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VARIABILITY OF SOUND PROPAGATION PREDICTION DUE TO ATMOSPHERIC VARIABILITY

by C. Eugene Buell

Prepared by
KAMAN SCIENCES CORPORATION
Colorado Springs, Colo.
for George C. Marshall Space Flight Center

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SUMMARY

This report is divided into two parts. The first covers topics related to estimating sound intensity from a stationary source in the natural atmosphere; the second covers some of the problems of estimating sound intensity from a moving sound source. In both cases the far field of sound propagation is of primary interest (1. to 40. km).

The major results in the first part deal with (a) the estimation of intensity by calculating a single ray and associated parameters and (b) the method of locating caustics above the ground. These two problems are closely related.

To estimate sound intensity at a point the rate of change of distance to the returning ray with respect to initial inclination angle is required. This may be obtained numerically from the distance of the returning ray when they have been calculated for several rays. It seems desirable to differentiate the integrals for distance with respect to inclination formally and compute these derivatives as part of the ray tracing procedure. The formal integration, however, leads to divergent integrals. It is found that if these formally divergent integrals are integrated by parts, the resulting integrals, though improper, are convergent.

The location of caustics for simple cases by algebraic manipulation of the equations is relatively easy, but the numerical processes required in an actual case lead to some computing problems. These problems arise because for an efficient ray tracing procedure integration over the largest convenient layers is necessary. The caustics aloft tend to appear near the reflection level (between the reflection level and the next lower data level). Use is made of the nearly improper character of the ray tracing integrals to obtain the height of the caustics.

The second part on estimating for field sound intensity from a moving source contains a discussion of several items to be considered that are usually omitted in the stationary source situation. Most of these items can be included with no problems. However, the calculation of the focusing factor for a moving sound source presents grave problems (and is a prime necessity for intensity estimates). In the first place, it is necessary to calculate not only the horizontal travel distance, but also the time of travel. If only this is done, the intensity calculation requires extensive storage of results and then interpolation (to put the ray picture in terms of uniform arrival time) and numerical differentiation (to get the focusing factor). The appropriate quantities for focus factor calculations may be obtained in convenient form from kinematical considerations of the propagation process.

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INTRODUCTION

The first part of this report deals with certain problems involving ray tracing methods from an essentially stationary sound source, though the results can be applied with suitable modification to the estimation of sound intensity from a moving source also. The principal results deal with the calculation of sound intensity along a ray and the calculation of the location of caustics (foci) aloft.

To calculate the sound intensity or focusing factor, it is necessary to have available the rate of change of horizontal travel with respect to the initial inclination angle of the ray. If intensity at the ground is the only item of interest, rays for several inclination angles may be computed to ground return and the resulting intensity may be obtained by numerical differentiation. When this is done, the effect of caustic surface aloft is neglected. If intensity can be calculated at various points along the sound ray, then intensity at the ground may be determined with no further effort from a single ray path (one initial inclination angle) and also the presence and locations of caustics aloft may be computed.

The integrals to obtain horizontal travel may be differentiated explicitly with respect to the initial inclination angle and this derivative determined for points along the ray. The initial inclination angle occurs only in the integrand of the ray integral. These integrals are improper but convergent at every reflection layer to start with, but after formal differentiation the results are improper but divergent at the reflection level. Formal integration by parts results in integrals that are again improper, but convergent. This change of form of the basic equations permits intensity evaluation along the ray as the ray tracing is carried out step by step.

The transformed ray tracing integrals for these derivatives clearly shows one of the reasons why the linear layer model of the atmosphere is so unsatisfactory. The technique of integrating by parts reduces the order of the discontinuity of the integrand at the reflection layer but introduces into the integrand the second derivative of the wind and speed of sound with respect to height. This is exactly the quantity handled in so cavalier a manner in the linear layer model of the atmosphere.

The location of caustics aloft presents some problems. These are expected to lie in an altitude range between Z_{n-1} and the reflection level. The ray travels an unusually long distance in this layer. Their location is obtained by using an approximate expression for the integral which gives an explicit form for the altitude of the caustic on the ray concerned. The horizontal distance to where this ray is tangent to the caustic may be obtained by interpolation, but is much more accurately found by explicit ray path integration using the caustic altitude as the upper limit of the ray integrals.

The estimation of sound intensity from a moving source presents many problems that can be omitted from consideration for a stationary source. When the moving source is the rocket jet, the source is highly directive, a factor that cannot be ignored. This means not only that directivity must be considered, but also the details of the sound ray azimuth and elevation to the pitch and yaw of the rocket engine. Doppler frequency shift must be taken into account. The coupling of the sound and receiver source with the atmosphere cannot be ignored because of the large altitude separations. These considerations merely require adjustment of the stationary source results. The biggest problem is that of adequately evaluating the focusing factor itself.

In the stationary source case, differences in arrival time from place to place can be ignored. They must be very carefully included where the source is in motion. Thus time of sound travel becomes a necessary calculation. It is possible to store all of the time and distance calculations for all source levels in the computer, then reduce the results to a fixed time of arrival at the ground by inverse interpolation methods, and finally carry out the required numerical differentiation to obtain a valid focus factor. The demanding storage of this method may be eliminated by consideration of the sound propagation kinematics. The result is that if certain derivatives of distance and time with respect to ray inclination and source altitude are calculated in the ray tracing steps, these are then combined with the components of the source velocity vector to yield a focusing factor for each ray. The computation steps are not as complex as it would seem because of some interrelations that exist between the required quantities. This shortens significantly the steps required.

A. MISCELLANEOUS RAY TRACING PROBLEMS

The items of this section comprise several topics of importance for application of the ray tracing method.

The first section is devoted to a few words on the basic difference between using a linear layer model of the atmosphere and a model with smoothly varying tangent. The burden of the discussion is not that the parabolic model that we have used introduces additional "interpolated" points, but that it is "smooth", a fact that shows up only in the form of the integrals evaluated. We have had an impression that this distinction has been misunderstood.

The calculation of the derivatives of the ray travel distance with respect to initial inclination angle makes it possible to calculate intensity from a single "ray". Otherwise a collection of rays is required. The routine differentiation of the expressions involved leads to integrals that cannot be evaluated. We have found a method that can be applied easily without encountering this difficulty. It requires integration by parts and consequent evaluation of the derivatives of wind and speed of sound with respect to height. Second derivatives of these quantities are required. These have infinite discontinuities at data points and further emphasize the fact that the linear layer model for the atmosphere is most inappropriate. This latter point is illustrated in the third section separately.

The location of caustic surfaces aloft is reasonably simple for elementary models that can be treated explicitly. This is not the case for actual ray-tracing practice in the atmosphere. The fourth section discusses these problems. The major problem is in locating the ray envelope when data levels are well spaced apart. A method of location is developed that makes use of the infinite integrals of the preceding paragraph.

The final section is devoted to a discussion of the evaluation of the focusing factor at a focus for a particular case.

1. A Philosophy of the Ray Tracing Technique

In any method of ray tracing, the problem is to obtain a numerical solution to the differential equations for the sound rays. There are many methods that can be used. When interest basically lies in the sound intensity on the ground, interest is confined to estimating the "size" of the ray tube where it reaches the ground, that is in the focusing factor

$$f = R^2 \cos \varphi_o / r(dr/d\varphi_o) \sin \varphi_p$$

where R = slant distance from source to receiver,

r = horizontal distance from source to receiver

φ_o = initial ray tangent of the ray (at the source)

φ_p = ray tangent of the ray at the receiver.

For practical considerations, there is little or no interest in how the ray gets from source to receiver as long as the focusing factor can be calculated.

Unfortunately, for most practical solutions of the problem, the ray path is calculated in arriving at the focusing factor. The reason for this lies in the fact that it is reasonably straightforward to treat the differential equations for the rays as an initial value problem.

In carrying out the solution of the initial value problem for the rays the various techniques reduce to the approximate integration through layers of the atmosphere which are assumed to be homogeneous in the horizontal, but changing in the vertical in a known way. The ray equations are such that, if the proper assumptions are made concerning the variations of wind and speed of sound through the layer the quadratures involved are exact. This leads to the possibility of using rather thick layers and consequently to a considerable reduction in the numerical work of calculating a ray path. The above is strictly true only in a stationary atmosphere. For a windy atmosphere it is only approximately true. The wind component along the ray plane and the speed of sound enter the integrals in somewhat different ways so that exact integration is scarcely possible or leads to exceedingly complex formal integration expressions. However, it is possible to separate the integrand into two factors, one of which varies much more slowly than the other. This technique permits reasonably accurate integration over large layers, in which the slowly varying part is assumed constant and the exact quadrature for the more rapidly varying part may be expressed rather simply.

The other alternative is to divide the atmosphere into very thin layers and to use reasonably crude methods of integrating over many such layers. In this instance, the exact character of the ray path is of little or no importance.

We have pointed out that the linear layer model of the atmosphere is inconsistent with the physical assumptions behind the ray tracing technique and that the results obtained from a linear layer model show certain peculiarities of the rays, such as bifurcated rays at a maximum of the speed of sound profile, that are inherent in the violation of these physical assumptions. In order to overcome this difficulty, the parabolic model was introduced. The use of a parabolic model has the great advantage that the basic assumptions of the ray tracing technique are much more realistically satisfied. In particular, the profile of speed of sound as a function of altitude does not have the inherent infinite point discontinuities of the second derivatives at the data points, a characteristic that cannot be avoided in the linear layer model.

The parabolic model has the virtues; (a) it is simply constructed in a unique way, (b) it leads to easy separation of the ray tracing integrand into a slowly changing factor that can be treated as approximately constant and a factor that changes rapidly but which may be integrated exactly, and finally (c) it is easily controlled to represent the structure of speed of sound (plus wind component) as a function of altitude.

Any other smoothly varying function of speed of sound (plus wind component) versus altitude would be satisfactory as far as the physical assumptions of the ray tracing method are concerned. Some are reviewed briefly with the reasons for their rejection.

(a) Standard interpolation formulas, such as LaGrange interpolation either become too complex to manipulate handily when the whole sounding is considered as a unit or, if applied in a piecewise manner, do not preserve the continuity of slope at the data points. In the first instance, some 25 or more data points require a polynomial of 24'th degree. This is not only silly in itself, but such a polynomial is also not guaranteed to reasonably represent the sounding in the sense that cases may occur where the interpolated values between data points would fluctuate wildly.

(b) Hermite interpolation may be used. This requires that a slope be assigned at each data point. This can be done handily by assigning as slope the secant across adjacent data values, $(dc/dz)_n \cong (c_{n+1} - c_{n-1})/(z_{n+1} - z_{n-1})$. The curve for the range from z_n to z_{n+1} would pass through points c_n and c_{n+1} with the assigned slopes $(dc/dz)_n$ and $(dc/dz)_{n+1}$. In this simplest case, the curve is cubic. (The wind component is omitted in the above expressions for the sake of simplicity.) The integral for ray tracing, when separated into its slowly varying factor and more rapidly varying factor, now becomes difficult to handle. The factor to be integrated involves the square root of a cubic expression. This can be integrated in terms of elliptic integrals but to do so a cubic of the most general form must be dealt with. The isolation and evaluation of all of the real roots is a first step. Then the location of the end point of the integration interval with respect to all of the roots is required. Each of the combined cases of root and end point locations leads to separate expressions in terms of ellipitc integrals. Finally these elliptic integrals are to be evaluated. The subroutines are not generally available in software so that evaluation by subroutines are required.

(c) Spline interpolation has the same disadvantages as Hermite interpolation or worse. There are handy spline methods, but the simplest is the cubic spline and we have seen that the cubic polynomial leads to problems.

Note in the above that Hermite and spline interpolation are objectionable only when it comes to evaluating the resulting integrals. Both methods serve admirably to obtain interpolated points from the sounding. But interpolation is not the problem. The problem is the exact integration of the radical that contains the interpolation function.

In order to obtain a simple quadratic interpolation function that would go through the data points with given slope, special additional points were interpolated at the halfway levels, $(z_n + z_{n+1})/2$. The values assigned at these points were fixed by the requirement that the jump of the second derivative at these points be a minimum while the parabolic arcs join smoothly (same first derivative). The use of these additional points does not lie in any requirement for more closely spaced information on wind or speed of sound. There is no implication that the accuracy of the ray trajectory is improved by using thinner layers. These points would have been avoided had we seen any way of fitting the parabolas without them. They were required to get a smooth parabolic fit from layer to layer.

Parabolas can be easily fitted to the data points for interpolation purposes but they generally do not conform to the requirements. Parabolas fitted to points by threes, (z_1, z_2, z_3), (z_3, z_4, z_5), (z_5, z_6, z_7), etc., will not generally have common tangents at the join points, z_3, z_5, z_7, etc., with the consequence that we are not better off than before. Many other parabolic formulae were considered, averaged overlapping parabolas, etc., but none were satisfactory.

The interpolated points at which the parabolic arcs are joined, the original data points, and the points at the apex of the parabolic arcs, are used in the parabolic model for ray tracing to determine limits of integrals and integrands for the evaluation of ray traced distance under the assumption that the sounding itself is a sampling from such parabolic arcs. In other words, the values that would have been observed had many more sounding points been obtained would lie on the parabolic arcs. It is obvious that this is a fiction, but much less of a fiction than the assumption that, had many more points been observed, they would all lie on straight lines joining the few that were observed.

The interpolated midway points and the apex points appear to augment the data somewhat (but in reality they do not). There may be occasions when it seems reasonable to join the original and augmented points by straight lines to improve the interpretation of the sounding. We feel that there is indeed some improvement when this is done. But it is as incorrect to then use this augmented linear layer model for ray tracing as it was to use the original data points in a linear layer model. The reason is exactly as it was originally. Any ray tracing method in which the integrals are evaluated as though speed of sound varies linearly with height are physically unacceptable: the basic assumption of a smoothly varying slope is violated at every point where the straight line segments join together.

If the parabolic model, or a Spline or Hermite interpolation formula were used to compute the interpolated values of speed of sound (plus wind component) at, say 10, or 100, or 1000, intermediate points, and if a linear layer model were used with integration over these individual finely divided layers, then an approach would be made to the results of using the parabolic model ray tracing technique. However, even in this case there are some objections because of the lack of continuity of the slope of the speed of sound as a function of height. When the calculations are carried out on a fine enough scale, (i.e., very small angle increments) the irregularity of the relation $r=r(\varphi_o)$, r=distance to ray return, φ_o = initial elevation angle would show up. Nothing will eliminate these except treating the parabolic model in the proper sense; of using the interpolation parabola in the important radical of the integrand.

2. Amplitude Intensity along a Ray

The amplitude of sound intensity as a modification of spherical spreading is determined by the focusing factor, f,

$$I = I_{sph} f, \quad f = R^2 \cos \varphi_o / r(dr/d\varphi_o) \sin \varphi_p,$$

where I_{sph} is the intensity due to spherical spreading from a sound source at O to a receiver location at P where the distance R is that of the line OP while the distance r is the distance between the projections of O and P on the horizontal plane. φ_o is the inclination of the ray at O and φ_p is the inclination at P.

The term $\partial r/\partial\varphi_o$ in the focusing factor makes it impossible to say anything about sound intensity from the geometry of a single ray; it takes at least two rays to calculate or estimate $\partial r/\partial\varphi_o$. On the other hand, the value of $\partial r/\partial\varphi_o$ may be calculated in the same manner as the ray itself. When this is done then both the ray geometry and intensity along the ray are known simultaneously. This kind of information is needed if the ray tracing technique is to be modified to take into account the effect of caustics and foci on the ray paths since modification is required only at those levels or regions where the focusing factor changes rapidly (or becomes exceedingly large).

