EFFECT OF SHALLOW MERIDIONAL CURVATURE ON THE VIBRATION OF NEARLY CYLINDRICAL SHELLS

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NASA TN D-5143

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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SUMMARY

The effect of the meridional curvature on the minimum natural frequencies of nearly cylindrical shells with either positive or negative Gaussian curvature is investigated. The positive Gaussian curvature shells have minimum natural frequencies well above those of corresponding cylindrical shells. The minimum natural frequencies of the negative Gaussian curvature shells generally are below those of the corresponding cylinders and evidence wide variations in value, large reductions in magnitude occurring at certain critical values of curvature. The large reductions in magnitude result from a condition of essentially inextensional vibration. It is demonstrated that membrane theory affords a simple method of determining the modal wavelength ratio and curvature combinations at which these reductions occur, whereas pure bending theory gives a good estimate of the magnitudes of the associated frequencies.

INTRODUCTION

Shells of double curvature are common elements in aerospace vehicle structures. Because of the complexity of the equations of motion governing their dynamic behavior, numerical methods are commonly employed for solution. (For example, see refs. 1 to 3.) Numerical methods are adequate and yield accurate results for many applications; however, they are not practical for extensive parameter studies of the behavior of shell systems even with present-day high-speed computers. On the other hand, if certain approximations are made to the equations of motion, closed-form solutions become possible in some instances and facilitate rapid calculations. The purpose of the present paper is to develop closed-form solutions of approximate equations for the natural vibrations of a specific class of shells of double curvature in order to determine the effect of the meridional curvature on the natural frequencies of vibration of these shells.

*The information presented herein is based on part of a thesis entitled "Vibration of Stressed Shells of Double Curvature," offered in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Engineering Mechanics, Virginia Polytechnic Institute, Blacksburg, Virginia, June 1968.

A set of nonlinear homogeneous equations governing the dynamic behavior of pre-stressed doubly curved shells of revolution with positive or negative shallow meridional curvature is developed. The procedure used in developing these equations is similar to that used in references 4 and 5 where Donnell-type shallow shell equations are derived. In the present paper, however, the shallowness assumption is made only in relation to the meridional direction. By imposing the assumption of zero static meridional rotation, linear vibration equations with constant coefficients are obtained. These approximate equations govern the infinitesimal natural vibrations of prestressed nearly cylindrical shells of revolution and have closed-form solutions. They are solved here for the natural vibration of shells with freely supported edges.

Results from the approximate analysis are compared with more accurate results obtained by the use of techniques and equations given in reference 3 to indicate the range of application of the present analysis. The approximate analysis is then used to investigate the effects of meridional curvature on the natural vibrations of freely supported shells as the length and thickness parameters are varied.

The equations of the membrane and inextensional shell theories are also solved for the natural vibration of freely supported shells. The solutions to the membrane theory give a simple relationship for determining vibration modes in which shells of negative meridional curvature undergo predominantly inextensional deformations, whereas the solutions to the inextensional theory yield good approximations to the frequencies of these modes.

SYMBOLS

B   extensional stiffness (eqs. (A14))

c   central rise of shell meridian

D   bending stiffness (eqs. (A14))

E   Young's modulus of elasticity

h   shell thickness

k_ξ nondimensional meridional curvature, R/R_ξ

m   number of axial half-waves

M_ξ, M_θ
$M_{\xi\theta}, M_{\theta\xi}$ } moment resultants associated with vibration state

n   number of circumferential waves

N_ξ, N_θ
$N_{\xi\theta}, N_{\theta\xi}$ } stress resultants associated with vibration state

$\overline{N}_\xi, \overline{N}_\theta$ static stress resultants

p   surface loading

r   circumferential radius of doubly curved shell

R   radius of cylinder; circumferential radius at midlength of doubly curved shell

R_ξ meridional radii of curvature

s   total meridional arc length

t   time

u,v,w   displacement variables defining the vibration state in meridional (ξ), circumferential (θ), and normal directions, respectively

z   coordinate measured normal to middle surface

β ratio of circumferential to axial wavelength

$\epsilon_\xi, \epsilon_\theta, \epsilon_{\xi\theta}$ middle-surface strains associated with vibration state

θ circumferential coordinate

$\kappa_\xi, \kappa_\theta, \kappa_{\xi\theta}$ middle-surface bending strains associated with the vibration state

λ thickness parameter, h/R

Λ_j coefficients of equation (B2) (j = 0, 1, 2)

μ Poisson's ratio (taken as $\mu = 0.3$ for all calculations)

ν mass density

ξ meridional coordinate

τ percent ratio of meridional rise to length, $\frac{c}{s}(100) \approx \frac{s}{8R_\xi}(100)$

ψ membrane stress function defined in equations (B5)

ω natural circular frequency

Ω^2 frequency parameter, $\dfrac{R^2\omega^2\nu\left(1 - \mu^2\right)}{E}$

∇^4 biharmonic differential operator, $R^4\dfrac{\partial^4}{\partial\xi^4} + 2R^2\dfrac{\partial^4}{\partial\xi^2\partial\theta^2} + \dfrac{\partial^4}{\partial\theta^4}$

Notations used to identify load and deformation variables:

$(\sim)$ indicates variables associated with deformation measured with respect to an unstressed shell of double curvature

$(\wedge)$ indicates variables associated with total deformation measured with respect to an unstressed undeformed cylinder

$(-)$ indicates variables associated with the prestress state only

$( )_0$ indicates variables associated with initial displacements

$(\_)$ indicates terms not appearing in Donnell-type theory

Unmarked variables indicate variables associated with the vibration state only.

DESCRIPTION OF ANALYSIS

To determine the effect of double curvature on the natural vibration behavior of shell structures, the class of shells of revolution shown in figure 1 is investigated. These shells, chosen for their simplicity, have a constant meridional curvature R_ξ throughout their length. The equation of the circumferential radius r which defines the meridional shape is

$$r = R + R_\xi\left(\cos\frac{\xi - \frac{s}{2}}{R_\xi} - 1\right) \tag{1}$$

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where R is the circumferential radius at midlength of the shell, R_ξ is the constant meridional radius of curvature, s is the total length of meridian, and ξ is the meridional length coordinate measured with the origin at one edge of the shell. Accurate results can be found for the natural frequencies of specific shells described by equation (1) by the use of numerical procedures such as that given in reference 3. However, even with high-speed computers, it is impractical to attempt an extensive parameter study of the shell behavior. To achieve a more rapid and economical analysis of the effects of various parameters governing the dynamic behavior of the shell configurations described by equation (1), an approximate set of equations of motion is developed in appendix A. These approximate equations are based on the assumption that the shells are shallow in the meridional direction. In this context, equation (1) is replaced by a simpler expression in which only the first two terms of a power series expansion for r is retained, that is,

$$r = R - \frac{\left(\xi - \frac{s}{2}\right)^2}{2R_\xi} \tag{2}$$

Solutions to the approximate equations are obtained in appendix B. The accuracy of these solutions is evaluated by comparison of results with more accurate results found by using the numerical procedure of reference 3. When the numerical procedure is employed, the meridional shape is described by equation (1) and 200 equally spaced intervals are used in a finite-difference scheme along the meridian.

To develop the approximate governing equations, the nonlinear strain-displacement relations given by reference 6 are specialized for a cylinder and modified slightly by using a criterion of consistency with first approximation shell theory as developed in reference 7. The cylinder is then given a small axisymmetric deformation and the strains caused by this initial geometric perturbation are removed to establish an unstrained state in the resultant doubly curved configuration. The nonlinear equilibrium equations corresponding to this unstrained state are developed and linearized.

The resulting linearized differential equations are

$$u_{,\xi\xi} + \frac{1-\mu}{2}\frac{u_{,\theta\theta}}{R^2} + \frac{1+\mu}{2}\frac{v_{,\theta\xi}}{R} + \left(\frac{1}{R_\xi} + \frac{\mu}{R}\right)w_{,\xi} - \frac{\nu h}{B}u_{,tt} = 0$$

$$\frac{1+\mu}{2}\frac{u_{,\xi\theta}}{R} + \frac{1}{R^2}\left(1 + \frac{\lambda^2}{12}\right)v_{,\theta\theta} + \frac{1-\mu}{2}\left(1 + \frac{\lambda^2}{3}\right)v_{,\xi\xi} + \left(\frac{1}{R} + \frac{\mu}{R_\xi}\right)\frac{w_{,\theta}}{R} - \frac{\lambda^2}{12}\frac{w_{,\theta\theta\theta}}{R^2} - \frac{(2-\mu)\lambda^2}{12}w_{,\xi\xi\theta} - \frac{\nu h}{B}v_{,tt} = 0$$

$$\left(\frac{1}{R_\xi} + \frac{\mu}{R}\right)u_{,\xi} + \left(\frac{1}{R} + \frac{\mu}{R_\xi}\right)\frac{v_{,\theta}}{R} - \frac{\lambda^2}{12R^2}v_{,\theta\theta\theta} - \frac{\lambda^2(2-\mu)}{12}v_{,\xi\xi\theta} + \frac{\lambda^2}{12R^2}\nabla^4 w + \left(\frac{1}{R_\xi^2} + \frac{2\mu}{R_\xi R} + \frac{1}{R^2}\right)w$$

$$- \frac{\overline{N}_\xi}{B}w_{,\xi\xi} - \frac{\overline{N}_\theta}{B}\frac{w_{,\theta\theta}}{R^2} + \frac{\nu h}{B}w_{,tt} = 0$$

$$\tag{3a}$$

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and the resulting boundary conditions at $\xi = $ Constant are

$$u_{,\xi} + \frac{w}{R_\xi} + \mu\left(\frac{v_{,\theta}}{R} + \frac{w}{R}\right) = 0 \qquad\qquad \text{or} \quad u = 0$$

$$\frac{u_{,\theta}}{R} + \left(1 + \frac{\lambda^2}{3}\right)v_{,\xi} - \frac{\lambda^2}{3}w_{,\theta\xi} = 0 \qquad\qquad \text{or} \quad v = 0$$

$$\frac{\lambda^2}{12}\left[-(2-\mu)v_{,\xi\theta} + R^2 w_{,\xi\xi\xi} + (2-\mu)w_{,\theta\theta\xi}\right] - \frac{\overline{N}_\xi}{B}w_{,\xi} = 0 \qquad\qquad \text{or} \quad w = 0$$

$$w_{,\xi\xi} + \frac{\mu}{R^2}\left(w_{,\theta} - v\right)_{,\theta} = 0 \qquad\qquad \text{or} \quad w_{,\xi} = 0$$

(3b)

The exact solution of equations (3) for freely supported boundary conditions $\left(N_\xi = v = w = M_\xi = 0\right)$, is given by

$$u = U_{mn} \cos \frac{m\pi\xi}{s} \sin n\theta \, e^{i\omega t}$$

$$v = V_{mn} \sin \frac{m\pi\xi}{s} \cos n\theta \, e^{i\omega t}$$

$$w = W_{mn} \sin \frac{m\pi\xi}{s} \sin n\theta \, e^{i\omega t}$$

(4)

where U_{mn}, V_{mn}, and W_{mn} are constant modal amplitudes, m is the number of meridional half-waves, n is the number of circumferential waves, and ω is the natural circular frequency. Substitution of equations (4) into equations (3a) leads to the following characteristic equation for natural frequencies:

$$-\Omega^6 + \Lambda_2\Omega^4 - \Lambda_1\Omega^2 + \Lambda_0 = 0$$

(5)

where Ω is a nondimensional frequency given by

$$\Omega = R\omega\sqrt{\frac{\nu\left(1 - \mu^2\right)}{E}}$$

(6)

The coefficients Λ_j depend upon the shell geometry and mode shape and are given by equations (B3).

The limiting cases of membrane and pure bending (inextensional) behavior are also solved in appendix B for freely supported shells. The membrane frequency, with in-plane

inertias neglected and with zero initial static stresses, is given by

$$\Omega^2_{\text{membrane}} = \frac{\left(1 - \mu^2\right)\left(k_\xi + \beta^2\right)}{\left(1 + \beta^2\right)^2} \tag{7}$$

where $k_\xi = \dfrac{R}{R_\xi}$ is the ratio of the circumferential radius at midlength to the meridional radius of curvature and $\beta = \dfrac{m\pi R}{ns}$ is the ratio of the circumferential to axial wavelength of the vibration mode. It is obvious from equation (7) that the membrane frequency vanishes for

$$k_\xi = -\beta^2 \tag{8}$$

The pure bending frequency, Ω_b with in-plane inertial forces neglected and with zero initial static stresses, is given by

$$\Omega_b^2 = \frac{\lambda^2}{12}\left(1 + \beta^2\right)^2 n^4 \tag{9}$$

where $\lambda = \dfrac{h}{R}$ is the ratio of thickness to circumferential radius. The pure bending frequencies are independent of meridional curvature and are equal to the natural frequencies of a simply supported rectangular plate with aspect ratio $\pi R/s$.

