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THE VINTI DYNAMICAL PROBLEM AND THE GEOPOTENTIAL

by Diarmuid O'Mathuna

Electronics Research Center

Cambridge, Mass.

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ABSTRACT

A solution of the Vinti dynamical problem is derived in a form that is a clear generalization of the standard form of solution of the Kepler problem. Some geometrical results on the orbits are given together with a physical model for the potential.

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SUMMARY

A new form of the solution of the Vinti dynamical problem is derived wherein all three coordinates are expressed in terms of one independent variable corresponding to the true anomaly of the Kepler problem.

The relation of the Vinti potential to the geopotential is discussed together with the implied inference on the density distribution in the geoid.

INTRODUCTION

In this report we are primarily concerned with the solution of the Vinti dynamical problem, namely, the motion of a particle in the field of the Vinti potential. A second concern is the implication for geodesy of the close relationship between the latter potential and the geopotential. The relevance of the dynamical problem to the prediction of orbits of artifical satellites, first pointed out by Vinti, is now well recognized. It is also of interest that the potential itself suggests a geodetic inference on the density distribution in the geoid.

If the constant c in the Vinti problem is set to zero, it reduces to the Kepler problem. Accordingly, motion in the Vinti field must be expressible in a form which is a generalization of the corresponding form for the Kepler problem. This will be a feature of the representation we obtain for the solution, and the motivation serves as a guide to our method of obtaining it.

In the case of the Kepler problem the motion is given in terms of the true anomaly (f) by the three formulas for the three spherical coordinates, as follows:*

$$\frac{1}{r} = u = \frac{1}{p}\left(1 + e \cos f\right) \tag{A}$$

$$\cos \theta = \sin i \cdot \sin (f + \omega) \tag{B}$$

$$\tan(\phi - \Omega) = \cos i \cdot \tan (f + \omega) \tag{C}$$

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*An index of symbols appears on pages 82 and 83.

in which e denotes the eccentricity, i the angle of inclination, and p the semilatus rectum. The angles ω and Ω represent the argument of perigee and the angle of the ascending node, respectively. The above system is completed by inclusion of the time-angle relationship between time (t) and true anomaly (f), namely:

$$M = n(t - t_o)$$

$$= \arctan \left[\frac{\sqrt{1 - e^2} \sin f}{e + \cos f} \right] - \frac{e\sqrt{1 - e^2} \sin f}{1 + e \cos f} \qquad \text{(D)}$$

In formula (D), t_o denotes the time of perigee passage and n, the mean motion, is related to the semi-major axis a by the relation

$$n = \sqrt{\frac{\mu}{a^3}}$$

where μ is the normalized gravitational constant. The quantities p, a, and e are related by the formula

$$p = a\left(1 - e^2\right) .$$

Our aim here is the derivation of appropriate generalizations of the above formulas for the Vinti problem.

Solutions of the Vinti dynamical problem have already been proposed by Vinti and Izsak (refs. 1-6) in which are derived the appropriate generalizations of formulas (A) and (B). A related analysis by Aksenov, Gribenikov and Demin has appeared in the Russian literature (ref. 7). In our derivation of the analogues of formulas (A) and (B), we follow a procedure which is substantially equivalent to that of Izsak; however, the preliminary algebraic manipulation is made explicit here and we feel that the algebraic relations between the various constants are exhibited more clearly. Anticipating the approximation procedure introduced at a later stage, we note here that from these algebraic formulas it is a straightforward matter to make systematic approximations.

Corresponding to formula (C), the third coordinate gives rise to an elliptic integral of the third kind and here we have recourse to an approximation procedure in order to get the desired representation of the solution. We note that the Vinti potential approximates the geopotential only to second order in the small parameter; therefore, from the viewpoint of using the Vinti model as a basis for Earth-satellite orbit prediction, there is nothing lost in making a second-order approximation in the solution of the dynamical problem. In deriving the formula for the third coordinate such an approximation scheme is adopted to give the analog of formula (C). The same procedure is appropriate in deriving the time-angle relationship corresponding to formula (D). It is therefore also permissible to make second-order approximations from the algebraic formulas mentioned above since doing so is consistent with the overall method.

It is in dealing with the latter features of the problem--namely, in deriving approximate formulas for the third coordinate and for the time-angle relationship--that the present treatment differs substantially from that of Izsak. In the latter treatment, besides defining the independent variable f corresponding to true anomaly, Iszak further introduces three new auxiliary "independent" variables which we here call ψ_1, ψ_2, and Γ. The last variable corresponds to eccentric anomaly and the first pair are the am-functions associated with the Jacobian elliptic functions which appear in the representations of the first two coordinates in terms of f. The third coordinate is then expressed in terms of ψ_1 and ψ_2, involving both secular and trigonometric terms, and the time-angle relationship is a generalization of Kepler's equation which now involves both ψ_1 and ψ_2 as well as Γ. Both secular and periodic terms appear. The system must then be supplemented by an equation giving the relation between ψ_1 and ψ_2.

In the present treatment we do not introduce any auxiliary variables. We derive the formula relating ϕ to f in a form that is a clear generalization of formula (C). To complete the solution we need the relation between f and t; this is derived by similar techniques and appears as a generalization of the time-angle relation (D). Both relations are approximate, valid to second order.

The above considerations occupy the first six sections of this report. In Section 1 the problem is formulated. The first integrals are derived in Section 2. In Sections 3 and 4 we go through the algebraic manipulation leading to the analogue of relations (A) and (B) for the first two coordinates. In Section 5 we derive the formula for the third coordinate corresponding to relation (C). The time-angle relationship corresponding to relation (D) is derived in Section 6.

All the relations involve the Jacobian elliptic functions--a fact which helps us write the relations in relatively compact form. However, it is both consistent with the approximations already made and expedient as a preparation for numerical calculations to replace these elliptic functions by their second-order approximate truncated trigonometric series representations; this is done in Section 7. In Section 8 we include a geometrical result which gives some qualitative insight on the orbits.

As far as we know there has not been proposed a real physical situation giving rise to the Vinti potential field. However, if we consider how closely the Vinti field can be matched to the geopotential, we are led to the construction of a hypothetical geoid which would give rise to an external potential field of the Vinti type. In Section 9 we take up this question and derive the density distribution inside a sphere, which induces an external Vinti potential field. More generally, we derive the density distribution consistent with an arbitrary rotationally symmetric geopotential, the coefficients of which are empirically determined. This suggests an acceptable "first-order" hypothesis for the density distribution within the Earth. Such an inference on the mass distribution suggests a starting point from which further refinement may be possible.

1. FORMULATION OF THE PROBLEM

We define spheroidal coordinates R, σ, and ϕ by the relations:

$$x = \left(R^2 + c^2\right)^{1/2} \sin \sigma \cos \phi \tag{1.1a}$$

$$y = \left(R^2 + c^2\right)^{1/2} \sin \sigma \sin \phi \tag{1.1b}$$

$$z = R \cos \sigma \tag{1.1c}$$

from which we note the relations with the spherical coordinates r, θ, and ϕ, namely:

$$r^2 = R^2 + c^2 \sin^2 \sigma \tag{1.2a}$$

$$r \cos \theta = R \cos \sigma = z \tag{1.2b}$$

The metric coefficients g_{ij}, for the coordinate system (1.1), are given by

$$
\begin{aligned}
g_{11} &= \frac{R^2 + c^2 \cos^2 \sigma}{R^2 + c^2} \\
g_{22} &= R^2 + c^2 \cos^2 \sigma \qquad (1.3) \\
g_{33} &= \left(R^2 + c^2\right) \sin^2 \sigma
\end{aligned}
$$

with

$$g_{ij} = 0 \text{ for } i \neq j.$$

In this coordinate system the Vinti potential has the form (ref. 1):

$$V_1 = -\mu \frac{\left(R - c_1 c \cos \sigma\right)}{\left(R^2 + c^2 \cos^2 \sigma\right)} \qquad (1.4)$$

in which μ is the gravitational constant, c is the constant of the spheroidal system, and c_1 is arbitrary. Note that when $c = 0$ the coordinate system becomes spherical and the potential becomes the Kepler potential. When the potential [Eq. (1.4)] is expressed in terms of spherical harmonics, the constants c and c_1 can be so chosen that it matches the geopotential up to the second zonal harmonic; in fact, by a further modification (ref. 4) c_1 can be adjusted so that Eq. (1.4) matches the geopotential up to the third zonal harmonic. The deviation of Eq. (1.4) from the geopotential is then in the higher harmonics and of higher order in the small (oblateness) parameter.

We shall confine our attention to the dominant perturbation due to the even zonal harmonics. Since we can then set $c_1 = 0$, the computations are reduced considerably; however, this is not necessary for our procedure. Accordingly, we have

$$V = - \frac{\mu R}{R^2 + c^2 \cos^2 \sigma} \quad . \qquad (1.5)$$

The kinetic energy T is given by

$$T = \frac{1}{2}\left[\frac{R^2 + c^2 \cos^2 \sigma}{R^2 + c^2}\right]\dot{R}^2 + \frac{1}{2}\left(R^2 + c^2 \cos^2 \sigma\right)\dot{\sigma}^2$$

$$+ \frac{1}{2}\left(R^2 + c^2\right)\left(\sin^2 \sigma\right)\dot{\phi}^2 \tag{1.6}$$

so that the Lagrangian $L = T - V$ is given by

$$L = \frac{1}{2}\left[\frac{R^2 + c^2 \cos^2 \sigma}{R^2 + c^2}\right]\dot{R}^2 + \frac{1}{2}\left(R^2 + c^2 \cos^2 \sigma\right)\dot{\sigma}^2$$

$$+ \frac{1}{2}\left(R^2 + c^2\right)\left(\sin^2 \sigma\right)\dot{\phi}^2 + \frac{\mu R}{R^2 + c^2 \cos^2 \sigma} . \tag{1.7}$$

When c in Eq. (1.5) is chosen to fit the second-zonal harmonic in the geopotential, then the coefficient of the fourth zonal harmonic in roughly two-thirds of the corresponding coefficient of the geopotential, and all other harmonics have coefficients of smaller order than the corresponding coefficients in the geopotential. In fact, the coefficient of P_{2n} is of order c^{2n}, whereas none of the known coefficients of the zonal harmonics in the geopotential are of smaller order than c^4. We shall return to this question in Section 9.

2. SEPARATION: THE FIRST INTEGRALS

Since ϕ does not appear explicitly in the Lagrangian, we first utilize the fact that it is an ignorable coordinate. The third Lagrangian equation reads:

$$\frac{d}{dt}\left[\left(R^2 + c^2\right)\left(\sin^2 \sigma\right)\dot{\phi}\right] = 0 \tag{2.1}$$

which immediately yields the integral:

$$\left(R^2 + c^2\right)\left(\sin^2 \sigma\right)\dot{\phi} = \lambda_3 , \tag{2.2}$$

where λ_3 is a constant representing the polar component of angular momentum.

We now follow the standard procedure for ignorable coordinates (ref. 8), that is, we form a new Lagrangian by setting

$$L = \mathcal{L} - \sum_{r=1}^{k} \dot{q}_r \frac{\partial \mathcal{L}}{\partial \dot{q}_r}$$

where the q_r $(r = 1,\ldots,k)$ are the ignorable coordinates. We find that:

$$L = \frac{1}{2}\left[\frac{R^2 + c^2 \cos^2 \sigma}{R^2 + c^2}\right]\dot{R}^2 + \frac{1}{2}\left(R^2 + c^2 \cos^2 \sigma\right)\dot{\sigma}^2$$

$$+ \frac{\mu R}{R^2 + c^2 \cos^2 \sigma} - \frac{\lambda_3^2}{2} \cdot \frac{1}{\left(R^2 + c^2\right) \sin^2 \sigma} , \qquad (2.3)$$

that is, we have a modified Lagrangian with two degrees of freedom. The modified kinetic and potential energies are

$$T = \frac{1}{2}\left[\frac{R^2 + c^2 \cos^2 \sigma}{R^2 + c^2}\right]\dot{R}^2 + \frac{1}{2}\left(R^2 + c^2 \cos^2 \sigma\right)\dot{\sigma}^2 \qquad (2.4a)$$

$$V = - \frac{\mu R}{R^2 + c^2 \cos^2 \sigma} + \frac{\lambda_3^2}{2} \cdot \frac{1}{\left(R^2 + c^2\right) \sin^2 \sigma} \qquad (2.4b)$$

and

$$L = T - V \qquad (2.5)$$

To achieve separability, we first write the Lagrangian in the form:

$$L = \left(R^2 + c^2 \cos^2 \sigma\right)\left[\frac{1}{2} \frac{\dot{R}^2}{R^2 + c^2} + \frac{1}{2}\dot{\sigma}^2\right]$$

$$+ \frac{1}{R^2 + c^2 \cos^2 \sigma}\left[\mu R - \frac{\lambda_3^2}{2}\left(\frac{1}{\sin^2 \sigma} - \frac{c^2}{R^2 + c^2}\right)\right] \quad (2.6)$$

which, if we set

$$R = c \sinh \xi \quad (2.7)$$

takes the form:

$$L = c^2\left(\sinh^2 \xi + \cos^2 \sigma\right)\left[\frac{1}{2}\dot{\xi}^2 + \frac{1}{2}\dot{\sigma}^2\right]$$

$$+ \left[\frac{1}{c^2 \sinh^2 \xi + \cos^2 \sigma}\right]$$

$$\cdot \left[\left(\mu c \sinh \xi + \frac{\lambda_3^2}{2} \frac{1}{\cosh^2 \xi}\right) - \frac{\lambda_3^2}{2} \frac{1}{\sin^2 \sigma}\right] \quad (2.8)$$

This is now in standard Liouville form. For convenience, in subsequent manipulations, we set:

$$Q_1(\xi) = c^2 \sinh^2 \xi , \qquad Q_2(\sigma) = c^2 \cos^2 \sigma \quad (2.9a)$$

$$V_1(\xi) = -\left[\mu c \sinh \xi + \frac{\lambda_3^2}{2} \frac{1}{\cosh^2 \xi}\right], \quad V_2(\sigma) = \frac{\lambda_3^2}{2} \frac{1}{\cos^2 \sigma} \quad (2.9b)$$

$$Q(\xi,\sigma) = Q_1(\xi) + Q_2(\sigma) , \quad (2.9c)$$

so that

$$T = Q\left(\frac{1}{2}\dot{\xi}^2 + \frac{1}{2}\dot{\sigma}^2\right) , \qquad V = \frac{1}{Q}\left(V_1 + V_2\right) \quad (2.10)$$

and

$$L = Q\left(\frac{1}{2}\dot{\xi}^2 + \frac{1}{2}\dot{\sigma}^2\right) - V \tag{2.11a}$$

$$= Q\left(\frac{1}{2}\dot{\xi}^2 + \frac{1}{2}\dot{\sigma}^2\right) - \frac{1}{Q}\left(V_1 + V_2\right) . \tag{2.11b}$$

The Lagrangian equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) = \frac{\partial L}{\partial \xi} \quad , \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\sigma}}\right) = \frac{\partial L}{\partial \sigma} \tag{2.12a,b}$$

Before integrating these equations, we first derive the energy integral. Multiply Eq. (2.12a) by $\dot{\xi}$ and Eq. (2.12b) by $\dot{\sigma}$ and add to get

$$\dot{\xi}\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) + \dot{\sigma}\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\sigma}}\right) = \dot{\xi}\ \frac{\partial L}{\partial \xi} + \dot{\sigma}\ \frac{\partial L}{\partial \sigma} \tag{2.13}$$

which, after rearrangement, gives:

$$\frac{d}{dt}\left[\dot{\xi}\ \frac{\partial L}{\partial \dot{\xi}} + \dot{\sigma}\ \frac{\partial L}{\partial \dot{\sigma}}\right] = \dot{\xi}\ \frac{\partial L}{\partial \xi} + \dot{\sigma}\ \frac{\partial L}{\partial \sigma} + \ddot{\xi}\ \frac{\partial L}{\partial \dot{\xi}} + \ddot{\sigma}\ \frac{\partial L}{\partial \dot{\sigma}} = \frac{dL}{dt}$$

which integrates to give

$$\dot{\xi}\ \frac{\partial L}{\partial \dot{\xi}} + \dot{\sigma}\ \frac{\partial L}{\partial \dot{\sigma}} - L = E(\text{constant}) . \tag{2.14}$$