The following is an abbreviated analysis of some aspects of the calculation of this factor.

The basic ray equations in the form

$$\begin{aligned} dx/dt &= c \cos \varphi + u \\ dz/dt &= c \sin \varphi \end{aligned} \tag{1}$$

and Snell's law in the form

$$c/\cos \varphi + u = c_o/\cos \varphi_o + u_o = K = \text{constant} \tag{2}$$

lead to the integral for the ray displacement in the horizontal when penetrating a layer in the vertical

$$x_2 - x_1 = \int_{z_1}^{z_2} [(c \cos \varphi + u)/c \sin \varphi] dz. \qquad (3)$$

If the layer z_1, z_2 is completely penetrated, the phase normal inclination, φ, does not become zero within the layer and the integral is perfectly proper. If the ray is refracted earthward within a layer, z_2 is the level at which the ray becomes horizontal. In this case $\varphi(z_2) = 0$, and the integral is improper. Under most circumstances the way in which $\varphi(z_2)$ approaches zero is such that the improper integral is convergent and may be evaluated by elementary methods. (The exceptional cases are of no importance as far as the problem being discussed is concerned).

It is obvious that one may obtain $\partial r/\partial\varphi$ from $\partial x_2/\partial\varphi_o - \partial x_1/\partial\varphi_o$ by addition of values through the layers penetrated. Using (3) for this purpose, the result is

$$\partial x_2/\partial\varphi_o - \partial x_1/\partial\varphi_o = -(c_o \sin\varphi_o/\cos^2\varphi_o) \int_{z_1}^{z_2} [(c+u\cos\varphi)\cos^2\varphi/c^2\sin^3\varphi] dz \qquad (4)$$

where use has been made of

$$\partial\varphi/\partial\varphi_o = c_o \sin\varphi_o \cos^2\varphi/c \sin\varphi \cos^2\varphi_o \qquad (5)$$

from (2). The integrand may also be expressed as

$$(c+u \cos\varphi)\cos^2\varphi/c^2\sin^3\varphi = Kc/[(K-u)^2-c^2]^{3/2}. \qquad (6)$$

As long as the ray penetrates the layer (z_1, z_2) the integral (4) is proper and is evaluated with no difficulty. When z_2 is the level at which the ray becomes horizontal, then $\varphi = 0$ at that level and the integral is improper at the upper limit. It is readily seen from (6), if $(dc/dz)+(du/dz)$ is not zero at the level z_2, that $\sin\varphi$ approaches zero proportionally to $(z_2-z)^{1/2}$ and consequently the integrand behaves like $(z_2-z)^{3/2}$. This means that the integral is divergent.

Another item that needs consideration is the fact that at the crest of the ray, the value of z_2 is dependent on φ_o. In other words, the process of differentiating under the integral sign to obtain (4) is no longer valid. To avoid the difficulties, the process is started anew for a ray that becomes horizontal at z_2. Let $z^* = z_2 - \epsilon$ and consider the limit for $\epsilon \to 0$. Then

$$\partial x_2/\partial\varphi_o - \partial x_1/\partial\varphi_o = \lim_{\epsilon \to 0} [(c \cos\varphi + u)/c \sin\varphi]_{z^*}(\partial z^*/\partial\varphi_o)$$

$$- \lim_{\epsilon\to 0} \{c_o \sin\varphi_o/\cos^2\varphi_o \int_{z_1}^{z_*} [\cos^2\varphi(c+u \cos\varphi)/c^2\sin^{-3}\varphi]dz. \tag{7}$$

Since $z_2 = z_2(\varphi_o)$ is given by Snell's Law in the form

$$c(z_2) + u(z_2) = c_o/\cos\varphi_o + u_o$$

then

$$\partial z_2/\partial\varphi_o = [c_o/(c' + u')]\sin\varphi_o/\cos^2\varphi_o$$

and replace $\partial z_*/\partial\varphi_o$ by its limit, $\partial z_2/\partial\varphi_o$ so that the expression becomes

$$\partial x_2/2\varphi_o - \partial x_1/\partial\varphi_o = [c_o \sin\varphi_o/\cos^2\varphi_o]_{z^*}$$

$$- \lim_{\epsilon\to 0} \left\{ \int_{z_1}^{z^*} [\cos^2\varphi(c+u \cos\varphi)/c^2\sin^{-3}\varphi]dz + \left\{(u+c \cos\varphi)/c'+u')c \sin\varphi\right\}_{z_*}\right.$$

The last expression contains two terms that become large in the limit for ϵ. Let the integral expression in the above be indicated by I. If the variable of integration is changed from z to φ, then

$$I = -\int_{\varphi_1}^{\varphi_*} [(c+u\ \cos\varphi)/(c'+u'\ \cos\varphi)][\cos\varphi/\sin^2\varphi]\,d\varphi$$

and

$$I = -\ [(c_o+u_o\cos\varphi_o)/\cos\varphi_o]\int_{\varphi_1}^{\varphi_2}[\cos\varphi/c(c(c'+u')\cos\varphi]\,d(1/\sin\varphi).$$

The last expression may be integrated by parts to give

$$I = [(c_o+u_o\cos\varphi_o)/\cos\varphi_o]\Big\{\Big[\cos\varphi/c(c'+u'\cos\varphi)\sin\varphi\Big]_{\varphi_1}^{\varphi_*}$$

$$-\int_{z_1}^{z_*}\Big[\frac{d}{dz}\Big(\frac{\cos\varphi}{c(c'+u'\cos\varphi)}\Big)\Big]\frac{dz}{\sin\varphi}\Big\}$$

where, in the second term on the right, z has been restored as the parameter of integration. Then carrying out the differentiation with respect to z in the integrand on the right

$$I = [(c_o+u_o\cos\varphi_o)/\cos\varphi_o]\ \Big\{\Big[\cos^2\varphi/c\ \sin\ \varphi(c'+u'\cos\varphi)\Big]_{z_1}^{z_2}$$

$$+\int_{z_1}^{z_2}[\cos\varphi(c''+u''\cos\varphi)/c\ \sin\varphi(c'+u'\cos\varphi)^2]\,dz\Big\}.$$

The final limit to be evaluated then becomes $\partial x_2/\partial\varphi_o-\partial x_1/\partial\varphi_o$ = $[c\ \sin\varphi/\cos^2\varphi]\ \lim_{\epsilon\to 0}$ (A-B+C-D)

where

$$A = [(u+c\ \cos\varphi)/c(c'+u')\sin\varphi]_{z^*}\ ,$$

$$B = [(c_o+u_o\cos\varphi_o/\cos\varphi_o][\cos\varphi/c(c'+u'\cos\varphi)\sin\varphi]_{z_*},$$

$$C = [(c_o+u_o\cos\varphi_o/\cos\varphi_o][\cos\varphi/c(c'+u'\cos\varphi)\sin\varphi]_{z_1},$$

$$D = \int_{z_1}^{z_*}\cos\varphi(c''+u''\cos\varphi)/c\ \sin^2\varphi(c'+u'\cos\varphi)\,dz.$$

It may be readily verified that $\lim_{\epsilon \to o} (A-B) = 0$ so that the final result is

$$\partial x_3/\partial\varphi_o - \partial x_1/\partial\varphi_o = (c_o \sin\varphi_o/\cos^2\varphi_o)[(c_o+u_o\cos\varphi_o)/\cos\varphi_o]$$

$$\times \left\{[\cos\varphi/c(c'+u'\cos\varphi)\sin\varphi]_{z_1} - \int_{z_1}^{z_2}[\cos\varphi(c''+u''\cos\varphi)/c(c'+u'\cos\varphi)^2\sin\varphi]dz\right\}. \quad (8)$$

In the above the integral is convergent provided that the denominator factor $c'+u'\cos\varphi$ is not zero.

Another formulation for (8) is

$$\partial x_2/\partial\varphi_o - \partial x_1/\partial\varphi_o = (c_o \sin\varphi_o/\cos^2\varphi)\left\{K(K-u)/[(K-u)c'+cu'][(K-u)^2-c^2]^{\frac{1}{2}}\right]_{z_1}$$

$$- \int_{z_1}^{z_2}\left\{K(K-u)[(K-u)c''+cu'']/[(K-u)c'+cu'']^2[(K-u)^2-c^2]\right\}dz$$

where K is the Snell's Law constant from (2) and the explicit dependence on $u=u(z)$ and $c=c(z)$ as functions of altitude is shown.

3. Remark on Linear Layer Models

For the case $u(z) \equiv 0$, the form

$$\partial r_2/\partial K - \partial r_1/\partial K = [K/c'\sqrt{K^2-c^2}]_{z_1} - \int_{z_1}^{z_2} \{Kc''/(c')^2\sqrt{K^2-c^2}\}dz$$

throws a significant light on the ray-tracing method. We have pointed out previously that in the "linear layer" model the derivative $\partial r/\partial K$ has an infinite discontinuity at each point where the "linear layer" model changes slope, i.e., at each data point.

Consider the case in which the ray is reflected at level z_3 and let levels z_2 be a data level. Then

$$\partial(r_3-r_1)/\partial K = \partial(r_2-r_1)/\partial K + \partial(r_3-r_2)/\partial K$$

and, on carrying out an integration by parts for the first term along lines outlined in the preceding sections, one obtains

$$\partial(r_3-r_1)/\partial K = K/m_{12}\sqrt{K^2-c^2}]_{z_1} + (1/m_{23}-1/m_{12})K/\sqrt{K^2-c^2}]_{z_2}$$

$$- \int_{z_1}^{z_2} [Kc''/(c')^2\sqrt{K^2-c^2}]dz - \int_{z_2}^{z_3} [Kc''/(c')^2\sqrt{K^2-c^2}]dz$$

$$= A+B-C-D$$

where A,---,D indicate the terms on the lines above in order. The symbols m_{12} and m_{23} stand for c' in layers (z_1,z_2) and (z_2,z_3).

For a linear layer model, $c'' \equiv 0$ and m_{12}, m_{23} are constants, generally different. In this model, $C \equiv D \equiv 0$. Since the value of K is associated with $c(z_3)$ directly, $K=c(z_3)$, so that for $z_3 \to z_2$, then $K \to c_2$ and since generally

$m_{12} \equiv m_{23}$ so that $B \to \pm \infty$ depending on the sign of $m_{12} - m_{23}$.

For the parabolic model (or any model with sufficiently smoothly turning tangent) one must note that m_{23} is no longer a constant and that $m_{12} = c_2'$, i.e., the slope of $c(z)$ at z_2. Then one may write (parabolic model)

$$m_{23} = c_2' + c_2'' (z_3 - z_2) + ---$$

and note that

$$K = K(z_3) = c_2 + c_2'(z_3 - z_2) + ---$$

so that

$$\sqrt{K^2 - c_2^2} = \sqrt{K + c_2}\ \sqrt{K - c_2} = \sqrt{2c_2 c_2'(z_3 - z_2) + ---}\ .$$

Then

$$B \cong c_2''(+)\ \sqrt{c_2(z_3 - z_2)} / \sqrt{2}\ (c_2')^{5/2} + ---$$

so that $\lim_{z_3 \to z_2} B = 0$. Also $D \to 0$ so that the only terms left are A–C which is exactly the same as the expression for $\partial(r_2 - \partial r_1)/\partial K$ given initially. The curve $r(\varphi_o)$ or $r(K)$ is then one with a continuous tangent at the routine data points where the slope of the curve $c(z)$ is not zero. At the points where $c'(z) = 0$ the distance $r(\varphi_o)$ becomes large without bound and the above c_o analysis is meaningless.

In the more general case where wind component is also considered the analysis is essentially unchanged. Thus, one would obtain

$$\partial(r_3 - r_1)/\partial K = \{K(K-u)/[(K-u)c' + cu']\sqrt{(K-u)^2 - c^2}\}_{z_1}$$

$$- \{K(K-u)/[(K-u)c' + cu']\sqrt{(K-u)^2 - c^2}\}_{z_2 -}$$

$$+ \{K(K-u)/[(K-u)c' + cu']\sqrt{(K-u)^2 - c^2}\}_{z_2 +}$$

$$- \int_{z_1}^{z_2} \frac{K(K-u)[(K-u)c'' + cu'']dz}{[(K-u)c' + cu']^2 \sqrt{(K-u)^2 - c^2}} - \int_{z_2}^{z_3} \frac{K(K-u)[(K-u)c'' + cu'']dz}{[(K-u)c' + cu']^2 \sqrt{(K-u)^2 - c^2}}$$

$$= A - B + C - D - E$$

where A,---,E denote the corresponding terms in shorter form. The symbols z_2- and z_2+ indicate that the expressions are evaluated at z_2 but in the first case with $c'$and u' evaluated from below z_2 but in the second case from above z_2. c and u have the same value at z_2 whether approached from above or below. The Snell's law constant, K, is dependent on the reflection level, z_3, from $K = c(z_3) + u(z_3)$. Then

$$C-B = \{K(K-u)/\sqrt{(K-u)^2-c^2}\}\{1/[(K-u)c'_+ + cu'_+]-1/[(K-u)c'_- + cu'_-]\}$$

where the +, - symbols have been move to c' and u' and c and u are evaluated at z_2. A bit more arithmetic gives

$$C-B = Q[(K-u)(c'_- - c'_+) + c(u'_- - u'_+)]/\sqrt{(K-u)^2-c^2}$$

where $Q = K(K-u)/[(K-u)c'_+ + cu'_+][(K-u)c' + cu'_-]$. It is readily seen that if $K \rightarrow c+u$, i.e., $z_3 \rightarrow z_2$, the large term in the numerator becomes $c[c'_- + u'_-) - (c'_+ + u'_+)]$ which will be non zero if the curve for c(z) + c(z) has a discontinuous tangent at z_2 (a data point on a "corner"). At the same time the radical in the denominator becomes large without bound. The result is that the derivative of the curve $r(\varphi_o)$ or r(K) has an infinite discontinuity for values of φ_o or K corresponding to reflection at the level z_2.

Similarly, if the curve c(z)+u(z) has a sufficiently smoothly turning tangent at z_2 the numerator will approach zero more rapidly than the denominator so the result is

$$\lim_{z_3 \rightarrow z_2} (C - B) = 0.$$

4. Location of Caustics Aloft

a) Calculation of $\partial x/\partial \varphi_o$ for Successive Ground Reflections

In the usual method of ray computation the horizontal travel of the ray, x_2-x_1, in the layer z_1, z_2 is calculated and added to the sum of previous distances traveled. The result is that only the total travel of the ray is available at any level, say z_n, i.e., only x_n-x_o. The ray is computed to the reflection level and then its distance is doubled to get the point of return of the ray to the ground. To locate the position of caustics aloft it is necessary to retain intermediate steps in storage so that the whole process of the ray travel is available and to also keep track of $\partial x_n/\partial \varphi_o$ at the same time. This is especially true for the descending part of the ray which is usually not calculated at all. The following steps are required for location of caustics.

Let z_n be the altitude of the top of the layer within which reflection occurs. Then by addition of successive layers the following tabulation of steps occurs

Level	CALCULATED DIFFERENCES		ACCUMULATED SUMS	
	Distance	Derivative	Distance	Derivative
z_1	x_1-x_o	$\partial x_1/\partial \varphi_o$	x_1-x_o	$\partial x_1/\partial \varphi_o$
z_2	x_2-x_1	$\partial x_2/\partial \varphi_o-\partial x_1/\partial \varphi_o$	x_2-x_o	$\partial x_2/\partial \varphi_o$
---	---	---	---	---
z_{n-1}	$x_{n-1}-x_{n-2}$	$\partial x_{n-1}/\partial \varphi_o-\partial x_{n-2}/\partial \varphi_o$	$x_{n-1}-x_o$	$\partial x_{n-1}/\partial \varphi_o$
$z*$	$x*-x_{n-1}$	$\partial x_*/\partial \varphi_o-\partial x_{n-1}/\partial \varphi_o$	x_*-x_o	$\partial x_*/\partial \varphi_o$

The usual calculation procedure stops at the last step, the intermediate steps being discarded, and the accumulated sums doubled (last line only) to obtain values corresponding to the return of the ray to the ground.