In order to give a more descriptive indication of the degree of shallowness of the meridian than that provided by the ratio of radii of curvature k_ξ an auxiliary parameter τ, the percent ratio of the meridional rise to length, is introduced. The central rise of the shell meridian (c in fig. 1) is given by

$$c = R_\xi\left(1 - \cos\frac{s}{2R_\xi}\right) \tag{10}$$

The central rise can be approximated by the first term of a series expansion of the right-hand side of equation (10) if R_ξ is large compared with s so that τ may be represented approximately by

$$\tau = \frac{c}{s}(100) \approx \frac{s}{8R_\xi}(100) \tag{11}$$

which is consistent with the accuracy of the equations developed in this paper.

ACCURACY OF SOLUTIONS

The accuracy of the approximate theory is established by comparing results obtained with this theory with corresponding results obtained with the more accurate numerical method of reference 3. This comparison is made for several specific doubly curved unstressed shells with the same length but different values of k_ξ and results are shown

in figure 2. In figure 2, the lowest frequencies are plotted as a function of successive circumferential mode numbers for various positive and negative Gaussian curvature shells. The solid symbols denote results of the approximate theory and the open symbols denote results from the numerical solution. The cylinder results, found by using the approximate theory, are shown by a dashed line and are repeated on each figure for purposes of comparison. For the particular case of an unstressed cylinder, the approximate method yields exact solutions since the only approximation is neglect of small nonlinear terms related to initial static stress prior to linearization.

For the slightly curved shells ($\tau = \pm 1.87\%$) of figure 2(a), the approximate theory is shown to be very accurate, a larger percentage error being evident for the negative Gaussian curvature shell than that for the positive Gaussian curvature shell. The accuracy diminishes as the curvature of the shell meridian increases as is shown in figures 2(b) ($\tau = \pm 3.75\%$) and 2(c) ($\tau = \pm 5.62\%$). The positive curvature shell results never differ from the more accurate results by more than 8 percent. The negative curvature shell results have larger errors but still indicate the general character of the more accurate solution.

The lowest natural frequency for the axisymmetric vibration mode $n = 0$ is associated with a pure torsional mode and is independent of the meridional curvature in the approximate theory. The more accurate results, however, indicate that this frequency is dependent on curvature to a small degree as can be seen by inspection of figure 2.

The approximate procedure has a modal solution of sinusoidal form along the meridian (see eqs. (4)) whereas the mode shape in the numerical procedure is calculated once the frequency is determined and need not necessarily be sinusoidal. Plots of the normal deflection shape w for the minimum frequency determined by using the numerical procedure are given in figure 3 for particular values of n for the negative curvature shells of figure 2. In each case, the number of axial half-waves m determined by the approximate theory agrees with the modal shapes given in figure 3. As the curvature increases, however, the mode shape begins to deviate from sinusoidal form; thus, the sinusoidal modal solution of the approximate theory becomes a less accurate representation of the true modal configuration.

A comparison of the results found by the approximate theory and the numerical procedure for a long negative Gaussian curvature shell with a rise-length ratio greater than 6 percent is given in figure 4. Large percentage errors are evident, the approximate theory, in general, overestimating the lowest natural frequencies. The corresponding normal displacement modes given in figure 5 are found by using the numerical procedure and exhibit noticeable deviations from the assumed sinusoidal form of the approximate solution especially as n increases. Nevertheless, the approximate solution still exhibits the general shell behavior with respect to both the modes and frequencies.

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Based on the comparisons made in this section, it is believed that the approximate theory gives reasonable estimates of frequencies and predicts trends adequately for values of τ between -5 percent and 5 percent. All analyses in the remainder of this report are limited to shells contained within this range.

EFFECTS OF MERIDIONAL CURVATURE

By using the approximate analysis, the influence of meridional curvature is assessed by performing a parameter study for shells with differing meridional curvatures as the length-radius and thickness-radius ratios are held constant. Results of additional calculations based on membrane theory and pure bending theory are presented to clarify the relative roles of membrane action and bending action as they affect the vibration of these shells. Although the results obtained are for a specific class of shell configurations, they may provide insight into the general behavior of more complex doubly curved shells of revolution.

In calculating the effect of meridional curvature on shell vibration frequencies, the length-radius and thickness-radius ratios are held constant while the curvature ratio k_ξ is varied. Although the mass distribution varies slightly as k_ξ is changed, the effect of this variation on the natural frequencies is negligible if meridional shallowness is maintained. Thus, any change in natural frequencies can be attributed solely to the change in curvature of the shell and the resultant shell stiffness change.

Effective stiffness of a shell is defined here to be that resistance per unit deformation which the shell provides in a particular vibration mode. The natural frequencies of a shell structure are closely related to the effective stiffness of the structure in the sense that as the minimum effective stiffness increases, the lowest natural frequency increases. The positive curvature shells exhibit a strong stiffening character, the lowest natural frequencies increasing as the curvature of the meridian increases. For example, the lowest natural frequency of the positive Gaussian curvature shell in figure 2(a) is 150 percent higher than that of the cylinder, whereas in figure 2(c), the lowest natural frequency is over 300 percent higher. On the other hand, negative curvature shells exhibit large decreases in frequencies as the circumferential mode number changes; these large decreases indicate an appreciable loss in effective stiffness.

The lowest frequencies for each circumferential wave number occur at the simplest meridional mode $m = 1$ for positive Gaussian curvature shells and cylinders. However, this condition is not necessarily true for negative curvature shells. This trend can be seen in figure 4 where the lowest frequency occurs for $m = 2$ and higher meridional modes are associated with lowest frequencies for $n > 3$. This figure shows an interesting phenomenon in the spacing of the frequencies. As n increases, the spacing between frequencies associated with successive meridional wave numbers decreases.

This behavior suggests that experimental resolution of individual natural modes would be difficult to achieve at the higher n range.

In order to interpret the results for negative Gaussian curvature shells shown in figure 2, a more extensive parameter study was conducted. The lowest natural frequencies and associated modes of a series of freely supported shells with a length-radius ratio of 3 (that is, $s/R = 3$) and with the rise-length ratio τ varying from -5 percent to 5 percent calculated with the approximate theory are presented in figure 6. The lowest natural frequencies for the shells of figures 2(a) and 2(b) are located in figure 6 by the vertical dashed lines. As the positive curvature increases, the lowest natural frequency and circumferential wave number increase monotonically. As the negative curvature increases from zero, the minimum frequency decreases and occurs in a progressively lower circumferential mode. For each mode, there is a distinct minimum in the variation of frequency with curvature and hence, there are specific values of curvature for which large decreases in effective stiffness occur. The lowest natural frequency of cylinders and positive curvature shells occurs for $m = 1$. However, an $m = 2$ meridional mode is associated with the lowest natural frequency for shells within a small range of negative curvature.

Figure 7 is a compilation of nine plots of the same type as that of figure 6. The length-radius ratio ranges from 1 to 10 and the radius-thickness ratio ranges from 100 to 1000. The minimum frequencies, in general, decrease as the length increases and as the thickness decreases. The reductions in stiffness in the negative Gaussian curvature range are more prominent for the thinner shells, but the effect is noticeable to some degree for all the shells. The minimum frequency for each branch of the envelope in the negative curvature range occurs at the same curvature for a given length independent of the thickness of the shell. This result suggests that the curvatures at which the minimums occur are related to membrane action. On the other hand, the magnitude of the frequencies at the minimums is highly dependent on the thickness; thus, bending action must be a prominent factor in the effective stiffness of the negative curvature shell in the region of these minimum frequencies.

MEMBRANE AND PURE BENDING ANALYSIS

The results of the previous section imply that membrane behavior is closely related to the large reductions in effective stiffness observed for negative Gaussian curvature shells. The membrane equations associated with these shells have been solved in closed form for the membrane natural frequencies by assuming freely supported edge conditions. The membrane natural frequencies are given by equation (7) in terms of the nondimensional meridional curvature k_ξ, modal wave length ratio β, and the circumferential wave number n. The frequencies plotted in figure 8 for particular wavelength ratios for

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unstressed shells are shown to be continuous linear functions of k_ξ which decrease to zero as k_ξ approaches the value given by equation (8). Since the total mass and mass distribution are essentially constant as k_ξ is varied, this decrease in frequency must correspond to a decrease in effective membrane stiffness. Therefore, for a given modal wavelength ratio, there exists a negative Gaussian curvature membrane shell with a non-dimensional curvature k_ξ (given by eq. (8)) which vibrates without developing any effective membrane resistance (that is, in an inextensional mode). As the meridional curvature increases negatively from this critical value, the membrane shell regains its stiffening characteristics.

The lowest natural frequency envelope given in figure 6 is replotted in figure 9. Vertical dashed lines locate the values of τ at which the membrane theory predicts inextensional behavior for the modes associated with the lowest natural frequencies. These values of τ very nearly coincide with the rise ratios locating the minimum frequencies of the envelope. Hence, these reductions in lowest natural frequency occur at curvatures for which membrane theory predicts inextensional behavior. This behavior is discussed in reference 8 for static analyses where it is shown that pure bending states can occur in negative Gaussian curvature shells when in-plane displacement freedom (u or v) is allowed at the boundaries, a situation which exists for freely supported shells. This behavior can similarly be shown to be true for all the minimums in figure 7. The higher frequencies on either side of these minimums must be due to the recovery of membrane stiffness for that wavelength ratio as the curvature changes as is shown in figure 8.

Since the calculations show that the effective membrane stiffness is negligible in negative Gaussian curvature shells for certain wavelength ratios, the frequency at these critical combinations of wavelength ratio and meridional curvature must be governed almost entirely by bending stiffness. Thus, the minimum natural frequencies found with a pure bending (inextensional) theory should yield values near the observed minimums. To verify this statement, the pure bending solution was obtained and is given in equation (9). Horizontal dashed lines in figure 9 locate the pure bending frequencies for the modes occurring in the lowest natural frequency envelope. In each case, even though the pure bending results are independent of meridional curvature, the pure bending frequency closely approximates the minimum frequencies obtained by using the approximate theory for corresponding m,n modes. Hence, it may be concluded that the large reductions in frequency observed in certain negative curvature ranges are due to a loss in membrane action and that for these critical combinations of modal wavelength and meridional curvature, the shell is very nearly vibrating in a pure bending mode. Furthermore, these combinations may be predicted from a simple membrane equation (eq. (8)).

It should be noted that every negative Gaussian curvature shell vibrates in a nearly pure bending mode for some particular wavelength ratio; however, if the m or n number is large, there may be enough effective bending stiffness present to maintain a natural frequency well above the lowest natural frequency of the shell.

CONCLUDING REMARKS

An approximate set of equations of motion governing the vibration behavior of nearly cylindrical shells with shallow meridional curvature is developed and a closed-form solution is obtained for freely supported edge conditions. A comparison of the solutions from the approximate theory to those obtained from a more accurate theory shows that within a certain range of meridional curvature, the approximate theory yields acceptable solutions.

The effect of the meridional curvature on the minimum natural frequencies of nearly cylindrical shells with either positive or negative Gaussian curvature is investigated. The positive Gaussian curvature shells have minimum natural frequencies well above those of corresponding cylindrical shells. The minimum natural frequencies of the negative Gaussian curvature shells generally are below those of the corresponding cylinders and evidence wide variations in value, large reductions in magnitude occurring at certain critical values of curvature. The corresponding membrane and pure bending equations are also solved for the same edge conditions. Comparisons of the membrane, pure bending, and complete shell analyses show that the very low frequencies at the critical values of negative curvature result from a condition of essentially negligible effective membrane stiffness and, as a consequence, are dependent almost entirely on the shell bending stiffness. The membrane theory affords a simple method of determining the modal wavelength ratio and curvature combination at which this essentially inextensional behavior can exist for a negative Gaussian curvature shell, whereas the pure bending theory gives a good estimate of the magnitudes of the associated frequencies.

Langley Research Center,
    National Aeronautics and Space Administration,
        Langley Station, Hampton, Va., January 21, 1969,
            124-08-06-11-23.

APPENDIX A

DERIVATION OF APPROXIMATE SHELL EQUATIONS

In this appendix, a set of linear homogeneous equations is developed which governs the dynamic behavior of doubly curved axisymmetric shells with constant shallow meridional curvature of either positive or negative Gaussian curvature. To achieve this, the nonlinear strain-displacement relations given by reference 6 are specialized for a cylinder and modified slightly by using the criterion of reference 7. The cylinder is then given a small axisymmetric deformation, and the strains caused by this initial geometric perturbation are removed to establish an unstrained state in the resultant doubly curved configuration. The nonlinear equilibrium equations corresponding to this unstrained state are developed and linearized.

