mathematical one, though not exhibited in the conventional form of mathematical symbols. I also found that these methods were capable of being expressed in the ordinary mathematical forms..."

James Clerk Maxwell, Preface to "A Treatise on Electricity and Magnetism", 1873

I. General Properties

Introduction

The concept of magnetic field lines is to a large degree due to Michael Faraday. Faraday's views were often more intuitive than mathematical, and he regarded magnetic field lines (or to use his term, magnetic lines of force) as a useful way of visualizing the magnetic field. Later generations of scientists, more inclined towards mathematical abstraction, replaced Faraday's qualitative description with the more precise terms of vector fields and of vector and scalar potentials, relegating the use of field lines mainly to elementary textbooks.

Recently, however, there has arisen renewed interest in such lines, following investigations of charged particle motion in magnetic fields and of plasma phenomena, effects in which the configuration of field lines is significant. The purpose of this article is to describe a mathematical tool which is appropriate for such applications and to review some of its uses.

Euler Potentials

A general vector field in three dimensions requires 3 scalar functions of position for its description, but the magnetic field $\mathbf{B}$ requires only two, since it satisfies $\nabla \cdot \mathbf{B} = 0$ (in this work, $\mathbf{B}$ rather than $\mathbf{H}$ will be regarded as the field vector). One such representation (which can be shown to be generally possible at least locally) is

$$\mathbf{B} = \nabla\alpha \times \nabla\beta \tag{1}$$

The scalars α and β are termed Euler Potentials. They naturally lead to the vector potential

$$\mathbf{A} = \alpha\nabla\beta$$

satisfying the gauge condition $(\mathbf{A} \cdot \mathbf{B}) = 0$. Clearly, they are far from unique, for an arbitrary function of α can always be added to β, or vice versa. More generally, given the Euler potentials (EP) α and β, one can show by inspection that any pair of functions $u(\alpha, \beta)$ and $v(\alpha, \beta)$ may replace them in Equation (1), provided

$$\partial(u, v)/\partial(\alpha, \beta) = 1 \tag{2}$$

A necessary (though not sufficient) condition for α and β to satisfy eq. (1) is that they be independent solutions of the same linear partial differential equation

$$\left.\begin{aligned} \mathbf{B} \cdot \nabla\alpha &= 0 \\ \mathbf{B} \cdot \nabla\beta &= 0 \end{aligned}\right\}$$

Since $\nabla\alpha$ and $\nabla\beta$ are perpendicular to $\mathbf{B}$, surfaces of constant α and β are tangential to the field at all points, and the same holds for the lines along which such surfaces intersect. Such lines are therefore magnetic field lines.

Suppose that we know the EP representing a given field in some region in space. We then have two families of surfaces

$$\alpha(x, y, z) = \alpha_i$$

$$\beta(x, y, z) = \beta_i$$

Each field line in the region is the intersection of two surfaces, one from each family. It is consequently characterized by two parameters (α_i, β_i), equaling the constant values assumed by α and β along it. In this manner, a formulation of the field in terms of EP affords a direct representation of the field lines, in a way not possible with the ordinary vector potential $\mathbf{A}$.

A Short History

Long before the nature of magnetic fields was understood, mathematicians investigated the velocity field $\mathbf{v}$ of incompressible fluids, which is likewise solenoidal, since the equation of continuity then reduces to

$$\nabla \cdot \mathbf{v} = 0$$

The field lines of $\mathbf{v}$ are usually termed streamlines. Leonhard Euler(1) was the first to introduce into the description of $\mathbf{v}$ "stream functions" F and G which are conserved along streamlines (Figure 1). Because of the difficulty in deriving such functions explicitly, subsequent developments of fluid dynamics showed relatively little interest in the representation of streamlines. An exception was the Stokes stream function[2], derived by Sir George G. Stokes early in his scientific career[3], which gives streamlines in the more tractable case of axisymmetrical flow.

The first application of EP to magnetic fields is due to Sweet[4] and was continued by Dungey[5]. In a work on solar magneto-hydrodynamics, Sweet represented the field as

$$\mathbf{B} = F(\nabla\phi \times \nabla\psi) \tag{4}$$

with F a function of position. The functions ϕ and ψ are evidently conserved along field lines, and must therefore be functions of the EP. Indeed, if one replaces (α, β) in eq. (1) by two functions $\phi(\alpha, \beta)$ and $\psi(\alpha, \beta)$ <u>not</u> bound by eq. (2), one obtains a relation of the above form, with F equaling the jacobian. Pairs of functions such as ϕ and ψ will be termed <u>unmatched</u> EP: the cross product of their gradients is parallel to **B**, but not proportional to it in magnitude.

The use of EP as given in eq. (1) originated in investigations of charged particle motion by Northrop and Teller[6], Grad[7,8], Gardner[9], Ray[10] and others. It has become customary to designate them in that case by α and β; if one of the potentials can be derived in a straightforward manner from considerations of symmetry (e.g. the azimuth angle ϕ in axisymmetrical fields) the symbol β is usually reserved for it.

While there has been considerable uniformity in notation, less of it exists in terminology (and on one occasion, at least(11), the symbol α has been used as a name). Grad has referred to α and β as "Euler Stream Functions" (7) and "Flux Coordinates"(8), while Truesdell, in a work on fluid dynamics(12), has called them "Euler's Potentials". Most authors, by and large, have studiously avoided using any special name.

Mention may also be made here to the term "Monge Potentials", applied to functions (α, β, γ) in the decomposition of an arbitrary vector field $\mathbf{V}$ in the form

$$\mathbf{V} = (\nabla\alpha \times \nabla\beta) + \nabla\gamma \tag{5}$$

which is implied in a 1784 work by Gaspard Monge. By this definition, α and β could be called Monge potentials. However, the application of eq. (5) to divergence-free fields is particularly ambiguous. For instance, the main geomagnetic field created by sources in the earth's interior can be represented either as

$$\mathbf{B} = -\nabla\gamma \tag{6}$$

(the customary representation, with γ expanded in spherical harmonics), or by means of a cross product as in eq. (1), and of course by a large variety of expressions containing terms of both types as well.

In order to avoid such ambiguities, it appears best to use a special term for α and β in divergence-free fields. The term Euler potentials, recently introduced in work on the geomagnetic field, seems to be the most appropriate here, giving credit to the originator of the formulation and at the same time stressing that it provides a representation of the field by means of auxiliary functions, equivalent to that provided by the vector potential **A** or (in source-free regions) by the scalar potential γ.