The distance and derivative tables may be continued for the descending leg of the ray and for the ray after successive reflections from the ground. For the descending leg of the ray, the symbols x' and z' will be used. Thus at the i'th level, on descent, z'_i, the distance is

$$x'_i - x_o = 2(x_* - x_o) - (x_i - x_o)$$

and the derivative is

$$\partial x'_i/\partial\varphi_o = 2(\partial x_*/\partial\varphi_o) - (\partial x_i/\partial\varphi_o) \ .$$

After k reflections from the ground the distance on the ascending leg is

$$_k x_i - x_o = 2k(x_* - x_o) + (x_i - x_o) \tag{1}$$

and the corresponding derivative is

$$\partial\, _k x_i/\partial\varphi_o = 2k(\partial x_*/\partial\varphi_o) + (\partial x_i/\partial\varphi_o) \tag{2}$$

while for the descending leg the distance is

$$_k x'_i - x_o = 2(k+1)(x_* - x_o) - (x_i - x_o) \tag{3}$$

and its derivative

$$\partial\, _k x'_i/\partial\varphi_o = 2(k+1)(\partial x_*/\partial\varphi_o) - (\partial x_i/\partial\varphi_o)_o \tag{4}$$

For k=0, the results for the initial ray are obtained. When k=1, 2,---, one obtains the results for the first, second, etc., reflections of the ray by the ground.

b) Location of Points $\partial x/\partial\varphi_o = 0$ Along the Ray (Arithmetical Example

The problems involved in locating the points $\partial x/\partial\varphi_o = 0$ may be illustrated by an arithmetical example. Consider the case of ducted rays along the ground from the speed of sound profile $c = c_o + \mu_o z$. The circular arcs for the rays are given by the equation

$$\left[x-(2k+1)\sqrt{K^2-c_o^2}\;/\mu_o \right]^2 + (z+c_o/\mu_o)^2 = (K/\mu_o)^2 \qquad (5)$$

which may be solved for x to give

$$x = (2k+1)\sqrt{K^2-c_o^2}/\mu_o \pm \left[(K/\mu_o)^2 - (z+c_o/\mu_o)^2\right]^{\frac{1}{2}}. \qquad (6)$$

The - sign corresponds to the ascending leg of the ray while the + sign corresponds to the descending leg. $k = 0$ corresponds to the ray before ground reflection, $k = 1$ after one ground reflection, etc. Only the arcs for which $x \geq 0$ and $z \geq 0$ are considered. Holding z constant and differentiating with respect to K, then

$$\partial x/\partial K = (2K+1)K/\mu_o\sqrt{K^2-c_o^2} \pm K/\mu_o \left[(K/\mu\partial^2-(z+c_o/\mu_p)^2\right]^{\frac{1}{2}}. \quad (7)$$

This is converted into $\partial x/\partial\varphi_o$ by the relation

$$\partial x/\partial\varphi_o = (\partial x/\partial K)\ (\partial K/\partial\varphi_o) = K \tan \varphi_o(\partial x/\partial K)$$

and since $K = c_o/\cos\varphi_o$, $\cos\varphi_o = c_o/K$, $\tan\varphi_o=\sqrt{K^2-c_o^2}/c_o$, then

$$\partial x/\partial\varphi_o = (2k+1)K^2/\mu_o c_o \pm K^2\sqrt{K^2-c^2}/c_o\mu_o^2 \left[(K/\mu_o)^2-(z+c_o/\mu\partial^2\right]^{\frac{1}{2}} \quad (8)$$

The above corresponds to the values of $\partial x/\partial\varphi_o$ that would be calculated by the ray tracing integrals at the fixed levels z_i when fixed values of z are substituted in the last term.

The point of reflection at the top of the ray, x_*, is given by

$$x_* = (2k+1)\sqrt{K^2-c_o^{\,2}}/\mu_o$$

so that the displacement of the reflection point with respect to φ_o is

$$\partial x_*/\partial\varphi_o = (K\sqrt{K^2-c_o^{\,2}}/c_o)(\partial x_*/\partial K) = (2k+1)K^2/c_o\mu_o \quad . \qquad (9)$$

This is the first term of the expression for $\partial x/\partial\varphi_o$, Equation (8). The expression for $\partial x/\partial\varphi_o$ then consists of two parts, the first term corresponding to the rate of x-displacement of the center of the circular arcs and the rate of change of the x-coordinate of these arcs as reflected by their change of radius, negative on the up-leg and positive on the down-leg.

This last term of $\partial x/\partial\varphi_o$ becomes large without bound as z approaches $(K-c_o)/\mu_o$, the ray crest (reflection level).

Now consider the ray on the first ascending leg. Here

$$x = \sqrt{K^2-c_o^{\,2}}\;/\mu_o - \left[\,(K/\mu_o)^2 - (z+c_o/\mu_o)^2\right]^{\frac{1}{2}}$$

and

$$\partial x/\partial\varphi_o = (K^2/c_o\mu_o)\left\{1 - \sqrt{K^2-c_o^{\,2}}/\mu_o\left[\,(K/c_o)^2-(z+c_o/\mu_o)^2\right]^{\frac{1}{2}}\right\}.$$

For the corresponding level on the descending leg, denoted by x',

$$x' = \sqrt{K^2-c_o}/\mu_o + \left[\,(K/c_o)^2 - (z+c_o/\mu_o)^2\,\right]^{\frac{1}{2}}$$

$$\partial x'/\partial\varphi_o = (K^2/c_o\mu_o)\left\{1 + \sqrt{K^2-c_o^{\,2}}/\mu_o\left[\,(K/c_o)^2-(z+c_o/\mu_o)^2\right]^{\frac{1}{2}}\right\}$$

so that

$$\partial x/\partial\varphi_o + \partial x'/\partial\varphi_o = 2K^2/c_o\mu_o = 2(\partial x_*/\partial\varphi_o),$$

$$\partial x/\partial\varphi_o = 2(\partial x_*/\partial\varphi_o) - (\partial x/\partial\varphi_o).$$

If successive ascending and descending legs are considered, $k > 0$, the above would become

$$\partial x'/\partial\varphi_o = 2(2k+1)\ (\partial x_*/\partial\varphi_o) - (\partial x/\partial\varphi_o)$$

where now $(\partial x_*/\partial\varphi_o)_o$ refers to the initial ray for which $k = 0$, while $\partial x'/\partial\varphi_o$ and $\partial x/\partial\varphi_o$ refer to $k \geq 0$ (initial or ground reflected rays).

The algebraic solution for the caustic in this particular example is reasonably elementary. Let $\partial x/\partial\varphi_o = 0$ so that

$$(2k+1)\left[(K/\mu_o)^2 - (z+c_o/\mu_o)^2\right]^{\frac{1}{2}} = \sqrt{K^2-c_o^{\ 2}}\ /\mu_o$$

from which it follows that

$$z = -c_o/\mu_o + \left[(K^2/\mu_o)^2 - (K^2-e_o^{\ 2})/\mu_o^{\ 2}(2k+1)^2\right]^{\frac{1}{2}}.$$

(only the positive root has meaning for our problem). The corresponding x-coordinate is obtained from the original equations

$$x = (2k+1)\sqrt{K^2-c_o^{\ 2}}/\mu_o^{\ 2} - \sqrt{K^2-c_o^{\ 2}}/\mu_o(2k+1)$$

or

$$x = \left[(2k+1) - (2k+1)^{-1}\right]\ (\sqrt{K^2-c_o^{\ 2}}/\mu_o)$$

where only the negative root has physical meaning for this problem.

The above are parametric equations for the x, z coordinates of the caustics in terms of the parameter K. For the original rays, $k = 0$, there are no caustics but for $k \geq 1$, there is one parameter family of caustics, K as parameter. All of the caustics lie on the ascending legs of the rays after reflection at least once from the ground.

c) Caustics from the General Ray Method

In application to the practical ray-tracing problem the preceding algebraic relations are not available and the only recourse available is the sequence of values for $\partial_k x_i/\partial\varphi_o$, $\partial_k x'_i/\partial\varphi_o$. For the various levels i = 0,1,---- and for the several ground reflections k = 0,1,---. It would seem reasonable that all that is required is to inspect this sequence of values for a change of sign. That this is not the case is illustrated by the preceding algebraic example. The equations are

Ascending leg:

$$\partial_k x/\partial\varphi_o = (2k+1)K^2/c_o\mu_o - K^2\sqrt{K^2-c_o^2}/c_o\mu_o^2\left[(K/\mu_o)^2-(z+c_o/\mu_o)^2\right]^{\frac{1}{2}},$$

Reflection Point:

$$\partial_k x_*/\partial\varphi_o = (2k+1)K^2/c_o\mu_o,$$

Descending Leg:

$$\partial_k x/\partial\varphi_o = (2k+1)K^2/c_o\mu_o+K^2\sqrt{K^2-c_o^2}/c_o\mu_o^2\left[(K/\mu_o)^2-(z+c_o/\mu_o)^2\right]^{\frac{1}{2}}.$$

Consider what happens when z ranges from 0 to $\sqrt{K^2-c_o^2}/\mu_o=z_*$ and back to zero following a single ray (K fixed) through its various reflections:

Initial ray, k = 0:

Ascending, 0 to z_*, $\partial_o x/\partial\varphi_o$ decreases from 0 to $-\infty$

Crest , $z = z_*$, $\partial_o x_*/\partial\varphi_o = K^2/c_o\mu_o$

Descending, z_* to 0, $\partial_o x'/\partial\varphi_o$ decreases from $+\infty$ to $2K^2/c_o\mu_o$

First Reflection, $k = 1$

Ascending, 0 to z_*, $\partial_1 x/\partial\varphi_o$ decreases from $2K^2/c_o\mu_o{}^2$ to $-\infty$ (A root for some z on this range)

Crest, $z = z_*$, $\partial_1 x_*/\partial\varphi_o = 3K^2/c_o\mu_o$

Descending, z_* to 0, $\partial_1 x'/\partial\varphi_o$ decreases from $+\infty$ to $4K^2/c_o\mu_o$

Second Reflection, $k = 2$

Ascending, 0 to z_*, $\partial_2 x/\partial\varphi_o$ decreases from $4K^2/c_o\mu_o$ to $-\infty$ (A root for some z on this range)

Crest, $z = z_*$, $\partial_2 x_*/\partial\varphi_o = 5K^2/c_o\mu_o$

Descending, z_* to 0, $\partial_2 x'/\partial\varphi_o$ decreases from $+\infty$ to $6K^2/c_o\mu_o$

The situation is illustrated in Figure 1 where z is considered as a periodic variable.

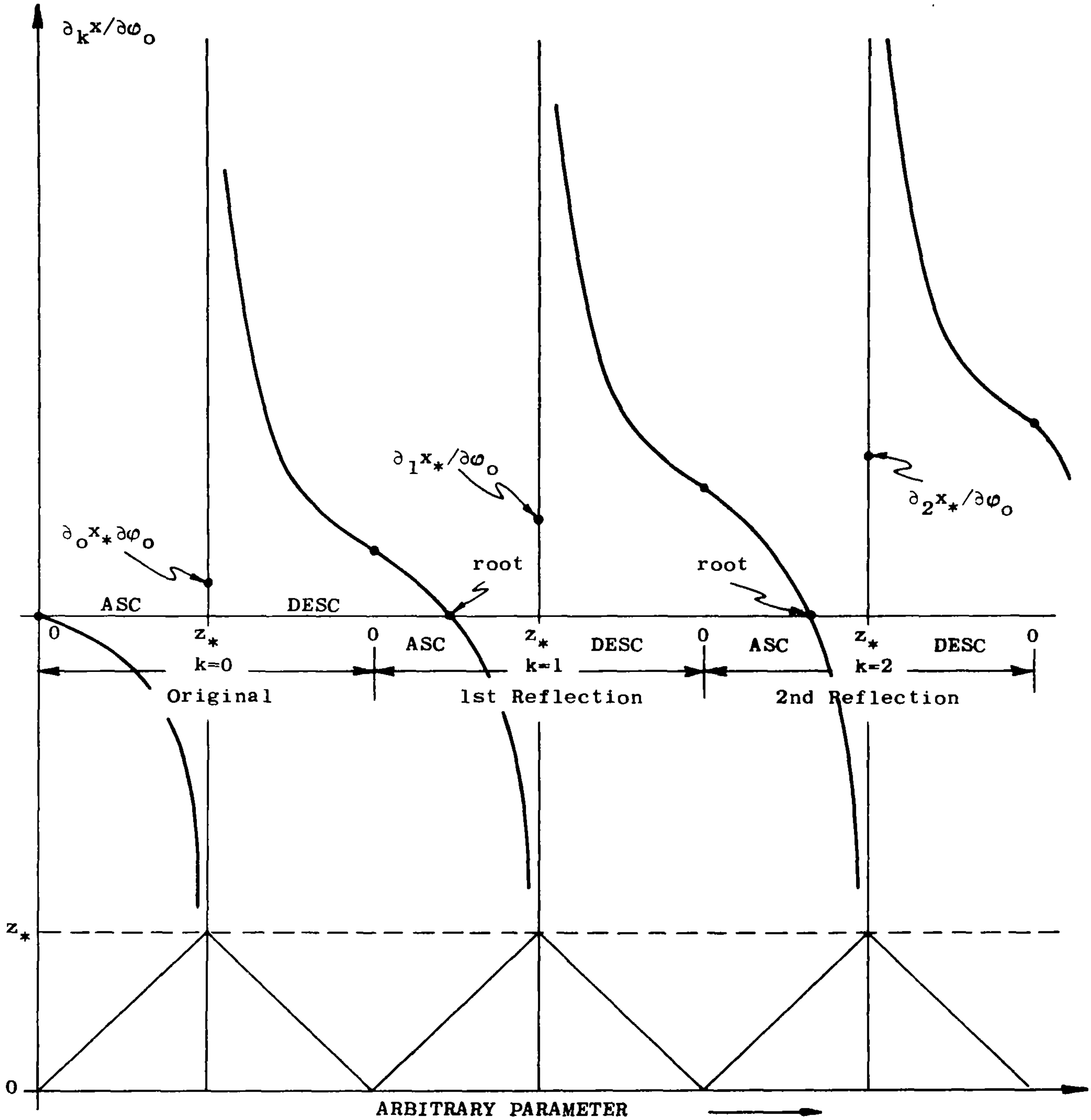

FIGURE 1. The derivative, $\partial x/\partial\varphi_o$, expressed as a function of an arbitrary parameter that increases indefinitely while altitude oscillates between 0 and z. Lower part shows variation of z from 0 to z_* and back again. Upper part shows the infinite discontinuity of $\partial x/\partial\varphi_o$ at the points where $z=z_*$.