Nonlinear Strain Displacement Relations

The first-approximation nonlinear strain-displacement relations developed in reference 6 are based on the assumption of small strains and moderately small rotations. For a cylinder with radius R and coordinates and positive displacement directions as shown in figure 10, the resulting equations become

$$
\left.
\begin{aligned}
\hat{\epsilon}_{\xi} &= \hat{u}_{,\xi} + \frac{1}{2}\left(\hat{w}_{,\xi}\right)^2 + \frac{1}{8}\left(\hat{v}_{,\xi} - \frac{\hat{u}_{,\theta}}{R}\right)^2 \\[2mm]
\hat{\epsilon}_{\theta} &= \frac{\hat{v}_{,\theta}}{R} + \frac{\hat{w}}{R} + \frac{1}{2R^2}\left(\hat{w}_{,\theta} - \hat{v}\right)^2 + \frac{1}{8}\left(\hat{v}_{,\xi} - \frac{\hat{u}_{,\theta}}{R}\right)^2 \\[2mm]
\hat{\epsilon}_{\xi\theta} &= \frac{1}{2}\left[\hat{v}_{,\xi} + \frac{\hat{u}_{,\theta}}{R} + \frac{\hat{w}_{,\xi}}{R}\left(\hat{w}_{,\theta} - \hat{v}\right)\right] \\[2mm]
\hat{\kappa}_{\xi} &= -\hat{w}_{,\xi\xi} \\[2mm]
\hat{\kappa}_{\theta} &= -\frac{1}{R^2}\left(\hat{w}_{,\theta} - \hat{v}\right)_{,\theta} \\[2mm]
\hat{\kappa}_{\xi\theta} &= -\frac{1}{R}\left[\left(\hat{w}_{,\theta} - \frac{3\hat{v}}{4}\right)_{,\xi} + \frac{\hat{u}_{,\theta}}{4R}\right]
\end{aligned}
\right\}
\tag{A1}
$$

where the symbol $\hat{\ }$ is introduced to indicate quantities measured with respect to an unstressed cylindrical surface.

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To permit the method of solution which follows, the term $\hat{\kappa}_{\xi\theta}$ must be modified so that the term $-\dfrac{\hat{u}_{,\theta}}{4R^2}$ is eliminated. The quantity $\hat{\kappa}_{\xi\theta}$ can be rewritten as

$$\hat{\kappa}_{\xi\theta} = -\frac{1}{R}\left(\hat{w}_{,\theta} - \hat{v}\right)_{,\xi} - \frac{\left(\hat{\epsilon}_{\xi\theta}\right)_L}{2R} \tag{A2}$$

where $\left(\hat{\epsilon}_{\xi\theta}\right)_L$ contains only the linear terms of $\hat{\epsilon}_{\xi\theta}$. It is shown in reference 7 that linear terms of the type ϵ_L/R (where ϵ_L is any physical linear middle-surface strain) multiplied by a numerical factor of the order of unity may be added to the linear expressions for bending strains without introducing an error greater than that originally introduced through application of the Kirchhoff-Love hypothesis. Hence, $\hat{\kappa}_{\xi\theta}$ may be defined by

$$\hat{\kappa}_{\xi\theta} = -\frac{1}{R}\left(\hat{w}_{,\theta} - \hat{v}\right)_{,\xi} \tag{A3}$$

In addition, only the Von Karman type of nonlinear terms are retained in the strain-displacement relations (that is, nonlinear terms composed of products of derivatives of the normal displacement $\hat{w}$). The final form of the cylinder strain-displacement relations reduces to

$$\left.\begin{aligned}
\hat{\epsilon}_{\xi} &= \hat{u}_{,\xi} + \frac{1}{2}\left(\hat{w}_{,\xi}\right)^2 \\[6pt]
\hat{\epsilon}_{\theta} &= \frac{\hat{v}_{,\theta}}{R} + \frac{\hat{w}}{R} + \frac{1}{2R^2}\left(\hat{w}_{,\theta}\right)^2 \\[6pt]
\hat{\epsilon}_{\xi\theta} &= \frac{1}{2}\left(\hat{v}_{,\xi} + \frac{\hat{u}_{,\theta}}{R} + \frac{1}{R}\,\hat{w}_{,\xi}\hat{w}_{,\theta}\right) \\[6pt]
\hat{\kappa}_{\xi} &= -\hat{w}_{,\xi\xi} \\[6pt]
\hat{\kappa}_{\theta} &= -\frac{1}{R^2}\left(\hat{w}_{,\theta} - \hat{v}\right)_{,\theta} \\[6pt]
\hat{\kappa}_{\xi\theta} &= -\frac{1}{R}\left(\hat{w}_{,\theta} - \hat{v}\right)_{,\xi}
\end{aligned}\right\} \tag{A4}$$

A set of strain-displacement relations governing the behavior of a shell of revolution with a shallow meridional curvature is developed in a fashion similar to that of references 4 and 5 by the introduction of an initial displacement $w_0(\xi)$ to the cylinder middle surface. Let

$$\hat{w}(\xi,\theta,t) = \tilde{w}(\xi,\theta,t) + w_0(\xi) \tag{A5}$$

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where $\tilde{w}$ is the displacement measured from the initially deformed surface along the normal to the cylindrical surface. Since the meridional curve is assumed to be shallow, it is sufficient as a first approximation to represent the initially deformed curve by

$$w_0(\xi) = -\frac{\left(\xi - \frac{s}{2}\right)^2}{2R_\xi} \tag{A6}$$

where s is the total meridional arc length and R_ξ is the constant radius of curvature of the initially deformed meridional curve. (See, for example, ref. 8.) This representation restricts the subsequent analysis to shells of the form shown in the sketches in figure 1. As a consequence of equation (A6), if R_ξ is large, the meridional displacement measured along a tangent to the initially deformed surface $\tilde{u}$ and the circumferential displacement $\tilde{v}$ can be written as

$$\left. \begin{aligned} \tilde{u} &= \hat{u} - \frac{\left(\xi - \frac{s}{2}\right)\tilde{w}}{R_\xi} \\ \tilde{v} &= \hat{v} \end{aligned} \right\} \tag{A7}$$

With application of equations (A5), (A6), and (A7), the strains caused by the total deformation can be written as

$$\left. \begin{aligned} \hat{\epsilon}_\xi &= \tilde{u}_{,\xi} + \frac{\tilde{w}}{R_\xi} + \frac{1}{2}\left(\tilde{w}_{,\xi}\right)^2 + \frac{1}{2}\left(\frac{\xi - \frac{s}{2}}{R_\xi}\right)^2 \\ \hat{\epsilon}_\theta &= \frac{\tilde{v}_{,\theta}}{R} + \frac{\tilde{w}}{R} + \frac{1}{2}\left(\frac{\tilde{w}_{,\theta}}{R}\right)^2 - \frac{\left(\xi - \frac{s}{2}\right)^2}{2R_\xi R} \\ \hat{\epsilon}_{\xi\theta} &= \frac{1}{2}\left(\tilde{v}_{,\xi} + \frac{\tilde{u}_{,\theta}}{R} + \frac{\tilde{w}_{,\xi}\tilde{w}_{,\theta}}{R}\right) \\ \hat{\kappa}_\xi &= -\tilde{w}_{,\xi\xi} - \frac{1}{R_\xi} \\ \hat{\kappa}_\theta &= -\frac{1}{R^2}\left(\tilde{w}_{,\theta} - \tilde{v}\right)_{,\theta} \\ \hat{\kappa}_{\xi\theta} &= -\frac{1}{R}\left(\tilde{w}_{,\theta} - \tilde{v}\right)_{,\xi} \end{aligned} \right\} \tag{A8}$$

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The strains caused by the initial deformation w_0 are

$$
\left.
\begin{aligned}
\epsilon_{\xi_0} &= \frac{1}{2}\left(\frac{\xi - \frac{s}{2}}{R_\xi}\right)^2 \\[2ex]
\epsilon_{\theta_0} &= -\frac{\left(\xi - \frac{s}{2}\right)^2}{2R_\xi R} \\[2ex]
\epsilon_{\xi\theta_0} &= 0 \\[2ex]
\kappa_{\xi_0} &= -\frac{1}{R_\xi} \\[2ex]
\kappa_{\theta_0} &= 0 \\[2ex]
\kappa_{\xi\theta_0} &= 0
\end{aligned}
\right\}
\tag{A9}
$$

The initially deformed system is now taken to be in an unstrained equilibrium state. Any deformation $\tilde{u}, \tilde{v}, \tilde{w}$ away from this new equilibrium state introduces strains given approximately by subtracting the initial deformation strain of equations (A9) from the total strains of equations (A8) $\left(\text{for example, } \tilde{\epsilon}_\xi = \hat{\epsilon}_\xi - \epsilon_{\xi_0}\right)$. Hence,

$$
\left.
\begin{aligned}
\tilde{\epsilon}_\xi &= \tilde{u}_{,\xi} + \frac{\tilde{w}}{R_\xi} + \frac{1}{2}\left(\tilde{w}_{,\xi}\right)^2 \\[2ex]
\tilde{\epsilon}_\theta &= \frac{\tilde{v}_{,\theta}}{R} + \frac{\tilde{w}}{R} + \frac{1}{2}\left(\frac{\tilde{w}_{,\theta}}{R}\right)^2 \\[2ex]
\tilde{\epsilon}_{\xi\theta} &= \frac{1}{2}\left(\tilde{v}_{,\xi} + \frac{\tilde{u}_{,\theta}}{R} + \frac{\tilde{w}_{,\xi}\tilde{w}_{,\theta}}{R}\right) \\[2ex]
\tilde{\kappa}_\xi &= -\tilde{w}_{,\xi\xi} \\[2ex]
\tilde{\kappa}_\theta &= -\frac{1}{R^2}\left(\tilde{w}_{,\theta} - \tilde{v}\right)_{,\theta} \\[2ex]
\tilde{\kappa}_{\xi\theta} &= -\frac{1}{R}\left(\tilde{w}_{,\theta} - \tilde{v}\right)_{,\xi}
\end{aligned}
\right\}
\tag{A10}
$$

where ξ is now measured along the meridian.

APPENDIX A

Equilibrium Equations

Nonlinear equations.- Equations (A10) are the strain-displacement relations for a shell of revolution with a shallow meridional curvature. To obtain a set of equilibrium equations consistent with equations (A10), the principle of minimum potential energy is applied.

The potential energy Π for an isotropic, linearly elastic, homogeneous, vibrating shell with a uniform lateral conservative surface load p is

$$\Pi = \frac{E}{2\left(1 - \mu^2\right)} \int_0^S \int_0^{2\pi} \int_{-h/2}^{h/2} \left[\left(\tilde{\epsilon}_\xi + z\tilde{\kappa}_\xi\right)^2 + \left(\tilde{\epsilon}_\theta + z\tilde{\kappa}_\theta\right)^2 + 2\mu\left(\tilde{\epsilon}_\xi + z\tilde{\kappa}_\xi\right)\left(\tilde{\epsilon}_\theta + z\tilde{\kappa}_\theta\right) \right.$$

$$\left. + 2(1 - \mu)\left(\tilde{\epsilon}_{\xi\theta} + z\tilde{\kappa}_{\xi\theta}\right)^2 \right] R \, dz \, d\theta \, d\xi - p \int_{\xi_1}^{\xi_2} \int_0^{2\pi} \tilde{w} R \, d\theta \, d\xi$$

$$- \frac{1}{2}\nu \int_{\xi_1}^{\xi_2} \int_0^{2\pi} \int_{-h/2}^{h/2} \left[\left(\tilde{u}_{,t}\right)^2 + \left(\tilde{v}_{,t}\right)^2 + \left(\tilde{w}_{,t}\right)^2 \right] R \, dz \, d\theta \, d\xi \qquad \text{(A11)}$$

Performing the integration over z and requiring the variation of the energy to vanish yields

$$\delta\Pi = \int_0^S \int_0^{2\pi} \left[\tilde{N}_\xi \, \delta\tilde{\epsilon}_\xi + \tilde{N}_\theta \, \delta\tilde{\epsilon}_\theta + 2\tilde{N}_{\xi\theta} \, \delta\tilde{\epsilon}_{\xi\theta} + \tilde{M}_\xi \, \delta\tilde{\kappa}_\xi + \tilde{M}_\theta \, \delta\tilde{\kappa}_\theta + 2\tilde{M}_{\xi\theta} \, \delta\tilde{\kappa}_{\xi\theta} \right.$$

$$\left. - p \, \delta w - \nu h\left(\tilde{u}_{,t} \, \delta\tilde{u}_{,t} + \tilde{v}_{,t} \, \delta\tilde{v}_{,t} + \tilde{w}_{,t} \, \delta\tilde{w}_{,t}\right) \right] R \, d\theta \, d\xi = 0 \qquad \text{(A12)}$$

where

$$\left. \begin{aligned} \tilde{N}_\xi &= B\left(\tilde{\epsilon}_\xi + \mu\tilde{\epsilon}_\theta\right) \\ \tilde{N}_\theta &= B\left(\tilde{\epsilon}_\theta + \mu\tilde{\epsilon}_\xi\right) \\ \tilde{N}_{\xi\theta} &= B(1 - \mu)\tilde{\epsilon}_{\xi\theta} \\ \tilde{M}_\xi &= D\left(\tilde{\kappa}_\xi + \mu\tilde{\kappa}_\theta\right) \\ \tilde{M}_\theta &= D\left(\tilde{\kappa}_\theta + \mu\tilde{\kappa}_\xi\right) \\ \tilde{M}_{\xi\theta} &= D(1 - \mu)\tilde{\kappa}_{\xi\theta} \end{aligned} \right\} \qquad \text{(A13)}$$

$$\tilde{M}_\xi = \overline{M}_\xi(\xi) + M_\xi(\xi,\theta,t)$$

$$\tilde{M}_\theta = \overline{M}_\theta(\xi) + M_\theta(\xi,\theta,t) \qquad\qquad\text{(A17)}$$