Properties of Euler Potentials

Given the magnetic field **B** in the region surrounding a point P, it is always possible, in principle, to derive a set of Euler potentials describing it in some vicinity of the point. The proof of this[13] will now be sketched.

The linear homogeneous partial differential equation (3) admits, in the vicinity of P, two independent solutions. Let u and v denote two such solutions (obviously, any well-behaved function f(u, v) is also a solution). Since they are independent, they may be supplemented by a 3rd function w so that (u, v, w) can be used as curvilinear coordinates in the vicinity of P.

Since ∇u and ∇v satisfy (3), their cross product is tangential to **B** and satisfies (4)

$$\mathbf{B} = F(\nabla u \times \nabla v)$$

The divergence-free character of **B** limits the choice of F:

$$\nabla \cdot \mathbf{B} = 0 = (\partial F/\partial w)\, \nabla w \cdot (\nabla u \times \nabla v)$$

$$= (\partial F/\partial w)\ \partial(w, u, v)/\partial(x, y, z)$$

Since the functions are independent, the derivative of F must vanish, so that F is a function of (u, v) alone. One may now form

$$\chi(u, v) = \int_{v_0}^{v} F(u, v')\,dv'$$

with v_0 some arbitrarily chosen constant. Then

$$\nabla \chi = \lambda \nabla u + F(u, v) \nabla v$$

with λ some irrelevant function related to v_0. The function χ constructed in this manner is thus indeed the EP matching u, which completes the proof.

For certain symmetrical fields, Euler potentials are readily obtained. For instance, for a poloidal axisymmetric field (e.g. that of a dipole), the vector potential may be chosen in the azimuthal direction

$$\mathbf{A} = A(r, \theta) \hat{\boldsymbol{\phi}}$$

leading to the choice

$$\left. \begin{aligned} \alpha &= A(r, \theta)\ r \sin \theta \\ \beta &= \phi \end{aligned} \right\} \qquad (7)$$

Similarly, one may represent by Euler potentials pure toroidal fields

$$\mathbf{B} = \nabla \times \mathbf{r} \psi = \nabla \psi \times \nabla (r^2/2)$$

and two-dimensional fields (with neither dependence on cartesian z nor a component in its direction)

$$\mathbf{B} = \nabla \times A \hat{\mathbf{z}} = \nabla A(x, y) \times \nabla z$$

In more general cases, however, the analytic derivation of EP is quite difficult, since they enter eq. (1) in a nonlinear fashion involving products of derivatives. This non-linearity prevents the superposition of solution (and thus the knowledge of EP for $\mathbf{B}_1$ and $\mathbf{B}_2$ does not help one find those of $[\mathbf{B}_1 + \mathbf{B}_2]$), except for the case when the superposed fields all have one Euler potential in common — e.g., when they all belong to one of the symmetric classes previously listed. In fields deviating only slightly from symmetry, perturbation techniques may be employed, and these have been successfully used in connection with the representation of the main geomagnetic field[14].

The non-linearity is a distinct disadvantage compared with the ordinary vector potential $\mathbf{A}$ which may be derived by linear superposition of contributions from the various field sources, thus providing a general "brute force" method of numerical derivation. However, as will be shown in what follows (and may, indeed, be inferred from the existence proof), somewhat similar methods also do exist for Euler potentials.

Suppose that one is given the line pattern of a magnetic field in a certain region, as well as the field $\mathbf{B}$ on some surface σ through which each field line in the region passes exactly once. In that case, we shall now show, there exists a numerical procedure for deriving α and β which, indeed, could be the most straightforward way of obtaining $\mathbf{B}$ from such data.

The problem, incidentally is not without practical interest. Consider the earth's magnetic field: one can observe it with great precision over the earth's surface, while satellite observations[15] provide us with a general field line pattern (influenced by factors that are not too well understood) far away from earth (the satellites also give information about the field's magnitude, but due to the steep intensity fall-off and the appreciable variations observed, such information tends to be less useful). The problem of reconstructing the field on the basis of such information is very similar to the one previously described, except that due allowance must be made for the fact that most field lines cross the earth's surface twice.

Let (λ, μ, ν) be curvilinear coordinates, such that the surface σ is characterized by a constant value ν_0 of ν (for the case of the magnetic field observed outside a spherical earth, ν will be the radial distance r or some function of it). Then surface points may be labeled by two functions of the remaining coordinates, e.g. by

$$u = u(\lambda, \mu)$$

$$v = v(\lambda, \mu)$$

Since the field line pattern is given, one may associate with each point in it two quantities U and V, equal to the values of u and v at the intersection with σ of the field line passing through the point. By this definition, U and V are spatial functions, numerically derivable and having the property

$$U(\lambda, \mu, \nu_0) = u(\lambda, \mu)$$

$$V(\lambda, \mu, \nu_0) = v(\lambda, \mu)$$

Both U and V are conserved along field lines and thus constitute a set of EP, in general unmatched. For some particular choices of (u, v), however, they will be matched, and it will be now shown that such choices can be obtained using only information about the field on σ.

Given an unmatched set (U, V), let $[U, \beta(U, V)]$ be a matched one; then the grid on σ generated by $[u, \beta(u,v)]$ generates matched potentials. On σ, every vector quantity may be resolved into components tangential and normal to the surface, denoted by subscripts $\parallel$ and $\perp$ respectively, with the latter component oriented along $\nabla\nu$. One gets

$$B_\perp = (\partial\beta/\partial V)\left[\nabla U_\parallel \times \nabla V_\parallel\right]_\perp$$

The gradient components involved in the cross product are readily derived from the known functions u and v, since (λ, μ, ν not necessarily orthogonal)

$$\begin{aligned}\nabla U_\parallel &= (\partial U/\partial\lambda)_\sigma \nabla\lambda_\parallel + (\partial U/\partial\mu)_\sigma \nabla\mu_\parallel \\ &= (\partial u/\partial\lambda)\nabla\lambda_\parallel + (\partial u/\partial\mu)\nabla\mu_\parallel\end{aligned}$$

and similarly for V. Since **B** on σ is assumed given, one may derive a function of u and v (to which U, V reduce on σ)

$$f(u, v) = \left(B_\perp / \left[\nabla U_\parallel \times \nabla V_\parallel\right]_\perp\right)_\sigma$$

and obtain β by integration

$$\beta(u, v) = \int^{v} f(u, v')\, dv'$$

This gives β within an arbitrary function of u, but such a function does not contribute to the cross product.