In terms of the ray tracing integrals, one evaluates for the original ascending ray

$$\partial_o x_i/\partial\varphi_o - \partial_o x_{i-1}/\partial\varphi_o = -\int_{z_{i-1}}^{z_i} Kc/\left[(K-u)^2-c^2\right]^{3/2} dz$$

$$\partial_o x_*/\partial\varphi_o - \partial_o x_{n-1}/\partial\varphi_o = \left\{ K(K-u)/\left[(K-u)c'+cu'\right]\sqrt{(K-u)^2-c^2} \right\}_{z_{n-1}}$$
$$- \int_{z_{n-1}}^{z_*} \frac{K(K-u)\left[(K-u)c''+eu''\right]dz}{\left[(k-u)e'+cu'\right]^2\sqrt{(K-u)^2-c^2}}$$

and all other values of $\partial_k x_i/\partial\varphi_o$ and $\partial_k x_*/\partial\varphi_o$ are computed in terms of these values by the recurrence relations of the previous section. The first of the above expressions becomes large without bound if z_i approaches z_*, the reflection level, corresponding to the behavior of the z-dependent term in the expression for $\partial x/\partial\varphi_o$ of our elementary example. The finite character of the second term of the second expression above corresponds to the term independent of z in the elementary example.

The basic problem is that inspection of the sequence of values for a change of sign of $\partial_k x_i/\partial\varphi_o$ is complicated by the fact that this sequence also contains terms that are from a function that has an infinite discontinuity and sign change at each reflection layer aloft, but at that layer the finite value $\partial_k x_*/\partial\varphi_o$ has been substituted. Since the sequence is at reasonably wide spacing of levels z_i, it is quite likely that no appropriate sign change will appear explicitly. To locate the sign change it will then be necessary to make use of the fact that $\partial x_{n-1}/\partial \varphi_o^- \to \infty$.

The integral expression for the derivative difference may be written as

$$\int_{z_{n-1}}^{z} \left\{ Kc/\left[(k-u)^2-c^2\right]^{3/2} dz = F_o \int_{z_{n-1}} \left[(K-u)^2-c^2\right]^{-3/2} dz = F_o I\right.$$

where F_o is a suitable average value for the slowly varying factor of the integrand. If $c(z)+u(z) = A+B(z-z_{\#})^2$, the integration may be carried out explicitly. The integrals involved may be reduced by considering the two cases:

$B > 0,\ K - A > 0,\ z > z_{n-1} > z_{\#}$

$$I = \left\{ 1/(K-A)\sqrt{B} \right\} \left\{ (z-z_{\#})/\left[(K-A)/B-(z-z_{\#})^2\right]^{\frac{1}{2}} - (z_{n-1}-z_{\#})/\left[(k-A)/B-(z_{n-1}-z_{\#})^2\right]^{\frac{1}{2}} \right\}$$

and $B < 0,\ K-A < 0,\ z_{n-1} < z < z_{\#}$

$$I = \left\{ -1/(K-A) \right\} \left\{ (z-z_{\#})/\left[(z-z_{\#})^2-(A-K)/(-B)\right]^{\frac{1}{2}} - (z_{n-1}-z_{\#})/\left[(z_{n-1}-z_{\#})^2-(A-K)/(-B)\right]^{\frac{1}{2}} \right. .$$

For the other possible cases, the ray would not have a reflection level in the layer (z_{n-1}, z_n).

We are now ready to consider the sequences, $i = 1, ---, n-1$,

$\partial_k x_i/\partial\varphi_o$: ascending after k ground reflections

$\partial_k x'_i/\partial\varphi_o$: descending after k ground reflections

for the possible location of caustics.

If either sequence exhibits a change of sign, this locates the caustic between the two levels adjacent to the sign change and no further analysis, except that interpolation for coordinates, is required.

If no change of sign occurs, note the value of this derivative at the reflection point, $\partial_k x_*/\partial\varphi_o$. If $\partial_k x_*/\partial\varphi_o > 0$, the caustic may be located on the ascending leg of the ray between the level z_{n-1} and the reflection level z_*. One now extends the sequence into the layer (z_{n-1}, z_*) by adding one or the other of the integrals above to the sequence to obtain

$$\partial_k x/\partial\varphi_o = 2k(\partial x_*/\partial\varphi_o) + I(z) - I\ (z_{n-1}) + \partial x_{i-1}/\partial\varphi_o$$

set this expression to zero, and solve for z

$$I(z) = I(z_{n-1}) - \partial k(\partial x_*/\partial\varphi_o) - \partial x_{i-1}/\partial\varphi_o.$$

The expressions for I(z) are of the form

$$I(z) = \alpha(z-z_\#)/\sqrt{(z-z_\#)^2-\beta^2} \quad \text{or} \quad \alpha(z-z_\#)/\sqrt{\beta^2-(z-z_\#)^2}$$

so one has, for example,

$$\alpha(z-z_\#)/\sqrt{(z-z_\#)^2-\beta^2} = Q$$

so that

$$z = z_{\#} \pm \left[-Q^2\beta^2/(\alpha^2-Q^2)\right]^{\frac{1}{2}}.$$

If $\partial_k x_*/\partial\varphi_o < 0$, then the caustic is suspected to lie on the descending leg. The interpolated value for the level of the caustic, z, is determined from

$$\partial_k x'/\partial\varphi_o = 2(k+1)(2x_*/\partial\varphi_o)-I(z)+I(z_{n-1})-\partial x_{n-1}/\partial\varphi_o$$

in the same way as before.

The corresponding values of x for the caustic cannot be accurately determined from interpolation from z_{n-1} to z_* using linear methods. The rays travel a long distance in level (z_{n-1}, z_*'). A better value of x may be obtained from the original integral expressions for horizontal travel using the z value at the caustic from the above analyses as the upper limit.

5. Comparison of Focus Factors for an Ideal Example

a. The Example

Consider a two layer atmosphere in which the speed of sound is constant c_o, through a depth next to the ground and above the layer increases with a constant rate dc/dz. The sound propagation will take place in such a way that there will be focusing. The basic relations are tabulated without derivation.

The distance from the source to the point at which the ray returns to the ground, r, is given by[1)]

$$ar/2N = \tan \varphi_o + p/\tan \varphi_o \quad (1)$$

$$p = aH = (H/c_o)(dc/dz) \quad (2)$$

$$a = (1/c_o)(dc/dz) \quad (3)$$

and where φ_o is the inclination angle of the initial ray and N is one more than the number of reflections from the ground. The location of the focus is obtained from the zero of $dr/d\varphi_o$. Since

$$a(dr/d\varphi_o)/2N = (\tan^2\varphi_o - p)/\sin^2\varphi_o \quad (4)$$

the inclination angle for rays at the focus is given by

$$\varphi_* = \tan^{-1}\sqrt{p} \quad (5)$$

and the distance to the focus is

$$r_* = 4N\sqrt{p}/a = 4\ NH/\sqrt{p} \ . \quad (6)$$

The focusing factor at any distance at which rays return is given by

$$f = (\tan^2\varphi_o + p)/(1+\tan^2\varphi_o)(\tan^2\varphi_o - p) \quad (7)$$

Whether rays return at a given distance r is determined by solving (1) for φ_o

$$\tan \varphi_o = (ar/4N) \pm [(ar/4N)^2 - p]^{\frac{1}{2}} \qquad (8)$$

which must have positive values.

If distance is expressed in terms of the height of the uniform layer, H, say $\xi = r/H$, then the above relations contain only the parameter p instead of both a and p

$$p\xi/2N = \tan \varphi_o + p_o/\tan \varphi_o, \qquad (1a)$$

$$\xi_* = 4N\sqrt{p}, \qquad (6a)$$

$$\tan\varphi_o = (p\xi/4N) \pm [(p\xi/4N)^2 - p]^{\frac{1}{2}}. \qquad (8a)$$

If the roots of (8) or (8a) are indicated by $\tan\varphi_1$ and $\tan\varphi_2$, then at this distance two rays return so that the sound intensities are added (incoherent noise is assumed). The total intensity will involve the sum of the individual focus factors which will then be given as the sum

$$f = f_1 + f_2$$

where the right hand terms are from (7) using both of the roots of (8) or (8a).

For reference later, the second derivative $d^2r/d\varphi_o^2$ is tabulated here

$$d^2r/d\varphi_o^2 = (4N/a)[\cos \varphi_o(\tan^4\varphi_o + p)/\tan^3\varphi_o]. \qquad (9)$$

The height of penetration of a ray into the upper layer is of some interest in considering the physical limitations that are involved. This is

$$Z - H = (H/p)[\sec\varphi_o - 1] \qquad (10)$$

The conditions for the physical reality of the treatment are given by

$$H >> (\lambda^2/4\pi^2 a)^{\frac{1}{3}}$$

or

$$p \gg (\lambda/H)^2/4\pi^2$$

(The first condition is stronger than the second, if in the first the factor is 10^{+1}, the factor in the second is 10^{+3}.)

b. The Field Near the Focus

The ratio of the field at the caustic to that at the same point in an homogeneous medium is given by[2)]

$$S = v(o)2^{5/6}R \exp\{i[\pi/4+w(\xi_a)-k_oR]\}/[kr_o n(z)\tan\varphi_o \sin\varphi_p]^{\frac{1}{2}}[\partial^2 r/\partial\xi^2] \tag{9}$$

and we consider only the focus or the intersection of the caustic with the ground. Then $R=r$, $\varphi_p=\varphi_o$, $n(z)=1$ and the focusing factor is given by

$$f = |S|^2$$

whence

$$f = [v/o^2][2^{5/3}r/k_o\tan\varphi_o\sin\varphi_o(\partial^2 r/\partial\xi^2)_*^{2/3}] \tag{10}$$

The value of $v(o)$ is $0.62927 = \sqrt{\pi}/3^{2/3}\Gamma(2/3)$. The independent variable, ξ, is given by $\xi = k_o\cos\varphi_o$. The second derivative is obtained from

$$\partial r/\partial\xi = (\partial r/\partial\varphi_o)(\partial\varphi_o/\partial\xi)$$

$$\partial^2 r/\partial\xi^2 = (\partial^2 r/\partial\varphi_o)(\partial\varphi_o/\partial\xi)^2+(\partial r/\partial\varphi_o)(\partial^2\varphi_o/\partial\xi^2)$$

where

$$\partial\varphi_o/\partial\xi = -1/k_o\sin\varphi_o$$

$$\partial^2\varphi_o/\partial\xi^2 = -\cos\varphi_o/k_o^2\sin^{-3}\varphi_o$$

so that

$$\partial r/\partial\xi = -(\partial r/\partial\varphi_o)/k_o \sin\varphi_o \tag{11a}$$

$$\partial^2 r/\partial\xi^2 = [\sin\varphi_o(\partial^2 r/\partial\varphi_o^2) - \cos\varphi_o(\partial r/\partial\varphi_o)]/k_o^2 \sin^{-3}\varphi_o \tag{11b}$$

Since the derivative is evaluated at a focal point where $\partial r/\partial\varphi_o$ vanishes,

$$(\partial^2 r/\partial\xi^2)_* = (\partial^2 r/\partial\varphi_o^2)_*/k_o^2 \sin^2\varphi_o$$

and the focusing factor becomes

$$f = [(v/o)]^2\ 2^{5/3} k_o^{1/3} r \cos\varphi_o/\sin^{2/3}\varphi_o (d^2 r/d\varphi_o{}^2)_*^{2/3}. \tag{12}$$

It may be readily verified that

$$d^2 r/d\varphi_o^2 = (4N/a)\cos\varphi_o(\tan^4\varphi_o + p)/\tan^3\varphi_o \tag{13}$$

Substituting into the above and using the expression for r to eliminate N, the focusing factor becomes

$$\begin{aligned} f &= [(v/o)]^2\ 2^{5/3}(k_o r p)^{\frac{1}{3}}/(1+p)^{\frac{1}{2}} \\ &= 1.2574\ (k_o r p)^{\frac{1}{3}}/(1+p)^{\frac{1}{2}} \end{aligned} \tag{14}$$

It is pointed out that the field in the neighborhood of the focus is proportional to v(t)/v(o) so that this field is then represented by

$$f(t) = f[v(t)/v(o)]^2 \tag{15}$$

where t is the parameter

$$t = \pm 2^{\frac{1}{3}}(\Delta r)/(\partial^2 r/\partial\xi)_*^{\frac{1}{3}} \tag{16}$$

The expression in the denominator of (16), as in (10), comes from (11) and (13). Thus

$$\alpha = -k_o \sin^2\varphi_o (dr/d\varphi_o)^3 / 2[\sin\varphi(d^2r/d\varphi_o^2) - \cos\varphi_o(dr/d\varphi_o)]^2 \qquad (24)$$

Applying (4) and (13) for the particular atmospheric conditions considered

$$\alpha = -2\pi(H/\lambda_o)(N/p)(\tan^2\varphi_o - p)^3 \sin\varphi_o / [\tan^3\varphi_o(2\tan^2\varphi_o - 1) + 3p]^3. \qquad (25)$$

In view of the asymptotic behavior of $|J(\alpha)|$, the fact that $\alpha \to \infty$ for $\lambda_o \to 0$ and $|J(\alpha)| \to 1$ to give the geometric approximation and $\alpha \to 0$ for $\varphi_o \to \tan^{-1}\sqrt{p}$ (focus) so that $|J(\alpha)|^2 \to f_*$ to give the focus factor in Section 2 have already been noted.

B. SOUND INTENSITY FROM A MOVING SOURCE

The several sections of this part are devoted to topics that pertain to determining the ground intensity of sound from a moving source. The source is thought of as a launch vehicle following a near vertical trajectory. However, several examples for other flight paths are used to illustrate some particular point where a vertically rising sound source would be inconvenient.

In the first section on shocks associated with a moving source, the item of importance is the "following" shock that is not usually considered. This shock, fortunately, is extremely weak at the flight path intersection, but becomes significant where it joins the "Mach Conoid," the surface of revolution that would correspond to the Mach Cone in the usual case.

The several factors that must be taken into consideration for a moving source are briefly discussed together. Downward propagation presents no problems here except that the rays must reach the ground. The jet noise is of prime concern and its pronounced directivity must be considered which implies very careful coupling of the sound ray orientation and vehicle orientation. The shift in frequency due to doppler effect is a consideration that must be included. Acoustical coupling of the source with the air would be important even for a stationary source if there were a significant difference in elevation between source and receiver. Since this is always the case for a vertically rising source it cannot be overlooked.

The ray tracing procedures for a moving sound source (third section) are essentially the same as for a stationary source, but the calculation of the focusing factor is totally different. This lies in the fact that for a stationary source the differences in arrival time can be overlooked. When the source is moving, corrections must be made for arrival time differences. These have been carried out as a part of the ray tracing method as additional ray computations. There are other possible approaches, but a bare minimum requires that travel time must be calculated, a quantity not essential for the stationary source case.

1. Shocks Associated with Moving Sources in a Uniform Medium

The following sections are devoted to a brief survey of the elementary geometry of shock waves from a vehicle moving in a straight line. The case of uniform supersonic motion is first considered to review the basic ideas. The case of a uniformly accelerated motion more nearly approximates that of a vertically rising aerospace vehicle. In this case the shock configuration takes on a more complicated pattern.

Note that throughout we consider the vehicle that carries the sound source as having no volume so that we are not considering "shocks" in the usual sense.

Shock waves from a moving sound source may be considered as the envelope of the surfaces of constant phase. This elementary concept starts with the equation for the spheres of constant phase

$$[x-x_s(t-\tau)]^2+[y-y_s(t-\tau)]^2+[z-z_s(t-\tau)]^2 = c^2\tau^2$$

where the center of the sphere is located at the sound source with time lag τ

$$x_s(t-\tau), y_s(t-\tau), \; z_s(t-\tau)$$

where t is the total time concerned. The radius of the sphere is the speed of sound times the lag time, $c\tau$, or the distance traveled since emission.