$$\tilde{M}_{\xi\theta} = M_{\xi\theta}(\xi,\theta,t)$$

Also let

$$\tilde{u} = \bar{u}(\xi) + u(\xi,\theta,t)$$

$$\tilde{v} = \bar{v}(\xi,\theta,t) \qquad\qquad\text{(A18)}$$

$$\tilde{w} = \overline{w}(\xi) + w(\xi,\theta,t)$$

where the barred terms are associated with the stresses and deformations due to an axisymmetric static loading and the unbarred terms are associated with stresses and deformations due to a subsequent infinitesimal linear vibration about the prestressed state. The equations governing the linear vibration about the prestressed state are found by introducing equations (A17) and (A18) into equations (A15), applying equilibrium of the prestressed state, and neglecting nonlinear vibration terms. (See, for example, ref. 3.) The resulting equations are

$$N_{\xi,\xi} + \frac{N_{\xi\theta,\theta}}{R} - \nu h u_{,tt} = 0$$

$$\frac{N_{\theta,\theta}}{R} + N_{\xi\theta,\xi} + \frac{M_{\theta,\theta}}{R^2} + \frac{2M_{\xi\theta,\xi}}{R} - \nu h v_{,tt} = 0 \qquad\qquad\text{(A19)}$$

$$M_{\xi,\xi\xi} + \frac{2M_{\xi\theta,\xi\theta}}{R} + \frac{M_{\theta,\theta\theta}}{R^2} - \frac{N_\xi}{R_\xi} - \frac{N_\theta}{R} + \overline{N}_\xi w_{,\xi\xi} + \frac{\overline{N}_\theta}{R^2} w_{,\theta\theta} - \nu h w_{,tt} = 0$$

where the assumption $\dfrac{d\overline{w}}{d\xi} = 0$ has been made (that is, meridional static rotations are neglected) and where it is assumed that the static loading is such that $\overline{N}_{\xi\theta} = 0$. These assumptions lead to constant membrane static stress resultants $\overline{N}_\xi$ and $\overline{N}_\theta$.

The corresponding boundary conditions at $\xi = $ Constant are

$$N_\xi = 0 \qquad\qquad\qquad\qquad \text{or} \quad u = 0$$

$$N_{\xi\theta} + \frac{2M_{\xi\theta}}{R} = 0 \qquad\qquad \text{or} \quad v = 0$$

(Equations continued on next page)

$$M_{\xi,\xi} + \frac{2M_{\xi\theta,\theta}}{R} + \overline{N}_{\xi}w_{,\xi} = 0 \qquad\qquad \text{or} \quad w = 0$$

$$M_{\xi} = 0 \qquad\qquad\qquad\qquad\qquad \text{or} \quad w_{,\xi} = 0$$

$$\tag{A20}$$

The equilibrium equations (eqs. (A19)) and boundary conditions (eqs. (A20)) apply to shells of revolution with shallow meridional curvature but have no such limitation in the circumferential direction. The distinction between these equations and a Donnell-type formulation (shallow in both directions) involves the underlined terms in equations (A19) and (A20) which would be omitted for a Donnell-type approximation. The terms which would be omitted for the Donnell approximation are similarly underlined in all subsequent derivations.

The vibratory stress and moment resultants are found from equations (A10), (A13), (A17), and (A18). Since $\frac{d\overline{w}}{d\xi} = 0$, the resultants are

$$
\left.
\begin{aligned}
N_{\xi} &= B\left(\epsilon_{\xi} + \mu\epsilon_{\theta}\right)\\
N_{\theta} &= B\left(\epsilon_{\theta} + \mu\epsilon_{\xi}\right)\\
N_{\xi\theta} &= B(1 - \mu)\epsilon_{\xi\theta}\\
M_{\xi} &= D\left(\kappa_{\xi} + \mu\kappa_{\theta}\right)\\
M_{\theta} &= D\left(\kappa_{\theta} + \mu\kappa_{\xi}\right)\\
M_{\xi\theta} &= D(1 - \mu)\kappa_{\xi\theta}
\end{aligned}
\right\}
\tag{A21a}
$$

where

$$
\left.
\begin{aligned}
\epsilon_{\xi} &= u_{,\xi} + \frac{w}{R_{\xi}}\\
\epsilon_{\theta} &= \frac{v_{,\theta}}{R} + \frac{w}{R}\\
\epsilon_{\xi\theta} &= \frac{1}{2}\left(\frac{u_{,\theta}}{R} + v_{,\xi}\right)\\
\kappa_{\xi} &= -w_{,\xi\xi}\\
\kappa_{\theta} &= -\frac{1}{R^2}\left(w_{,\theta} - \underline{v}\right)_{,\theta}\\
\kappa_{\xi\theta} &= -\frac{1}{R}\left(w_{,\theta} - \underline{v}\right)_{,\xi}
\end{aligned}
\right\}
\tag{A21b}
$$

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With the aid of equations (A21), equations (A19) are written in terms of displacements as

$$u_{,\xi\xi} + \frac{1-\mu}{2}\frac{u_{,\theta\theta}}{R^2} + \frac{1+\mu}{2}\frac{v_{,\theta\xi}}{R} + \left(\frac{1}{R_\xi} + \frac{\mu}{R}\right)w_{,\xi} - \frac{\nu h}{B}u_{,tt} = 0$$

$$\frac{1+\mu}{2}\frac{u_{,\xi\theta}}{R} + \frac{1}{R^2}\left(1 + \frac{\lambda^2}{12}\right)v_{,\theta\theta} + \frac{1-\mu}{2}\left(1 + \frac{\lambda^2}{3}\right)v_{,\xi\xi} + \left(\frac{1}{R} + \frac{\mu}{R_\xi}\right)\frac{w_{,\theta}}{R}$$

$$-\frac{\lambda^2}{12}\frac{w_{,\theta\theta\theta}}{R^2} - \frac{(2-\mu)\lambda^2}{12}w_{,\xi\xi\theta} - \frac{\nu h}{B}v_{,tt} = 0 \tag{A22a}$$

$$\left(\frac{1}{R_\xi} + \frac{\mu}{R}\right)u_{,\xi} + \left(\frac{1}{R} + \frac{\mu}{R_\xi}\right)\frac{v_{,\theta}}{R} - \frac{\lambda^2}{12R^2}v_{,\theta\theta\theta} - \frac{\lambda^2(2-\mu)}{12}v_{,\xi\xi\theta} + \frac{\lambda^2}{12R^2}\nabla^4 w$$

$$+ \left(\frac{1}{R_\xi^2} + \frac{2\mu}{R_\xi R} + \frac{1}{R^2}\right)w - \frac{\overline{N}_\xi}{B}w_{,\xi\xi} - \frac{\overline{N}_\theta}{B}\frac{w_{,\theta\theta}}{R^2} + \frac{\nu h}{B}w_{,tt} = 0$$

where

$$\lambda = \frac{h}{R}$$

Similarly, the corresponding boundary conditions at ξ = Constant are

$$u_{,\xi} + \frac{w}{R_\xi} + \mu\left(\frac{v_{,\theta}}{R} + \frac{w}{R}\right) = 0 \qquad\qquad \text{or} \quad u = 0$$

$$\frac{u_{,\theta}}{R} + \left(1 + \frac{\lambda^2}{3}\right)v_{,\xi} - \frac{\lambda^2}{3}w_{,\theta\xi} = 0 \qquad\qquad \text{or} \quad v = 0$$

$$\frac{\lambda^2}{12}\left[-(2-\mu)v_{,\xi\theta} + R^2 w_{,\xi\xi\xi} + (2-\mu)w_{,\theta\theta\xi}\right] - \frac{\overline{N}_\xi}{B}w_{,\xi} = 0 \qquad \text{or} \quad w = 0 \tag{A22b}$$

$$w_{,\xi\xi} + \frac{\mu}{R^2}\left(w_{,\theta} - v\right)_{,\theta} = 0 \qquad\qquad \text{or} \quad w_{,\xi} = 0$$

The constant membrane prestress stress resultants are defined by

$$\left.\begin{array}{l}\overline{N}_\xi = \overline{N}\\[2mm]\overline{N}_\theta = R\left(p - \dfrac{\overline{N}}{R_\xi}\right)\end{array}\right\} \tag{A23}$$

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where $\overline{N}$ is an applied meridional stress resultant and p is the applied constant lateral internal pressure. Equations (A22) are partial differential equations with constant coefficients. If prestress deformations had been retained, additional terms with variable coefficients would occur and numerical methods of solution similar to those used in reference 3 would be required.

APPENDIX B

SOLUTIONS OF APPROXIMATE SHELL EQUATIONS

General Solution

A closed-form solution to equations (A22) is available for the freely supported boundary conditions $\left(N_\xi = v = w = M_\xi = 0\right)$. The solution can be expressed as

$$\left.\begin{aligned} u &= U_{mn} \cos \frac{m\pi\xi}{s} \sin n\theta \, e^{i\omega t} \\[2mm] v &= V_{mn} \sin \frac{m\pi\xi}{s} \cos n\theta \, e^{i\omega t} \\[2mm] w &= W_{mn} \sin \frac{m\pi\xi}{s} \sin n\theta \, e^{i\omega t} \end{aligned}\right\} \tag{B1}$$

where m is the number of meridional half-waves, n is the number of circumferential waves, and ω is the natural circular frequency. This solution satisfies the freely supported conditions at $\xi = 0$ and $\xi = s$. Equations (B1) are substituted into equations (A22) to yield a set of linear homogeneous equations. For a nontrivial solution to exist, the determinant of the coefficient matrix of the resultant set of equations must vanish. This procedure leads to the characteristic equation

$$-\Omega^6 + \Lambda_2 \Omega^4 - \Lambda_1 \Omega^2 + \Lambda_0 = 0 \tag{B2}$$

where Ω is a nondimensional frequency parameter and where the coefficients Λ_0, Λ_1, and Λ_2 are as follows:

$$\left.\begin{aligned} \Lambda_0 &= b_{11}\left[b_{22}b_{33} - \left(b_{23}\right)^2\right] - b_{12}\left(b_{12}b_{33} - b_{13}b_{23}\right) + b_{13}\left(b_{12}b_{23} - b_{13}b_{22}\right) \\[2mm] \Lambda_1 &= b_{22}b_{33} - \left(b_{23}\right)^2 + b_{33}b_{11} - \left(b_{13}\right)^2 + b_{11}b_{22} - \left(b_{12}\right)^2 \\[2mm] \Lambda_2 &= b_{11} + b_{22} + b_{33} \end{aligned}\right\} \tag{B3}$$

where

$$b_{11} = \left(\frac{m\pi R}{s}\right)^2 + \frac{1 - \mu}{2} n^2$$

$$b_{12} = \frac{1 + \mu}{2} \frac{m\pi R}{s} n$$

$$b_{13} = - \frac{m\pi R}{s}\left(k_\xi + \mu\right)$$

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$$b_{22} = \left(1 + \frac{\lambda^2}{12}\right)n^2 + \frac{1 - \mu}{2}\left(1 + \frac{\lambda^2}{3}\right)\left(\frac{m\pi R}{s}\right)^2$$

$$b_{23} = -n\left(1 + \mu k_\xi\right) - \frac{\lambda^2}{12} n\left[n^2 + (2 - \mu)\left(\frac{m\pi R}{s}\right)^2\right]$$

$$b_{33} = \frac{\lambda^2}{12}\left[\left(\frac{m\pi R}{s}\right)^2 + n^2\right]^2 + k_\xi^2 + 2\mu k_\xi + 1 + \frac{\overline{N}}{B}\left(\frac{m\pi R}{s}\right)^2 + \frac{\overline{N}_\theta}{B}n^2$$

and where the ratio of curvatures has been introduced as $k_\xi = \dfrac{R}{R_\xi}$. The natural frequencies of a freely supported doubly curved shell of revolution with shallow meridional curvature can be found for any specific mode by solving equation (B2). This equation leads to three frequencies for each set of m,n considered, since in-plane inertia terms have been retained.