The preceding existence proof and construction methods, unfortunately, are only valid locally, enabling one to derive (α, β) only in some neighborhood of a given point and not necessarily for all space or even for the entire region of interest. When one attempts to extend such construction over larger regions, one often finds that the EP are no longer single valued.

As an example[16], consider the field of a current in a ring filament. This field is axisymmetrical, its α-surfaces are toroidal rings nested inside each other (all of them enclosing the conductor) and its β-surfaces are meridional planes. By adding an infinitely long current filament perpendicular to the ring and passing through its center, the field lines acquire a slant, causing each to spiral around the ring (figure 2).

Let a point P be given in this field, and let Euler potentials (α, β) be derived (numerically or otherwise) in some small region τ surrounding it. This labels each field line with a pair of numbers, and one may extend this labeling to points outside τ by simply following labeled field lines after they have left the region. Ultimately, however, these field lines will have circled the central wire and will have returned into τ, encountering previously labeled points, in general with different values of (α, β). Such points will then have more than one pair of values (α, β) associated with them; the number of such pairs can usually be made arbitrarily large by taking into account field lines that have circled the wire more than once.

Mathematically, this ambiguity can be sidestepped by introducing a cut — a surface of discontinuity at which the labeling is artificially terminated. In the example cited, the plane surface (or any other) bounded by a suitable section of the ring and two radii can serve this purpose, so that each (α, β) line begins on the surface and ends there as well.

In actual practice, this artifice has limited value, for given problems often involve following field lines for long distances[17][18]. For example, particles trapped in a stellarator field (which resembles that of Fig. 2) may, under suitable conditions, execute many circuits around the central wire while following a field line. One may still use a surface of discontinuity to achieve unique characterization; on such a surface, then, to each (α, β) line-segment corresponds some point P. After one circuit, the representative point is usually shifted some distance, so that one can regard each circuit as a mapping of one point on the surface into another, and repeated circuits as iterated mappings. The study of long-term behavior of field lines — at least, in situations where the above-mentioned shift is not too wild — is then transformed into an analysis of iterated mappings[19][20].

As was noted before, Euler potentials are useful in problems where the physical situation involves the field line configuration. In such cases it often turns out that the equations are easier to handle when of the 3 coordinates of position needed to define a point in space, two are chosen as its Euler potentials α and β. The 3rd coordinate is usually chosen as the distance s along a field line, measured from some arbitrary surface.

By its definition, ∇s has a component of magnitude unity along $\mathbf{B}$. However, it is not generally parallel to $\mathbf{B}$: if it were, this would mean

$$\mathbf{B} = \lambda \nabla s$$

$$\mathbf{B} \cdot (\nabla \times \mathbf{B}) = 0$$

a condition not always fulfilled. A transformation of (α, β) will not in general affect s, since that latter quantity depends only on the field line structure, not on its representation by Euler potentials. Similarly, without affecting the EP, s may be transformed by changing the surface from which it is measured. The general form of such a transformation is

$$s \rightarrow s' = s + f(\alpha, \beta)$$

with f an arbitrary function.

With these preliminaries, we now proceed to an example of the application of Euler potentials.

II. Application

Guiding Center Motion

As an example of the application of Euler potentials we shall now carry out the canonical derivation of the nonrelativistic guiding center motion of a charged particle, following a course outlined in general form by Gardner[9].

Consider a particle of change e and mass m in a time-independent magnetic field that is given in terms of Euler potentials, with canonical variables of position x_i and their conjugate cononical momenta Π_i. The Hamiltonian will be

$$H = \left\{ \Pi - (e/c)\alpha\nabla\beta \right\}^2 / 2m \tag{8}$$

We assume that the Euler potentials α and β, as well as the arc length s along field lines are known functions of the x_i. Following Gardner, we introduce a canonical transformation to variables (Q_i, P_i), generated by

$$F(x_i, P_i) = s P_2 + \beta P_1 + \alpha P_3 - (c/e) P_1 P_3 \tag{9}$$

From the transformation equations

$$Q_i = \partial F / \partial P_i \tag{10}$$

One obtains

$$\alpha = (c/e) P_1 + Q_3 \tag{11}$$

$$\beta = Q_1 + (c/e) P_3 \tag{12}$$

$$s = Q_2 \tag{13}$$

while the remaining equations

$$\Pi_i = \partial F / \partial x_i \tag{14}$$

yield, when substituted in (8)

$$H = (1/2m) \left\{ P_2 \nabla s - (e/c)\, Q_3 \nabla \beta + P_3 \nabla \alpha \right\}^2 \tag{15}$$

Homogeneous Field

Assume at first that the field is in the z direction and has a constant intensity B. We may then take

$$\begin{aligned} \alpha &= x \\ \beta &= B\,y \\ s &= z \end{aligned} \tag{16}$$

The Hamiltonian separates into two parts, representing motion parallel and perpendicular to the field

$$H_{\parallel} = (1/2m)\, P_2^2 = E_{\parallel} \tag{17}$$

$$H_{\perp} = (1/2m) \left\{ (eB/c)^2\, Q_3^2 + P_3^2 \right\} = E_{\perp} \tag{18}$$

The parallel part simply gives constant motion along the z axis

$$P_2 = m\, v_{\parallel} = \text{constant} \tag{19}$$

while $H_{\perp}$ resembles the Hamiltonian of the harmonic oscillator and yields

$$Q_3 = (2\, E_{\perp} / m\omega^2)^{1/2} \sin \omega t = R_g \sin \omega t \tag{20}$$

$$P_3 = (2mE_\perp)^{1/2} \cos \omega t \tag{21}$$

where

$$\omega = eB/mc \tag{22}$$

is the well-known gyration frequency and R_g is the particle's gyration radius. The motion is periodic, and the associated action variable is

$$J = \oint P_3 \, dQ_3 = 2\pi E_\perp/\omega \tag{23}$$

Let

$$W = W_0(Q_3, J) - E_\perp t \tag{24}$$

be the solution of the Hamilton-Jacobi equation for H; W_0 generates a transformation to action-angle variables (J, Ω) and satisfies

$$(eB/c)^2 Q_3^2 + (\partial W_0/\partial Q_3)^2 = 2mE_\perp \tag{25}$$

from which (J substituted for $E_\perp$)

$$W_0 = \int \left\{ (JBe/\pi c) - (eB/c)^2 Q_3^2 \right\}^{1/2} dQ_3 \tag{26}$$

Differentiation beneath the integral sign gives

$$\Omega = \partial W_0/\partial J = (2\pi)^{-1} \arcsin Q_3 (\pi eB/Jc)^{1/2} \tag{27}$$

which, compared to (20), identifies $2\pi\Omega$ as the accumulated gyration angle, so that Ω grows linearly in time and increases by 1 each period, as required of a canonical angle variable.