If the first expression for surfaces of constant phase is written in the form

$$F(\tau) = 0,$$

Then the envelope of surfaces of constant phase is obtained by eliminating τ between the pair

$F(\tau) = 0$, $\partial F(\tau)/\partial\tau = 0$.

To perform the steps required to find the envelope, specific formulas for the flight path as a function of time are required.

a. Vehicle Moving with Constant Speed

For a uniformly moving source, let $x_s = vt$, $y_s = 0$, $z_s = 0$, so that

$$F(\tau) = [x-v(t-\tau)]^2+\rho^2-c^2\tau^2 = 0 \qquad (1)$$

and

$$\partial F(\tau)/\partial\tau = 2\{v[(x-v(t-\tau)]-c^2\tau\} = 0. \qquad (2)$$

From (2)

$$\tau = v(x-vt)/c^2-v^2) = M(x-vt)/c(1-M^2) \qquad (3)$$

where $M = v/c$. On substituting into (1)

$$(x-vt)^2(1-M^2)+\rho^2 = 0.$$

If $M < 1$, this is not a real surface. If $M > 1$ the surface is the Mach Cone.

b. Vehicle with Constant Acceleration

For an accelerated source $x_s = at^2/2$, $y_s = 0$, $z_s = 0$, then

$$F(\tau) = [x-a(t-\tau)^2/2]^2+\rho^2-c^2\tau^2 = 0, \qquad (1)$$

and

$$\partial F(\tau)/\partial\tau = a(t-\tau)[x-a(t-\tau)^2/2] -2c^2\tau = 0. \qquad (2)$$

From (2),

$$x=a(t-\tau)^2/2 + c^2\tau/a(t-\tau) \tag{3}$$

and using (3) in (1)

$$\rho = c\tau[a^2(t-\tau)^2-c^2]^{\frac{1}{2}}/a(t-\tau). \tag{4}$$

The pair (3) and (4) are parametric equations of the envelope in the parameter τ.

Differentiating these with respect to the parameter,

$$dx/d\tau = [c^2t - a^2(t-\tau)^3]/a(t-\tau)^2 \tag{5}$$

and

$$d\rho/d\tau = c[a^2(t-\tau)^3-c^2t]/a(t-\tau)^2[a^2(t-\tau)^2-c^2]^{\frac{1}{2}} \tag{6}$$

so that

$$d\rho/dx = (d\rho/d\tau)/(dx/d\tau) = -c/[a^2(t-\tau)^2-c^2]^{\frac{1}{2}}. \tag{7}$$

It is readily seen from (4) that the envelope is real only if $a(t-\tau) \geq c$ which corresponds to a vehicle flight time at which the vehicle speed becomes supersonic (t=total time, τ = travel time of sound wave, $t-\tau$ = vehicle flight time to time of sound emission).

Also from (4), $\rho=0$ at $\tau_1=0$ and $\tau_2=t-c/a$. In the first case the shock is on the vehicle and forms an apex with half angle φ where, from (7),

$$\tan\varphi = c/[a^2t^2-c^2]^{\frac{1}{2}},$$

corresponding to the Mach angle for a vehicle traveling at the same but constant speed. In the second case (7) indicates that $d\rho/dx = \infty$ so that this part of the shock is perpendicular to the flight path. The location of this rearward shock, from (2), is at

$$x = c(t-c/2a)$$

or its distance behind the vehicle is

$$x-a^2t^2/2 = -(at-c)^2/2a.$$

Both $dx/d\tau = 0$ and $d\rho/d\tau = 0$ at $\tau_3 = t-(c^2t/a^2)^{\frac{1}{3}}$ which corresponds to a local maximum of ρ and a local minimum of x, but the expression (7) for $d\rho/dx$ is well defined at this point and has the value

$$(d\rho/dx)_{\tau_3} = - [(at/c)^{2/3}-1]^{-\frac{1}{2}}$$

and also

$$x(\tau_3) = (c^2/a)[(3/2)(at/c)^{2/3}-1],$$

$$\rho(\tau_3) = (c^2/a)[(at/c)^{2/3}-1]^{3/2}$$

Thus the two branches of the shock meet in a cusp at this point.

The shock configuration is shown schematically in Figure 2.

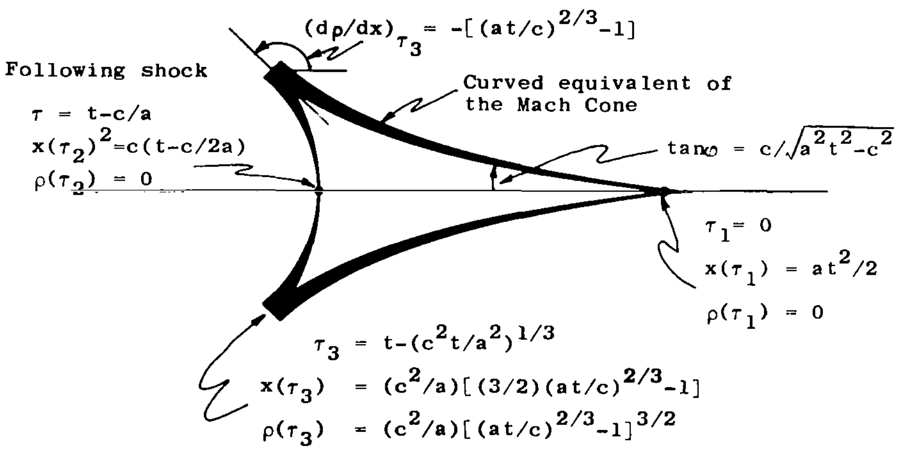

FIGURE 2. Shock configuration for a vehicle moving at constant acceleration in a straight line in a homogeneous medium.

The shock strength may be measured by the density of τ - points along the envelope. This is accomplished by

$$S = (d\tau/ds)^2$$

where

$$(ds/d\tau)^2 = (dx/d\tau)^2 + (d\rho/d\tau)^2$$
$$= [c^2t-a^2(t-\tau)^3]^2/(t-\tau)^2[a^2(t-\tau)^2-c^2].$$

It is readily seen that $S = \infty$ at $\tau = \tau_3$, $S = 0$ at $\tau = \tau_2$ and $S = t^2/(a^2t^2-c^2)$ at $\tau = 0$. Thus the "strongest" shock is at the cusp of the wave envelope and decreases to zero at the center of the rear branch and to a finite value at the vehicle location. The width of the line in Figure 1 roughly represents this variation.

The shock forward of the cusp to the accelerating vehicle is formed from the envelope of emitted sound in much the same way as the Mach cone is formed for the vehicle in uniform motion. (Of course, in the real case the shock is basically derived from the air displacement of the vehicle itself and its strength would depend on vehicle aerodynamics, but these details are of no importance for the treatment here.) The shock that crosses the flight path from cusp to cusp behind the accelerating vehicle is the envelope of emitted sound waves originating back to the time that the vehicle first became supersonic and which have been "out run" by the accelerating vehicle.

The expressions summarized in Figure 2 for all of the geometric properties of the shock configuration are dependent on only the total flight time. While the vehicle is accelerating, the point where the following shock intersects the flight path moves only at one speed and consequently falls farther and farther behind the vehicle. The edge of the skirt of the "cone" or cusp point moves forward and outward in a direction normal to the cusp tangent at sonic velocity. Consequently it follows the vehicle motion with a component less than the speed of sound and falls behind even more rapidly than the "following" shock on the flight path. [Note, this elementary treatment neglects all such items like the fact that shock speed is dependent on shock strength and is somewhat unrealistic in these respects.]

The implications for the transmission of sound from the moving vehicle to the earth are reasonably apparent. Only sound emitted within an angle of $\tan^{-1}[c/(a^2t^2c^2)^{\frac{1}{2}}]$ of the flight path can effectively be transmitted rearward. The rearward transmitted sound must pass through the "following" shock. Near the vehicle axis this shock is negligibly weak, but toward the skirt or cusp it becomes quite strong and may modify the sound transmission.

2. Factors Affecting Sound from a Moving Source

Several factors that must be considered in estimating sound intensity from a moving source, but which may be neglected when the source is stationary are discussed briefly.

The downward propagation of sound is subject to some limitations due to the vertical profile of combined speed of sound and wind component. It is possible for some rays to be refracted upward and never reach the ground. As with the source on the ground, some initially upward rays may be refracted back down to the ground.

The directivity of the initial source is much more pronounced for the moving jet noise source. This directivity factor must be accounted for so that a ray on a given azimuth and elevation must be tied to the jet engine. This requires that engine pitch and yaw be known as a vehicle trajectory parameter in addition to the coordinates and velocity components.

As the source speed increases, doppler effects cause pronounced shifts to lower frequencies.

The source and receiver are at widely different altitudes so that density ratios (and speed of sound ratios) modify the intensity calculations.

a. Downward Sound Propagation

One is interested primarily in the downward propagation of sound from a rising vehicle, the limiting elevation angle for phase normals at the vehicle is of some importance. These limits are readily evaluated from the modified form of Snell's Law. For an horizontal phase normal (and ray tangent), $\varphi = 0$, so that

$$c+u = c_o/\cos\varphi_o + u_o$$

for any particular azimuth. Sound source values are indicated by the zero subscript. If then the value c+u exceeds c_o+u_o, there will be some initial phase normal, φ_o, satisfying the above relation.

Let B be the largest value of c+u at or below the level Z_o of the source and let A be the largest at or above the level Z_o. Let φ_A or φ_B be the values of the phase normal obtained from the modified Snell's Law with the values of c+u taken as A or B respectively. The solution in the range $(0,\pi/2)$ will be considered. These values are all functions of the source level, Z_o. Then the rays may be classed as follows:

1.	$-\pi/2 < \varphi_o < -\varphi_B$	descending rays that reach the ground
2.	$-\varphi_B < \varphi_o < 0$	descending rays refracted upward
3.	$0<\varphi_o<\varphi_B$, $\varphi_B < \varphi_A$	ascending rays that are trapped
4.	$\varphi_B<\varphi_o<\varphi_A$, $\varphi_B < \varphi_A$	ascending rays that are refracted to earth
5.	$0<\varphi_o<\varphi_A$, $\varphi_A < \varphi_B$	ascending rays that are trapped
6.	$\varphi_A < \varphi_o$, $\varphi_A < \varphi_B$	ascending rays that continue upward

It is readily seen that of the above 6 cases, only two result in rays reaching the ground; Cases 1 and 4. Once the values φ_B and φ_A are obtained, it is sufficient to check rays $-\pi/2 < \varphi_o < -\varphi_B$ and $\varphi_B < \varphi_o < \varphi_A$, when $\varphi_B < \varphi_A$, to locate the phase normal range within which a ray will return to earth.

b. Directivity of the Moving Jet Source

The mean square sound pressure radiated in the direction ψ from a moving jet source is given by Ribner[4] as

$$\overline{p^2(x,\theta)} = \frac{3\omega_f^4 (\overline{p_o^2}) L^3}{4\pi^2 c_o^4 x^2} \; \frac{1+M_o \cos\psi_e}{(1-M_o^2 \sin^2\psi) C^5}$$

where

$$\cos\psi_e = \cos\psi (1-M_o^2 \sin^2\psi)^{\frac{1}{2}} - M_o \sin^2\psi$$

and

$$x^2 (1-M_o^2 \sin^2\psi) = x_e^2 (1+M_o \cos\psi_e)^2$$

and in which the symbols have the significance

ω_f	= radian frequency of turbulence in the correlation $\exp(-\omega_f \lvert\tau\rvert)$
$\overline{p_o^2}$	= mean square sound pressure at the source
$\overline{p^2}$	= mean square sound pressure at the receiver
L	= turbulence scale in the jet
M_o	= flight mach number
M_e	= (eddy convection speed)/c_o = $U_j/2c_o$
U_j	= nozzle gas speed
c_o	= ambient speed of sound in the air
ψ	= angle from the jet axis to the observer (apparent)
x	= (apparent) source receiver distance
x_e	= effective source receiver distance
C	= convection factor

The relations between X, x_e, θ, θ_e are shown in Figure 3.

Note that in this illustration the source is at A when the sound is received at O and has apparently traveled the distance x. Actually, the sound was emitted at E and traveled the distance x_e. The distance traveled by the jet is AE = $U_o t^*$ while the distance traveled by the sound in the same time is $c_o t^* = x_e$.

The convection factor is given by

$$C^2 = (1-M_c \cos\psi)^2 + \alpha^2 M_c^2$$

where

$$\alpha^2 = \omega_f^2 L^2/\pi c_o^2 M_c^2 = 4\omega_f^2 L^2/\pi U_j^2$$

and is a function of the jet nozzle characteristics. M_c is the jet mach number $M_c = U_j/c_o$.

This may be written a bit more conveniently in terms of effective angle and distance as

$$\overline{p^2(x,\psi)} = 3\omega_f^4(\overline{p_o^2})L^3/4\pi^2 c_o^4 x_e^2(1+M_o\cos\psi_e)$$

or in the usual standard form for an inverse square radiator

$$\overline{p^2(x,\psi)} = \{3\omega_f^4(p_o^2)L^3/4\pi^2 c_o^4\} x_e^{-2}\{C^5(1+M_o\cos\psi_e)\}^{-1}$$

where the first factor is an initial sound strength, the second corresponds to the inverse square radiator, and the last factor involves only the directivity of the sound source. The angle ψ_e is to be used in the expression for C.