This is the energy integral. However, when the Lagrangian has the form (2.11), we have:

$$\dot{\xi}\ \frac{\sigma L}{\partial \dot{\xi}} + \dot{\sigma}\ \frac{\partial L}{\partial \dot{\sigma}} = \dot{\xi}\ \frac{\partial T}{\partial \dot{\xi}} + \dot{\sigma}\ \frac{\partial T}{\partial \dot{\sigma}} = 2T$$

and the energy integral takes the form

$$T + V = E \ . \tag{2.15}$$

We now introduce the form (2.11a) into Eqs. (2.12) and get:

$$\frac{d}{dt}\left(Q\dot{\xi}\right) = \frac{dQ_1}{d\xi}\left(\frac{1}{2}\dot{\xi}^2 + \frac{1}{2}\dot{\sigma}^2\right) - \frac{\partial V}{\partial \xi} \tag{2.16a}$$

$$\frac{d}{dt}\left(Q\dot{\sigma}\right) = \frac{dQ_2}{d\sigma}\left(\frac{1}{2}\dot{\xi}^2 + \frac{1}{2}\dot{\sigma}^2\right) - \frac{\partial V}{\partial \sigma} \tag{2.16b}$$

If we multiply Eq. (2.16a) by $Q\dot{\xi}$, we find

$$\begin{aligned}
Q\dot{\xi}\ \frac{d}{dt}\left(Q\dot{\xi}\right) &= Q\left(\frac{1}{2}\dot{\xi}^2 + \frac{1}{2}\dot{\sigma}^2\right)\frac{dQ_1}{d\xi}\dot{\xi} - Q\ \frac{\partial V}{\partial \xi}\dot{\xi} \\
&= \left[T\ \frac{dQ_1}{d\xi} - Q\ \frac{\partial V}{\partial \xi}\right]\dot{\xi} \\
&= \left[\left(E - V\right)\frac{dQ_1}{d\xi} - Q\ \frac{\partial V}{\partial \xi}\right]\dot{\xi} \\
&= \left[E\ \frac{dQ_1}{d\xi} - \frac{\partial}{\partial \xi}\left(QV\right)\right]\dot{\xi} \\
&= \left[E\ \frac{dQ_1}{d\xi} - \frac{dV_1}{d\xi}\right]\dot{\xi} = E\ \frac{dQ_1}{dt} - \frac{dV_1}{dt}\ ,
\end{aligned}$$

which integrates to give:

$$\frac{1}{2}Q^2\dot{\xi}^2 = EQ_1 - V_1 + \lambda_1 \tag{2.17}$$

where λ_1 is a constant of integration. If we multiply Eq. (2.16b) by $Q\dot{\sigma}$ and proceed in a similar manner, we obtain:

$$\tfrac{1}{2}Q^2\dot{\sigma}^2 = EQ_2 - V_2 + \lambda_2 \qquad (2.18)$$

where λ_2 is a second constant of integration. Adding Eqs. (2.17) and (2.18), we have:

$$QT = Q(E - V) + \lambda_1 + \lambda_2 , \qquad (2.19)$$

which, in view of the energy integral of Eq. (2.15), implies that

$$\lambda_1 + \lambda_2 = 0 . \qquad (2.20)$$

We shall be primarily concerned with cases with negative energy, that is:

$$E = -\alpha^2 , \qquad (2.21)$$

and as a consequence λ_2 must be positive. This fact will be evident from Eq. (2.25b) below. Accordingly, we set

$$\lambda_1 = -\frac{\lambda^2}{2} , \quad \lambda_2 = \frac{\lambda^2}{2} \qquad (2.22)$$

and now writing Eqs. (2.17) and (2.18) in terms of α^2 and λ^2, we have:

$$\tfrac{1}{2}Q^2\dot{\xi}^2 = -\frac{\lambda^2}{2} - V_1 - \alpha^2 Q_1 \qquad (2.23a)$$

$$\tfrac{1}{2}Q^2\dot{\sigma}^2 = \frac{\lambda^2}{2} - V_2 - \alpha^2 Q_2 \qquad (2.23b)$$

If Eqs. (2.23) are added, we obtain

$$T + V = -\alpha^2 , \qquad (2.24)$$

so that the energy relation is implied by Eqs. (2.23) and so can henceforth be ignored.

We now use relations (2.9) to write Eqs. (2.23) explicitly and reintroduce the variable R from (2.7) to get

$$\frac{1}{2}\,\frac{\left(R^2 + c^2 \cos^2 \sigma\right)^2}{R^2 + c^2}\,\dot{R}^2 = -\frac{\lambda^2}{2} + \mu R + \frac{\lambda_3^2}{2}\,\frac{c^2}{R^2 + c^2} - \alpha^2 R^2 \qquad (2.25a)$$

$$\frac{1}{2}\left(R^2 + c^2 \cos^2 \sigma\right)^2 \dot{\sigma}^2 = \frac{\lambda^2}{2} - \frac{\lambda_3^2}{2}\,\frac{1}{\sin^2 \sigma} - \alpha^2 c^2 \cos^2 \sigma \qquad (2.25b)$$

The pair of Eqs. (2.25), together with Eq. (2.2), constitutes the set of equations, the solution of which is the complete solution of the dynamical problem. They are the first integrals -- or action integrals -- with constants α^2, λ_3, and λ. The first two represent the energy and polar component of angular momentum, respectively. The constant λ, which also has the dimension of angular momentum, does not have an obvious physical interpretation, except when $c = 0$, in which case it is the magnitude of the angular momentum vector.

We next introduce a true anomaly f defined by

$$\frac{df}{dt} = \frac{\Lambda}{R^2 + c^2 \cos^2 \sigma} \qquad (2.26)$$

where Λ is a constant, having the dimension of angular momentum, which will be defined later; in the Kepler case ($c = 0$), the angle f has a simple geometrical interpretation. If we let ' denote differentiation with respect to f, then Eqs. (2.25) take the form

$$\Lambda^2 R'^2 = -\left(R^2 + c^2\right)\left[\lambda^2 - 2\mu R + 2\alpha^2 R^2\right] + \lambda_3^2 c^2 \qquad (2.27a)$$

$$\Lambda^2 \sigma'^2 = \lambda^2 - \frac{\lambda_3^2}{\sin^2 \sigma} - 2\alpha^2 c^2 \cos^2 \sigma \qquad (2.27b)$$

and Eq. (2.2) becomes:

$$\phi' = \frac{\lambda_3}{\Lambda} \frac{R^2 + c^2 \cos^2 \sigma}{\left(R^2 + c^2\right) \sin^2 \sigma}$$

or

$$\phi' = \frac{\lambda_3}{\Lambda}\left[\frac{1}{\sin^2 \sigma} - \frac{c^2}{R^2 + c^2}\right] . \tag{2.28}$$

We now proceed to a consideration of the above equations individually.

3. ADJUSTMENT OF EQUATIONS (2.27)

In this section, we go through a lengthy procedure of algebraic manipulations in order to write Eqs. (2.27) in a form which immediately yields the solution we seek.

The constants appearing in Eq. (2.27) arose naturally in the mathematical separation, and are determined readily from initial conditions. The constants that will appear in the final form are more "natural" in the physical-geometrical sense. To determine them in terms of initial conditions, it is necessary to determine them in terms of the constants appearing in Eq. (2.27). This is the aim of this section.

The Equation for R

We first write Eq. (2.27a) in the form:

$$\Lambda^2 R'^2 = -2\alpha^2\left\{\left(R^2 + c^2\right)\left[R^2 - \frac{\mu}{\alpha^2}R + \frac{\lambda^2}{2\alpha^2}\right] - \frac{c^2\lambda^2}{2\alpha^2} \cdot \frac{\lambda_3^2}{\lambda^2}\right\} \tag{3.1}$$

We now associate with the energy and angular momentum constants two length scales a_o and p_o (corresponding to the semi-major axis and semi-latus rectum, respectively) by setting

$$a_o = \frac{\mu}{2\alpha^2} \quad , \quad p_o = \frac{\lambda^2}{\mu} \tag{3.2}$$

and Eq. (3.1) then reads:

$$\frac{\Lambda^2}{2\alpha^2} R'^2 = -\left[\left(R^2 + c^2\right)\left(R^2 - 2a_o R + a_o p_o\right) - \nu^2 c^2 a_o p_o\right] \tag{3.3}$$

where the dimensionless parameter ν is defined by

$$\nu = \frac{\lambda_3}{\lambda} . \tag{3.4}$$

We shall find it convenient for the subsequent analysis to have Eq. (3.3) written in dimensionless form. To do so we introduce dimensionless parameters ℓ_o and η by the relations

$$\ell_o^2 = \frac{p_o}{a_o} , \qquad \eta = \frac{c}{p_o} \tag{3.5}$$

and define the dimensionless independent variable y by the relation

$$R = a_o y .$$

Then, in these terms, Eq. (3.3) reads:

$$\begin{aligned}\frac{\Lambda^2}{2\alpha^2 a_o^2} y'^2 &= -\left[\left(y^2 + \ell_o^4 \eta^2\right)\left(y^2 - 2y + \ell_o^2\right) - \eta^2 \nu^2 \ell_o^6\right] \\ &= -\left[y^4 - 2y^3 + \ell_o^2\left(1+\eta^2 \ell_o^2\right) y^2 \right. \\ &\qquad \left. - 2\eta^2 \ell_o^4 y + \eta^2 \ell_o^6 \left(1-\nu^2\right)\right]\end{aligned} \tag{3.6}$$

Note that in the Kepler case the parameters ν and ℓ_o have a geometric interpretation. The ratio of polar angular momentum to "total" angular momentum is measured by ν which corresponds, therefore, to cos i where i is the inclination in the Kepler case. The parameter ℓ_o measures the ratio of "latus rectum" to "major axis" and so corresponds to $(1-e_k^2)$ where e_k is the Kepler eccentricity. The small parameter η measures the "oblateness" (or, alternatively, the inhomogeneity, c.f. Section 9) against the characteristic dimension of the orbit. We shall ultimately approximate the solution in terms of this small parameter. We first aim at a form of the solution suitable for making systematic approximations.

Once the initial conditions have been specified, the quantities λ_3, λ and α^2 are known from Eqs. (2.2) and (2.5) and so the length scales a_o and p_o are immediately determined. We therefore refer to a_o and p_o as the fundamental length scales. Similarly, we shall refer to ν, ℓ_o, and η as the fundamental parameters. Most of the subsequent algebraic manipulation is aimed at expressing the constants which shall appear in the representation of the solution in terms of these fundamental quantities.

We start by decomposing the quartic on the right side of Eq. (3.6) into two quadratic factors such that each factor is a "perturbation" of the corresponding factor in the Kepler case: we set

$$y^4 - 2y^3 + \ell_o^2\left(1 + \eta^2\ell_o^2\right)y^2 - 2\eta^2\ell_o^4 y + \eta^2\ell_o^6\left(1 - \nu^2\right)$$

$$= \left[y^2 - 2\eta^2\ell_o^2 h_o y + \frac{\eta^2\ell_o^4}{s_o}\,\frac{\left[s_o - h_o\left(1 - \eta^2\ell_o^2 h_o\right)^2\right]}{1 - \eta^2\ell_o^2 h_o}\right]$$

$$\cdot\left[y^2 - 2\left(1 - \eta^2\ell_o^2 h_o\right)y + \frac{\ell_o^2}{s_o}\left(1 - \eta^2\ell_o^2 h_o\right)^2\right] \quad . \tag{3.7}$$

With this choice the identity requirements on the first, second, and fourth coefficients are automatically satisfied. The identification of the third and fifth coefficients gives the pair of equations for the determination of h_o and s_o, namely:

$$\frac{\left(1 - \eta^2\ell_o^2 h_o\right)^2}{s_o} + \frac{\eta^2\ell_o^2}{s_o}\cdot\frac{\left[s_o - h_o\left(1 - \eta^2\ell_o^2 h_o\right)^2\right]}{1 - \eta^2\ell_o^2 h_o}$$

$$+ 4\eta^2 h_o\left(1 - \eta^2\ell_o^2 h_o\right) = 1 + \eta^2\ell_o^2 \tag{3.8a}$$

and

$$\frac{\left(1 - \eta^2\ell_o^2 h_o\right)\left[s_o - h_o\left(1 - \eta^2\ell_o^2 h_o\right)^2\right]}{s_o^2} = 1 - \nu^2 \tag{3.8b}$$

which after some rearrangement may be written:

$$\left(1 - \eta^2\ell_o^2 h_o\right)^2\left[\left(1 - \eta^2\ell_o^2 h_o\right)^2 - s_o\left(1 + \eta^2\ell_o^2\right)\right]$$

$$+ \eta^2 s_o\left[4h_o\left(1 - \eta^2\ell_o^2 h_o\right)^2 + \ell_o^2\left(1 - \nu^2\right)s_o\right] = 0 \tag{3.9a}$$

$$h_o\left(1 - \eta^2\ell_o^2 h_o\right)^3 = s_o\left(1 - \eta^2\ell_o^2 h_o\right) - s_o^2\left(1 - \nu^2\right) \tag{3.9b}$$

This pair of algebraic equations can be uncoupled if we associate with s_o the related quantity q_o defined by

$$s_o = \left(1 - \eta^2\ell_o^2 h_o\right)^2 q_o \ . \tag{3.10}$$

Inserting Eq. (3.10) into Eq. (3.9b) and rearranging, we get h_o in terms of q_o in the form

$$h_o = \frac{q_o\left[1 - q_o\left(1 - \nu^2\right)\right]}{\left[1 - \eta^2\ell_o^2\left(1 - \nu^2\right)q_o^2\right]} \tag{3.11a}$$

and hence

$$1 - \eta^2\ell_o^2 h_o = \frac{1 - \eta^2\ell_o^2 q_o}{\left[1 - \eta^2\ell_o^2\left(1 - \nu^2\right)q_o^2\right]} \tag{3.11b}$$

Substituting s_o and h_o from Eqs. (3.10) and (3.11) into Eq. (3.9a), we get the equation for q_o, namely:

$$\left[1 + \eta^2\ell_o^2\left(1 - \nu^2\right)q_o^2\right]\left[1 - \eta^2\ell_o^2\left(1 - \nu^2\right)q_o^2\right]^2$$
$$+ 4\eta^2 q_o^2\left(1 - \eta^2\ell_o^2 q_o\right)\left[1 - q_o\left(1 - \nu^2\right)\right]$$
$$= q_o\left(1 + \eta^2\ell_o^2\right)\left[1 - \eta^2\ell_o^2\left(1 - \nu^2\right)q_o^2\right]^2 \ . \tag{3.12}$$

This sixth-order equation cannot be solved algebraically. However, if we note that the root we seek is that which tends to 1 as η tends to zero so that

$$q_o = 1 + O\left(\eta^2\right) \ , \tag{3.13}$$

then we see that the determination of the root as a power series in η^2 is a straightforward matter. In fact, if we anticipate the later approximations when effects of order η^6 are neglected, it would be consistent to replace Eq. (3.12) by the cubic equation:

$$\left[1 - \eta^4\ell_o^4\left(1 - \nu^2\right)^2\right]\left[1 - \eta^2\ell_o^2\left(1 - \nu^2\right)q_o^2\right]$$
$$+ 4\eta^2\left(1 - \eta^2\ell_o^2\right)q_o^2\left[1 - q_o\left(1 - \nu^2\right)\right]$$
$$= q_o\left(1 + \eta^2\ell_o^2\right)\left[1 + \eta^4\ell_o^4\left(1 - \nu^2\right) - 2\eta^2\ell_o^2\left(1 - \nu^2\right)q_o^2\right]$$

(3.12*)

which can be solved algebraically. However, it is probably more economical to determine the root as a power series in η^2.