Membrane Solution

It is of value to inspect the extreme case of zero bending stiffness (D = 0). The vibratory behavior in this case is associated with only the extensional properties of the shell. The membrane equations are found directly from equations (A19) by deleting all moment terms to yield

$$\left.\begin{aligned}
N_{\xi,\xi} + \frac{N_{\xi\theta,\theta}}{R} &= 0 \\[2mm]
\frac{N_{\theta,\theta}}{R} + N_{\xi\theta,\xi} &= 0 \\[2mm]
\frac{N_\xi}{R_\xi} + \frac{N_\theta}{R} - \overline{N}_\xi w_{,\xi\xi} - \overline{N}_\theta \frac{w_{,\theta\theta}}{R^2} + \nu h w_{,tt} &= 0
\end{aligned}\right\} \tag{B4}$$

where, for convenience in this limiting case, the in-plane inertias have been neglected. The first two equilibrium equations are identically satisfied by the introduction of the stress function ψ defined by

$$\left.\begin{aligned}
\frac{\psi_{,\theta\theta}}{R^2} &= \frac{N_\xi}{B} = u_{,\xi} + \frac{w}{R_\xi} + \mu\left(\frac{v_{,\theta}}{R} + \frac{w}{R}\right) \\[2mm]
\psi_{,\xi\xi} &= \frac{N_\theta}{B} = \frac{v_{,\theta}}{R} + \frac{w}{R} + \mu\left(u_{,\xi} + \frac{w}{R_\xi}\right) \\[2mm]
-\frac{\psi_{,\xi\theta}}{R} &= \frac{N_{\xi\theta}}{B} = \frac{1 - \mu}{2}\left(v_{,\xi} + \frac{u_{,\theta}}{R}\right)
\end{aligned}\right\} \tag{B5}$$

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The third equilibrium equation becomes

$$\frac{\psi_{,\theta\theta}}{R_\xi} + R\psi_{,\xi\xi} - \frac{\overline{N}_\xi}{B} R^2 w_{,\xi\xi} - \frac{\overline{N}_\theta}{B} w_{,\theta\theta} + \frac{\nu h R^2}{B} w_{,tt} = 0 \tag{B6}$$

It can be seen from equation (B6) that for a negative Gaussian curvature $\left(R_\xi \text{ negative}\right)$ unstressed membrane shell, the governing equation has a hyperbolic character. The equation for the cylinder has a parabolic character and for the positive Gaussian curvature shell has an elliptic character as noted in reference 8 for static membrane theory.

Equation (B6) is satisfied by

$$\left.\begin{aligned} \psi &= \Psi_{membrane} \sin \frac{m\pi\xi}{s} \sin n\theta \, e^{i\omega t} \\[2ex] w &= W_{membrane} \sin \frac{m\pi\xi}{s} \sin n\theta \, e^{i\omega t} \end{aligned}\right\} \tag{B7}$$

where it follows from equations (B5) that

$$\left.\begin{aligned} u &= U_{membrane} \cos \frac{m\pi\xi}{s} \sin n\theta \, e^{i\omega t} \\[2ex] v &= V_{membrane} \sin \frac{m\pi\xi}{s} \cos n\theta \, e^{i\omega t} \end{aligned}\right\} \tag{B8}$$

This solution satisfies the freely supported boundary conditions.

With the application of equations (B7) and (B8) to equations (B5), equations (B5) may be written in matrix form as

$$\begin{bmatrix} -\dfrac{m\pi R}{s} & -\mu n & k_\xi + \mu \\[2ex] -\mu \dfrac{m\pi R}{s} & -n & 1 + \mu k_\xi \\[2ex] n & \dfrac{m\pi R}{s} & 0 \end{bmatrix} \begin{Bmatrix} U \\[2ex] V \\[2ex] W \end{Bmatrix}_{membrane} = -\frac{\Psi_{membrane}}{R} \begin{Bmatrix} n^2 \\[2ex] \left(\dfrac{m\pi R}{s}\right)^2 \\[2ex] \dfrac{2n\left(\dfrac{m\pi R}{s}\right)}{1-\mu} \end{Bmatrix} \tag{B9}$$

where from Cramer's rule

$$W_{membrane} = -\frac{\Psi_{membrane}}{R} \frac{\left[\left(\dfrac{m\pi R}{s}\right)^2 + n^2\right]^2}{\left(1 - \mu^2\right)\left[\left(\dfrac{m\pi R}{s}\right)^2 + n^2 k_\xi\right]} \tag{B10}$$

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Substitution of equations (B7), (B8), and (B10) into equation (B6) yields the following membrane frequency equation:

$$\Omega^2_{\text{membrane}} = \frac{\left(1 - \mu^2\right)\left(k_\xi + \beta^2\right)^2}{\left(1 + \beta^2\right)^2} + \left(\frac{\overline{N}_\theta}{B} + \frac{\overline{N}_\xi}{B} \beta^2\right)n^2 \qquad (B11)$$

where $\beta = \dfrac{m\pi R}{ns}$ is the ratio of the circumferential to the axial wavelength of the vibration mode. The frequency determined from equation (B11) vanishes for an initially unstressed membrane for meridional curvatures given by

$$k_\xi = -\beta^2 \qquad (B12)$$

Thus, it appears that for certain mode shapes the vibration of a negative curvature membrane shell with freely supported edges is not sustained by the membrane stiffness of the shell. This condition would suggest that the shell experiences inextensional behavior. The proof that this behavior occurs can be demonstrated by setting the middle surface strains of equations (A21b) equal to zero and substituting in equations (A21b) the solution given by equations (B1). The resulting conditions for inextensibility are that

$$\left.\begin{aligned} k_\xi &= -\beta^2 \\[6pt] \frac{V_{mn}}{W_{mn}} &= \frac{1}{n} \\[6pt] \frac{U_{mn}}{W_{mn}} &= -\frac{\beta}{n} \end{aligned}\right\} \qquad (B13)$$

It can be shown that these three conditions do occur when the membrane frequency is zero.

Pure Bending Solution

With the assumptions that B remains finite and that the middle surface strains equal zero, the pure bending (inextensional) equations are found from equations (A19) and (A21a). The resulting equations in terms of displacements can be written down directly from equations (A22a) by retaining all terms containing λ^2, and v and w inertia terms. Since in-plane displacements contribute mainly to in-plane stretching and, in general, have a negligible effect on inextensional deformation, a further simplification can be made by deleting the v inertia and the non-Donnell terms (that is, by deleting the second equation of (A22a) and by deleting all derivatives of v in the third equation of (A22a)). The following equation results:

$$\frac{\lambda^2}{12} \nabla^4 w + \frac{\nu R^2 \left(1 - \mu^2\right)}{E} w_{,tt} = 0 \tag{B14}$$

The meridional curvature $1/R_\xi$ does not appear in equation (B14); thus in the approximate formulation the pure bending state is independent of R_ξ.

The freely supported boundary conditions are satisfied by

$$w = W_b \sin \frac{m\pi \xi}{s} \sin n\theta \, e^{i\omega_b t} \tag{B15}$$

where the subscript b denotes bending.

The natural frequencies of pure bending Ω_b are given by

$$\Omega_b^2 = \frac{\lambda^2}{12} \left(1 + \beta^2\right)^2 n^4 \qquad (n \neq 0) \tag{B16}$$

where

$$\Omega_b^2 = \frac{R^2 \omega_b^2 \nu \left(1 - \mu^2\right)}{E}$$

and where again

$$\beta = \frac{m\pi R}{ns}$$

These frequencies are equivalent to the natural frequencies of a simply supported rectangular plate with aspect ratio $\pi R/s$.

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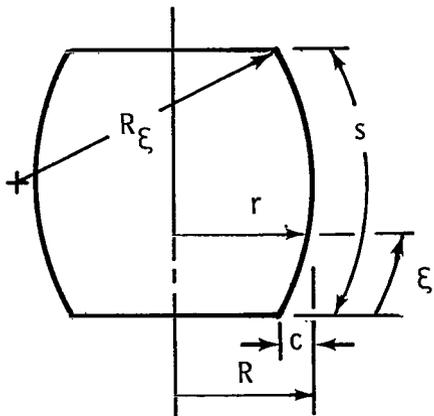
Positive Gaussian curvature shell $R_\xi > 0$
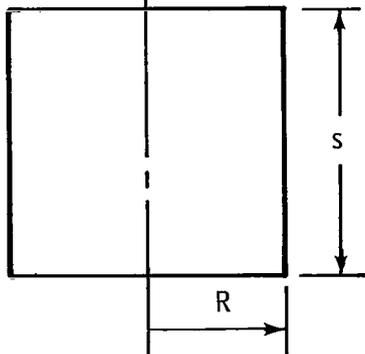
Zero Gaussian curvature shell   (cylinder)
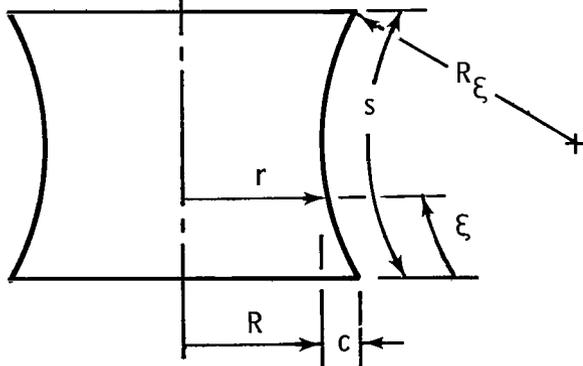
Negative Gaussian curvature shell $R_\xi < 0$

Figure 1.- Geometry of class of shell investigated.
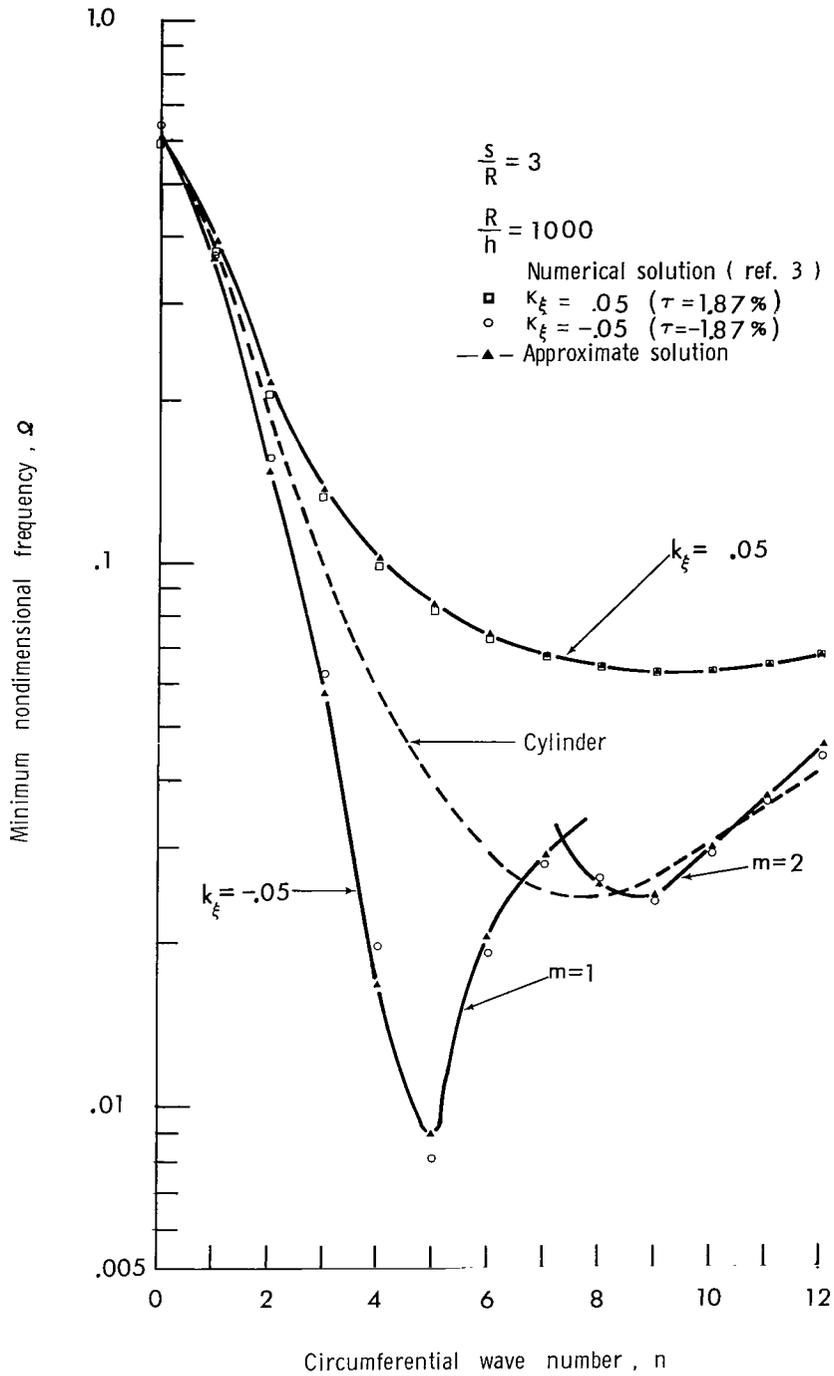
(a) $k_\xi = \pm 0.05$; s/R = 3; R/h = 1000.

Figure 2.- Comparison of minimum frequencies from approximate analysis with those from numerical analysis for freely supported shells.