The analysis summarized above for the mean square acoustical pressure from a moving jet source requires important modification for use in a non-uniform atmosphere. The preceding results pertain to a uniform atmosphere. The factors x_e^{-2} and $(1+M_o\cos\psi_e)^{-1}$ of the last expression involve the sound transmission through the atmosphere and should be deleted. In their place one substitutes R^{-2} and f, where R is the source receiver distance and f is the focusing factor. The result is expressed as

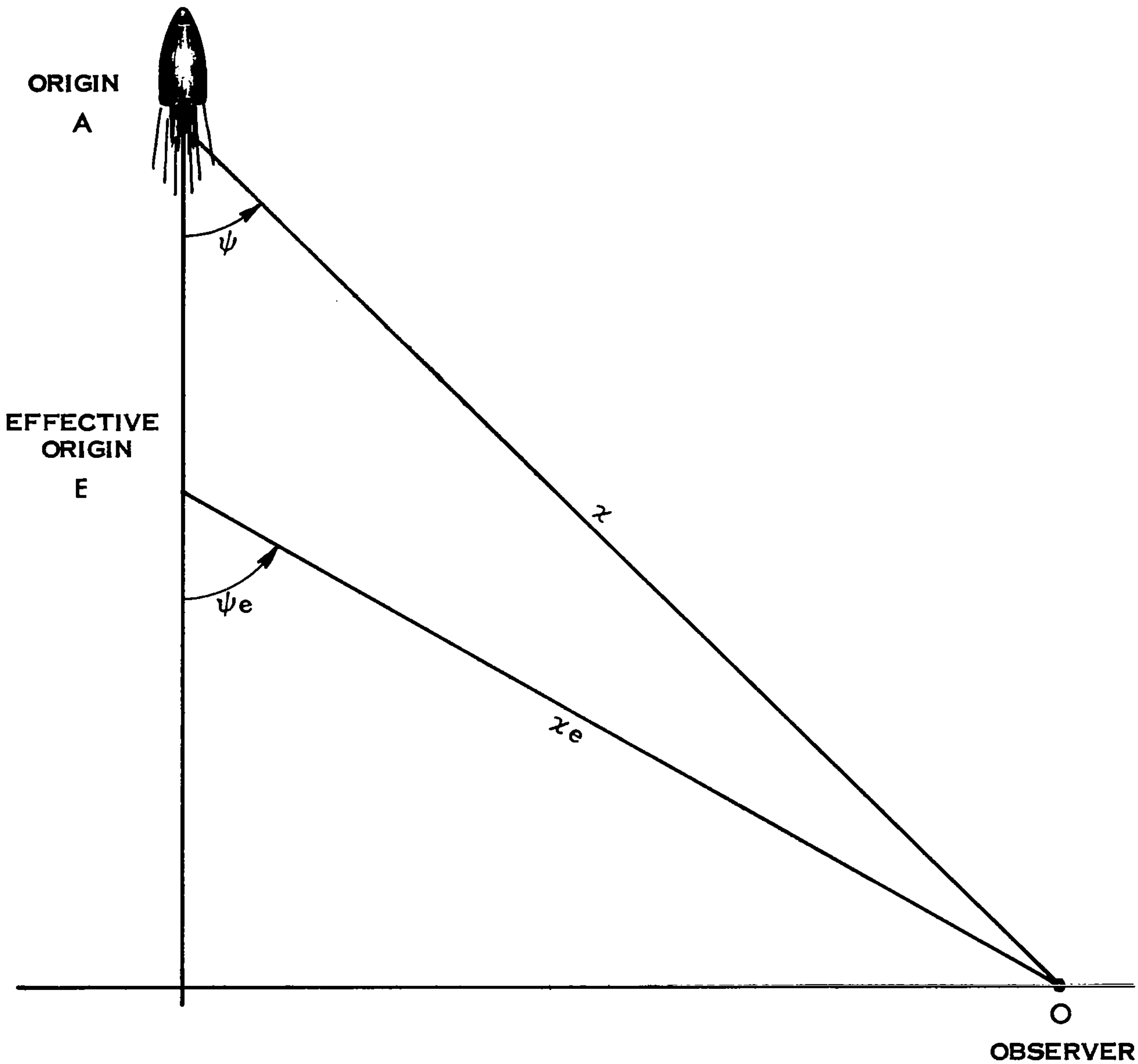

FIGURE 3

RELATION BETWEEN THE VEHICLE LOCATION (A), THE ORIGIN OF THE SOUND (E) AND THE OBSERVER (O).

$$\overline{p^2(r,\psi)} = [3\omega_f^4 \overline{(p_o^2)}\, L^3/4\pi^2 c_o^4 C^5](f/R^2).$$

The directivity of the jet source is taken into account in C. The focusing factor, f, will bear the burden of the effects of atmospheric nonhomogeniety and source velocity. The details required for computing the focusing factor under these conditions are discussed at length in a separate section.

c. Coupling the Ray Tracing to the Source

The ray tracing picture is calculated indepdendently of the source. To provide the correct input sound intensity and frequency, the rays must be coupled with the local geometry of the vehicle and its motion.

The ray parameters are φ_o and θ_o, the altitude and azimuth angles of the initial phase normal, respectively, referred to an earth coordinate system in which the reference plane (X_o, Y_o) is parallel to the earth's surface and the Z_o axis is directed vertically.

If the vehicle is assumed to be oriented along the tangent of the trajectory, then angle of the sound ray with respect to the vehicle, ψ, is given by the formula

$$\cos\psi = \sin\varphi_o \sin\varphi_v + \cos\varphi_o \cos\varphi_v \cos(\theta_v - \theta_o)$$

where θ_o = sound ray azimuth angle

φ_o = sound ray elevation angle

θ_v = vehicle trajectory azimuth

φ_v = vehicle trajectory elevation angle.

where the reference frame is an earth fixed system and ψ is the angle that the sound ray makes with respect to the vehicle trajectory.

The vehicle orientation is not necessarily that described above since for guidance purposes it does not always point along the trajectory tangent. When such departures from the trajectory tangent are appreciable, the vehicle orientation angles should be used in place of θ_v and φ_v above.

The angle of the sound ray with respect to the moving vehicle is important since the vehicle sound source is highly directional. Some details of these characteristics are discussed in the next section.

d. Doppler Effect (Uniform Case)

The change of frequency of sound from a moving source as received by a stationary observer is well known. The frequency of the sound as received, ω', is related to the initial frequency on the moving source ω by the relation

$$\omega' = \omega[1-(1/c)(dR/dt)]$$

where c is the ambient speed of sound and R is the distance from the sound source at time of emission and the receiver when this sound pulse is observed. Then, from Figure 4,

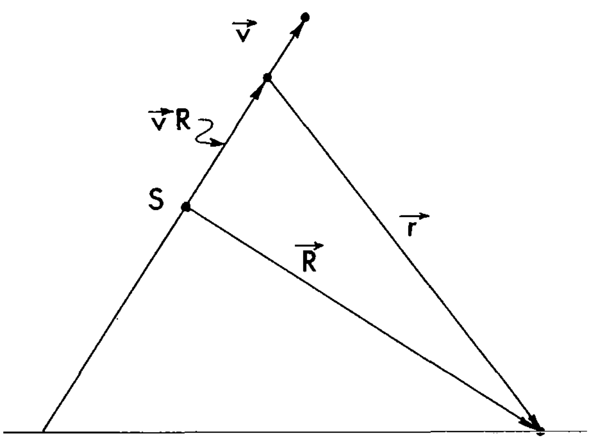

FIGURE 4

Geometrical Relations in Doppler Effect.

$$\vec{R} = \vec{r} + \vec{v}R/c.$$

Then

$$d\vec{R}/dt = d\vec{r}/dt + (\vec{v}/c)(dR/dt)$$

Now $d\vec{r}/dt = -\vec{v}$ where $\vec{v}$ is the (uniform) source speed so that

$$d\vec{R}/dt = -\vec{v}[1-(dR/dt)/c]$$

and

$$\vec{R}\cdot(d\vec{R}/dt) = R(dR/dt) = -(\vec{R}\cdot\vec{v})[1-(dR/dt)/c].$$

Then

$$(dR/dt)[R-(\vec{R}\cdot\vec{v})/c] = -(\vec{R}\cdot\vec{v})$$

and

$$(dR/dt)/c = -[(\vec{R}\cdot\vec{c})/c]/[R- (\vec{R}\cdot\vec{v})/c].$$

If $\vec{R}\cdot\vec{v} = Rv \cos \psi_e$ and $v/c = M$, then

$$\omega' = \omega/(1-M\cos\psi_e)$$

Note that the angle ψ_e is the angle between the direction of motion of the source $\vec{v}$ and the direction of sound transmission that will arrive at the receiver. In the subsonic case this is unique, but in the supersonic case there may be two such.

The second sound source that would be heard by an observer is not discussed here since it is physically nonexistent for a vehicle that is rising on a nearly vertical trajectory. If the vehicle were flying nearly horizontally it would be possible for both sound sources to be heard simultaneously. As the shock passes the observer the sound would come from a single source located at a point where the perpendicular to the shock at the observer meets the flight path. This source would instantly separate into two sources, one moving forward and the other backward. The backward moving apparent source is not effective for a ground observer in the case of a really vertically rising vehicle.

e. Acoustical Coupling with the Atmosphere

The conditions along a ray tube are characterized by the relation

$$\overline{p^2} V_s A/(\overline{p_1^2}) V_{s1} A_1 = \rho q c^2/\rho_1 q_1 c_1{}^2$$

where symbols with and without subscripts refer to two different points and

$\overline{(p^2)}$ = mean square acoustical pressure

V_s = speed of propagation along the ray tube $V_s = |\vec{V}_s|$, $\vec{V}_s = c\vec{n} + \vec{w}$

c = speed of sound

$\vec{n}$ = unit phase normal vector

$\vec{w}$ = wind speed vector

A = ray tube cross section

ρ = atmospheric density

q = $cc_o/(c+w_n)$

c_o = reference speed of sound

w_n = projection of wind speed vector on the phase normal

In a slightly different form

$$\overline{(p^2)} = \overline{(p_1^2)}(A_1/A)(\rho/\rho_1)(q/q_1)(c/c_1)^2(V_{s_1}/V_s)$$

The factor (A_1/A) is handled by the ray tracing method. The factor (ρ/ρ_1), the ratio of air density at the receiver to that at the source, is an easily obtained amplification factor. The product of the last three factors is nearly c/c_1, as follows,

$$V_s^2 = c^2+2cw_n+w^2$$

so

$$(V_{s_1}c^2q)/(V_sc_1^2q) = \left\{\frac{1+2(w_n/c_1)+(w_1/c_1)^2}{1+2(w_n/c)+(w/c)^2}\right\}^{\frac{1}{2}} \cdot \frac{1+w_n/c_1}{1+w_n/c} \quad \frac{c}{c_1}$$

$$\cong c/c_1$$

since all terms w/c, w_n/c are small compared with 1.

Consequently the mean square acoustical pressure is approximated by

$$(p^2) \cong (p_1^2)(A_1/A)(c\rho/c_1\rho_1)$$

where the last factor is a function of the atmospheric conditions at the source and receiver only. The first factor represents the initial source strength, including the directivity effects discussed previously. The second factor, the ray tube area ratio, is the part that is handled by ray tracing method.

3. Ray Tracing from a Moving Source

The basic problem for ray tracing from a stationary source is the estimation of the focusing factor which modifies the intensity from the inverse square law spreading to that of the rays curved and reflected by the atmosphere. In the stationary source case, it is sufficient to consider a "static" problem and disregard the fact that a "wave front" arrives at different locations at different times.

When the moving source is considered, the focusing factor must be put on a basis of simultaneous arrival of the sound. In order to do this, we consider first a more general form of the focusing factor. The general formulation of the ray tracing method is analyzed so that the derivations concerned are put on a total time basis. The detailed expressions for the derivative terms are then considered and expressed in terms of derivatives in which time is ignored.

a. Focusing Factor

The focusing factor may be defined as the ratio of the area of a ray tube reaching the ground to that of a ray tube that represents spherical spreading. The tube area for spherical spreading is

$$A_S = R^2 \cos\varphi \, \Delta\varphi \, \Delta\theta$$

where R is the source-receiver distance, φ is the ray elevation angle, and θ is the ray azimuth angle.

At any given instant, the ray has coordinates given by

$$x(\varphi,\theta\ \tau),\ y(\varphi,\theta,\tau),\ z(\varphi,\theta,\tau)$$

which represent points on an expanding surface where (φ,θ) are coordinates on the surface. For constant (φ,θ), the ray point moves along a space curve with parameter τ. The vector

$$\partial x/\partial\tau,\ \partial y/\partial\tau,\ \partial z/\partial\tau: \text{ velocity vector}$$

is the velocity vector along the ray. The vectors that are tangent to the surface are given by

$\partial x/\partial\varphi$, $\partial y/\partial\varphi$, $\partial z/\partial\varphi$: tangent to curves θ = const.

$\partial x/\partial\theta$, $\partial y/\partial\theta$, $\partial z/\partial\theta$: tangent to curves φ = const.

The ray tube intersects the ground in a mapping of the source coordinates (φ,θ) onto the ground. The intersection of the ray tube with the ground is identified with a quadrilateral determined by the points $x(\varphi,\theta)$, $y(\varphi,\theta)$; $x(\varphi+\Delta\varphi,\theta)$, $y(x+\Delta\varphi,\theta)$; $x(\varphi,\theta+\Delta\theta$, $y(\varphi,\theta+\Delta\theta)$; and $x(\varphi+\Delta\varphi,\theta+\Delta\theta)$, $y(\varphi+\Delta\varphi, \theta+\Delta\theta)$. These points are all associated with different times of arrival if the sound is initiated at a single time. Conversely, the same situation holds if the mapping of sound emission coordinates (φ,θ) on the ground is viewed at a single total time, but in this case the sound is emitted at different times for each point.

The area of the ground intersection is found from the usual differential approximation

$$A_g=(r_\varphi x r_\theta)\Delta\varphi\Delta\theta = [(\partial x/\partial\varphi)(\partial y/\partial\theta)-(\partial x/\partial\theta)(\partial y/\partial\varphi)]\Delta\varphi\Delta\theta.$$

It is necessary to correct this by multiplying by the sine of the inclination angle of the ray, φ_p, to obtain the area of the ray tube itself. Thus

$$A = A_g \sin\varphi_p.$$

is the area of the ray tube as it reaches the ground. The focusing factor is then given by

$$f = A_S/A = |R^2\cos\varphi_o/[(\partial x/\partial\varphi)(\partial y/\partial\theta)-(\partial x/\partial\theta)(\partial y/\partial\varphi)]\sin\varphi_p| \quad (1)$$

where φ_o = inclination angle at the source, φ_p = inclination angle at the receiver.

b) Reduction to Time Dependent Arrival

The preceding section on the focus factor was expressed in terms of the simultaneous arrival of sound everywhere on the ground. To facilitate computation, it is helpful to express the focusing factor in terms of quantities obtained for simultaneous emission time. The ray equations are used for this purpose.

In general form

$$x^* = X(t) + x(Z(t), \varphi, \theta) \quad (2)$$

$$y^* = Y(t) + y(Z(t), \varphi, \theta) \quad (3)$$

$$\tau = \tau(Z(t), \varphi, \theta) \quad (4)$$

$$T = t + \tau \quad (5)$$

where X(t), Y(t), Z(t) are coordinates of the source at emission time t, x, y, are the coordinates of the ray with inclination φ, azimuth θ, with respect to a stationary source when it reaches the ground. The travel time τ is computed as from a stationary source from (4). The total arrival time is given by (5).