Both P_1 and Q_1 are absent from either of the Hamiltonians, showing that they are constant. By equations (11) and (12), they may be identified with $(e/c)\,\alpha$ and β of the field line around which the particle spirals. In what follows, the point

$$(\alpha, \beta, s) = \Big((c/e)P_1, Q_1, Q_2\Big)$$

around which the particle instantaneously revolves will be termed its guiding center and quantities associated with it will be distinguished by subscript zero.

Inhomogeneous Field

In an inhomogeneous field, one can define ω and R_g at every point by using the local value of B. Suppose next that the ratio of R_g to the scale L on which the field varies — a ratio which will be loosely designated by ϵ — is everywhere much less than unity; this occurs when either the extent of the inhomogeneity or the perpendicular energy $E_\perp$ is small. Then to lowest degree of approximation, the particle on its local scale senses a homogeneous field, and its motion is the same as derived in the preceding section, with its canonical variables and guiding center defined as before.

To obtain a better approximation, one has to take into account the variation of field quantities over the course of a gyration, and this is usually done by expanding them around the guiding center. If β_g is the value of β at the guiding center, the expansion of a function $f(\beta)$ would be, by eq. 11

$$f(\beta) = f(\beta_g) + (c/e)\,P_3(\partial f/\partial \beta)_g + \ldots \qquad (28)$$

Here P_3 is of order R_g while $f/(\partial f/\partial \beta)$ is of order L, so that the ratio of the terms is $O(\epsilon)$ and, under the assumption made, the series provides a good approximation. The approximation is even better (by one order of ϵ) when we consider quantities averaged over one gyration, since in homogeneous fields such averages vanish and in slightly inhomogeneous ones it seems logical (and may be shown) that they are no larger than $O(\epsilon)$. The gyration-averaged ratio of the terms in (28) is thus $O(\epsilon^2)$.

By the same reasoning, since in a homogeneous field α and β of the guiding center are conserved, in a slightly inhomogeneous field their deviation from their initial values (α_0, β_0) will be $O(\epsilon)$, as long as the time scale is not too large. All this will be assumed a-priori in the following calculation but will also be borne out by the final results.

To extend the canonical formalism to slightly inhomogeneous fields, one replaces the transformation generator of eq. (9) by

$$F(x_i, P_i) = s_0 P_2 + \beta P_1 + \alpha P_3 - \tfrac{1}{2}\lambda P_3^2 - (c/e) P_1 P_3 \tag{29}$$

where s is replaced by

$$s_0 = s - (\alpha - \alpha_0)\, a - (\beta - \beta_0)\, b \tag{30}$$

and

$$\begin{aligned} a &= B^{-2}\left\{(\nabla\beta)^2(\nabla s \cdot \nabla\alpha) - (\nabla\alpha \cdot \nabla\beta)(\nabla s \cdot \nabla\beta)\right\} \\ b & B^{-2}\left\{(\nabla\alpha)^2(\nabla s \cdot \nabla\beta) - (\nabla\alpha \cdot \nabla\beta)(\nabla s \cdot \nabla\alpha)\right\} \\ \lambda &= (c/e)(\nabla\alpha \cdot \nabla\beta)/(\nabla\beta)^2 \end{aligned} \tag{31}$$

Retracing the transformation equations

$$\alpha = (c/e)P_1 + Q_3 + \lambda P_3 \tag{32}$$

$$\beta = Q_1 + (c/e)P_3 \tag{33}$$

$$s_0 = Q_2 \tag{34}$$

and

$$H = (1/2m)\left\{ P_2 \nabla s_0 - (e/c)(Q_3 + \lambda P_3)\nabla\beta + P_3 \nabla\alpha - \tfrac{1}{2} P_3^2 \nabla\lambda \right\}^2 \tag{35}$$

Let H_0 denote that part of the Hamiltonian that is contributed by the scalar products $(\nabla s_0)^2$, $(\nabla\alpha)^2$, $(\nabla\beta)^2$ and $(\nabla\alpha \cdot \nabla\beta)$. Using

$$B^2 = (\nabla\alpha)^2(\nabla\beta)^2 - (\nabla\alpha \cdot \nabla\beta)^2 \tag{36}$$

one finds

$$H_0 = (1/2m)\left\{ P_2^2(\nabla s_0)^2 + (e/c)^2(\nabla\beta)^2 Q_3^2 + P_3^2 B^2/(\nabla\beta)^2 \right\} \tag{37}$$

The remaining terms in H can be classified by the scalar products which they contain, of which there exist 6. It will now be shown that these terms are all of a lesser order of magnitude than those contained in H_0.

Consider first the terms involving $(\nabla s_0 \cdot \nabla\alpha)$ and $(\nabla s_0 \cdot \nabla\beta)$. If $\hat{\mathbf{B}}$ is the unit vector in the field's direction, one finds

$$\nabla s_0 = \hat{\mathbf{B}} - (\alpha - \alpha_0)\nabla a - (\beta - \beta_0)\nabla b$$

$$= \hat{\mathbf{B}} + O(\epsilon) \tag{38}$$

Thus these terms are $O(\epsilon)$ — and, since they turn out to have factors of Q_3 or P_3, their averages are $O(\epsilon^2)$. The reason for replacing s with s_0 now becomes evident: in a slightly inhomogeneous field, the term $P_2 \nabla s$ in (15) usually does not represent the momentum component parallel to $\mathbf{B}$, whereas the term $P_2 \nabla s_0$ in (35) does, to the lowest order of approximation.