With q_o determined from Eq. (3.12), h_o and s_o follow immediately from Eqs. (3.11a) and (3.10), respectively, and so the coefficients in the quadratic on the right of Eq. (3.7) are all determined. From now on we shall consider s_o, q_o, and h_o known in terms of the fundamental parameters. We add the remark that taking Eqs. (3.13) with (3.10) and (3.11) shows that

$$q_o = 1 + O\left(\eta^2\right) \quad , \qquad h_o = \nu^2\left[1 + O\left(\eta^2\right)\right]. \tag{3.14}$$

We now combine Eqs. (3.6) and (3.7) and factor out the constant term in the second factor. If we note from Eq. (3.10) that

$$\frac{s_o}{1 - \eta^2 \ell_o^2 h_o} = q_o\left(1 - \eta^2 \ell_o^2 h_o\right) = \sqrt{s_o q_o} \tag{3.15}$$

and substitute for α^2 from Eq. (3.2), we get:

$$\Lambda^2 y'^2 = -\frac{\mu a_o \ell_o^2}{q_o}\left[1 - 2\frac{\sqrt{s_o q_o}}{\ell_o^2} y + \frac{q_o}{\ell_o^2} y^2\right] \cdot \left[y^2 - 2\eta^2 \ell_o^2 h_o y + \eta^2 \ell_o^4 \frac{\left(q_o - h_o\right)}{\sqrt{q_o s_o}}\right] \tag{3.16}$$

where we have interchanged the order of the factors. Written in terms of the original variable R and again noting Eqs. (3.2) and (3.5), Eq. (3.16) becomes

$$\Lambda^2 R'^2 = -\frac{\lambda^2}{q_o}\left[1 - 2\frac{\sqrt{s_o q_o}}{p_o} R + \frac{q_o}{a_o p_o} R^2\right] \cdot \left[R^2 - 2\eta^2 h_o p_o R + \eta^2 \frac{\left(q_o - h_o\right)}{\sqrt{q_o s_o}} p_o^2\right] \tag{3.17}$$

At this point, still following the procedure for the Kepler problem, we introduce a new independent variable u, defined by

$$u = \frac{1}{R} \quad \text{so that} \quad u' = -\frac{R'}{R^2} \tag{3.18}$$

and Eq. (3.17) takes the form:

$$\Lambda^2 u'^2 = -\frac{\lambda^2}{q_o}\left[u^2 - 2\frac{\sqrt{s_o q_o}}{p_o}u + \frac{q_o}{a_o p_o}\right]$$
$$\cdot\left[1 - 2\eta^2 h_o p_o u + \eta^2 \frac{\left(q_o - h_o\right)}{\sqrt{q_o s_o}} p_o^2 u^2\right]$$

$$= \frac{\lambda^2}{q_o}\left[\frac{q_o s_o}{p_o^2}\left(1 - \frac{\ell_o^2}{s_o}\right) - \left(u - \frac{\sqrt{q_o s_o}}{p_o}\right)^2\right]$$
$$\cdot\left[1 - 2\eta^2 h_o p_o u + \eta^2 \frac{\left(q_o - h_o\right)}{\sqrt{q_o s_o}} p_o^2 u^2\right] \tag{3.19}$$

This suggests defining the quantity e_o (corresponding to the eccentricity) by the relation

$$e_o^2 = 1 - \frac{\ell_o^2}{s_o} \tag{3.20}$$

and Eq. (3.19) then reads:

$$\Lambda^2 u'^2 = \frac{\lambda^2}{q_o}\left[q_o s_o \frac{e_o^2}{p_o^2} - \left(u - \frac{\sqrt{q_o s_o}}{p_o}\right)^2\right]$$
$$\cdot\left[1 - 2\eta^2 h_o p_o u + \eta^2 \frac{\left(q_o - h_o\right)}{\sqrt{q_o s_o}} p_o^2 u^2\right] . \tag{3.21}$$

The next step is to set

$$u = \frac{\sqrt{q_o s_o}}{p_o}\left(1 + e_o w\right) \tag{3.22}$$

which, when introduced into Eq. (3.21), gives as the equation for w:

$$\Lambda^2 w'^2 = \frac{\lambda^2}{q_o}\left(1 - w^2\right)\left[1 - 2\eta^2 \sqrt{q_o s_o}\, h_o (1 + e_o w) + \eta^2 \sqrt{q_o s_o}\left(q_o - h_o\right)\left(1 + e_o w\right)^2\right]$$

$$= \frac{\lambda^2}{q_o}\left(1 - w^2\right)\left[\left[1 - \eta^2 \sqrt{q_o s_o}\left(3h_o - q_o\right)\right] - 2\eta^2 \sqrt{q_o s_o}\left(2h_o - q_o\right)e_o w + \eta^2 \sqrt{q_o s_o}\left(q_o - h_o\right)e_o^2 w^2\right]$$

$$= \lambda^2 \, \frac{1 - \eta^2 \sqrt{q_o s_o}\left(3h_o - q_o\right)}{q_o}\left[1 - w^2\right] \cdot \left[1 - \frac{2\eta^2 \sqrt{q_o s_o}\left(2h_o - q_o\right)}{1 - \eta^2 \sqrt{q_o s_o}\left(3h_o - q_o\right)}\, e_o w + \frac{\eta^2 \sqrt{q_o s_o}\left(q_o - h_o\right)}{1 - \eta^2 \sqrt{q_o s_o}\left(3h_o - q_o\right)}\, e_o^2 w^2\right] \qquad (3.23)$$

To facilitate subsequent manipulation, we set:

$$\left.\begin{aligned} h_1 &= \frac{\sqrt{q_o s_o}\left(2h_o - q_o\right)}{1 - \eta^2 \sqrt{q_o s_o}\left(3h_o - q_o\right)} \\ g_1^2 &= \frac{\sqrt{q_o s_o}\left(q_o - h_o\right)}{1 - \eta^2 \sqrt{q_o s_o}\left(3h_o - q_o\right)} \quad , \end{aligned}\right\} \qquad (3.24)$$

so that the above equation for w reads:

$$\Lambda^2 w'^2 = \lambda^2 \left[1 - \eta^2 \frac{q_o s_o\left(3h_o - q_o\right)}{q_o}\right]\left[1 - w^2\right]$$

$$\cdot\left[1 - 2\eta^2 h_1 e_0 w + \eta^2 g_1^2 e_0^2 w^2\right] . \tag{3.25}$$

To set this equation up for a straightforward solution, we make a final adjustment to the quadratics on the right of Eq. (3.25). We write:

$$\begin{aligned} 1 - w^2 &\equiv J^2\left[\left(1 - \delta w\right)^2 - \left(w - \delta\right)^2\right] \\ &= J^2\left(1 - \delta^2\right)\left(1 - w^2\right) , \end{aligned} \tag{3.26}$$

so that

$$J^2\left(1 - \delta^2\right) = 1 . \tag{3.27}$$

Seeking a similar decomposition for the second quadratic, we write:

$$1 - 2\eta^2 h_1 e_0 w + \eta^2 g_1^2 e_0^2 w^2 \equiv J^2\left[A\left(1 - \delta w\right)^2 + B\left(w - \delta\right)^2\right] \tag{3.28}$$

which yields the following relations

$$\left.\begin{aligned} J^2\left(A + B\delta^2\right) &= 1 \\ J^2\left(B + A\delta^2\right) &= \eta^2 e_0^2 g_1^2 \\ J^2\delta\left(A + B\right) &= \eta^2 e_0 h_1 \end{aligned}\right\} \tag{3.29}$$

from which, together, with Eq. (3.27), we derive the following:

$$A - B = 1 - \eta^2 e_0^2 g_1^2 \tag{3.30a}$$

$$\frac{\left(1 + \delta^2\right)}{\left(1 - \delta^2\right)}\left(A + B\right) = 1 + \eta^2 e_0^2 g_1^2 \tag{3.30b}$$

$$\frac{\delta}{1 - \delta^2}\left(A + B\right) = \eta^2 e_0 h_1 \tag{3.30c}$$

Combining Eqs. (3.30a) and (3.30b), we have the equation for δ, namely:

$$\frac{\delta}{1 + \delta^2} = \frac{\eta^2 e_0 h_1}{1 + \eta^2 e_0^2 g_1^2} \quad , \tag{3.31}$$

so that $\delta = O(\eta^2)$. We set

$$\delta = \eta^2 e_o d_o \quad , \qquad h = \frac{h_1}{1 + \eta^2 e_0^2 g_1^2} \quad , \tag{3.32}$$

and then Eq. (3.31) is equivalent to the quadratic equation for d_o:

$$\eta^4 e_o^2 h d_o^2 - d_o + h = 0 \tag{3.33}$$

with solution

$$d_o = \frac{1 - \sqrt{1 - 4\eta^4 e_o^2 h^2}}{2\eta^4 e_o^2 h}$$

$$= h\left[1 + \eta^4 e_o^2 h^2 + O\left(\eta^4 e_o^2\right)^2\right] \ . \tag{3.34}$$

If we substitute for δ from Eq. (3.32) and use Eq. (3.31) in Eq. (3.30b), we obtain:

$$A + B = 1 + \eta^2 e_0^2 g_1^2 - 2\eta^4 e_0^2 hd_0 \tag{3.35}$$

$$\approx 1 + \eta^2 e_0^2\left(g_1^2 - 2\eta^2 h^2\right) . \tag{3.35*}$$

If we now combine Eq. (3.30a) with Eq. (3.35), we obtain for A and B:

$$A = 1 - \eta^4 e_o hd_o, \qquad B = \eta^2 e_0^2\left(g_1^2 - \eta^2 hd_0\right) \tag{3.36}$$

Returning to Eq. (3.25) and using Eqs. (3.26) and (3.28), we have:

$$\Lambda^2 w'^2 = \lambda^2 \frac{\left[1 - \eta^2 \sqrt{q_o s_o}\left(3h_o - q_o\right)\right]}{q_o}$$

$$\cdot\ J^4\left[(1 - \delta w)^2 - (w - \delta)^2\right]$$

$$\cdot\left[A(1 - \delta w)^2 + B(w - \delta)^2\right] , \tag{3.37}$$

or dividing across by $J^4(1 - \delta w)^4$, we have

$$\Lambda^2\left[\frac{w'}{J^2(1 - \delta w)}\right]^2 = \lambda^2 \frac{\left[1 - \eta^2 \sqrt{q_o s_o}\left(3h_o - q_o\right)\right]}{q_o}$$

$$\cdot\left[1 - \left(\frac{w - \delta}{1 - \delta w}\right)^2\right]$$

$$\cdot\left[A + B\left(\frac{w - \delta}{1 - \delta w}\right)^2\right] \tag{3.38}$$

We therefore write:

$$v = \frac{w - \delta}{1 - \delta w} \qquad \text{or} \qquad w = \frac{v + \delta}{1 + \delta v} \quad , \tag{3.39a}$$

so that, again using Eq. (3.27), we have:

$$v' = \frac{w'}{J^2 (1 - \delta w)^2} \tag{3.39b}$$

and Eq. (3.38) takes the form:

$$\begin{aligned} \Lambda^2 v'^2 &= \frac{\lambda^2 \left[1 - \eta^2 \sqrt{q_o s_o}\left(3h_o - q_o\right)\right]}{q_o} \left[1 - v^2\right]\left[A + Bv^2\right] \\ &= \frac{\lambda^2 \left[1 - \eta^2 \sqrt{q_o s_o}\left(3h_o - q_o\right)\right]}{q_o} \left(A + B\right) \\ &\quad \cdot \left[1 - v^2\right]\left[1 - \frac{B}{A + B}\left(1 - v^2\right)\right] \quad ; \end{aligned} \tag{3.40}$$

introducing (A + B) and B from Eqs. (3.25) and (3.36), respectively, into Eq. (3.40) gives:

$$\begin{aligned} \Lambda^2 v'^2 = \frac{\lambda^2}{q_o} &\left[1 - \eta^2 \sqrt{q_o s_o}\left(3h_o - q_o\right)\right] \\ &\cdot \left[1 + \eta^2 e_0^2 \left(g_1^2 - 2\eta^2 h d_0\right)\right]\left[1 - v^2\right] \\ &\cdot \left[1 - \frac{\eta^2 e_0^2 \left(g_1^2 - 2\eta^2 h d_0\right)\left(1 - v^2\right)}{\left[1 + \eta^2 e_0^2 \left(g_1^2 - 2\eta^2 h d_0\right)\right]}\right] \end{aligned} \tag{3.41}$$

where, by Eqs. (3.24), (3.32), and (3.34), the constants g_1, h, and d_o are determined in terms of the initial parameters.

Next we do a corresponding analysis on the equation for σ.

The Equation for σ

Rewriting Eq. (2.27b), we have

$$\Lambda^2\sigma'^2 = \frac{\lambda^2}{\sin^2\sigma}\left[\sin^2\sigma - \frac{\lambda_3^2}{\lambda^2} - 2\frac{\alpha^2c^2}{\lambda^2}\sin^2\sigma\cos^2\sigma\right]$$

$$= \frac{\lambda^2}{\sin^2\sigma}\left[\sin^2\sigma - \nu^2 - \ell_o^2\eta^2\sin^2\sigma\cos^2\sigma\right]$$

$$= \frac{\lambda^2}{\sin^2\sigma}\left[\left(1-\nu^2\right) - \left(1+\ell_o^2\eta^2\right)\cos^2\sigma + \ell_o^2\eta^2\cos^4\sigma\right] \qquad (3.42)$$

where we have substituted from Eqs. (3.2), (3.4), and (3.5).

We now define a parameter m_o in terms of the fundamental parameters ℓ_o, ν, and η by the relation:

$$2m_o = 1 + \left[1 - 4\frac{\ell_o^2\eta^2\left(1-\nu^2\right)}{\left(1+\ell_o^2\eta^2\right)^2}\right]^{1/2} \qquad (3.43)$$

so that

$$m_o = 1 + O(\eta^2) \; . \qquad (3.43^*)$$

If we consider the quantity on the right side of Eq. (3.42) as a quadratic in $\cos^2\sigma$, its roots are neatly expressed in terms of m_o. In fact, it can be readily checked that we can write Eq. (3.42) in the form:

$$\Lambda^2\sigma'^2 = \frac{\lambda^2}{\sin^2\sigma}\left[\left(1 + \ell_o^2\eta^2\right)m_o - \eta^2\ell_o^2\cos^2\sigma\right]$$

$$\cdot\left[\frac{1-\nu^2}{\left(1 + \ell_o^2\eta^2\right)m_o} - \cos^2\sigma\right]$$

$$= \frac{\lambda^2\left(1 + \eta^2\ell_o^2\right)m_o}{\sin^2\sigma}\left[1 - \frac{\eta^2\ell_o^2}{\left(1 + \eta^2\ell_o^2\right)m_o}\cos^2\sigma\right]$$

$$\cdot\left[\frac{1-\nu^2}{\left(1 + \ell_o^2\eta^2\right)m_o} - \cos^2\sigma\right] . \tag{3.44}$$

We now make the substitution:

$$\cos\sigma = \sqrt{\frac{1-\nu^2}{\left(1 + \eta^2\ell_o^2\right)m_o}}\,\zeta \tag{3.45}$$

so that

$$-\sin\sigma\cdot\sigma' = \sqrt{\frac{1-\nu^2}{\left(1 + \eta^2\ell_o^2\right)m_o}}\,\zeta' \tag{3.46}$$

and Eq. (3.44) takes the form:

$$\Lambda^2\zeta'^2 = \lambda^2\left(1 + \eta^2\ell_o^2\right)m_o\left[1 - \zeta^2\right]\left[1 - \frac{\eta^2\ell_o^2\left(1-\nu^2\right)}{\left(1 + \eta^2\ell_o^2\right)^2 m_o^2}\zeta^2\right] \tag{3.47}$$

where we have reversed the order of the factors. Equation (3.47) is what suggests our choice for Λ.