Then

$$\partial x^*/\partial\varphi = (\partial t/\partial\varphi)[\partial X/\partial t + (\partial x/\partial Z)(\partial Z/\partial t)] + \partial x/\partial\varphi$$

$$\partial x^*/\partial\theta = (\partial t/\partial\theta)[\partial X/\partial t + (\partial x/\partial Z)(\partial Z/\partial t)] + \partial x/\partial\theta$$

$$\partial y^*/\partial\varphi = (\partial t/\partial\varphi)[\partial Y/\partial t + (\partial y/\partial Z)(\partial Z/\partial t)] + \partial y/\partial\varphi$$

$$\partial y^*/\partial\theta = (\partial t/\partial\theta)[\partial Y/\partial t + (\partial y/\partial Z)(\partial Z/\partial t)] + \partial y/\partial\theta$$

The first factor of the first term may be determined from (4) and (5)

$$t = T - \tau(Z(t)_1\varphi_1\theta)$$

so that for T = constant

$$-\partial t/\partial\varphi = (\partial\tau/\partial Z)(\partial Z/\partial t)(\partial t/\partial\varphi) + \partial\tau/\partial\varphi,$$

$$-\partial t/\partial\theta = (\partial\tau/\partial Z)(\partial Z/\partial t)(\partial t/\partial\theta) + \partial\tau/\partial\theta,$$

or

$$\partial t/\partial\varphi = -(\partial\tau/\partial\varphi)/[1+(\partial\tau/\partial Z)/(\partial Z/\partial t)],$$

$$\partial t/\partial\theta = -(\partial\tau/\partial\theta)/[1+(\partial\tau/\partial Z)/(\partial Z/\partial t)].$$

whence

$$\partial x_*/\partial\varphi = \partial x/\partial\varphi-(\partial\tau/\partial\varphi)[(\partial X/\partial t)+(\partial x/\partial Z)(\partial Z/\partial t)]/[1+(\partial\tau/\partial Z)(\partial Z/\partial t)]$$

$$\partial x_*/\partial\theta = \partial x/\partial\theta-(\partial\tau/\partial\theta)[(\partial X/\partial t)+(\partial x/\partial Z)(\partial Z/\partial t)]/[1+(\partial\tau/\partial Z)(\partial Z/\partial t)]$$

$$\partial y_*/\partial\varphi = \partial y/\partial\varphi-(\partial\tau/\partial\varphi)[(\partial Y/\partial t)+(\partial y/\partial Z)(\partial Z/\partial t)]/[1+(\partial\tau/\partial Z)(\partial Z/\partial t)]$$

$$\partial y_*/\partial\theta = \partial y/\partial\theta-(\partial\tau/\partial\theta)[(\partial Y/\partial t)+(\partial y/\partial Z)(\partial Z/\partial t)]/[1+(\partial\tau/\partial Z)(\partial Z/\partial t)]$$

If one writes $U = \partial X/\partial t$, $V=\partial Y/\partial t$, $W=\partial Z/\partial t$, then

$$\partial x_*/\partial\varphi = \partial x/\partial\varphi-(\partial\tau/\partial\varphi)(U+W\partial x/\partial Z)/(1+W\partial\tau/\partial Z)$$

$$\partial x_*/\partial\theta = \partial x/\partial\theta-(\partial\tau/\partial\theta)(U+W\partial x/\partial Z)/(1+W\partial\tau/\partial Z)$$

$$\partial y_*/\partial\varphi = \partial y/\partial\varphi-(\partial\tau/\partial\varphi)(V+W\partial y/\partial Z)/(1+W\partial\tau/\partial Z)$$

$$\partial y_*/\partial\theta = \partial y/\partial\theta-(\partial\tau/\partial\theta)(V+W\partial y/\partial Z)/(1+W\partial\tau/\partial Z).$$

Note that U, V, W are evaluated at the emission time t rather than at the total time T. The first terms, $\partial x/\partial\varphi$,---,$\partial y/\partial\theta$ are evaluated from the rays as computed from the ray trace method neglecting vehicle motion (stationary source at X,Y,Z). The terms $\partial\tau/\partial\varphi$,$\partial r/\partial\theta$ are computed from the same only the sound travel time is required. U,V,W are the vehicle velocity components. The terms $\partial x/\partial Z$, $\partial y/\partial Z$, $\partial\tau/\partial Z$ are changes of ground coordinate and time per unit change of the source height. Thus, five ray tacing quantities are needed in addition to the usual terminal ray coordinates: $\partial\tau/\partial\varphi$, $\partial\tau/\partial\theta$, $\partial x/\partial Z$, $\partial y/\partial Z$, $\partial\tau/\partial Z$.

c) Simplified System

If the rays are propagated in a plane, then one can write

$$x = r(Z(t),\varphi)\cos\theta,$$

$$y = r(Z(t),\varphi)\sin\theta.$$

Then

$$\partial x/\partial\varphi = (\partial r/\partial\varphi)\cos\theta,$$

$$\partial y/\partial\varphi = (\partial r/\partial\varphi)\sin\theta,$$

$$\partial x/\partial\theta = -r\ \sin\theta,$$

$$\partial y/\partial\theta = r\ \cos\theta,$$

$$\partial x/\partial Z = (\partial r/\partial Z)\cos\theta,$$

$$\partial y/\partial Z = (\partial r/\partial Z)\sin\theta,$$

$$\partial\tau/\partial\theta = 0,$$

so that

$$\partial x_*/\partial\varphi = (\partial r/\partial\varphi)\cos\theta-(\partial\tau/\partial\varphi)(U+W(\partial r/\partial Z)\cos\theta)/(1+W\partial\tau/\partial Z),$$

$$\partial y_*/\partial\varphi = (\partial r/\partial\varphi)\sin\theta-(\partial\tau/\partial\varphi)(V+W(\partial r/\partial Z)\sin\theta)/(1+W\partial\tau/\partial Z),$$

$$\partial x_*/\partial\theta = -r\ \sin\theta,$$

$$\partial y_*/\partial\theta = r\ \cos\theta.$$

The denominator factor in the focus factor (1) involving these derivatives becomes

$$(\partial x_*/\partial\varphi)(\partial y_*/\partial\theta)-(\partial x_*/\partial\theta)(\partial y_*/\partial\varphi)=r[\partial r/\partial\varphi-(\partial\tau/\partial\varphi)W(\partial r/\partial Z)/(1+W\partial\tau/\partial Z)]$$

$$-r(\partial\tau/\partial\varphi)(U\cos\theta+V\sin\theta)/(1+W\partial\tau/\partial Z)$$

The factor of the second term which involves the azimuth is exactly the horizontal component of source velocity in the azimuth plane of sound propagation. This term is replaced by U_θ so the expression shortens somewhat to

$$[r(\partial r/\partial\varphi_o)]_T = \partial r/\partial\varphi-(\partial\tau/\partial\varphi)(U_\theta-W\partial r/\partial Z)/(1+W\partial\tau/\partial Z)] \qquad (6)$$

in which all quantities are found in the azimuth plane of propagation. It is readily seen that if U = W = 0, stationary source, the factor above reduces to $r(\partial r/\partial\varphi)$ and the focus factor is of the standard type for a stationary source.

The full expression for the focusing factor is then

$$f = R^2\cos\varphi_o/[r(\partial r/\partial\varphi_o)]_T\sin\varphi_p \qquad (7)$$

where by $r[dr/d\varphi_o]_T$ is meant that the expression is to be evaluated as above at constant arrival time.

i) A Simple Example

Consider a source moving with a velocity U parallel to the ground at an height H. The parametric equations for a ray point are

$$x = Ut+c\tau\cos\varphi$$

$$z = H - c\tau\sin\varphi$$

$$T = t+\tau$$

where t is the time of emission of sound, τ the time of travel along a ray path, T the total time and φ the angle of emission measured downward from the source path. The sound reaches the ground in a travel time $\tau = H/c\sin\varphi$ determined by z = 0 and solving the second relation for τ. The time of emission along a ray with initial angle φ is obtained from $t=T-H/c\sin\varphi$.

These expressions for t and τ substituted in the first equation give the expression for ground coordinate of arriving sound at total time T as a function of the angle of emission, φ

$$x = UT + H(\cos\varphi - M)/\sin\varphi$$

where $M = U/c$ is the source Mach number.

The focusing factor is

$$f = |R^2\cos\varphi/x(\partial x/\partial\varphi)_T\sin\varphi|$$

where $(\partial x/\partial\varphi)_T$ is calculated on the basis of simultaneous arrival time T and it is to be noted that $R^2 = H^2 + x^2$ where x is calculated on the basis of the horizontal travel distance in the time τ. Thus, $x = c\tau\cos\varphi = H\cos\varphi/\sin\varphi$ and $R^2 = H^2/\sin^2\varphi$. It is easily verified that

$$(\partial x/\partial\varphi)_T = -H(1-M\cos\varphi)/\sin^2\varphi$$

and so

$$f = 1/|1-M\cos\varphi|.$$

For subsonic speeds the focusing factor is largest for $\varphi=0$, $1/(1-M)$, and smallest at $\varphi=\pi$, $1/(1+M)$ and has the value 1 at $\varphi=\pi/2$. In general, f, decreases monotonically from $\varphi=0$ to $\varphi=\pi$ so that the sound at the ground is of larger intensity ahead of the source and of smaller intensity at some distance behind the source than would have been the case for a stationary source. Note, however, that the crossover point is not beneath the source at time T (closest distance), but at a distance HM behind the source position UT (by substituting $\varphi=0$ in the expression for x).

In the case of supersonic speed, the focusing factor becomes large without bound at $\cos\varphi = 1/M$ or at a distance $H\sqrt{M^2-1}$ behind the source. This corresponds to the intersection of the Mach cone with the ground. Note further that the expression for x_* contains the fact that to the rearward of the point $H\sqrt{M^2-1}$ behind the present location of the source, UT, each surface point receives sound from two different angles φ_1, φ_2. This is readily seen from the expression for x which is large and negative for small φ and for φ near π. As φ increases from 0 to $\cos^{-1}(1/M)$ the point x moves from $-\infty$ to $UT-H\sqrt{M^2-1}$ and as it increases from $\cos^{-1}(1/M)$ to π the point x retreats again to $-\infty$.

The above simply arithmetical example presupposes knowledge of the ray path which is used extensively in the manipulations. To compute the focusing factor without using such extensive knowledge, we consider the focusing factor as derived previously

$$f = |R^2\cos\varphi_o/[r(\partial r/\partial\varphi_o)]_T\sin\varphi_p|$$

where

$$[r(\partial r/\partial\varphi_o)]_T=r[\partial r/\partial\varphi_o-(\partial\tau/\partial\varphi_o)(W\partial r/\partial T + U_r)/(1+W\partial\tau/\partial Z)].$$

In this simple example, W=0, V_r=V so that

$$[r(\partial r/\partial\varphi_o)]_T = r[\partial r/\partial\varphi - U\partial\tau/\partial\varphi]$$

and $\varphi_o = \varphi_p = \varphi$. If the source were stationary, the ray intersections with the ground would be given by $r=H\cot\varphi$ and the time of sound travel would be given by $\tau = H/c\sin\varphi$. These are quantities that would be calculated by the ray tracing method in the stationary source case. Then $\partial r/\partial\varphi = -H/\sin^2\varphi$, $\partial\tau/\partial\varphi = -H\cos\varphi/c\sin^2\varphi$ and $R^2=H^2+r^2=H^2/\sin^2\varphi$. Substituting these expressions, the focusing factor becomes $f = 1/|1 - M\cos\varphi|$ as before.

d) The Ray Tracing Terms

The terms required from ray tracing procedure are thence $\partial r/\partial\varphi_o$, $\partial r/\partial Z$, $\partial\tau/\partial\varphi_o$, and $\partial\tau/\partial Z$. The source motion implies that time of sound transmission is required. Both radial distance and sound travel time are differentiated with respect to the initial ray inclination and the source height. These are the two basic parameters that enter into Snell's Law constant

$$K = c_o/\cos\varphi_o + u_o,\ c_o = c_o(Z),\ u_o = u_o(Z).$$

Since the Snell's law constant is the only place where these initial conditions occur, the derivatives can be replaced by derivatives with respect to Snell's Law constant, K, and the other derivatives computed using an appropriate factor. Thus

$$\partial r/\partial\varphi_o = (\partial r/\partial K)(\partial K/\partial\varphi_o),$$
$$\partial r/\partial Z = (\partial r/\partial K)(\partial K/\partial Z),$$
$$\partial\tau/\partial\varphi_o = (\partial\tau/\partial K)(\partial K/\partial\varphi_o),$$
$$\partial\tau/\partial Z = (\partial\tau/\partial K)(\partial K/\partial Z).$$

One then needs $\partial r/\partial K$, $\partial\tau/\partial K$ and the derivative factors, $\partial K/\partial\varphi_o$, $\partial K/\partial Z$. It is readily seen that

$$\partial K/\partial\varphi_o = c_o\sin\varphi_o/\cos^2\varphi_o,$$
$$\partial K/\partial Z = c_o'/\cos\varphi_o + u_o'.$$

i) The Distance Derivatives

The ray integral through a layer

$$x_2 - x_1 = \int_{z_1}^{z_2} [(c\cos\varphi + u)/c\sin\varphi]\,dz \qquad (8)$$

is transformed using Snell's Law

$$c/\cos\varphi + u = c_o\cos\varphi_o + u_o = K = \text{constant}$$

or

$$\cos\varphi = c/(K-u), \quad \cos\varphi_o = c_o/(K-u_o) \qquad (9)$$

into

$$x_2-x_1 = \int_{z_1}^{z_2}\{[c^2+u(K-u)]/c\sqrt{(K-u)^2-c^2}\}dz. \qquad (10)$$

The relation between φ_o, K, and z_o permits the following

$$\partial-/\partial\varphi_o = (\partial-/\partial K)(\partial K/\partial\varphi_o), \qquad (11)$$

$$\partial-/\partial Z_o = (\partial-/\partial K)(\partial K/\partial z_o), \qquad (12)$$

where

$$\partial K/\partial\varphi_o = c_o\sin\varphi_o/\cos^2\varphi_o,$$

$$\partial K/\partial z_o = c_o'/\cos\varphi_o + u_o',$$

and

$$c_o' = \partial c_o/\partial z_o, \quad u_o' = \partial u_o/\partial z_o.$$

Consequently, once $\partial x_2/\partial K - \partial x_1/\partial K$ has been obtained, the expressions for $\partial x_2/\partial\varphi_o - \partial x_1/\partial\varphi_o$ and $\partial x_2/\partial z_o - \partial x_1/\partial z_o$ are easily obtained by using the appropriate multiplier. The differentiation of x_2-x_1 with respect to K involves two distinct cases depending on whether or not the ray is reflected at the upper limit of integration. If no reflection occurs at z_2, i.e., the ray penetrates the layer (z_1,z_2), the upper limit, z_2, is fixed and the parameter K is contained only in the integrand.

<u>CASE I</u>: z_2 independent of K (ray penetrates the layer (z_1,z_2). Then, if $\Delta x = x_2-x_1$

$$\partial(\Delta x)/\partial K = \int_{z_1}^{z_2}\left[\partial\{[c^2+u(K-u)]/c\sqrt{(K-u)^2-c^2}\}/\partial K\right]dz$$

or

$$\partial(\Delta x)/\partial K = -\int_{z_1}^{z_2}\{Kc/[(K-u)^2-c^2]^{3/2}\}dz \qquad (13)$$

In this case the integration poses no problems since the integral is proper through the entire layer.

When reflection occurs at z_2, the upper limit of the integral is also dependent on K so that another term is involved.

CASE II: z_2 depends on K (ray reflected at z_2). Then formal differentiation leads to

$$\partial(\Delta x)/\partial K = \{[c^2+u(K-u)]/c\sqrt{(K-u)^2-c^2}\}_{z_2}\ \partial z_2/\partial K$$

$$-\int_{z_1}^{z_2}[Kc/\sqrt{(K-u)^2-c^2}]\,dz$$

It is readily seen that, since $K-u = c$ at z_2, both terms of this expression are meaningless at that level. To avoid this difficulty, reduce the range of integration by a small amount, ϵ, so that the upper limit becomes $z_2-\epsilon$ and consider the limit for $\epsilon \to 0$

$$\lim_{\epsilon \to 0}\ \partial(\Delta x)/\partial K = \lim_{z_* \to z_2}[A-B] \tag{14}$$

where

$$A = \{[c^2+u(K-u)]/c\sqrt{(K-u)^2-c^2}\}_{z_*}(\partial z_*/\partial K),$$

$$B = K\int_{z_1}^{z_*}\{c/[(K-u)^2-c^2]^{\frac{1}{2}}\}dz.$$

Since $\partial z_*/\partial K = c' + u'$ evaluated at z_*, then

$$A = \{[c^2+u(K-u)]/c\sqrt{(K-u)^2-c^2}(c'+u')\}_{z_*} \tag{15}$$

The integral in B requires some manipulation. This is integrated by parts. Note that

$$\frac{d}{dz}\{(K-u)/\sqrt{(K-u)^2-c^2}\} = c[(K-u)c'+cu']/[(K-u)^2-c^2]^{3/2}$$

so that

$$B = \int_{z_1}^{z_*} \frac{K}{(K-u)c'+cu'} \left[\frac{d}{dz}\left(\frac{K-u}{\sqrt{(K-u)^2-c^2}}\right)\right] dz$$

$$= \left[K(K-u)/[(K-u)c'+cu'] \sqrt{(K-u)^2-c^2} \right]_{z_1}^{z_*}$$

$$- \int_{z_1}^{z_*} \frac{K-u}{\sqrt{(K-u)^2-c^2}} \left[\frac{d}{dz}\left(\frac{K}{(K-u)c'+cu'}\right)\right] dz.$$

Let B be written in the form B=C-D+E where

$$C = \left[K(K-u)/[(K-u)c'+cu'] \sqrt{(K-u)^2-c^2} \right]_{z_*} , \qquad (16)$$

$$D = \left[K(K-u)/[(K-u)c'+ cu'] \sqrt{(K-u)^2-c^2} \right]_{z_1} , \qquad (17)$$

$$E = - \int_{z_1}^{z_*} \frac{K-u}{\sqrt{(K-u)^2-c^2}} \left[\frac{d}{dz}\left(\frac{K}{(K-u)c'+cu'}\right)\right] dz , \qquad (18)$$

so that (7) becomes

$$\lim_{\epsilon \to 0} [\partial(\Delta x)/\partial K] = \lim_{z_* \to z_2} (A-C) + D - \lim_{z_* \to z_2} E. \qquad (19)$$

The first term on the right of (19) requires some manipulation.