The other terms all involve $\nabla\lambda$, which is of order L^{-1}, and they therefore belong to the same order, except for the term involving $(\nabla\lambda)^2$, which is $O(\epsilon^2)$ even without any averaging. To this collection of $O(\epsilon)$ terms with $O(\epsilon^2)$ averages we now add several others by expanding the functions of position contained in H_0 around the guiding center in the general manner of eq. (28), using the relations (32) and (33). Only the lowest order terms of this expansion are retained in H_0.

If we wish to solve the motion in Q_3 and P_3 to order ϵ — for instance, to derive the first-order correction to the magnetic moment (see later) — the higher order terms have to be retained and no averaging is permitted. Such a calculation, obviously, is quite lengthy. Here, instead, we shall only investigate the average first-order motion of the guiding center coordinates, in which case H_0 is all that is needed. We begin with a transformation to new canonical variables

$$(J, \Omega, p_1, q_1, p_2, q_2)$$

with a generator that is a cross between the one of eq. (26) and that of the identity transformation, namely

$$W(\mathbf{p}, \mathbf{Q}) = e(\nabla\beta)^2/cB \int \left\{ \left(JBc/\pi e(\nabla\beta)^2 \right) - Q_3^2 \right\}^{1/2} dQ_3 + p_1 Q_1 + p_2 Q_2 \quad (39)$$

where B and $(\nabla\beta)^2$ are defined for mixed guiding center variables

$$\alpha = c p_1 / e$$

$$\beta = Q_1$$

$$s_0 = Q_2$$

Applying

$$P_3 = \partial W / \partial Q_3$$

one finds that the terms in H_0 involving P_3 and Q_3 condense to

$$JBe/\pi c = \mu B \quad (40)$$

The expression obtained for Ω resembles that of (27), except that B there is replaced by $(\nabla\beta)^2 / B$.

To obtain the rest of the variables, one first has to derive the integral I_1 in (39). One gets two terms — one containing a factor Q_3, the other proportional to Ω. Fortunately, in the coefficient multiplying the latter term, the dependence on position cancels out, so that when one derives quantities such as

$$p_1 = \partial W / \partial Q_1$$

I_1 contributes no terms proportional to Ω, only such ones that resemble in form the second part of (28), i.e. of order ϵ with $O(\epsilon^2)$ averages (a possible vanishing denominator also cancels). Apart from such terms, which will be neglected, the remaining new canonical variables are the same as the corresponding old ones. We get, to our order of approximation

$$H = (1/2m)\left\{ p_2^2(\nabla s_0)^2 + \mu B \right\} \tag{41}$$

Since Ω appears only in $O(\epsilon)$ terms, μ is constant to the lowest order: this is the well-known adiabatic invariance of the magnetic moment to its lowest order. If $p_\perp$ is the perpendicular component of the momentum, then

$$\mu = p_\perp^2/B$$

We now use (41) to derive the first-order variation of q_1 and p_1, the so-called guiding center drifts. The velocity component normal to **B** is defined

$$\mathbf{v}_\perp = B^{-2}\Big(\mathbf{B} \times (\mathbf{v} \times \mathbf{B})\Big)$$

Using the identification of $(cp_1/e,\ q_1)$ as (α, β) of the guiding center, Hamilton's equations give

$$(\mathbf{v} \times \mathbf{B}) = \nabla\alpha(\mathbf{v} \cdot \nabla\beta) - \nabla\beta(\mathbf{v} \cdot \nabla\alpha)$$

$$= \dot{\beta}\,\nabla\alpha - \dot{\alpha}\,\nabla\beta$$

$$(c/e)\,(\mathbf{v} \times \mathbf{B}) = \nabla\alpha(\partial H/\partial \alpha) + \nabla\beta(\partial H/\partial \beta)$$

$$= \nabla H - \nabla s_0(\partial H/\partial s_0) \tag{42}$$

with (38), this gives to lowest order

$$\mathbf{v}_\perp = (c/eB^2)\,(\mathbf{B} \times \nabla H) \tag{43}$$

Referring to (41), we find that $\mathbf{v}_\perp$ consists of two parts. The second term in H contributes the so-called gradient drift, proportional to ∇B

$$\mathbf{v}_g = (\mu c/2meB^2)\ (\mathbf{B} \times \nabla B) \tag{44}$$

The first term contributes the curvature drift, proportional to the curvature of field lines

$$\mathbf{v}_c = (c/meB^2)\ p_2^2 \left(\mathbf{B} \times (\nabla s_0 \cdot \nabla\nabla s_0)\right)$$

From (38), to lowest orders

$$\begin{aligned}\nabla s_0 \cdot \nabla\nabla s_0 &= \hat{\mathbf{B}} \cdot \left(\nabla\hat{\mathbf{B}} - (\nabla\alpha\nabla a) - (\nabla\beta\nabla b) + O(\epsilon^2)\right) \\ &= \hat{\mathbf{B}} \cdot \nabla\hat{\mathbf{B}} + O(\epsilon^2)\end{aligned} \tag{45}$$

If $E_\parallel = E - \mu B$ denotes the energy associated with motion parallel to the field lines, this leads to

$$\mathbf{v}_c = (2cE_\parallel/eB)\left(\hat{\mathbf{B}} \times (\hat{\mathbf{B}} \cdot \nabla\hat{\mathbf{B}})\right) \tag{46}$$

The drifts thus derived are local and vary from point to point; however, making use of the constancy of μ and E, they are easily derived if the initial conditions and structure of the field are given. On time scales of $O(\epsilon^{-1})$ gyration times, the accumulated drift will be of order R_g, justifying eq. (38) for such time scales.

The 2nd Adiabatic Invariant J_2

If the dependence of $\mathbf{B}$ on the coordinates is known, one may deduce its dependence on s_0 and use this to integrate the motion along field lines. In particular, if this motion turns out to be periodic, it will have an action variable

$$J_2 = \oint p_2 \, dq_2 \tag{47}$$

which may be shown from general principles[21] to be adiabatically invariant under slow perturbations, e.g. those arising from the drift motion. This may also be shown directly, as will be done in the following calculation, which resembles one by Northrop[22] but is shorter due to the canonical formulation.