Definition of Λ

The only requirement on the quantity Λ is that it have the dimension of angular momentum. As suggested by Eq. (3.47), we define Λ by making the identification:

$$\Lambda^2 = \lambda^2\left(1 + \eta^2\ell_o^2\right)m_o \tag{3.48}$$

and the pair of equations for v and ζ [(3.41) and (3.47)] take the form:

$$v'^2 = \frac{\left[1 - \eta^2\sqrt{q_0 s_0}\left(3h_0 - q_0\right)\right]\left[1 + \eta^2 e_0^2\left(g_1^2 - 2\eta^2 h d_0\right)\right]}{q_o m_o\left(1 + \eta^2\ell_o^2\right)} \cdot \left[1 - v^2\right]\left[1 - \frac{\eta^2 e_0^2\left(g_1^2 - 2\eta^2 h d_0\right)}{1 + \eta^2 e_0^2\left(g_1^2 - 2\eta^2 h d_0\right)}\left(1 - v^2\right)\right] \tag{3.49}$$

$$\zeta'^2 = \left[1 - \zeta^2\right]\left[1 - \frac{\eta^2\ell_o^2\left(1 - \nu^2\right)}{\left(1 + \eta^2\ell_o^2\right)^2 m_o^2}\,\zeta^2\right] . \tag{3.50}$$

The next step is to obtain explicit representations for the solutions of the above equations.

4. EXPLICIT SOLUTION FOR R AND σ

Considering Eqs. (3.49) and (3.50), we set

$$j_1^2 = \frac{\left[1 - \eta^2\sqrt{q_0 s_0}\left(3h_0 - q_0\right)\right]\left[1 + \eta^2 e_0^2\left(g_1^2 - 2\eta^2 h d_0\right)\right]}{q_o m_o\left(1 + \eta^2\ell_o^2\right)} \tag{4.1}$$

$$k_1^2 = \frac{\eta^2 e_0^2\left(g_1^2 - 2\eta^2 h d_0\right)}{1 + \eta^2 e_0^2\left(g_1^2 - 2\eta^2 h d_0\right)}\ , \quad k_2^2 = \frac{\eta^2\ell_o^2\left(1 - \nu^2\right)}{\left(1 + \eta^2\ell_o^2\right)^2 m_o^2} \tag{4.2a,b}$$

and the pair of equations takes the form:

$$v'^2 = j_1^2\left[1 - v^2\right]\left[1 - k_1^2\left(1 - v^2\right)\right] \tag{4.3a}$$

$$\zeta'^2 = \left[1 - \zeta^2\right]\left[1 - k_2^2\zeta^2\right] \tag{4.3b}$$

with solutions:

$$v = \mathrm{cn}\left[j_1\left(f + \omega_1\right), k_1\right] \tag{4.4a}$$

$$\zeta = \mathrm{sn}\left[f + \omega_2, k_2\right] \tag{4.4b}$$

respectively, where ω_1 and ω_2 are arbitrary constants introduced by the integration.

If we define "perigee" as the points at which $v' = 0$ and $v'' > 0$, and if we make the "angle" f have origin at perigee, then it follows immediately from Eqs. (4.3a) and (4.4a) that

$$\omega_1 = 0 \ . \tag{4.5}$$

If we define the "angle of perigee" ω as minus the value of f at the first equatorial crossing ($\zeta = 0$), then from Eq. (4.4b) we have

$$\omega_2 = \omega \ , \tag{4.6}$$

and the solution [Eq. (4.4)] is now

$$v = \mathrm{cn}\left[j_1 f, k_1\right] \ , \qquad \zeta = \mathrm{sn}\left[f + \omega, k_2\right] \ . \tag{4.7a}$$

It remains to express R and σ in terms of f.

For the R expression we first take Eq. (4.7a) and insert it into Eq. (3.39a). Then the resulting expression for w is substituted into Eq. (3.22) to give:

$$\frac{1}{R} = u = \frac{\sqrt{q_o s_o}}{p_o}\left[1 + e_o \, \frac{\mathrm{cn}\left[j_1 f, k_1\right] + \delta}{1 + \delta \, \mathrm{cn}\left[j_1 f, k_1\right]}\right]$$

$$= \frac{\sqrt{q_o s_o}\left(1 + \partial e_o\right)}{p_o}\left[\frac{1 + \left(\frac{e_o + \delta}{1 + \delta e_o}\right) \mathrm{cn}\left[j_1 f, k_1\right]}{1 + \delta \, \mathrm{cn}\left[j_1 f, k_1\right]}\right] \tag{4.8}$$

In a similar manner, if we insert Eq. (4.7b) into Eq. (3.45), we have

$$\cos \sigma = \sqrt{\frac{1 - \nu^2}{\left(1 + \eta^2 \ell_o^2\right) m_o}} \; \mathrm{sn}\left[f + \omega, k_2\right]. \tag{4.9}$$

We now see how to define the final constants of the problem, namely those appearing in the solutions [Eqs. (4.8) and (4.9)]. We shall refer to these constants as the semilatus rectum p, the eccentricity e, and the inclination parameter N, defined by

$$p = \frac{p_o}{\sqrt{q_o s_o}\left(1 + \delta e_o\right)} = \frac{p_o}{\sqrt{q_o s_o}\left(1 + \eta^2 e_o^2 d_o\right)} \tag{4.10a}$$

$$e = \frac{e_o + \delta}{1 + \delta e_o} = \frac{e_o\left(1 + \eta^2 d_o\right)}{\left(1 + \eta^2 e_o^2 d_o\right)} \tag{4.10b}$$

$$1 - N^2 = \frac{1 - \nu^2}{\left(1 + \eta^2 \ell_o^2\right) m_o} \tag{4.10c}$$

The solutions [Eqs. (4.8) and (4.9)] in terms of these read:

$$\frac{1}{R} = u = \frac{1}{p}\left[\frac{1 + e \, \mathrm{cn}\left[j_1 f, k_1\right]}{1 + \delta \, \mathrm{cn}\left[j_1 f, k_1\right]}\right] \tag{4.11a}$$

$$\cos\sigma = \sqrt{1 - N^2}\ \mathrm{sn}\left[f + \omega, k_2\right] \tag{4.11b}$$

We shall find useful the following alternative expressions for k_1, k_2, and δ, namely:

$$k_1^2 = \eta^2 e^2 \cdot \frac{\left(1 + \eta^2 e_0^2 d_0\right)^2}{\left(1 + \eta^2 d_0\right)^2} \cdot \frac{g_1^2 - 2\eta^2 h d_0}{1 + \eta^2 e_0^2\left(g_1^2 - 2\eta^2 h d_0\right)} = \eta^2 e^2 g^2 \tag{4.12a}$$

$$k_2^2 = \eta^2\left(1 - N^2\right)\frac{\ell_0^2}{\left(1 + \eta^2 \ell_0^2\right) m_0} = \eta^2\left(1 - N^2\right)\ell^2 \tag{4.12b}$$

$$\delta = \eta^2 e \cdot d_o \frac{\left(1 + \eta^2 e_o^2 d_o\right)}{1 + \eta^2 d_o} = \eta^2 e \cdot d \tag{4.12c}$$

where the quantities g, ℓ, and d are defined by the above relations.

Note the phenomenon of perigee precession is here indicated by the fact that the right-hand side of Eq. (4.11b) has period $4K_2 \neq 2\pi$ where

$$\frac{2K_2}{\pi} = 1 + \frac{k_2^2}{4} + \frac{9k_2^4}{64} + O\left(k_2^6\right) . \tag{4.13}$$

The deviation of the right-hand side from unity measures the perigee precession rate.

5. THE INTEGRATION OF THE ϕ EQUATION

Turning to Eq. (2.28), we substitute for Λ from Eq. (3.48) and noting Eq. (3.4), we obtain (setting $R = 1/u$):

$$\phi' = \frac{\nu}{\left[\left(1 + \eta^2 \ell_o^2\right) m_o\right]^{1/2}} \left[\frac{1}{\sin^2 \sigma} - \frac{c^2 u^2}{1 + c^2 u^2}\right] . \tag{5.1}$$

Using relations (3.5) and (4.10), we rewrite the above equation in the form:

$$\phi' = \frac{\nu}{\left[\nu^2 + \left(1 + \eta^2 \ell_o^2\right) m_o - 1\right]^{1/2}} \cdot \left[\frac{N}{\sin^2 \sigma} - N \cdot \frac{\eta^2 q_o s_o \left(1 + \eta^2 e_o^2 d_o\right)^2 (pu)^2}{1 + \eta^2 q_o s_o \left(1 + \eta^2 e_o^2 d_o\right)^2 (pu)^2}\right] \tag{5.2}$$

It is now convenient to introduce the symbols:

$$q_1 = \frac{\left[\nu^2 + \left(1 + \eta^2 \ell_o^2\right) m_o - 1\right]^{1/2}}{\nu} = 1 + O\left(\eta^2\right) \tag{5.3a}$$

$$s_1 = q_o s_o \left(1 + \eta^2 e_o^2 d_o\right)^2 = 1 + O\left(\eta^2\right) \tag{5.3b}$$

so that Eq. (5.2) reads:

$$q_1 \phi' = \frac{N}{\sin^2 \sigma} - N \cdot \frac{\eta^2 s_1 (pu)^2}{1 + \eta^2 s_1 (pu)^2} \tag{5.4}$$

or, anticipating the approximation procedure, we write

$$q_1 \phi' = \frac{N}{\sin^2 \sigma} - N s_1 \left[\eta^2 (pu)^2 - \eta^4 (pu)^4\right] + O\left(\eta^6\right) . \tag{5.5}$$

In the subsequent computation, the evaluation of each term must be done separately. We therefore introduce the auxiliary quantities ϕ_1, ϕ_2, and ϕ_3 as follows:

$$\phi_1' = \frac{N}{\sin^2 \sigma} = \frac{N}{1 - \cos^2 \sigma} \tag{5.6a}$$

$$\phi_2' = (pu)^2 , \qquad \phi_3' = (pu)^4 \tag{5.6b,c}$$

so that Eq. (5.5) reads:

$$q_1\phi' = \phi_1' - Ns_1\left(\eta^2\phi_2' - \eta^4\phi_3'\right) + O\left(\eta^6\right) \tag{5.7}$$

which integrates as

$$q_1\left(\phi - \Omega_o^*\right) = \phi_1 - Ns_1\eta^2\left(\phi_2 - \eta^2\phi_3\right) + O\left(\eta^6\right) \tag{5.7*}$$

where Ω_o^* is the constant introduced by the integration.

We now consider Eqs. (5.6) individually. Starting with Eq. (5.6a), we introduce cos σ from Eq. (4.11b). If we write:

$$F_2 = f + \omega , \tag{5.8}$$

we have

$$\begin{aligned}\frac{d\phi_1}{dF_2} &= \frac{N}{1 - \left(1 - N^2\right)\mathrm{sn}^2\left(F_2,k_2\right)} \\ &= \frac{N\ \mathrm{nc}^2\left(F_2,k_2\right)}{1 + N^2\ \mathrm{sc}^2\left(F_2,k_2\right)}\end{aligned} \tag{5.9}$$

and so

$$\frac{d\phi_1}{dF_2} = \frac{N\ nc^2(F_2,k_2)dn(F_2,k_2)}{1 + N^2\ sc^2(F_2,k_2)} + \frac{N\ nc^2(F_2,k_2)\left[1 - dn(F_2,k_2)\right]}{1 + N^2\ sc^2\ F_2,k_2}$$

$$= \frac{N\ nc^2(F_2,k_2)dn(F_2,k_2)}{1 + N^2\ sc^2(F_2,k_2)} + k_2^2 \frac{N\ sc^2(F_2,k_2)}{\left[1 + N^2\ sc^2(F_2,k_2)\right]\left[1 + dn(F_2,k_2)\right]}$$

$$= \frac{N\ nc^2(F_2,k_2)dn(F_2,k_2)}{1 + N^2\ sc^2(F_2,k_2)} + \frac{k_2^2 N}{1 - N^2} \cdot \left[\frac{nc^2(F_2,k_2)}{1 + N^2\ sc^2(F_2,k_2)} - 1\right] \cdot \frac{1}{1 + dn(F_2,k_2)}$$

$$= \frac{N\ nc^2(F_2,k_2)dn(F_2,k_2)}{1 + N^2\ sc^2(F_2,k_2)} + \eta^2 \ell^2 \left[\frac{d\phi_1}{dF_2} - N\right] \cdot \frac{1}{1 + dn(F_2,k_2)} \tag{5.10}$$

where we have substituted from relation (4.12b) and utilized Eq. (5.9).

For the integration of Eq. (5.10), we first make some observations. In connection with the second term on the right, we note that

$$\frac{1}{1+\mathrm{dn}\left(F_2,k_2\right)} = \frac{1}{2}\left[1+\frac{1-\mathrm{dn}\left(F_2,k_2\right)}{1+\mathrm{dn}\left(F_2,k_2\right)}\right] \tag{5.11a}$$

$$= \frac{1}{2}\left[1+k_2^2\,\frac{\mathrm{sn}^2\left(F_2,k_2\right)}{\left[1+\mathrm{dn}\left(F_2,k_2\right)\right]^2}\right] \tag{5.11b}$$

while for the first term on the right we note that

$$\frac{d}{dF_2}\left[\tan^{-1}\left[N\,\mathrm{sc}\left(F_2,k_2\right)\right]\right] = \frac{N\,\mathrm{nc}^2\left(F_2,k_2\right)\mathrm{dn}\left(F_2,k_2\right)}{1+N^2\,\mathrm{sc}^2\left(F_2,k_2\right)} \tag{5.12}$$

so that Eq. (5.10) may be written in the form:

$$\frac{d}{dF_2}\left[\phi_1-\tan^{-1}\left[N\,\mathrm{sc}\left(F_2,k_2\right)\right]\right]$$

$$= \frac{\eta^2\ell^2}{2}\left(\frac{d\phi_1}{dF_2}-N\right)\left[1+k_2^2\,\frac{\mathrm{sn}^2\left(F_2,k_2\right)}{\left[1+\mathrm{dn}\left(F_2,k_2\right)\right]^2}\right] \tag{5.13}$$

Again, with a view to integrating the right-hand side, we note:

$$k_2^2\,\mathrm{sn}^2\left(F_2,k_2\right)\cdot\frac{d\phi_1}{dF_2} = k_2^2\,\frac{N\,\mathrm{sc}^2\left(F_2,k_2\right)}{1+N^2\,\mathrm{sc}^2\left(F_2,k_2\right)}$$

$$= \frac{k_2^2 N}{1-N^2}\left[\frac{\mathrm{nc}^2\left(F_2,k_2\right)}{1+N^2\,\mathrm{sc}^2\left(F_2,k_2\right)}-1\right] = \eta^2\ell^2\left(\frac{d\phi_1}{dF_2}-N\right) \tag{5.14}$$

where we have again used Eq. (5.9) and the trick used in arriving at Eq. (5.10). Inserting Eq. (5.14) into Eq. (5.13), we obtain:

$$\frac{d}{dF_2}\left[\phi_1\left(1 - \frac{\eta^2 \ell^2}{2}\right) - \tan^{-1}\left[N\ sc\left(F_2,k_2\right)\right]\right] + \frac{\eta^2 \ell^2}{2} N$$

$$= \frac{\eta^4 \ell^4}{2} \cdot \frac{1}{\left[1 + dn\left(F_2,k_2\right)\right]^2}\left[\frac{d\phi_1}{dF_2} - N - N\left(1 - N^2\right) sn^2\left(F_2,k_2\right)\right]. \quad (5.15)$$