Thus

$$\sqrt{(K-u)^2-c^2}(A-C) = [c^2+u(K-u)]/c(c'+u') - K(K-u)/[K-u)c'+cu']$$

whence on combining fractions and simplifying

$$(A-C) = - \frac{[c'(K-u)(c-u) + u'c(K-u+c)]\sqrt{K-u-c}}{c(c'+u')[(K-u)c'+cu']\sqrt{K-u+c}}$$

so that

$$\lim_{z_* \to z_2} (A-C) = 0$$

and

$$\lim_{\epsilon \to 0} [\partial(\Delta x)/\partial K] = D - \lim_{z_* \to z_2} E \quad . \tag{20}$$

The remaining limit consists of formally substituting z_2 for z_* in (18). Then from (17) and (18)

$$\partial x_2/\partial K - \partial x_1/\partial K = \left[K(K-u)/[(K-u)c' + cu'] \sqrt{(K-u)^2 - c^2} \right]_{z_1} - \int_{z_1}^{z_2} \frac{K(K-u)[(K-u)c'' + cu'']}{[(K-u)c' + cu']^2 \sqrt{(K-u)^2 - c^2}} dz \tag{21}$$

This limit is used in place of the formal expression for $\partial x_2/\partial K - \partial x_1/\partial K$. The first term of (14) is acceptable since it is evaluated at z_1 where $K \neq u + c$. The integral term is acceptable because, though improper, it may be evaluated under almost the same circumstances as the integral for $x_2 - x_1$ itself. We are not concerned in the above with the case $c' = u' = 0$ since under these circumstances the ray concerned may not have its reflection level at z_2.

SPECIAL CASE U = 0

The presence of u(z) complicates the arithmetic of the above reduction. When $u(z) \equiv 0$, the situation is straightforward. We consider only the case of reflection at z_2 so that with $z_* = z_2 - \epsilon$ one obtains

$$\partial(\Delta x)/\partial K = c/\sqrt{K^2-c^2}]_z (\partial z/\partial K) + \int_{z_1}^{z_*} [\partial(c/\sqrt{K^2-c^2})/\partial K] dz.$$

Snell's Law at the reflection level is c = K so that $\partial z_*/\partial K = 1/c'$ and so, differentiating the integrand

$$\partial(\Delta x)/\partial K = c/c'\sqrt{K^2-c^2}\,]_{z_*} - \int_{z_1}^{z_*} [cK/(K^2-c^2)^{3/2} \quad dz.$$

Note that $d[(K^2-c^2)^{-1/2}]/dz = cc'/(K^2-c^2)^{3/2}$ so that the integrand may be written in as

$$(K/c')[d(K-c)^{-\frac{1}{2}}/dz].$$

Integrating by parts

$$\partial(\Delta x)/\partial K = [c/c'\sqrt{K^2-c^2}]_{z_*} - [K/c'\sqrt{K^2-c^2}\,]_{z_1}^{z_*}$$

$$- \int_{z_1}^{z_*} [Kc''/(c')^2 \sqrt{K^2-c^2}] dz \quad ,$$

$$= -[\sqrt{K-c}/c'\sqrt{K+c}]_{z_*} + K/c'\sqrt{K^2-c^2}]_{z_1}$$

$$- \int_{z_1}^{z_*} [Kc''/(c')^2 \sqrt{K^2-c^2}] \; dz \; .$$

The first term vanishes for $z_* \rightarrow z_2$. The result in the limit is

$$\partial x_2/\partial K - \partial x_1/\partial K = K/c' \sqrt{K^2-c^2}\Big]_{z_1} - \int_{z_1}^{z_2} [Kc''/ \ (c')^2 \ \ K^2-c^2] \, dz$$

which checks the previous result for $u(z) \equiv 0$.

THE TIME DERIVATIVES

The evaluation of $\partial\tau/\partial K$ also requires the consideration of the same details. The time to traverse the layer z_1, z_2 is

$$\tau_2 - \tau_1 = \int_{z_1}^{z_2} [K-u)/c \sqrt{(K-u)^2-c^2}\ ]\ dz$$

If the ray is not reflected at z_2, then z_2 is independent of Snell's constant, K,

$$\partial\tau_2/\partial K - \partial\tau_1/\partial K = -\int_{z_1}^{z_2} \{c/[\ (K-u)^2-c^2]^{3/2}\} dz. \qquad (22)$$

To obtain the result in the case the ray is reflected at the level z_2, we integrate to the level $z_* = z_2 - \epsilon$. This level will depend on the Snell's constant K via $z_2(K)$. Thus one has

$$\partial\tau_*/\partial K - \partial\tau_1/\partial K = [\ (K-u)/c \sqrt{(K-u)^2-c^2}]_{z_*} (\partial z_*/\partial K) - \int_{z_1}^{z_*} \{c/[\ (K-u)^2-c^2]^{3/2}\}\ dz$$

now since $K = u(z_*) + c(z_*)$ for the reflected ray, $\partial z_*/\partial K = 1/(c'+u')]_{z*}$ and the integrand of the last term is rewritten using

$$d[\ (K-u)/\sqrt{(K-u)^2-c^2}]/dz = c[\ (K-u)c'+cu']/[\ (K-u)^2-c^2]^{3/2}$$

to give

$$\partial\tau_*/\partial K - \partial\tau_1/\partial K = [\ (K-u)/c(c'+u') \sqrt{(K-u)^2-c^2}]_{z_*} - \int_{z_1}^{z_*} \frac{1}{(K-u)c'+cu'} \left[\frac{\partial}{\partial z}\left(\frac{K-u}{\sqrt{(K-u)^2\ -c^2}}\right)\right] dz\ .$$

Integrating by parts and collecting terms evaluated at z_* together

$$\partial\tau_*/\partial K-\partial\tau_1/\partial K=\Big[c'(K-u)\sqrt{K-u-c/\bar{c}}\;\;c'+u')\quad[(K-u)c'+cu']\sqrt{K-u+c}\Big]_{z_*}$$

$$+(K-u)/[(K-u)c'+cu']\sqrt{(K-u)^2-c^2}\Big]_{z_1}$$

$$-\int_{z_1}^{z_*}\frac{(K-u)\;[(K-u)c''+cu'']}{[(K-u)c'+cu']^2\quad(K-u)^2-c^2}\,dz\quad .$$

Then for $z_* \rightarrow z_2$ or $\epsilon \rightarrow 0$ the first term becomes zero with the result that

$$\partial\tau_2/\partial K-\partial\tau_1/\partial K = (K-u)/[(K-u)c'+cu']\sqrt{(K-u)^2-c^2}\Big]_{z_1}$$

$$-\int_{z_1}^{z_*}\frac{(K-u)[K-u)c''+cu'']}{[(K-u)c'+cu']^2\sqrt{(K-u)^2-c^2}}\,dz\,. \qquad (23)$$

SUMMARY

The focus factor for estimating sound intensity is given by the more general expression

$$f = \left[R^2\cos\varphi_o / \left[r\partial r/\partial\varphi_o\right]_T \sin\varphi_p\right]$$

where

$$\left[r\partial r/\partial\varphi_o\right]_T = r\left[\partial r/\partial\varphi_o - (\partial\tau/\partial\varphi_o)(W\partial r/\partial Z + U_\theta)/(1+W\partial\tau/\partial Z)\right]$$

stands for the usual expression but corrected for simultaneous time of arrival of the sound rays at the ground. The remaining symbols are

R = distance of source (at emission time) to receiver (at reception time)

φ_o = initial ray inclination angle

Z = source altitude

r = horizontal projection of the radial source-receiver distance, R

φ_p = ray inclination at the ground

τ = sound travel time

U_θ = horizontal component of source velocity in propagation plane

W = vertical source velocity component

All of the partial derivatives $\partial r/\partial\varphi_o$, $\partial\tau/\partial\varphi$, $\partial r/\partial Z$, $\partial\tau/\partial Z$ are obtained from the ray tracing results in usual form (i.e., the fact that the rays do not arrive simultaneously is ignored).

The radial distance and travel time are obtained from layer wise addition of the layer integrals

$$\Delta r = r_2 - r_1 = \int_{z_1}^{z_2} \left\{ \left[c^2 + a(K-u)\right] / c\sqrt{(K-a)^2 - c} \right\} dz$$

$$\Delta\tau = \tau_2 - \tau_1 = \int_{z_1}^{z_2} \left\{ (K-u/c\sqrt{(K-u)^2 - c^2} \right\} dz$$

where K is the Snell's law constant from

$$c/\cos\varphi + u = c_o/\cos\varphi_o + u_o = K$$

where c is the speed of sound, u is the wind component in the plane of propagation (both functions of altitude, Z). When the ray is reflected, the upper limit of integration is obtained from Snell's law by setting $\varphi = 0$ so that

$$c(z_2) + u(z_2) = K$$

and solution for z_2 is required.

The derivatives with respect to both φ_0 and Z are obtained from derivatives with respect to K from the relations

$$\partial-/\partial\varphi_o = (\partial-/\partial K)(\partial K/\partial\varphi_o) \ ,$$

$$\partial-/\partial Z = (\partial-/\partial K)(\partial K/\partial Z),$$

where

$$\partial K/\partial \varphi_o = c_o \sin\varphi_o / \cos^2\varphi_o ,$$

$$\partial K/\partial Z = c_o'/\cos\varphi_o + u_o' .$$

When the ray penetrates the layer concerned,

$$\partial(\Delta r)/\partial K = -\int_{z_1}^{z_2} \left\{ Kc/\left[(K-u)^2-c^2\right]^{3/2} \right\} dz$$

and

$$\partial(\Delta \tau)/\partial K = -\int_{z_1}^{z_2} \left\{ c/\left[(K-u)^2-c^2\right]^{3/2} \right\} dz$$

or

$$\partial(\Delta r)/\partial K = K(\partial(\Delta\tau)/\partial K).$$

When the ray is reflected at the upper boundary of the layer, these integrals cannot be evaluated. The resulting integrals are substituted:

$$\partial(\Delta r)\partial K = \left[K(K-u)/\left[(K-u)c'+cu'\right]\sqrt{(K-u)^2-c^2} \right]_{z_1}$$

$$-\int_{z_1}^{z_2} \frac{K(K-u)\left[(K-u)c''+cu''\right]}{\left[(K-u)c'+cu'\right]^2\sqrt{(K-u)^2-c^2}} dz,$$

and

$$\partial(\Delta\tau)/\partial K = (K-u)/\left[(K-u)c' + cu'\right]\sqrt{(K-u)^2-c^2}\,\Big]_{z'}$$

$$-\int_{z_1}^{z_2} \frac{(K-u)\left[(K-u)c'' + cu''\right]}{\left[(K-u)c' + cu'\right]^2 \sqrt{(K-u)^2-c^2}}\,dz$$

and again

$$\partial(\Delta r)/\partial K = K\left[\partial(\Delta\tau)/\partial K\right]$$

The ray tracing quantities to be calculated are then r, τ, $\partial\tau/\partial K$ with the additional quantities coming from

$$\partial\tau/\partial\varphi = (\partial\tau/\partial K)(\partial K/\partial\varphi_o),$$

$$\partial\tau/\partial Z = (\partial\tau/\partial K)(\partial K/\partial Z),$$

$$\partial r/\partial Z = K(\partial\tau/\partial K)(\partial K/\partial Z).$$

SHORT-CUTTING THE RAY TRACING ROUTINE

The process outlined for calculation of rays requires ray calculation for altitudes Z_o from the surface upward as separate operations. There is a possibility of shortening the calculations somewhat.

When one is dealing with an atmosphere made up of horizontally homogeneous layers the ray path is uniquely determined by the conditions at the start of a ray through the layers. Consider the source with elevation Z_o at some level say z_k and that rays have been computed through $z_k, z_{k-1}, ---, z_2, z_1, z_o$ = surface level = 0. Now consider the source at Z_o on some other level $z_\ell, \ell > k$. It is required to calculate the ray through the levels $z_\ell, ---, z_k, ---, z_1, z_o$.

In the first ray calculations at level $Z_o = z_k$, the rays have been calculated with initial elevation angles $\varphi^k_{o1}, ---, \varphi^k_{on}$, where n is the number of elevation angles used. These rays extend to the ground.

The second calculation with $Z_o = z_\ell$ requires the rays for elevation angles $\varphi^\ell_{o1}, ---, \varphi^\ell_{om}$. These rays are then calculated to the level z_k. The calculated data will consist of the horizontal travel $x_{\ell k}$ the time, $\tau_{\ell k}$, and the phase normal $\varphi_{k\ell}$. The phase normals may now be compared with the phase normals of the previous calculations. If $\min(\varphi^k_{oi}) < \varphi_{k\ell} < \max(\varphi^k_{oi})$, that is if $\varphi_{k\ell}$ lies within the range of angles φ^k_{oi} used to start rays at level z_k, then the remaining horizontal travel and time of travel may be calculated by interpolation. Thus, if we have $\varphi^k_{oi} < \varphi_{k\ell} < \varphi^k_{oi+1}$, then the remaining horizontal travel and time of travel for the ray be obtained by interpolation between the corresponding quantities associated with the values φ^k_{oi} and φ^k_{oi+1}.

When $\varphi_{i\ell}$ lies outside this range, then the calculations for the individual ray must be continued to the surface.

The continued use of the interpolation procedure for successively higher levels of the source Z_o requires that suitable precautions be taken to avoid loss of accuracy. Internal machine numbers usually contain far more significant digits than the situation really warrants. These, together with a reasonably high order interpolation formula (i.e., one that uses several values of φ^k_{oi} in the interpolation process) should suffice to preserve a reasonable degree of accuracy. Linear interpolation will probably result in quickly accumulated errors. In any case, a sophisticated interpolation will be more efficient than continuation of the ray tracing through the lower layers. An occasional check by comparing interpolated values with a complete ray trace should serve to keep errors under positive control and determine when the point of excessive error has been reached. At this point the rays can be all traced in detail again and the process resumed with fresh values.

REFERENCES

1. Brekhovskikh, L. M., Waves in Layered Media, (tr by D. Lieberman and R. T. Beyer), Academic Press, N. Y., 1960, xi+561 pp. (The example used here follows that of p. 499ff with some extensions to this particular application).

2. See reference 1 above, p. 483ff.

3. Marsh, H. W., "The Use of Ray Methods and First Order Diffraction Corrections," USL Tech. Memo No. 1100-61-54, U. S. Navy Underwater Sound Laboratory, New London, Connecticut.

4. Ribner, H.S., "The Generation of Sound by Turbulent Jets," pp. 103-181 in Advances in Applied Mechanics, Vol. 8, Academic Press, N. Y., 1964 (H. L. Dryden and Th. von Karman, eds.).

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