We begin by associating the particle with guiding center at

$$(\alpha, \beta, s_0) = (cp_1/e, q_1, q_2)$$

with an "instantaneous adiabatic invariant"

$$\begin{aligned} J_2'(E, \mu, p_1, q_1) &= \oint \left\{ \sqrt{2mE - \mu B} \,/\, |\nabla s_0| \right\} dq_2 \\ &= \oint \left\{ P_{\parallel} \,/\, |\nabla s_0| \right\} dq_2 \end{aligned} \tag{48}$$

where E is the energy integral and $P_{\parallel}$ is defined by the last relation. Qualitatively, J_2' may be viewed as the value J_2 would assume if the particle ceased drifting.

Actually, J_2' will slowly change, due to its dependence on p_1 and q_1 which vary in the course of the drift motion. The purpose of this calculation is to show that, to the lowest order of expansion, this variation is periodic and averages zero.

By equation (38)

$$\begin{aligned}(\nabla s_0)^2 &= 1 - 2(\alpha - \alpha_0)(\nabla a \cdot \nabla s_0) - 2(\beta - \beta_0)(\nabla b \cdot \nabla s_0) \\ &= 1 - 2(cp_1/e - \alpha_0)(\partial a/\partial s_0) - 2(q_1 - \beta_0)(\partial b/\partial s_0) \\ &= 1 - 2u \end{aligned} \tag{49}$$

To the same order

$$J_2' = \oint P_\parallel (1 + u + (3/2)u^2)\, dq_2 \tag{50}$$

Hamilton's equations give

$$dJ_2'/dt = \dot{q}_1(\partial J_2'/\partial q_1) + \dot{p}_1(\partial J_2'/\partial p_1) \tag{51}$$

$$= (\partial H/\partial p_1)(\partial J_2'/\partial q_1) - (\partial H/\partial q_1)(\partial J_2'/\partial p_1)$$

Explicitly

$$2m(\partial H/\partial q_1) = \mu(\partial B/\partial q_1) - 2p_2^2(\partial u/\partial q_1) \tag{52}$$

with an analogous expression for $(\partial H/\partial p_1)$ and similar ones for primed quantities. Since $\dot{q}_1$ and $\dot{p}_1$ are associated with the drift motion, they are $O(\epsilon)$ and so are the terms of eq. (52).

To express dJ_2'/dt, we have to differentiate the integrand of (50), discarding higher order terms. In this integrand, if $P_\parallel$ is $O(1)$, then u is $O(\epsilon)$ (see eq. 38);

however, taking $\partial P_{\parallel}/\partial q_1$ as $O(\epsilon)$ (it contains a term of eq. 52), we find that $\partial u/\partial q_1$ is also of the same order, due to the factor q_1 in eq. (49). Thus, keeping the two lowest orders

$$\begin{aligned}
\partial J_2'/\partial q_1 &= \oint \left\{ -\mu \partial B/\partial q_1 (1+u) + 2P_{\parallel}^2 (\partial u/\partial q_1 + 3u \partial u/\partial q_1) \right\} /2P_{\parallel}\, dq_2 \\
&= \oint \left\{ -\mu \partial B/\partial q_1 (1+u) + 2p_2^2 (1-2u)(\partial u/\partial q_1 + 3u \partial u/\partial q_1) \right\} /2P_{\parallel}\, dq_2 \\
&= -m \oint \left\{ (1+u)(\partial H/\partial q_1) + O(\epsilon^3) \right\} /P_{\parallel}\, dq_2 \qquad (53)
\end{aligned}$$

By Hamilton's equations

$$\dot{q}_2 = \partial H/\partial p_2 = p_2(1-2u)/m$$

$$dq_2 = P_{\parallel} dt/(1+u)m \qquad (54)$$

If τ is the period of the motion in q_2, one may write

$$\partial J_2'/\partial q_1 = -\int_0^{\tau} (\partial H/\partial q_1)\, dt$$

where among the arguments of H, q_2 and p_2 vary with t but q_1 and p_1 are held fixed. Substituting this in (51) and using Hamilton's equations

$$dJ_2'/dt' = \int_0^{\tau} \left\{ (\partial H'/\partial q_1)(\partial H/\partial p_1) - (\partial H'/\partial p_1)(\partial H/\partial q_1) \right\} dt \qquad (55)$$

where q_2 and p_2 that appear in H are functions of t and participate in the integration, while those in H' relate to the particle's parameters at time t' at which $\partial J_2/dt$ is evaluated and do not participate. The expression, in general, does not vanish; however, let us calculate the total change $\Delta J_2'$ of J_2' in one period

$$\Delta J_2' = \int_0^\tau (dJ_2'/dt')\, dt'$$

$$= \int_0^\tau \int_0^\tau (\partial H'/\partial q_1)(\partial H/\partial p_1) - (\partial H'/\partial p_1)(\partial H/\partial q_1)\, dt\, dt' \qquad (56)$$

If H were the same function of t as H' of t', this vanishes by symmetry, since an interchange of the dummy variables then reverses the sign. This, however, is not strictly true, for two reasons. First, the arguments $q_2(t)$ and $p_2(t)$ of H follow the motion of a particle the drift of which has been "turned off", whereas the corresponding variables in H' — call them $q_2'(t')$ and $p_2'(t')$ — follow the actual trajectory. Secondly, both H and H' depend on q_1 and p_1 of the drifting particle, which in turn depend on t' alone.

However, due to the slowness of the drift, the integrand may be expanded in the small differences between (q_1, p_1) and their initial values (q_{10}, p_{10}) at $t = 0$, and between (q_2, p_2) and (q_2', p_2'). The leading term of this expansion is completely symmetric, so that to lowest order ΔJ_2 vanishes.

Let us evaluate what this means. If E is the energy, by (48) and (55)

$$J_2' = O(E\tau) \qquad (57)$$

The integrand of (56) is proportional to the energy involved in the drift motion and thus $O(\epsilon^2 E)$; however, since its lowest order does not contribute,

$$\Delta J_2' = O(\epsilon^3 E \tau^2)$$

Combining, and assuming τ is of order ϵ^{-1}

$$\Delta \log J_2' = O(\epsilon^2) \tag{58}$$

Thus in the course of ϵ^{-1} periods, one would expect J_2' to vary by no more than $O(\epsilon)$ of its value. The integral J_2 of (47) closely approximates J_2' and its long-term behavior is the same.

Motion in the Earth's Magnetic Field

An example of the preceding is afforded by radiation-belt particles moving in the earth's magnetic field, which may be viewed as a dipole field with external and internal distortions added. A typical field line starts from one hemisphere and ends in the other; it has a high field intensity B near its ends and a minimum of B somewhere near the point at which it is farthest away from earth.