The procedure separates out the dominant contribution and it appears that it can be continued indefinitely. However, in our approximation, we do not seek accuracy beyond the second power of η^2 and so we terminate the manipulation at this point. We note that*

$$dn\left(F_2,k_2\right) = 1 + O\left(\eta^2\right) \quad (5.16a)$$

$$sn\left(F_2,k_2\right) = \sin G_2 + O\left(\eta^2\right) \quad (5.16b)$$

where

$$G_2 = \frac{\pi}{2K_2} F_2 = F_2\left[1 - \frac{k_2^2}{4} - \frac{5k_2^4}{64} + O\left(k_2^6\right)\right] \quad (5.16c)$$

If we now carry the approximation to the second power in η^2, it is consistent to replace Eq. (5.15) by

$$\frac{d}{dF_2}\left[\phi_1\left(1 - \frac{\eta^2 \ell^2}{2} - \frac{\eta^4 \ell^4}{8}\right) - \tan^{-1}\left[N\ sc\left(F_2,k_2\right)\right]\right]$$

$$+ \frac{\eta^2 \ell^2}{2} N\left(1 + \frac{\eta^2 \ell^2}{4}\right) = - \frac{\eta^4 \ell^4}{8} N\left(1 - N^2\right) \sin^2 G_2$$

$$= - \frac{\eta^4 \ell^4}{16} N\left(1 - N^2\right)\left(1 - \cos 2G_2\right) \quad (5.17)$$

*Approximations such as these used in sections 5 and 6 are derived in section 7.

or, on rearrangement:

$$\frac{d}{dF_2}\left[\phi_1\left(1 - \frac{\eta^2\ell^2}{2} - \frac{\eta^4\ell^4}{8}\right) - \tan^{-1}\left[N\ \mathrm{sc}\left(F_2,k_2\right)\right]\right]$$

$$+ \frac{\eta^2\ell^2}{2}N\left[1 + \frac{\eta^2\ell^2}{8}\left(3 - N^2\right)\right] = \frac{\eta^4\ell^4}{16}N\left(1 - N^2\right)\cos\ 2G_2\ , \qquad (5.18)$$

which integrates to give:

$$\phi_1\left(1 - \frac{\eta^2\ell^2}{2} - \frac{\eta^4\ell^4}{8}\right) = \tan^{-1}\left[N\ \mathrm{sc}\left(F_2,k_2\right)\right]$$

$$- \frac{\eta^2\ell^2}{2}N\left[1 + \frac{\eta^2\ell^2}{8}\left(3 - N^2\right)\right]F_2$$

$$+ \frac{\eta^4\ell^4}{32}N\left(1 - N^2\right)\sin\ 2G_2\ . \qquad (5.19)$$

In Eq. (5.19) we have not included the arbitrary additive constant since we can consider it already absorbed in the constant Ω_o^* appearing in Eq. (3.7*).

We next consider Eq. (5.6b). If we introduce u from Eq. (4.11a) and set

$$F_1 = j_1 f\ , \qquad (5.20)$$

we have:

$$j_1\,\frac{d\phi_2}{dF_1} = \left[\frac{1 + e\ \mathrm{cn}\left(F_1,k_1\right)}{1 + \delta\ \mathrm{cn}\left(F_1,k_1\right)}\right]^2\ . \qquad (5.21)$$

Since in Eq. (5.7*) the factor ϕ_2 is multiplied by η^2, it is sufficient for our approximation to compute ϕ_2 up to the first power in η^2. Accordingly, using relation (4.12c), we note that

$$\frac{1 + e\ \mathrm{cn}\left(F_1,k_1\right)}{1 + \delta\ \mathrm{cn}\left(F_1,k_1\right)} = 1 + e\left(1 - \eta^2 d\right)\mathrm{cn}\left(F_1,k_1\right) - \eta^2 d e^2\ \mathrm{cn}^2\left(F_1,k_1\right) + O\left(\eta^4\right)$$

and so, inserting in Eq. (5.21), we have:

$$j_1 \frac{d\phi_2}{dF_1} = 1 + 2e\left(1 - \eta^2 d\right)\mathrm{cn}\left(F_1,k_1\right) + e^2\left(1 - 4\eta^2 d\right)\mathrm{cn}^2\left(F_1,k_1\right) - 2\eta^2 d e^3\ \mathrm{cn}^3\left(F_1,k_1\right) + O\left(\eta^4\right)$$

$$= 1 + 2e\left(1 - \eta^2 d\right)\left[\mathrm{cn}\left(F_1,k_1\right)\mathrm{dn}\left(F_1,k_1\right) - k_1^2\ \frac{\mathrm{sn}^2\left(F_1,k_1\right)\mathrm{cn}\left(F_1,k_1\right)}{1 + \mathrm{dn}\left(F_1,k_1\right)}\right] + e^2\left(1 - 4\eta^2 d\right) \cdot \left[\frac{\mathrm{cn}\left(2F_1,k_1\right) + \mathrm{dn}\left(2F_1,k_1\right)}{1 + \mathrm{dn}\left(2F_1,k_1\right)}\right] - 2\eta^2 d e^3\ \mathrm{cn}^3\left(F_1,k_1\right) \quad (5.22)$$

where we have now omitted terms of order η^4. The first bracket in Eq. (5.22) was obtained by a manipulation similar to that done below Eq. (5.9), while the second bracket was obtained by expressing cn^2 in terms of the elliptic functions of the double argument.

In the first bracket in Eq. (5.22) we set

$$\frac{1}{1 + \mathrm{dn}\left(F_1, k_1\right)} = \frac{1}{2} + O\left(\eta^2\right) , \tag{5.23a}$$

and in the second bracket we set:

$$\frac{1}{1 + \mathrm{dn}\left(2F_1, k_1\right)} = \frac{1}{2}\left[\mathrm{dn}\left(2F_1, k_1\right) + \frac{3k_1^2}{4}\,\mathrm{sn}^2\left(2F_1, k_1\right) + O\left(\eta^4\right)\right] \tag{5.23b}$$

and we can replace Eq. (5.22) by

$$\begin{aligned} j_1 \frac{d\phi_2}{dF_1} = {} & 1 + 2e\left(1 - \eta^2 d\right)\left[\mathrm{cn}\left(F_1, k_1\right)\mathrm{dn}\left(F_1, k_1\right)\right. \\ & \left. - \frac{k_1^2}{2}\,\mathrm{sn}^2\left(F_1, k_1\right)\mathrm{cn}\left(F_1, k_1\right)\right] + \frac{e\left(1 - 4\eta^2 d\right)}{2} \\ & \cdot \left[1 + \mathrm{cn}\left(2F_1, k_1\right)\mathrm{dn}\left(2F_1, k_1\right) - k_1^2\,\mathrm{sn}^2\left(2F_1, k_1\right)\right. \\ & \left. \cdot \left[1 - \frac{3}{4}\left(\mathrm{cn}\left(2F_1, k_1\right) + \mathrm{dn}\left(2F_1, k_1\right)\right)\right]\right] \\ & - 2\eta^2 d e^3\,\mathrm{cn}^3\left(F_1, k_1\right) \end{aligned} \tag{5.24a}$$

$$\begin{aligned} = {} & 1 + \frac{e}{2}\left(1 - 4\eta^2 d\right) + \frac{d}{dF_1}\left[2e\left(1 - \eta^2 d\right)\mathrm{sn}\left(F_1, k_1\right)\right. \\ & \left. + \frac{e}{4}\left(1 - 4\eta^2 d\right)\mathrm{sn}\left(2F_1, k_1\right)\right] - k_1^2 e \\ & \cdot \left[\left(1 - \eta^2 d\right)\mathrm{sn}^2\left(F_1, k_1\right)\mathrm{cn}\left(F_1, k_1\right)\right. \\ & + \frac{1}{2}\left(1 - 4\eta^2 d\right)\mathrm{sn}^2\left(2F_1, k_1\right)\left[1 - \frac{3}{4}\left(\mathrm{cn}\left(2F_1, k_1\right)\right.\right. \\ & \left.\left.\left. + \mathrm{dn}\left(2F_1, k_1\right)\right)\right]\right] - 2\eta^2 d e^3\,\mathrm{cn}^3\left(F_1, k_1\right) \end{aligned} \tag{5.24b}$$

In the latter terms of the above expansion (namely, the terms with k_1^2 as a factor) we now take:

$$\mathrm{dn}\left(F_1, k_1\right) = \mathrm{dn}\left(2F_1, k_1\right) = 1 + O\left(\eta^2\right) \tag{5.25a}$$

$$\left.\begin{aligned} \mathrm{sn}\left(F_1, k_1\right) &= \sin G_1 + O\left(\eta^2\right), \\ \mathrm{sn}\left(2F_1, k_1\right) &= \sin 2G_1 + O\left(\eta^2\right) \end{aligned}\right\} \tag{5.25b}$$

$$\left.\begin{aligned} \mathrm{cn}\left(F_1, k_1\right) &= \cos G_1 + O\left(\eta^2\right), \\ \mathrm{cn}\left(2F_1, k_1\right) &= \cos 2G_1 + O\left(\eta^2\right) \end{aligned}\right\} \tag{5.25c}$$

where

$$G_1 = \frac{\pi}{2K_1} F_1 = F_1\left[1 - \frac{k_1^2}{4} - \frac{5k_1^2}{64} + O\left(k_1^6\right)\right] \tag{5.25d}$$

and also, noting relations (4.12), it is consistent with the approximation to replace Eq. (5.24) by

$$\begin{aligned} j_1 \frac{d\phi_2}{dF_1} = 1 &+ \frac{e}{2}\left(1 - 4\eta^2 d\right) + \frac{d}{dF_1}\left[2e\left(1 - \eta^2 d\right)\mathrm{sn}\left(F_1, k_1\right)\right. \\ &\left. + \frac{e}{4}\left(1 - 4\eta^2 d\right)\mathrm{sn}\left(2F_1, k_1\right)\right] - \eta^2 e^3 \\ &\cdot \left\{g^2\left[\sin^2 G_1 \cos G_1 - \frac{3}{8}\sin^2 2G_1 \cos 2G_1\right.\right. \\ &\left.\left. + \frac{1}{16}\left(1 - \cos 4G_1\right)\right] - \frac{d}{2}\left[3\cos G_1 + \cos 3G_1\right]\right\} \end{aligned}$$

and

$$j_1 \frac{d\phi_2}{dF_1} = \left[1 + \frac{e}{2} - 2\eta^2 e\left(d - \frac{g^2e^2}{32}\right)\right] + \frac{d}{dF_1}\left[2e\left(1 - \eta^2 d\right)\operatorname{sn}\left(F_1,k_1\right)\right.$$

$$\left. + \frac{e}{4}\left(1 - 4\eta^2 d\right)\operatorname{sn}\left(2F_1,k_1\right)\right] - \eta^2 e^3$$

$$\cdot \frac{d}{dF_1}\left[g^2\left(\frac{\sin^3 G_1}{3} - \frac{\sin^3 2G_1}{16} - \frac{\sin 4G_1}{64}\right)\right.$$

$$\left. - \frac{d}{2}\left(\frac{\sin 3G_1}{3} + 3\sin G_1\right)\right] \qquad (5.26)$$

We next note from Eqs. (4.12), (3.34), (3.32) and (3.24) that

$$d = 2N^2 - 1 + O\left(\eta^2\right) \ , \quad g^2 = 1 - N^2 + O\left(\eta^2\right) \ , \qquad (5.27)$$

so that integrating Eq. (5.26) and rearranging, we get:

$$\phi_2 = \left[1 + \frac{e}{2} + 2\eta^2 e\left[1 + \frac{e^2}{32} - 2N^2\left(1 + \frac{e^2}{64}\right)\right]\right]F_2$$

$$+ 2e\,\frac{\left[1 + \eta^2\left(1 - 2N^2\right)\right]}{j_1}\operatorname{sn}\left(F_1,k_1\right)$$

$$+ e\,\frac{\left[1 + 4\eta^2\left(1 - 2N^2\right)\right]}{j_1}\operatorname{sn}\left(2F_1,k_1\right)$$

$$- \eta^2 e^3\left[\frac{7 - 13N^2}{4}\sin G_1 - \frac{3\left(1 - N^2\right)}{4}\sin 2G_1\right.$$

$$\left. + \frac{1 - 3N^2}{12}\sin 3G_1 - \frac{1 - N^2}{64}\sin 4G_1 + \frac{1 - N^2}{64}\sin 6G_1\right]$$

$$(5.28)$$

where we have used the additive constant to replace f by F_2 in the secular term. The reason for doing this will appear later.

It remains to calculate ϕ_3. If we substitute for u from Eq. (4.11a) into Eq. (5.6c) we have:

$$\phi_3' = \left[\frac{1 + e\ \mathrm{cn}\left(j_1 f, k_1\right)}{1 + \delta\ \mathrm{cn}\left(j_1 f, k_1\right)}\right]^4 . \tag{5.29}$$

We note that in Eq. (5.7*) ϕ_3 has a factor η^4. Accordingly, in the calculation of ϕ_3, we may neglect terms of order η^2. For our approximation, therefore, it is consistent to replace Eq. (5.29) by

$$\frac{d\phi_3}{dF_1} = \left(1 + e\cos G_1\right)^4 . \tag{5.30}$$

If we multiply out the right-hand side of the above equation and then express the powers of the cosine in terms of cosines of the multiple angles, and neglect term of order η^2, we have:

$$\frac{d\phi_3}{dF_1} = \left(1 + 3e^2 + \frac{3}{8}e^4\right) + e\left(4 + 3e^2\right)\cos G_1 + e^2\left(3 + \frac{e^2}{2}\right)\cos 2G_1$$
$$+ e^3\cos 3G_1 + \frac{e^4}{8}\cos 4G_1 \tag{5.31}$$

which, on integrating, gives:

$$\phi_3 = \left(1 + 3e^2 + \frac{3}{8}e^4\right)F_2 + e\left(4 + 3e^2\right)\sin G_1 + \frac{e^2}{4}\left(6 + e^2\right)\sin 2G_1$$
$$+ \frac{e^3}{3}\sin 3G_1 + \frac{e^4}{32}\sin 4G_1 \tag{5.32}$$

where, as in Eq. (5.28), we have used the additive constant to replace f by F_2 in the secular term.