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$\frac{s}{R} = 3$

$\frac{R}{h} = 1000$

Numerical solution (ref. 3)

□ $K_\xi = .1$ $(\tau = 3.75\%)$
○ $K_\xi = -.1$ $(\tau = -3.75\%)$
—▲— Approximate solution

$k_\xi = .1$

Cylinder

$m=4$

$m=3$

$k_\xi = -.1$

$m=1$

$m=2$

Minimum nondimensional frequency , Ω

1.0

0.1

0.01

0   2   4   6   8   10   12

Circumferential wave number , n
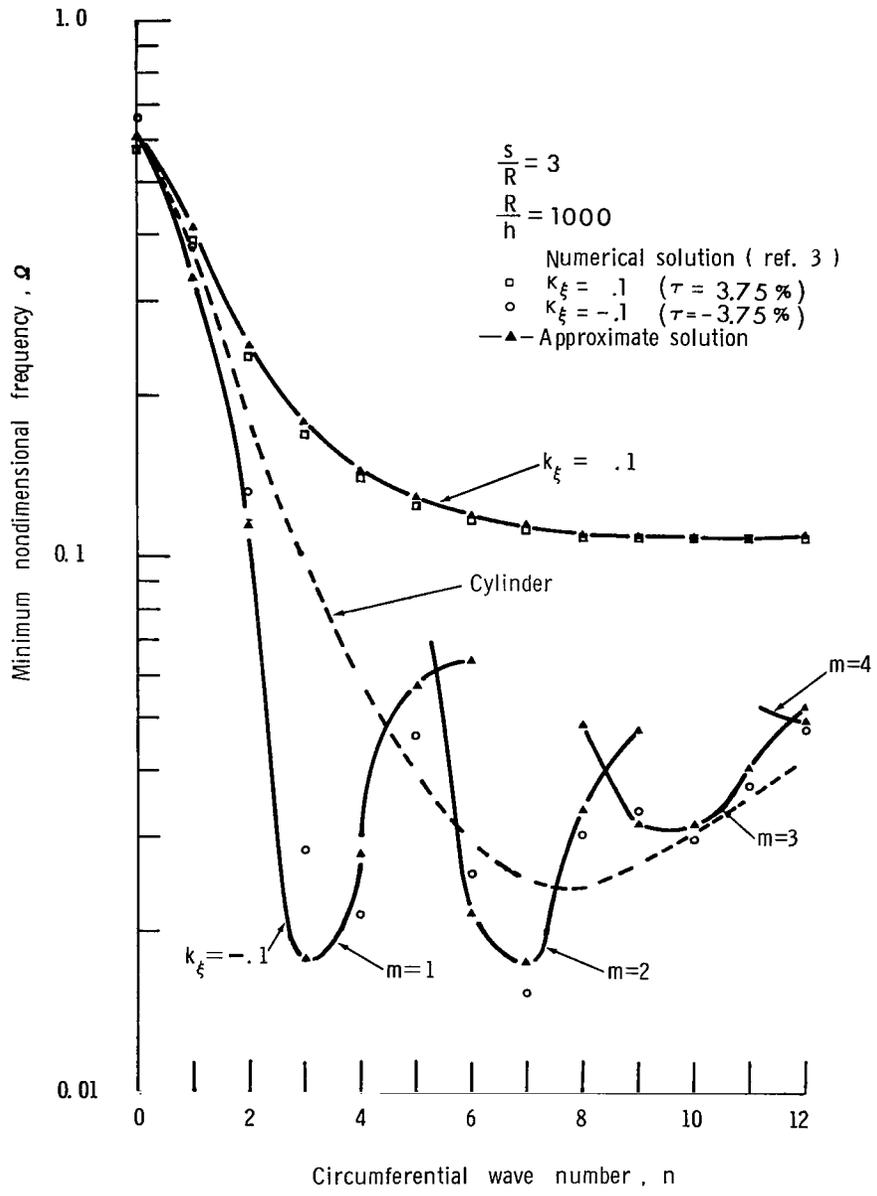
(b) $k_\xi = \pm 0.1$; s/R = 3; R/h = 1000.

Figure 2.- Continued.

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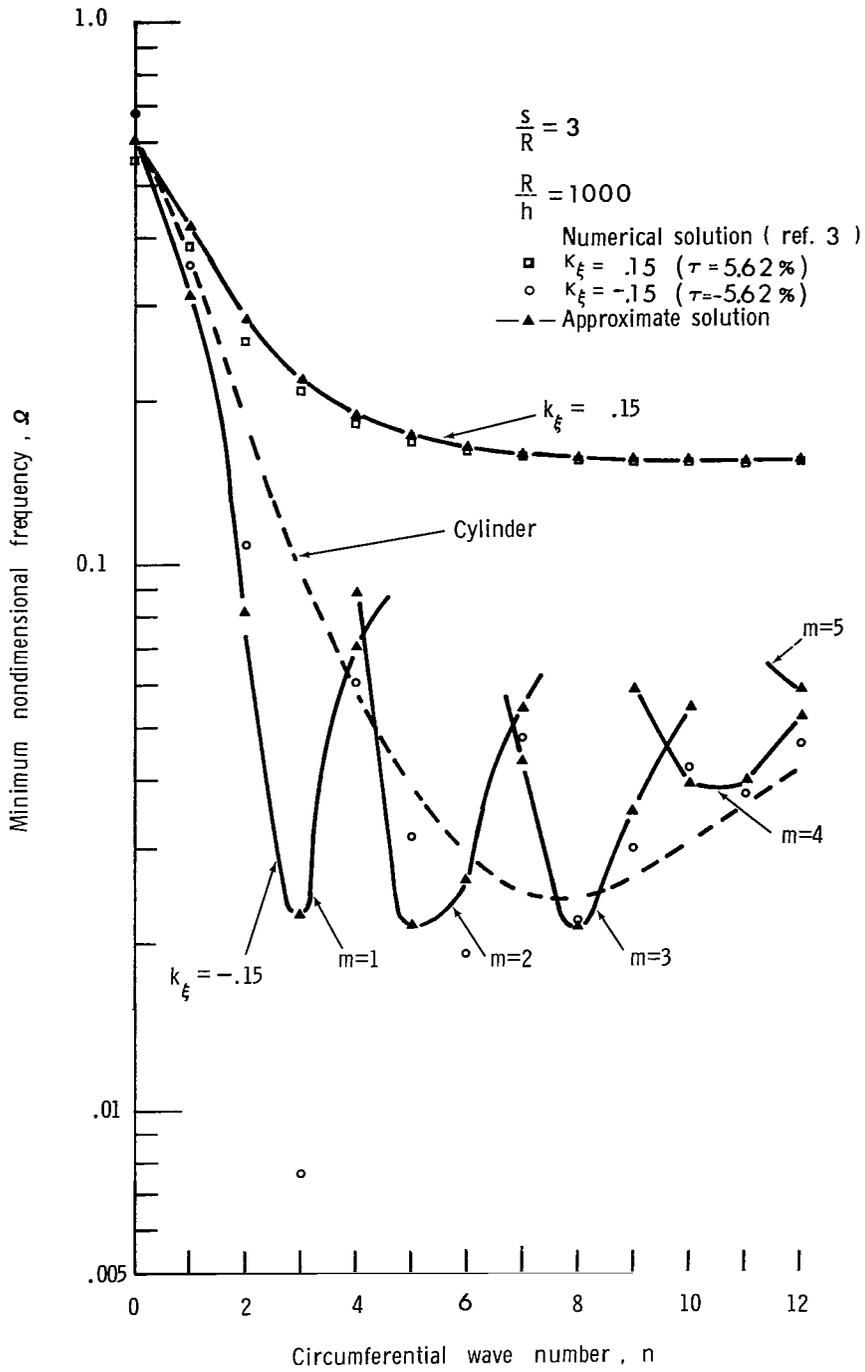
(c) $k_\xi = \pm 0.15$; $s/R = 3$; $R/h = 1000$.

Figure 2.- Concluded.

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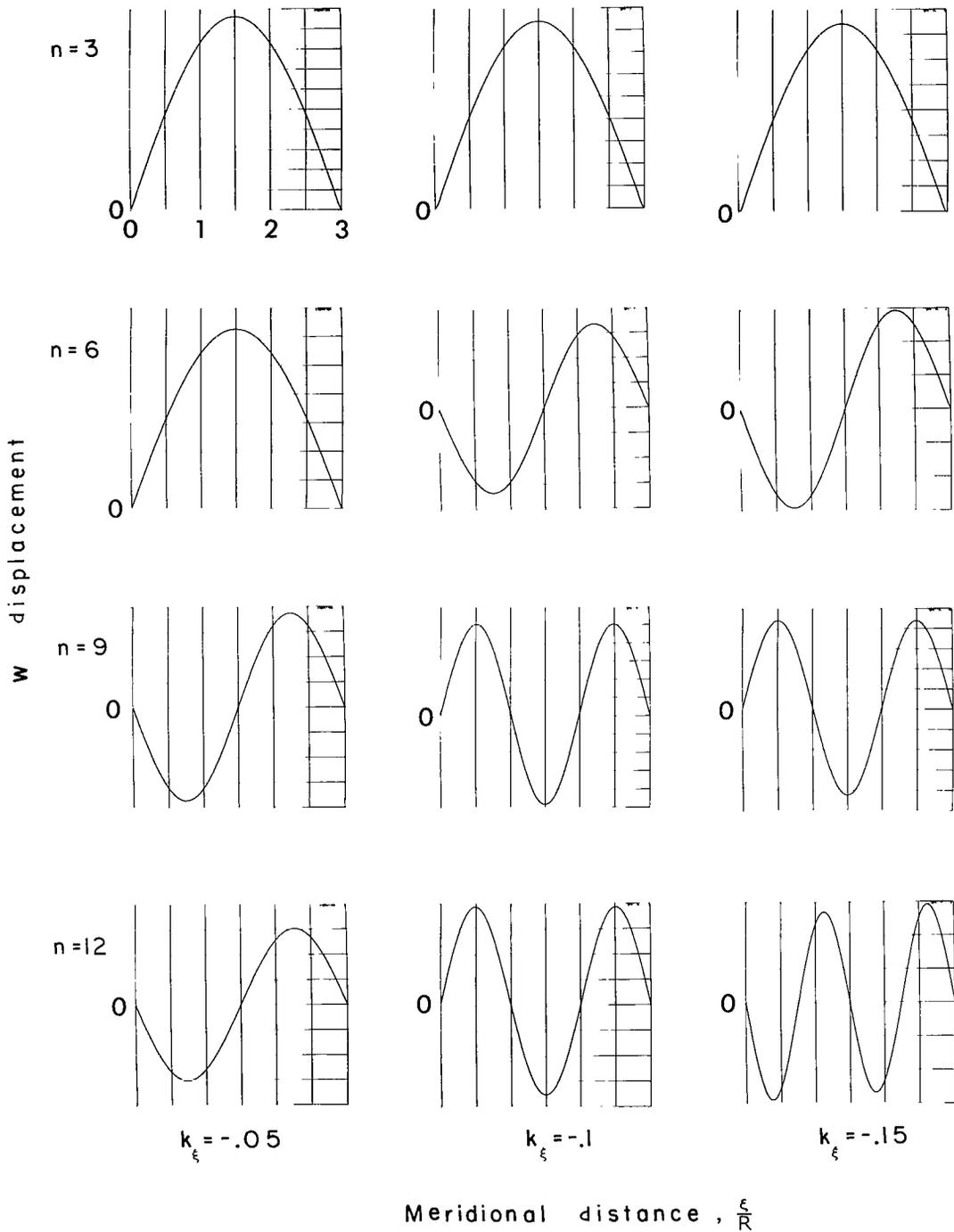
Figure 3.- Comparison of normal deflection shape w for several values of n for successive degrees of curvature. k_ξ = -0.05, -0.1, -0.15; s/R = 3; R/h = 1000. All calculations are based on the numerical analysis of reference 3 and are evaluated at minimum frequencies.