A particle trapped on such a line and conserving μ will bounce back and forth between "mirror points" at which

$$B = 2mE/\mu = B_m$$

slowly drifting azimuthally at the same time and conserving J_2 while doing so (Figure 3).

The bounce motion is evidently periodic in s, if s is defined to be measured from the mirror surface $B(x,y,z) = B_m$, but it is not immediately evident that it can also be regarded as periodic in s_0, as required by the preceding formalism. This, however, may be ensured by a proper choice of s. By substitution in equations (21), it may be shown that the general transformation

$$s \to s' = s + f(\alpha, \beta)$$

leads to

$$s_0 \to s_0' = s_0 + f(\alpha, \beta) + (\alpha - \alpha_0)(\partial f/\partial \alpha)$$
$$+ (\beta - \beta_0)(\partial f/\partial \beta)$$

Given some function $g(\alpha,\beta)$, one can in principle find a solution f for

$$(\alpha - \alpha_0)(\partial f/\partial \alpha) + (\beta - \beta_0)(\partial f/\partial \beta) + f(\alpha, \beta) + g(\alpha, \beta) = 0$$

which means that for a prescribed transformation of s_0

$$s_0 \to s_0' = s_0 - g(\alpha, \beta) \tag{59}$$

there usually exists some transformation of s that leads to it. Now suppose that s is given as vanishing on one mirror surface. Every point on that surface can be labeled by a pair of values (α,β), and there will in general exist a function $g(\alpha,\beta)$ giving the values assumed by s_0 on that surface at each point. If we now derive and apply that transformation of s which makes s_0 transform as in eq. (59), we will have arrived at an (α,β, s_0) system in which s_0 vanishes on the mirror surface. The motion is then periodic in q_2 and all preceding arguments apply.

In the geomagnetic field, the drift motion may carry the particle all the way around the earth — as in the case of the dipole field — or else the particle may drift off and eventually become "untrapped" before completing its full circuit[23].

In the former case, the motion will have a third periodicity and an associated adiabatic invariant, the so-called Flux Invariant $\Phi^{(6)}$. In the present case there exists no need for Φ, since μ, J_2 and the energy integral E are sufficient to describe the motion; it comes into its own in the case of time dependent magnetic fields, which will generally have an associated electric field as well. It is possible to extend the Hamiltonian formalism to include such fields, but this is beyond the scope of the present work.

Another application, which will be only briefly mentioned here, involves the Liouville equation corresponding to the guiding center Hamiltonian. Writing down this equation leads almost immediately to the gyration averaged version of the equation describing the behavior of a collisionless plasma, the so-called CGL (for Chew, Goldberger and Low) approximation to the Vlasov equation.

Conclusion

The preceding review of the properties and applications of Euler potentials is in no sense complete. The example worked out, involving the canonical description of guiding center motion, will provide those persistent enough to follow it with some practice in their application, but it is only one of a rapidly growing number of applications of Euler potentials. Because of this increasing importance, it appears that the time has come to include at least a brief introduction to Euler potentials in the standard course on Classical Electrodynamics. Hopefully, this article will provide the material and the background for such an introduction.

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SCHOLION 2

49. Ex solutione problematis, dum per gradus ad aequationem propositam sumus progressi, casum, quo fluidi densitas q est constans et prima aequatio ita se habet

$$\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0,$$

resolvere poterimus, quod eo magis est notandum, quod huius solutionem ex generali, quam dedimus, derivare non licet. Quamquam autem hic tres tantum variabiles x, y et z considerantur, tamen nihil impedit, quominus in solutione ibi data etiam quartum t introducamus, eam quasi constantem spectando. Sumtis ergo pro lubitu duabus functionibus F, G quatuor variabilium x, y, z et t, ex iis ternae celeritates u, v et w ita determinabuntur, ut sit

$$u=\left(\frac{dF}{dy}\right)\left(\frac{dG}{dz}\right)-\left(\frac{dF}{dz}\right)\left(\frac{dG}{dy}\right)+\Gamma:(y,z,t)$$

$$v=\left(\frac{dF}{dz}\right)\left(\frac{dG}{dx}\right)-\left(\frac{dF}{dx}\right)\left(\frac{dG}{dz}\right)+\Delta:(x,z,t)$$

$$w=\left(\frac{dF}{dx}\right)\left(\frac{dG}{dy}\right)-\left(\frac{dF}{dy}\right)\left(\frac{dG}{dx}\right)+\Sigma:(x,y,t),$$

ubi totum momentum iterum in eo est situm, quod cuivis membro cuiusque formae in reliquis respondeat aliquod, quod cum eo datum factorem habeat communem et signo contrario sit affectum. Totus autem hic casus, quo densitas fluidi est quantitas constans, meretur, ut seorsim diligentius evolvatur,