Again noting Eq. (5.7*), we next evaluate the combination $(\phi_2 - \eta^2\phi_3)$. Combining Eqs. (5.28) and (5.32) in this manner we have:

$$\phi_2 - \eta^2\phi_3 = \left[1 + \frac{e}{2} - \eta^2\left[\left(1 - 2e + 3e^2 - \frac{e^3}{32} + \frac{3e^4}{8}\right) + 4N^2 e\left(1 + \frac{e^2}{64}\right)\right]\right]F_2$$
$$+ e\left[\frac{\left[1 + \eta^2\left(1 - 2N^2\right)\right]}{j_1}\,\mathrm{sn}\left(F_1, k_1\right)\right.$$
$$\left. + \frac{\left[1 + 4\eta^2\left(1 - 2N^2\right)\right]}{j_1}\,\mathrm{sn}\left(2F_1, k_1\right)\right]$$
$$- \eta^2 e\left[\left[4 + 3e^2 + \frac{e^2}{4}\left(7 - 13N^2\right)\right]\sin G_1\right.$$
$$+ \frac{e}{4}\left[6 + e^2 - \frac{3e}{16}\left(1 - N^2\right)\right]\sin 2G_1 + \frac{e^2}{3}\left(1 + \frac{1 - 3N^2}{4}\right)$$
$$\left.\cdot \sin 3G_1 + \frac{e^2}{4}\left(e + N^2 - 1\right)\sin 4G_1 + \frac{1 - N^2}{64}\sin 6G_1\right]$$

(5.33)

which, using an obvious notation, we may write as

$$\phi_2 - \eta^2\phi_3 = b_0 F_2 + e\left[\sum_{n=1}^{2} b_n \mathrm{sn}\left(nF_1, k_1\right) - \eta^2 \sum_{n=1}^{6}\beta_n \sin nG_1\right]. \quad (5.34)$$

We now multiply Eq. (5.7*) by the factor

$$1 - \frac{\eta^2\ell^2}{2} - \frac{\eta^4\ell^4}{8}$$

and then substitute from Eqs. (5.19) and (5.34). After combining the secular terms we have:

$$\left(1 - \frac{\eta^2\ell^2}{2} - \frac{\eta^4\ell^4}{8}\right)q_1\left(\phi - \Omega_0\right) = \tan^{-1}\left[N\ \mathrm{sc}\left(F_2,k_2\right)\right]$$

$$- \eta^2 N\left\{\left[\frac{\ell^2}{2}\left(1 + \frac{\eta^2\ell^2}{8}\left(3 - N^2\right)\right) + s_1 b_0\left(1 - \frac{\eta^2\ell^2}{2}\right)\right]F_2\right.$$

$$\left. + s_1\left(1 - \frac{\eta^2\ell^2}{2}\right)e\left[\sum_{n=1}^{2} b_n\left(\mathrm{sn}\left(nF_1,k_1\right) - \mathrm{sn}\left(nj_1\omega,k_1\right)\right)\right]\right\}$$

$$+ \eta^4 N\left\{s_1\left(1 - \frac{\eta^2\ell^2}{2}\right)e\left[\sum_{n=1}^{6} \beta_n\left(\sin nG_1 - \sin \frac{n\pi}{2K_1} j_1\omega\right)\right]\right.$$

$$\left. + \frac{\ell^4}{32}\left(1 - N^2\right) \sin 2G_2\right\} \qquad (5.35)$$

where we have replaced Ω_o^* by Ω_o — an adjustment to compensate for the constant terms introduced on the right. The angle Ω_o can now be interpreted as the angle of the "first" nodal crossing, that is, when $f = -\omega$, we have $\phi = \Omega_o$.

We now introduce the final notation for the expression of the formula for ϕ. We set*

$$j_3 = \left(1 - \frac{\eta^2\ell^2}{2} - \frac{\eta^4\ell^4}{8}\right)q_1 \qquad (5.36a)$$

$$j_3 m_s = \frac{\ell^2}{2}\left(1 + \frac{\eta^2\ell^2}{8}\left(3 - N^2\right)\right) + s_1 b_0\left(1 - \frac{\eta^2\ell^2}{2}\right) \qquad (5.36b)$$

$$j_3 m_{p1} = \left(1 - \frac{\eta^2\ell^2}{2}\right)s_1 \qquad (5.36c)$$

$$j_3 m_{p2} = \frac{\ell^4}{32} \qquad (5.36d)$$

so that Eq. (5.35) may be rewritten:

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*In fact, by making the necessary series expansions we find at least to second order that $j_3 = 1$.

$$j_3\left(\phi - \Omega_0\right) = \tan^{-1}\left[N\ \mathrm{sc}\left(F_2,k_2\right)\right] - \eta^2 N j_3$$

$$\cdot\left\{m_s F_2 + m_{p1} e\left[\sum_{n=1}^{2} b_n\left(\mathrm{sn}\left(nF_1,k_1\right) - \mathrm{sn}\left(nj_1\omega,k_1\right)\right)\right]\right\}$$

$$+ \eta^4 N j_3\left\{m_{p1} e\left[\sum_{n=1}^{6} \beta_n\left(\sin nG_1 - \sin \frac{n\pi}{2K_1} j_1\omega\right)\right]\right.$$

$$\left. + m_{p2}\left(1 - N^2\right) \sin 2G_2\right\} . \tag{5.37}$$

If we define the angle Ω by the relation

$$\Omega = \Omega_0 - \eta^2 N\left\{m_s F_2 + m_{p1} e\left[\sum_{n=1}^{2} b_n\left(\mathrm{sn}\left(nF_1,k_1\right) - \mathrm{sn}\left(nj_1\omega,k_1\right)\right)\right]\right\}$$

$$+ \eta^4 N\left\{m_{p1} e\left[\sum_{n=1}^{6} \beta_n\left(\sin nG_1 - \sin \frac{n\pi}{2K_1} j_1\omega\right)\right]\right.$$

$$\left. + m_{p2}\left(1 - N^2\right)\sin 2G_2\right\} , \tag{5.38}$$

then the relation [Eq. (5.37)] takes the compact form:

$$\tan\left[j_3(\phi - \Omega)\right] = N\ \mathrm{sc}\left(F_2,k_2\right) , \tag{5.39}$$

which is a clear generalization of the corresponding relation in the Kepler problem.

Note from Eq. (4.11b) that the crossing of the equatorial plane ($\cos\sigma = 0$) corresponds to the vanishing of $\mathrm{sn}(F_2,k_2)$ and so, from Eq. (5.39), to the vanishing of $\tan\left[j_3(\phi - \Omega)\right]$, so that Ω represents the angle of the ascending node and the secular and periodic variations of Ω can be read off directly from Eq. (5.38).

6. THE TIME-ANGLE RELATIONSHIP

To integrate relation (2.26), we first write it in the form:

$$\Lambda \frac{dt}{df} = \frac{1}{u^2} + c^2 \cos^2 \sigma \tag{6.1}$$

where we have replaced R by 1/u in accordance with Eq. (3.18). We now introduce u and cos σ from Eq. (4.11) and, noting the relations (4.10a) and (3.5), we get:

$$\frac{\Lambda}{p^2} \frac{dt}{df} = \left[\frac{1 + \delta \; cn\left(F_1, k_1\right)}{1 + e \; cn\left(F_1, k_1\right)}\right]^2$$

$$+ \eta^2 q_0 s_0 \left(1 + \eta^2 e_0^2 d_0\right)^2 \left(1 - N^2\right) sn^2\left(F_2, k_2\right). \tag{6.2}$$

If we integrate Eq. (6.2) we have:

$$j_1 \frac{\Lambda}{p^2}\left(t - t_0\right) = H_1 + \eta^2 q_o s_o \left(1 + \eta^2 e_0^2 d_0\right)^2 j_1 \left(1 - N^2\right) H_2 \tag{6.3}$$

where j_1 is given by relation (4.1) and t_o is the constant introduced by the integration. H_1 and H_2 are given by

$$H_1 = \int \left[\frac{1 + \delta \; cn\left(F_1, k_1\right)}{1 + e \; cn\left(F_1, k_1\right)}\right]^2 dF_1 \tag{6.4a}$$

$$H_2 = \int sn^2\left(F_2, k_2\right) \, dF_2 \tag{6.4b}$$

we must evaluate H_1 and H_2 individually.

In considering H_1 we first note from straightforward decomposition that

$$\left[\frac{1 + \delta \operatorname{cn}(F_1,k_1)}{1 + e \operatorname{cn}(F_1,k_1)}\right]^2 = \frac{(1 - \eta^2 d)^2}{\left[1 + e \operatorname{cn}(F_1,k_1)\right]^2} + \frac{2\eta^2 d(1 - \eta^2 d)}{1 + e \operatorname{cn}(F_1,k_1)} + \eta^4 d^2 \tag{6.5}$$

where we use relation (4.12a) for δ. Also it can be readily checked that

$$\frac{1}{\left[1 + e \operatorname{cn}(F_1,k_1)\right]^2} = \frac{1}{1 - e^2}\left\{\frac{1}{1 + e \operatorname{cn}(F_1,k_1)} - \frac{1}{\operatorname{dn}(F_1,k_1)} \cdot \frac{d}{dF_1}\left[\frac{e \operatorname{sn}(F_1,k_1)}{1 + e \operatorname{cn}(F_1,k_1)}\right]\right\} \tag{6.6}$$

so that, combining relations (6.5) and (6.6), we have

$$\left[\frac{1 + \delta \operatorname{cn}(F_1,k_1)}{1 + e \operatorname{cn}(F_1,k_1)}\right]^2 = \frac{1 - 2\eta^2 de^2 - \eta^4 d^2(1 - 2e^2)}{1 - e^2} \cdot \frac{1}{1 + e \operatorname{cn}(F_1,k_1)} - \frac{(1 - \eta^2 d)^2}{1 - e^2} \frac{1}{\operatorname{dn}(F_1,k_1)} \cdot \frac{d}{dF_1}\left[\frac{e \operatorname{sn}(F_1,k_1)}{1 + e \operatorname{cn}(F_1,k_1)}\right] + \eta^4 d^2 . \tag{6.7}$$

Accordingly we set:

$$L_1 = \int \frac{dF_1}{1 + e\ \mathrm{cn}(F_1,k_1)} \tag{6.8a}$$

$$L_2 = \int \frac{1}{\mathrm{dn}(F_1,k_1)} \frac{d}{dF_1}\left[\frac{e\ \mathrm{sn}(F_1,k_1)}{1 + e\ \mathrm{cn}(F_1,k_1)}\right] dF_1 \tag{6.8b}$$

and, therefore, from relations (6.4a), (6.7), and (6.8) we have:

$$\begin{aligned} H_1 = \frac{1}{1-e^2}\Big\{&\left[1 - 2\eta^2 de^2 - \eta^4 d^2\left(1 - 2e^2\right)\right] L_1 \\ &- \left(1 - \eta^2 d\right)^2 L_2\Big\} + \eta^4 d^2 F_1 \ , \end{aligned} \tag{6.9}$$

so that the determination of H_1 is reduced to the evaluation of L_1 and L_2.

To proceed with the calculation we first observe that if we set:

$$I_1 = \frac{1}{\sqrt{1 - e^2}} \arctan\left[\frac{\sqrt{1 - e^2}\ \mathrm{sn}(F_1,k_1)}{e + \mathrm{cn}(F_1,k_1)}\right] , \tag{6.10}$$

it can be immediately verified that

$$\frac{dI_1}{dF_1} = \frac{\mathrm{dn}(F_1,k_1)}{1 + e\ \mathrm{cn}(F_1,k_1)} \quad . \tag{6.11}$$

If we also introduce the notation

$$I_{12} = \int \frac{\mathrm{sn}^2(F_1,k_1)\ dF_1}{\left[1 + e\ \mathrm{cn}(F_1,k_1)\right]\left[1 + \mathrm{dn}(F_1,k_1)\right]} , \tag{6.12}$$

then, from relation (6.8a), it follows that

$$L_1 = \int \frac{dF_1}{1 + e\ \mathrm{cn}\left(F_1,k_1\right)}$$

$$= \int \frac{\mathrm{dn}\left(F_1,k_1\right)\ dF_1}{1 + e\ \mathrm{cn}\left(F_1,k_1\right)} + \int \frac{\left[1 - \mathrm{dn}\left(F_1,k_1\right)\right] dF_1}{1 + e\ \mathrm{cn}\left(F_1,k_1\right)}$$

$$= I_1 + k_1^2 I_{12}\ . \tag{6.13}$$

Also we set:

$$I_2 = \frac{1}{\mathrm{dn}\left(F_1,k_1\right)} \cdot \frac{e\ \mathrm{sn}\left(F_1,k_1\right)}{1 + e\ \mathrm{cn}\left(F_1,k_1\right)} \tag{6.14a}$$

and

$$I_{21} = \int \frac{e\ \mathrm{sn}^2\left(F_1,k_1\right)\mathrm{cn}\left(F_1,k_1\right)dF_1}{\left[1 + e\ \mathrm{cn}\left(F_1,k_1\right)\right]\mathrm{dn}^2\left(F_1,k_1\right)} \tag{6.14b}$$

Then an integration by parts on relation (6.8b) gives:

$$L_2 = I_2 - k_1^2 I_{21}\ . \tag{6.15}$$

If we introduce relations (6.13) and (6.15) into relation (6.9) and neglect terms of order η^6, we have:

$$H_1 = \frac{1}{1 - e^2}\left\{\left[1 - 2\eta^2 de^2 - \eta^4 d^2\left(1 - 2e^2\right)\right] I_1 - \left(1 - \eta^2 d\right)^2 I_2 \right.$$

$$\left. + k_1^2\left[\left(1 - 2\eta^2 de^2\right) I_{12} + \left(1 - 2\eta^2 d\right) I_{21}\right]\right\} + \eta^4 d^2 F_1$$

and so

$$H_1 = \frac{1}{1-e^2}\left\{\left[1 - 2\eta^2 de^2 - \eta^4 d^2\left(1 - 2e^2\right)\right] I_1 - \left(1 - \eta^2 d\right)^2 I_2 \right.$$

$$\left. + \eta^2 g^2 e^2 \left[\left(1 - 2\eta^2 d\right)\left(I_{12} + I_{21}\right) + 2\eta^2 d\left(1 - e^2\right) I_{12}\right]\right\}$$

$$+ \eta^4 d^2 F_1 \tag{6.16}$$

For consistency, we need to calculate $(I_{12} + I_{21})$ up to first order in η^2, but it will suffice to calculate I_{12} to zero-th order in η^2.

From relations (6.12) and (6.14b), we have

$$I_{12} + I_{21} = \int \frac{sn^2(F_1,k_1)}{1 + e\ cn(F_1,k_1)} \cdot \left[\frac{1}{1 + dn(F_1,k_1)} + \frac{e\ cn(F_1,k_1)}{dn^2(F_1,k_1)}\right] dF_1$$

$$= \int \frac{sn^2(F_1,k_1)\,dF_1}{dn^2(F_1,k_1)} - \int \frac{sn^2(F_1,k_1)}{1 + e\ cn(F_1,k_1)}$$

$$\cdot \left[\frac{1}{dn^2(F_1,k_1)} - \frac{1}{1 + dn(F_1,k_1)}\right] dF_1$$

$$= \int \frac{sn^2(F_1,k_1)\,dF_1}{dn^2(F_1,k_1)} - \frac{1}{e^2}\int\left[1 - e\ cn(F_1,k_1)\right.$$

$$\left. - \frac{1-e^2}{1 + e\ cn(F_1,k_1)}\right]\left[\frac{1}{dn^2(F_1,k_1)} - \frac{1}{1 + dn(F_1,k_1)}\right] dF_1 \tag{6.17}$$

Next we note that

$$\frac{1}{\mathrm{dn}^2\left(F_1,k_1\right)} = \mathrm{dn}\left(F_1,k_1\right) + k_1^2\,\frac{\mathrm{sn}^2\left(F_1,k_1\right)}{\mathrm{dn}^2\left(F_1,k_1\right)}\left[1 + \frac{\mathrm{dn}^2\left(F_1,k_1\right)}{1 + \mathrm{dn}\left(F_1,k_1\right)}\right]$$

$$= \mathrm{dn}\left(F_1,k_1\right) + \frac{3k_1^2}{2}\,\mathrm{sn}^2\left(F_1,k_1\right) + O\left(\eta^4\right) \tag{6.18a}$$

and, alternatively,

$$\frac{1}{\mathrm{dn}^2\left(F_1,k_1\right)} = 1 + k_1^2\,\frac{\mathrm{sn}^2\left(F_1,k_1\right)}{\mathrm{dn}^2\left(F_1,k_1\right)}$$

$$= 1 + k_1^2\,\mathrm{sn}^2\left(F_1,k_1\right) + O\left(\eta^4\right) \tag{6.18b}$$

From a similar calculation we note that

$$\frac{1}{1 + \mathrm{dn}\left(F_1,k_1\right)} = \frac{1}{2}\left[\mathrm{dn}\left(F_1,k_1\right) + k_1^2\,\mathrm{sn}^2\left(F_1,k_1\right)\cdot\frac{\left[2 + \mathrm{dn}\left(F_1,k_1\right)\right]}{\left[1 + \mathrm{dn}\left(F_1,k_1\right)\right]^2}\right]$$

$$= \frac{1}{2}\left[\mathrm{dn}\left(F_1,k_1\right) + \frac{3k_1^2}{4}\,\mathrm{sn}^2\left(F_1,k_1\right) + O\left(\eta^4\right)\right] \tag{6.19a}$$

and, alternatively,

$$\frac{1}{1 + \mathrm{dn}\left(F_1,k_1\right)} = \frac{1}{2}\left[1 + k_1^2\,\frac{\mathrm{sn}^2\left(F_1,k_1\right)}{\left[1 + \mathrm{dn}\left(F_1,k_1\right)\right]^2}\right]$$

$$= \frac{1}{2}\left[1 + \frac{k_1^2}{4}\,\mathrm{sn}^2\left(F_1,k_1\right) + O\left(\eta^4\right)\right]\ . \tag{6.19b}$$