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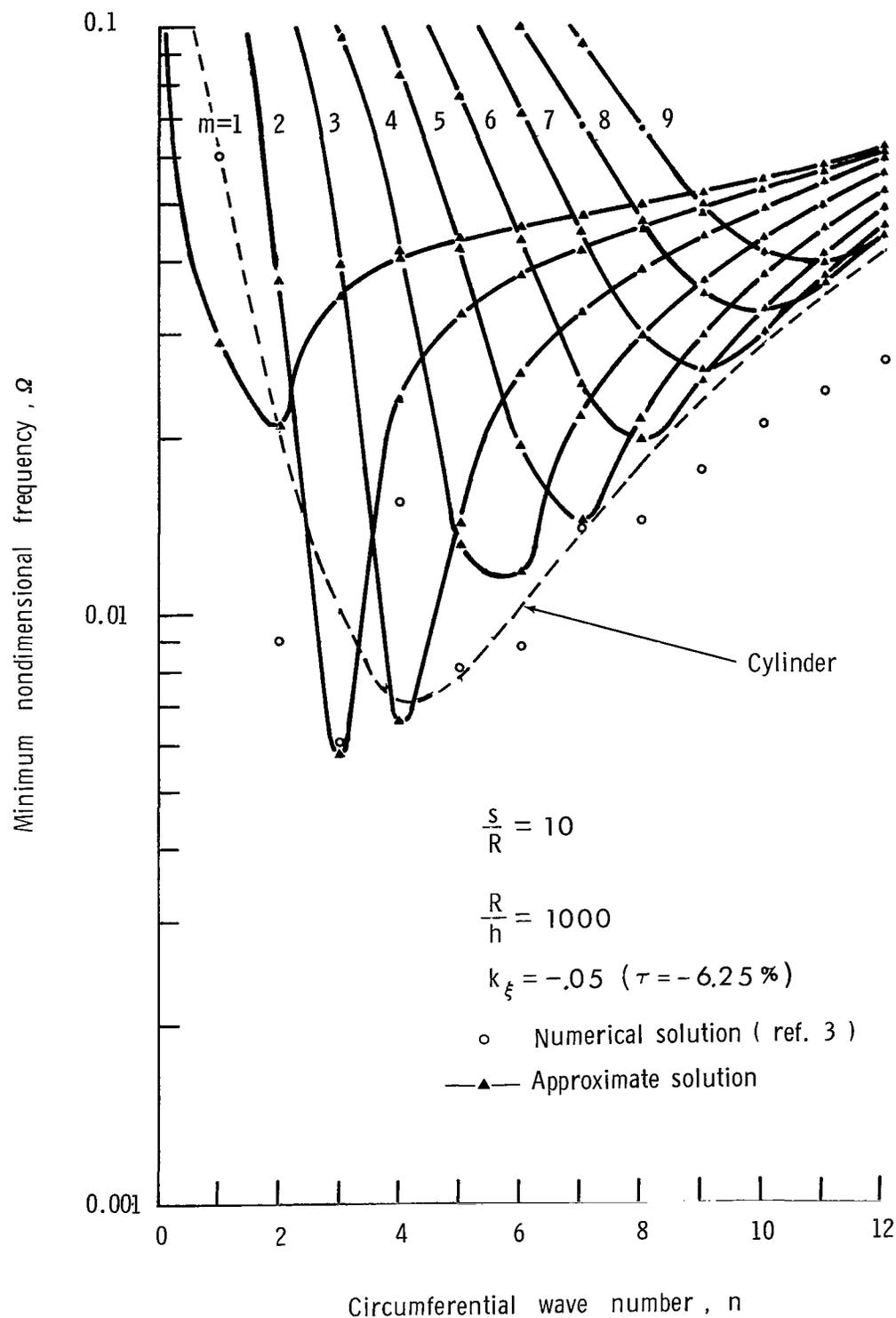
Figure 4.- Comparison of minimum frequencies from approximate analysis with those from numerical analysis for freely supported shells. $k_\xi = -0.05$; $s/R = 10$; $R/h = 1000$.
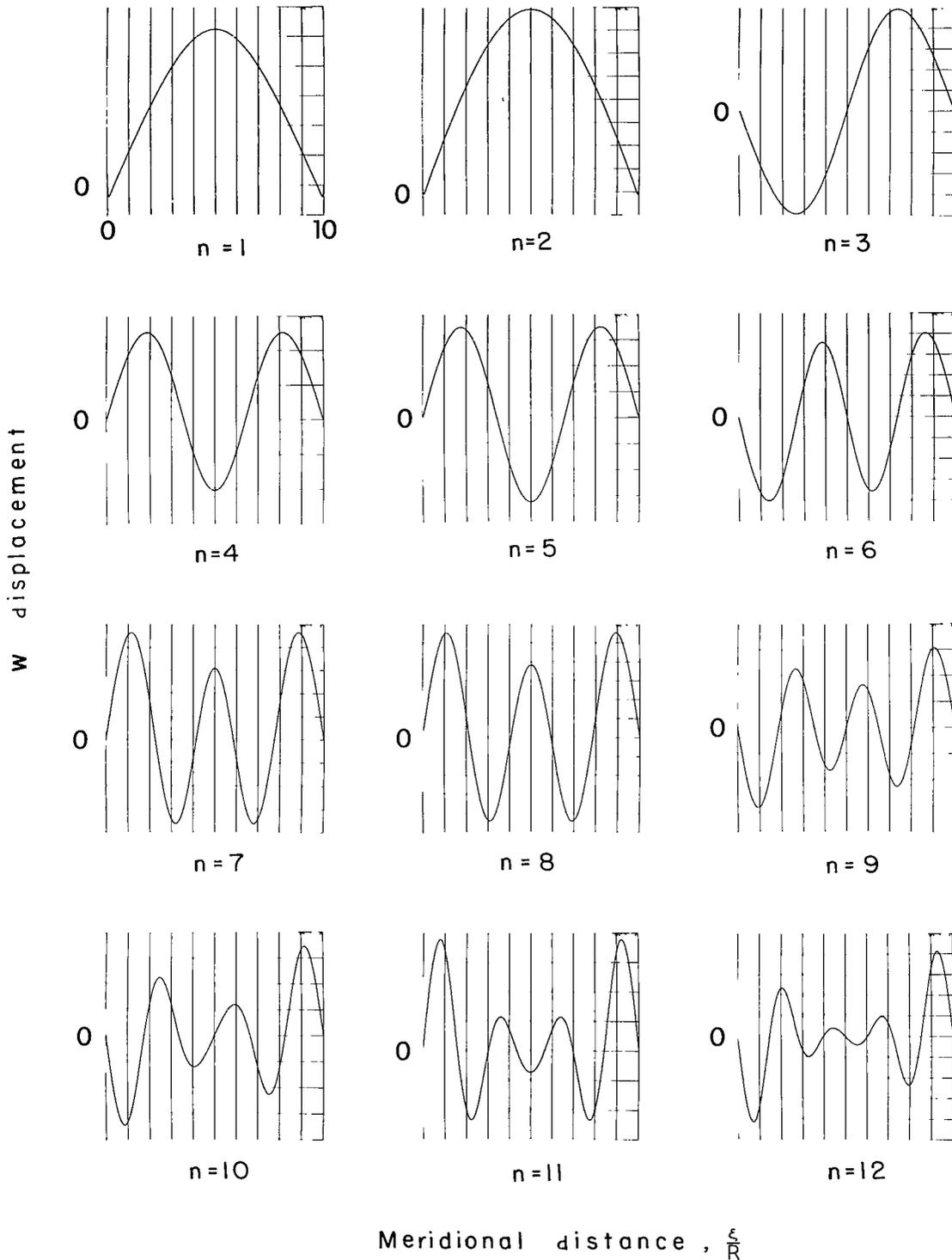
Figure 5.- Comparison of normal deflection shape w for several values of n for long shell with moderate curvature. k_ξ = -0.05; s/R = 10; R/h = 1000. All calculations are based on the numerical analysis of reference 3 and are evaluated at minimum frequencies.
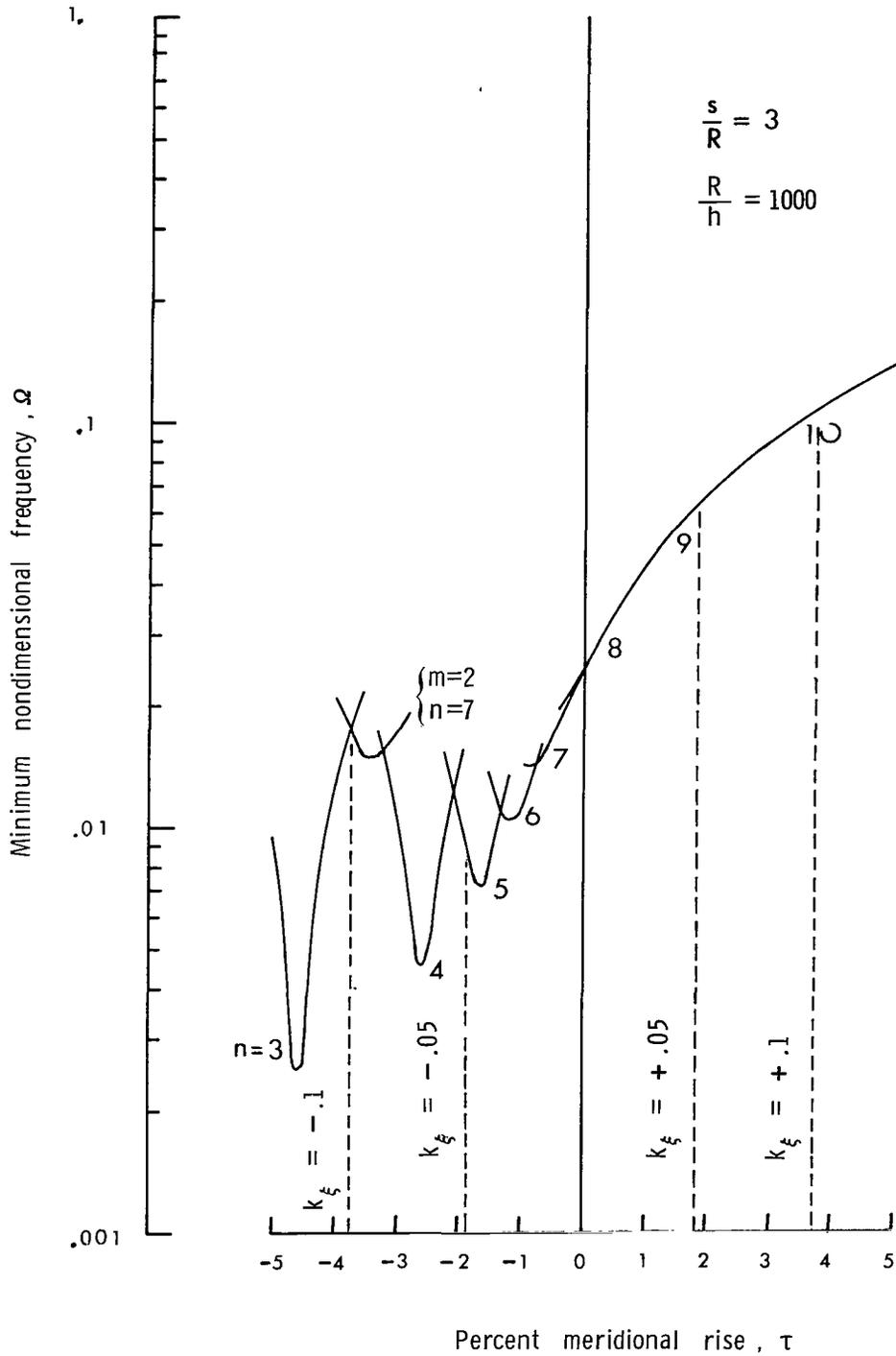
Figure 6.- Effect of meridional curvature on the minimum frequencies of freely supported unstressed shells. $\tau = -5\%$ to 5%; $s/R = 3$; $R/h = 1000$. The meridional wave number m is 1 unless otherwise noted; the circumferential wave number n is given on curves. All calculations based on approximate theory.

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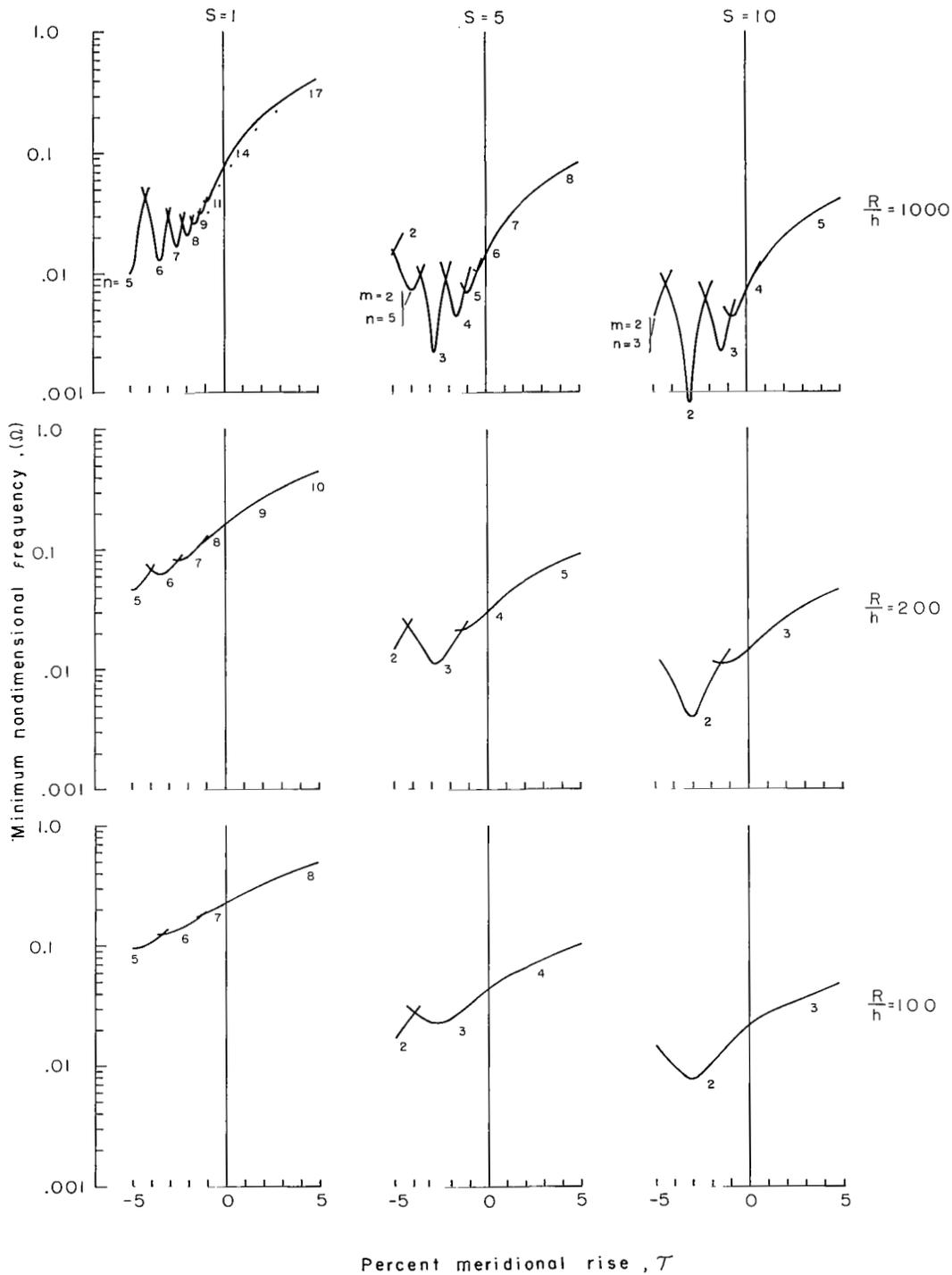
Figure 7.- Effect of meridional curvature on the minimum frequencies of freely supported unstressed shells for various length and thicknesses. τ = -5% to 5%; s/R = 1, 5, 10; R/h = 100, 200, 1000. The meridional wave number m is 1 unless otherwise noted, the circumferential wave number n is given on curves. All calculations based on approximate theory.