Figure 1

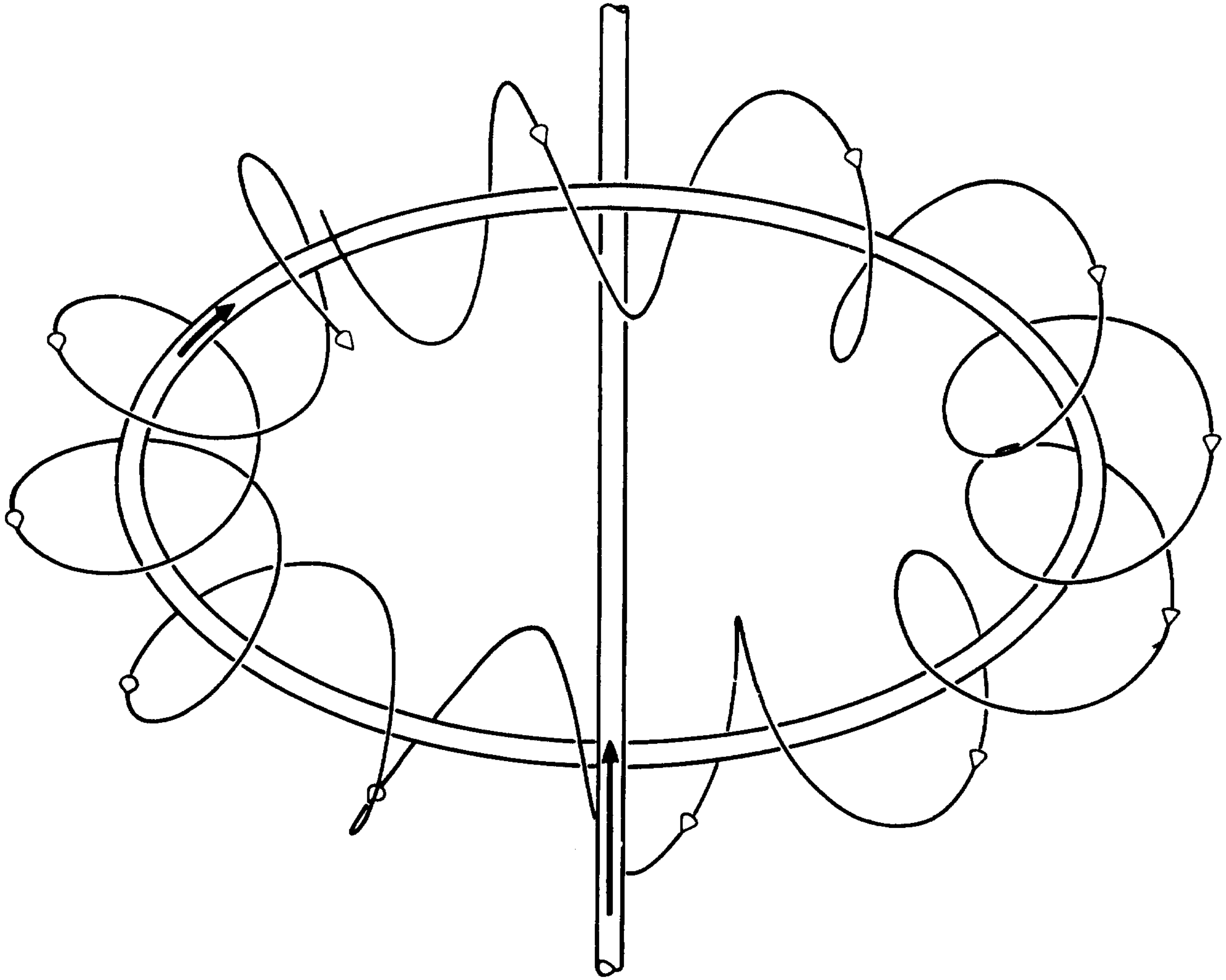

Figure 2

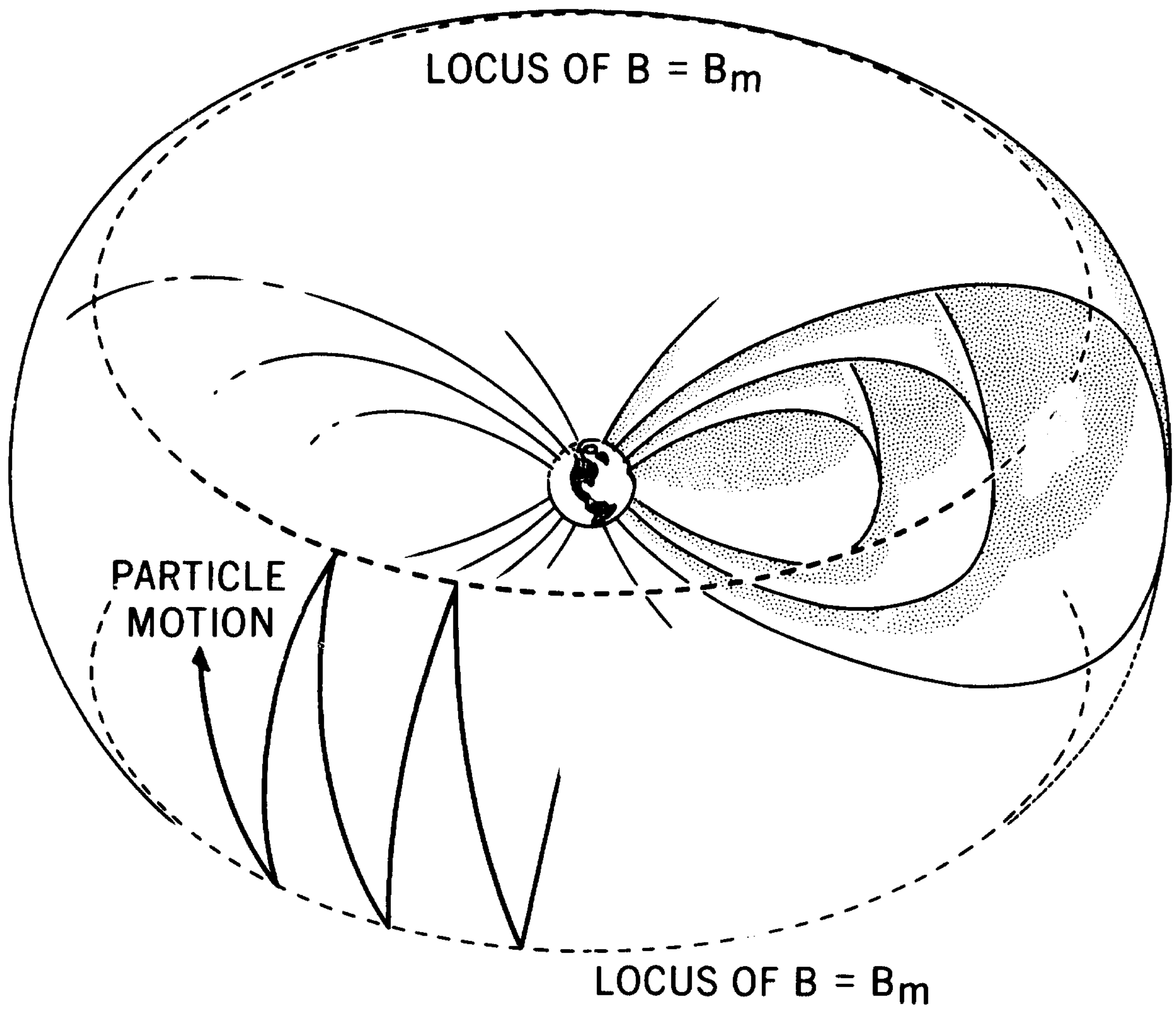

Figure 3