By combining relation (6.18a) with relation (6.19a) and relation (6.18b) with relation (6.19b) and omitting terms of order η^4, we get, respectively:

$$\frac{1}{dn^2(F_1,k_1)} - \frac{1}{1+dn(F_1,k_1)} = \frac{1}{2}\left[dn(F_1,k_1) + \frac{9k_1^2}{4}\,sn^2(F_1,k_1)\right] \quad (6.20a)$$

$$= \frac{1}{2}\left[1 + \frac{7k_1^2}{4}\,sn^2(F_1,k_1)\right] \quad . \quad (6.20b)$$

In the second integral in relation (6.17), we now use relation (6.20). With the first term in the first bracket we use relation (6.20b), and with the second and third terms we use relation (6.20a). If we also use relation (6.18b) in the first integral and neglect terms of order η^4, we get:

$$\begin{aligned}
I_{12} + I_{21} &= \int sn^2(F_1,k_1)\,dF_1 + k_1^2\int sn^4(F_1,k_1)\,dF_1 \\
&\quad - \frac{1}{2e^2}\int\left[1 + \frac{7k_1^2}{4}\,sn^2(F_1,k_1)\right]dF_1 \\
&\quad + \frac{1}{2e^2}\int\left[\frac{1-e^2}{1+e\,cn(F_1,k_1)} + e\,cn(F_1,k_1)\right] \\
&\quad \cdot\left[dn(F_1,k_1) + \frac{9k_1^2}{4}\,sn^2(F_1,k_1)\right]dF_1 \\
&= \left(1 - \frac{7}{8}\eta^2 g^2\right)\int sn^2(F_1,k_1)\,dF_1 \\
&\quad + k_1^4\int sn^4(F_1,k_1)\,dF_1 - \frac{F_1}{2e^2} + \frac{sn(F_1,k_1)}{2e} \\
&\quad + \frac{1-e^2}{2e^2}I_1 + \frac{9\eta^2 g^2}{8}\left[(1-e^2)\int\frac{sn^2(F_1,k_1)\,dF_1}{1+e\,cn(F_1,k_1)}\right. \\
&\quad \left. + e\int sn^2(F_1,k_1)\,cn(F_1,k_1)\,dF_1\right]
\end{aligned} \quad (6.21)$$

If we observe from relation (6.12) that, except for terms of order η^2, we may write:

$$I_{12} = \frac{1}{2}\int \frac{\mathrm{sn}^2\left(F_1,k_1\right)dF_1}{1 + e\ \mathrm{cn}\left(F_1,k_1\right)} \tag{6.22}$$

then it follows that

$$\begin{aligned}\left(1 - 2\eta^2 d\right)\left(I_{12} + I_{21}\right) + 2\eta^2 d\left(1 - e^2\right)I_{12} &= \left[1 - \eta^2\left(2d + \tfrac{7}{8}g^2\right)\right] \\ &\cdot \int \mathrm{sn}^2\left(F_1,k_1\right)dF_1 + \left(1 - 2\eta^2 d\right)\left[\frac{1 - e^2}{2e^2}I_1 - \frac{F_1}{2e^2} + \frac{\mathrm{sn}\left(F_1,k_1\right)}{2e}\right] \\ &+ k_1^2\int \mathrm{sn}^4\left(F_1,k_1\right)dF_1 + \eta^2\left(1 - e^2\right)\left(d + \tfrac{9}{8}g^2\right)\int \frac{\mathrm{sn}^2\left(F_1,k_1\right)dF_1}{1 + e\ \mathrm{cn}\left(F_1,k_1\right)} \\ &+ \frac{9\eta^2 g^2}{8} e\int \mathrm{sn}^2\left(F_1,k_1\right)\mathrm{cn}\left(F_1,k_1\right)dF_1\end{aligned} \tag{6.23}$$

We next note that except for terms of order η^2 we may write

$$\begin{aligned}\int \mathrm{sn}^4\left(F_1,k_1\right)dF_1 = \int \sin^4 G_1\ dF_1 &= \int\left[\frac{3}{8} - \frac{\cos 2G_1}{2} + \frac{\cos 4G_1}{8}\right]dF_1 \\ &= \frac{3}{8}F_1 - \frac{\sin 2G_1}{4} + \frac{\sin 4G_1}{32}\end{aligned} \tag{6.24a}$$

$$\begin{aligned}\int \frac{\mathrm{sn}^2\left(F_1,k_1\right)dF_1}{1 + e\ \mathrm{cn}\left(F_1,k_1\right)} &= \frac{1}{e^2}\int\left[1 - e\ \mathrm{cn}\left(F_1,k_1\right) - \frac{1 - e^2}{1 + e\ \mathrm{cn}\left(F_1,k_1\right)}\right]dF_1 \\ &= \frac{F_1}{e^2} - \frac{\mathrm{sn}\left(F_1,k_1\right)}{e} - \frac{1 - e^2}{e^2}I_1\end{aligned} \tag{6.24b}$$

$$\int \mathrm{sn}^2\left(F_1,k_1\right)\mathrm{cn}\left(F_1,k_1\right)dF_1 = \int \sin^2 G_1 \cos G_1 \, dG_1$$

$$= \frac{\sin^3 G_1}{3} = \frac{\sin G_1}{4} - \frac{\sin 3G_1}{12} \qquad (6.24c)$$

where G_1 above is that given by relation (5.25d). If we substitute relations (6.24) into relation (6.23) and rearrange, we see that

$$\left(1 - 2\eta^2 d\right)\left(I_{12} + I_{21}\right) + 2\eta^2 d\left(1 - e^2\right)I_{12} = \left[1 - \eta^2\left(2d + \tfrac{7}{8}g^2\right)\right]$$

$$\cdot\int \mathrm{sn}^2\left(F_1,k_1\right)dF_1 - \frac{F_1}{2e^2}\left\{1 - \eta^2\left[2d\left(2 - e^2\right) + \tfrac{9}{8}g^2\left(1 - e^2 + \tfrac{2}{3}e^4\right)\right]\right\}$$

$$+ \left\{1 - \eta^2\left[2d\left(2 - e^2\right) + \tfrac{9}{8}g^2\left(1 - e^2\right)\right]\right\}\left[\frac{1 - e^2}{2e^2} I_1 + \frac{\mathrm{sn}\left(F_1,k_1\right)}{e}\right]$$

$$+ \eta^2 g^2 e\left[\tfrac{9}{32}\sin G_1 - \tfrac{e}{4}\sin 2G_1 - \tfrac{3}{32}\sin 3G_1 + \tfrac{e}{32}\sin 4G_1\right] \qquad (6.25)$$

Finally we evaluate the first integral on the right of relation (6.25). Noting that

$$\mathrm{sn}^2\left(F_1,k_1\right) = \frac{1 - \mathrm{cn}\left(2F_1,k_1\right)}{1 + \mathrm{dn}\left(2F_1,k_1\right)} , \qquad (6.26)$$

we have

$$\int \mathrm{sn}^2\left(F_1,k_1\right)dF_1 = \int \frac{dF_1}{1 + \mathrm{dn}\left(2F_1,k_1\right)} - \int \frac{\mathrm{cn}\left(2F_1,k_1\right)dF_1}{1 + \mathrm{dn}\left(2F_1,k_1\right)} . \qquad (6.27)$$

Using formulas corresponding to relation (6.19b) in the first integral and the analog of relation (6.19a) in the second integral, we see, after neglecting terms of order η^4, that

$$\int \mathrm{sn}^2(F_1,k_1)dF_1 = \frac{1}{2}\int dF_1 - \frac{1}{2}\int \mathrm{cn}(F_1,k_1)\,\mathrm{dn}(F_1,k_1)dF_1$$

$$+ \frac{k_1^2}{8}\left[\int \sin^2 2G_1\, dG_1 - 3\int \sin^2 2G_1 \cos 2G_1\, dG_1\right]$$

$$= \left(1 + \frac{k_1^2}{8}\right)\frac{F_1}{2} - \frac{\mathrm{sn}(2F_1,k_1)}{2} - \frac{k_1^2}{32}\left[3\sin G_1 - \sin 3G_1 + \frac{\sin 4G_1}{2}\right] \tag{6.28}$$

If we substitute from relation (6.28) into relation (6.25), we get:

$$\left(1 - 2\eta^2 d\right)\left(I_{12} + I_{21}\right) + 2\eta^2 d\left(1 - e^2\right)I_{12}$$

$$= \left[1 - \eta^2\left[2d\left(2 - e^2\right) + \frac{9}{8}g^2\left(1 - e^2\right)\right]\right]\left[\frac{1 - e^2}{2e^2}I_1 + \frac{\mathrm{sn}(F_1,k_1)}{e}\right]$$

$$- \frac{1 - e^2}{2e^2}\left[1 - \eta^2\left[4d + \frac{9 - 7e^2}{8}\,g^2\right]\right]F_1$$

$$- \left[1 - \eta^2\left(2d + \frac{7}{8}g^2\right)\right]\mathrm{sn}(2F_1,k_1)$$

$$+ \frac{\eta^2 g^2 e}{32}\left[3(3 - e)\sin G_1 - 8e\sin 2G_1 - (3 - e)\sin 3G_1 + \frac{e}{2}\sin 4G_1\right], \tag{6.29}$$

which we now insert into relation (6.16). To make the resulting formula more compact we set:

$$D = \left[de^2 - \frac{g^2}{4}\left(1 - e^2\right)\right] + \frac{\eta^2}{2}\left[d^2\left(1 - 2e^2\right) + g^2\left(1 - e^2\right) \cdot \left[d\left(2 - e^2\right) + \frac{9g^2}{16}\left(1 - e^2\right)\right]\right] , \qquad (6.30)$$

and insertion of relation (6.29) into relation (6.16) yields:

$$\begin{aligned} H_1 = \frac{1}{1 - e^2}\Bigg[&\left(1 - 2\eta^2 D\right) I_1 - \left(1 - \eta^2 d\right)^2 I_2 \\ &- \eta^2 F_1\left[\frac{g^2}{2} - \eta^2\left[d^2 + 2g^2 d + \frac{9 - 7e^2}{16} g^4\right]\right] \\ &+ \frac{\eta^2 g^2 e}{1 - e^2}\left\{\left[1 - \eta^2\left[2d\left(2 - e^2\right) + \frac{9}{8}g^2\left(1 - e^2\right)\right]\right] \operatorname{sn}\left(F_1, k_1\right)\right. \\ &\left. - e\left[1 - \eta^2\left(2d + \frac{7}{8}g^2\right)\right] \operatorname{sn}\left(2F_1, k_1\right)\right\} \\ &+ \frac{\eta^4 g^4 e^3}{32\left(1 - e^2\right)}\Big[3(3 - e) \sin G_1 - 8e \sin 2G_1 \\ &- (3 - e) \sin 3G_1 + \frac{e}{2} \sin 4G_1\Big]\Bigg] . \end{aligned} \qquad (6.31)$$

Before we evaluate H_2 we first note that in Eq. (6.3) we wish to interpret t_o as the time of first perigee passage. With this in mind we have kept H_1 free of any constant term, that is, H_1 satisfies the condition:

$$f = 0 \Rightarrow H_1 = 0 . \qquad (6.32)$$

We shall therefore also require that

$$f = 0 \Rightarrow H_2 = 0 \tag{6.33a}$$

or in terms of F_2, this means

$$F_2 = \omega \Rightarrow H_2 = 0 \tag{6.33b}$$

In the evaluation of H_2, we proceed as we did in going from relation (6.27) to relation (6.28) and introduce constant terms to satisfy relation (6.33). On neglecting terms of order η^4, we get:

$$\begin{aligned} H_2 &= \int \mathrm{sn}^2\left(F_2, k_2\right) \, dF_2 \\ &= \left(1 + \frac{k_2^2}{8}\right) \frac{f}{2} - \frac{1}{2}\left[\mathrm{sn}\left(2F_2, k_2\right) - \mathrm{sn}\left(2\omega, k_2\right)\right] \\ &\quad - \frac{k_2^2}{32}\left\{3\left[\sin G_2 - \sin \gamma_2\right] - \left[\sin 3G_2 - \sin 3\gamma_2\right]\right. \\ &\qquad\qquad \left. + \frac{1}{2}\left[\sin 4G_2 - \sin 4\gamma_2\right]\right\}. \end{aligned} \tag{6.34}$$

where G_2 is as given by relation (5.16c) and γ_2 is as given by

$$\gamma_2 = \left(1 - \frac{k_2^2}{4} - \frac{5k_2^4}{64}\right) \omega \, . \tag{6.35}$$

If we now divide Eq. (6.3) by j_1 and introduce H_1 and H_2 from relations (6.31) and (6.34), respectively, we get, after combining the secular terms:

$$
\begin{aligned}
\frac{\Lambda}{p^2}\left(t - t_o\right) = {} & \frac{1}{1-e^2}\left[\frac{1-2\eta^2 D}{j_1} I_1 - \frac{\left(1-\eta^2 d\right)^2}{j_1} I_2\right] \\
& + \frac{\eta^2 f}{2}\left[q_o s_o\left(1-N^2\right) - g^2 + 2\eta^2\left[d^2 + 2g^2 d + \frac{9-7e^2}{16} g^4\right.\right. \\
& \left.\left. + e^2 d + \frac{\ell^2\left(1-N^2\right)}{16}\right]\right] + \eta^2 \frac{g^2 e}{1-e^2} \\
& \cdot \frac{1-\eta^2\left[2d\left(2-e^2\right) + \frac{9}{8} g^2\left(1-e^2\right)\right]}{j_1} \operatorname{sn}\left(F_1, k_1\right) \\
& - \eta^2\left[\frac{g^2 e^2}{1-e^2} \cdot \frac{1-\eta^2\left(2d + \frac{7}{8} g^2\right)}{j_1} \operatorname{sn}\left(2F_1, k_1\right)\right. \\
& \left. - \frac{q_o s_o}{2}\left(1 + 2\eta^2 e_o^2 d_o\right)\left(1-N^2\right)\left[\operatorname{sn}\left(2F_2, k_2\right) - \operatorname{sn}\left(2\omega, k_2\right)\right]\right] \\
& + \frac{\eta^4}{32}\left\{\frac{g^4 e^3}{1-e^2}\left[3(3-e)\sin G_1 - 8e \sin 2G_1\right.\right. \\
& \left. - (3-e)\sin 3G_1 + \frac{e}{2}\sin 4G_1\right] - \ell^2\left(1-N^2\right) \\
& \cdot \left[3\left(\sin G_2 - \sin \gamma_2\right) - \left(\sin 3G_2 - \sin 3\gamma_2\right)\right. \\
& \left.\left. + \frac{1}{2}\left(\sin 4G_2 - \sin 4\gamma_2\right)\right]\right\} \qquad (6.36)
\end{aligned}
$$

where we have substituted for k_2 from relation (4.12). Before attempting to simplify the above, we first consider the multiplying factor on the left.