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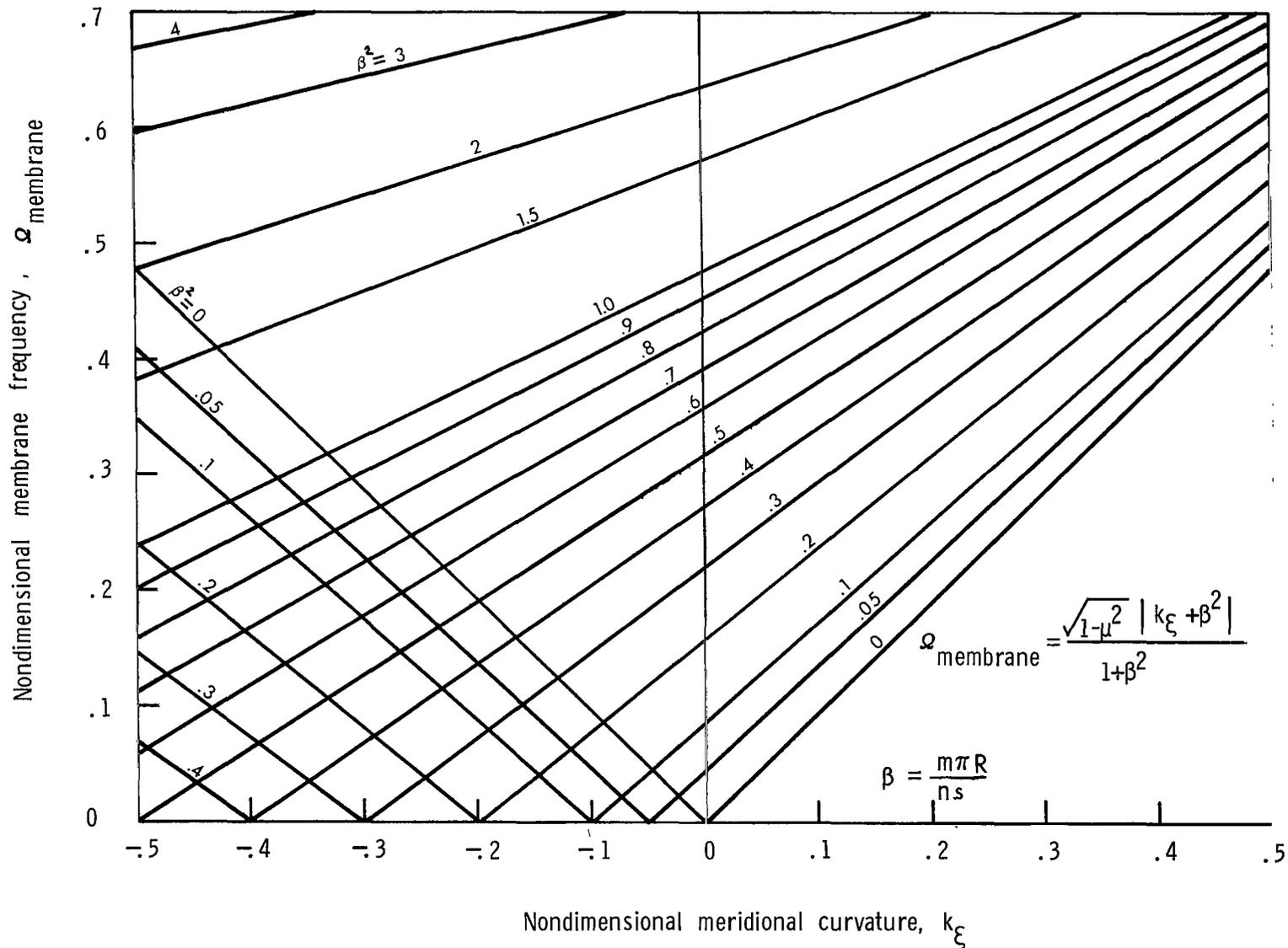
Figure 8.- Effect of meridional curvature on the natural frequencies of freely supported unstressed membrane shells. k_ξ = -0.5 to 0.5; μ = 0.3.

Nondimensional meridional curvature, k_ξ

Nondimensional membrane frequency, $\Omega_{membrane}$

$$\Omega_{membrane} = \frac{\sqrt{1-\mu^2}\ |k_\xi + \beta^2|}{1+\beta^2}$$

$$\beta = \frac{m\pi R}{n s}$$
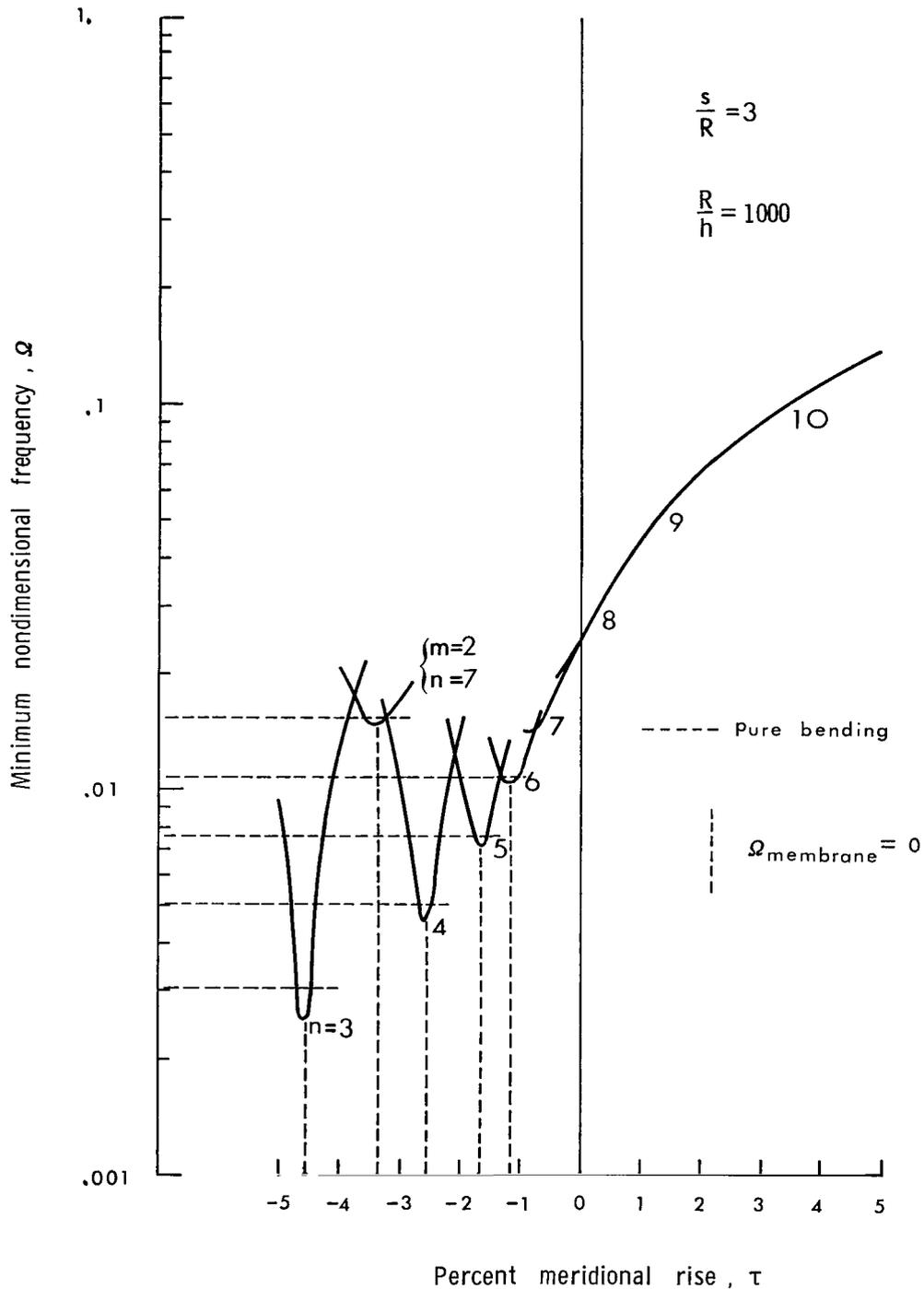
Figure 9.- Comparison of membrane and pure bending results with complete approximate theory. τ = -5% to 5%; s/R = 3; R/h = 1000. The meridional wave number m is 1 unless otherwise noted; the circumferential wave number n is given on curves. All calculations based on approximate theory.

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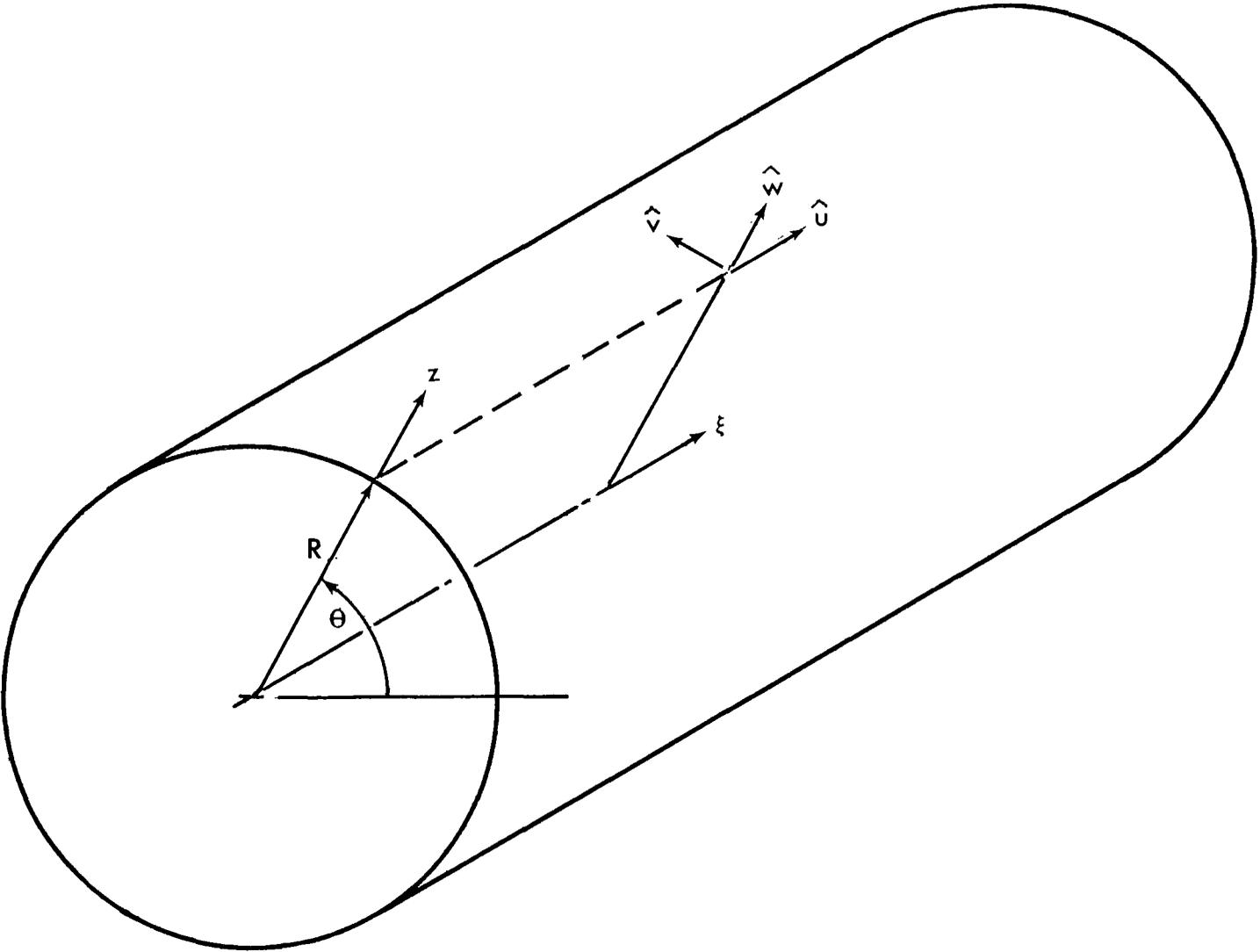
Figure 10.- Coordinate system and displacements associated with unstressed and deformed cylinder.

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