We observe that

$$
\frac{\Lambda}{p^2} = \frac{p_o^2}{p^2} \cdot \frac{\Lambda}{\lambda} \cdot \frac{\lambda}{p_o^2} \quad . \qquad (6.37)
$$

We use relation (4.10a) to substitute for the first factor on the right and relations (3.48) and (3.2) to substitute for the second and third factors, respectively, to get

$$\frac{\Lambda}{p^2} = q_o s_o \left(1 + \eta^2 e_o^2 d_o\right)^2 \sqrt{1 + \eta^2 \ell_o^2 \, m_o} \sqrt{\frac{\mu}{p_o^3}} \tag{6.38}$$

which, if we substitute for p_o from relation (3.5), reads:

$$\frac{\Lambda}{p^2} = q_o s_o \sqrt{m_o} \left(1 + \eta^2 e_o^2 d_o\right)^2 \frac{\sqrt{1 + \eta^2 \ell_o^2}}{\ell_o^3} \sqrt{\frac{\mu}{a_o^3}} \; . \tag{6.39}$$

We note from relation (4.10b) that

$$e = e_o \, \frac{1 + \eta^2 d_o}{1 + \eta^2 e_o^2 d_o} \tag{6.40}$$

and so, after some manipulation,

$$1 - e^2 = \frac{\left(1 - e_o^2\right)\left(1 - \eta^4 e_o^2 d_o^2\right)}{\left(1 + \eta^2 e_o^2 d_o\right)^2} \quad , \tag{6.41}$$

and hence

$$1 - e_o^2 = \frac{\left(1 + \eta^2 e_o^2 d_o\right)^2}{1 - \eta^4 e_o^2 d_o^2} \left(1 - e^2\right) \tag{6.42}$$

and so, from relation (4.20), we have:

$$\ell_o^2 = s_o\left(1 - e_o^2\right) = \frac{s_o\left(1 + \eta^2 e_o^2 d_o\right)^2}{1 - \eta^4 e_o^2 d_o^2} \left(1 - e^2\right) \; , \tag{6.43}$$

which, with relation (6.39), gives:

$$\frac{\Lambda^2}{p^2} = \frac{q_o}{1+\eta^2 e_o^2 d_o}\sqrt{\frac{m_o}{s_o}\left(1+\eta^2 \ell_o^2\right)\left(1-\eta^4 e_o^2 d_o^2\right)} \cdot \frac{1}{\left(1-e^2\right)^{3/2}}\sqrt{\frac{\mu}{a_o^3}} \tag{6.44}$$

If we write:

$$j_T = \frac{q_o}{1+\eta^2 e_o^2 d_o}\sqrt{\frac{m_o}{s_o}\left(1+\eta^2 \ell_o^2\right)\left(1-\eta^4 e_o^2 d_o^2\right)} \tag{6.45}$$

and define a "mean motion" n by setting

$$n \equiv \sqrt{\frac{\mu}{a_o^3}} \quad , \tag{6.46}$$

then Eq. (6.44) reads:

$$\frac{\Lambda}{p^2} = \frac{j_T}{\left(1 - e^2\right)^{3/2}}\, n \ . \tag{6.47}$$

If we further define a mean anomaly M by the relation:

$$M = n(t - t_o) \quad , \tag{6.48}$$

then, by multiplying Eq. (6.36) by the factor

$$\frac{\left(1 - e^2\right)^{3/2}}{j_T}$$

we obtain the following relation between the two anomalies M and f, namely:

$$
\begin{aligned}
M = {} & \frac{1 - 2\eta^2 D}{j_1 j_T}\left(1 - e^2\right)^{1/2} I_1 - \frac{\left(1 - \eta^2 d\right)^2}{j_1 j_T}\left(1 - e^2\right)^{1/2} I_2 \\
& + \frac{\eta^2}{2}\left(1 - e^2\right)^{3/2}\left\{\frac{q_o s_o\left(1 - N^2\right) - g^2}{j_T} + 2\eta^2\left[d^2 + \left(2g^2 + e^2\right)d\right.\right. \\
& \left.\left. + \frac{1}{16}\left[g^4\left(9 - 7e^2\right) + \ell^2\left(1 - N^2\right)\right]\right]\right\} f + \eta^2 g^2 e\left(1 - e^2\right)^{1/2} \\
& \cdot \frac{1 - \eta^2\left[2d\left(2 - e^2\right) + \frac{9}{8}g^2\left(1 - e^2\right)\right]}{j_1 j_T}\,\mathrm{sn}\left(F_1, k_1\right) \\
& - \eta^2\left\{g^2 e^2\left(1 - e^2\right)^{1/2} \cdot \frac{1 - \eta^2\left(2d + \frac{7}{8}g^2\right)}{j_1 j_T}\,\mathrm{sn}\left(2F_1, k_1\right)\right. \\
& - \left(1 - N^2\right)\left(1 - e^2\right)^{3/2} \cdot \frac{q_o s_o\left(1 + 2\eta^2 e_o^2 d_o\right)}{2 j_T} \\
& \left.\cdot\left[\mathrm{sn}\left(2F_2, k_2\right) - \mathrm{sn}\left(2\omega, k_2\right)\right]\right\} + \frac{\eta^4}{32}\left\{g^4 e^3\left(1 - e^2\right)^{1/2}\right. \\
& \cdot\left[3\left(3 - e\right)\sin G_1 - 8e\sin 2G_1 - \left(3 - e\right)\sin 3G_1 + \frac{e}{2}\sin 4G_1\right] \\
& - \left(1 - e^2\right)^{3/2}\ell^2\left(1 - N^2\right)\left[3\left(\sin G_2 - \sin\gamma_2\right)\right. \\
& \left.\left. - \left(\sin 3G_1 - \sin 3\gamma_1\right) + \frac{1}{2}\left(\sin 4G_2 - \sin 4\gamma_2\right)\right]\right\} . \qquad (6.48^*)
\end{aligned}
$$

This formula can be made a little simpler by using the algebraic formulas for the constants as developed in sections 3 and 4 and by exploiting the fact that we neglect terms of order η^6. We also set:

$$j_4 = \frac{1 - 2\eta^2 D}{j_1 j_T} \quad , \qquad j_5 = \frac{\left(1 - \eta^2 d\right)^2}{j_1 j_T} \tag{6.49}$$

and write I_1 and I_2 explicitly. After some manipulation with the above coefficients, we finally obtain:

$$\begin{aligned}
M = {} & j_4 \arctan\left[\frac{\sqrt{1-e^2}\ \mathrm{sn}\left(j_1 f, k_1\right)}{e + \mathrm{cn}\left(j_1 f, k_1\right)}\right] - j_5 \frac{1}{\mathrm{dn}\left(j_1 f, k_1\right)} \cdot \frac{e\ \mathrm{sn}\left(j_1 f, k_1\right)}{1 + e\ \mathrm{cn}\left(j_1 f, k_1\right)} \\
& + \frac{\eta^4}{16}\left(1 - e^2\right)^{3/2}\left[2 - 84N^2 + 130N^4 + \left(1 - e^2\right)N^2\left(21 - 37N^2\right)\right] f \\
& + \eta^2 e\left(1 - e^2\right)^{1/2}\left[\left(1 - N^2\right) - \frac{\eta^2}{8}\left[41 - 93N^2 + 69N^4\right.\right. \\
& \left.\left. - e^2\left(1 + 58N^2 - 75N^4\right)\right]\right] \mathrm{sn}\left(j_1 f, k_1\right) + \eta^2\left(1 - e^2\right)^{1/2} \\
& \cdot \left\{e^2\left[\left(1 - N^2\right) - \frac{\eta^2}{8}\left[39 - 122N^2 + 99N^4 + e^2\left(8 - 76N^2 + 68N^4\right)\right]\right]\right. \\
& \cdot \mathrm{sn}\left(2j_1 f, k_1\right) - \frac{\left(1 - e^2\right)\left(1 - N^2\right)}{2}\left[1 + \eta^2\left[5N^2 + 3e^2\left(3N^2 - 1\right)\right]\right] \\
& \left.\cdot \left[\mathrm{sn}\left[2\left(f + \omega\right)k_2\right] - \mathrm{sn}\left(2\omega, k_2\right)\right]\right\} + \frac{\eta^4}{32}\left(1 - N^2\right)\left(1 - e^2\right)^{1/2} \\
& \cdot \left\{\left(1 - N^2\right)e^3\left[3\left(3 - e\right) \sin G_1 - 8e \sin 2G_1 - (3 - e) \sin 3G_1\right.\right. \\
& \left. + \frac{e}{2} \sin 4G_1\right] - \left(1 - e^2\right)^2\left[3\left(\sin G_2 - \sin \gamma_2\right)\right. \\
& \left.\left. - \left(\sin 3G_2 - \sin 3\gamma_2\right) + \frac{1}{2}\left(\sin 4G_2 - \sin 4\gamma_2\right)\right]\right\} .
\end{aligned} \tag{6.50}$$

When $\eta = 0$, only the first two terms survive in which case the above relation reduces to the well known time-angle relation of the Kepler problem.

7. APPROXIMATE FORMULAS

Equations (4.11) and (5.39), with Ω given by Eq. (5.38), give the formulas for the three coordinates in terms of f. These are complemented by the time-angle relationship (6.50).

In arriving at relations (5.38), (5.39) and (6.50), we have made second-order approximations. It is therefore consistent to introduce such approximations into all the relations. Though this means that the formulas will, to a certain extent, lose their compact form, they will become simpler in that they will no longer involve elliptic functions. The latter will be replaced by their second-order approximations in terms of truncated trigonometric series. For clarification we list the sequence of steps leading to these approximate formulas; the analysis and derivation are available in the standard treatises, e.g., Whittaker and Watson (ref. 9) or Davis (ref. 10).

For the modulus k of the elliptic functions appearing in either Eqs. (4.11a) or (4.11b), we note that

$$k^2 = O\left(\eta^2\right) \quad , \tag{7.1}$$

and, if we define the complementary modulus k' by

$$k' = \left(1 - k^2\right)^{1/2} \quad , \tag{7.2}$$

then we have that

$$\begin{aligned}
\sqrt{k'} &= \left(1 - k^2\right)^{1/4} = 1 - \frac{k^2}{4} - \frac{3}{32}k^4 + O\left(\eta^6\right) \\
1 - \sqrt{k'} &= \frac{k^2}{4}\left[1 + \frac{3}{8}k^2 + \frac{7}{32}k^4 + O\left(\eta^6\right)\right] \\
1 + \sqrt{k'} &= 2\left[1 - \frac{k^2}{8} - \frac{3}{64}k^4 + O\left(\eta^6\right)\right]
\end{aligned} \tag{7.3}$$

Defining ε by the relation:

$$\varepsilon = \frac{1}{2}\,\frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} \quad , \tag{7.4}$$

then, after expanding, we find that

$$\varepsilon = \frac{k^2}{16}\left[1 + \frac{k^2}{2} + \frac{21}{64}k^4 + O\left(\eta^6\right)\right] = O\left(\eta^2\right) \ . \tag{7.5}$$

When the modulus q of the associated Theta functions is written in terms of ε, we have

$$\begin{aligned} q &= \varepsilon\left[1 + O\left(\varepsilon^4\right)\right] \\ &= \frac{k^2}{16}\left[1 + \frac{k^2}{2} + \frac{21}{64}k^4 + O\left(\eta^6\right)\right] \ , \end{aligned} \tag{7.6}$$

and so we have:

$$\begin{aligned} q^{1/2} &= \frac{k}{4}\left[1 + \frac{k^2}{4} + \frac{17}{128}k^4 + O\left(\eta^6\right)\right] \\ \frac{q^{1/2}}{k} &= \frac{1}{4}\left[1 + \frac{k^2}{4} + \frac{17}{128}k^4 + O\left(\eta^6\right)\right] \end{aligned} \tag{7.7}$$

The period 4K is related to 2π as follows:

$$\begin{aligned} \left(\frac{2K}{\pi}\right)^{1/2} &= 1 + 2q + O\left(\eta^6\right) \\ &= 1 + \frac{k^2}{8} + \frac{k^4}{16} + O\left(\eta^6\right) \ , \end{aligned} \tag{7.8}$$

and so we have:

$$\frac{2K}{\pi} = 1 + \frac{k^2}{4} + \frac{9k^4}{64} + o\left(\eta^6\right)$$
$$\frac{\pi}{2K} = 1 - \frac{k^2}{4} - \frac{5k^4}{64} + o\left(\eta^6\right) \qquad (7.9)$$

If the angle G is then defined in terms of F by the relation

$$G = \frac{\pi}{2K}F = \left(1 - \frac{k^2}{4} - \frac{5k^4}{64} + \ldots\right)F \,, \qquad (7.10)$$

then the trigonometric series representations of the Jacobian elliptic functions take the form:

$$\mathrm{sn}\left(F,k\right) = \frac{2\pi}{Kk}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1 - q^{2n+1}} \sin\ (2n + 1)G$$

$$= 4\frac{q^{1/2}}{k} \cdot \frac{\pi}{2K}\sum_{n=0}^{\infty}\frac{q^n}{1 - q^{2n+1}} \sin\ (2n + 1)G$$

$$= \left(1 - \frac{k^4}{128} + \ldots\right)\sum_{n=0}^{\infty}\frac{q^n}{1 - q^{2n+1}} \sin\ (2n + 1)G \qquad (7.11a)$$

$$\mathrm{cn}\left(F,k\right) = \left(1 - \frac{k^4}{128} + \ldots\right)\sum_{n=0}^{\infty}\frac{q^n}{1 - q^{2n+1}} \cos\ (2n + 1)G \qquad (7.11b)$$

$$\mathrm{dn}\left(F,k\right) = \left(1 - \frac{k^2}{4} - \frac{5k^4}{64} + \ldots\right)\left[1 + 4\sum_{n=0}^{\infty}\frac{q^n}{1 + q^{2n+1}} \cos\ 2nG\right] \qquad (7.11c)$$

$$\mathrm{am}(F,k) = G + \sum_{n=1}^{\infty} \frac{2q^n}{n\left(1 + q^{2n}\right)} \cos 2nG \ . \tag{7.11d}$$

We also include the trigonometric series representation of sc(F,k), namely:

$$\mathrm{sc}(F,k) = \left(1 + \frac{k^2}{4} + \frac{11k^4}{64} + \ldots\right) \cdot \left[\tan G + 4\sum_{n=1}^{\infty} \frac{(-)^n q^{2n}}{1 + q^{2n}} \sin 2nG\right] . \tag{7.11e}$$

In a second-order approximation the above series can be truncated. The approximate forms we shall use are:

$$\mathrm{sn}(F,k) = \left(1 + \frac{k^2}{16} + \frac{7k^4}{256}\right) \sin G + \frac{k^2}{16}\left(1 + \frac{k^2}{2}\right) \sin 3G + \frac{k^4}{256} \sin 5G \tag{7.12a}$$

$$\mathrm{cn}(F,k) = \left(1 - \frac{k^2}{16} - \frac{9k^4}{256}\right) \cos G + \frac{k^2}{16}\left(1 + \frac{k^2}{2}\right) \cos 3G + \frac{k^4}{256} \cos 5G \tag{7.12b}$$

$$\mathrm{dn}(F,k) = \left[1 - \frac{k^2}{4} - \frac{5k^4}{64}\right] + \frac{k^2}{4}\left(1 + \frac{k^2}{4}\right)\cos 2G + \frac{k^4}{64} \cos 4G \tag{7.12c}$$

$$\mathrm{sc}(F,k) = \left(1 + \frac{k^2}{4} + \frac{11k^4}{64}\right) \tan G - \frac{k^4}{64} \sin 2G \tag{7.12d}$$

with

$$G = \frac{\pi}{2K}F = \left(1 - \frac{k^2}{4} - \frac{5k^4}{64}\right)F \ . \tag{7.12e}$$

We shall now introduce these approximations into the formulas for the three coordinates.

We first note that, in accordance with relation (7.12e), we have:

$$G_1 = \left(1 - \frac{k_1^2}{4} - \frac{5k_1^4}{64}\right)F_1 = \left(1 - \frac{k_1^2}{4} - \frac{5k_1^4}{64}\right)j_1 f \tag{7.13a}$$

$$G_2 = \left(1 - \frac{k_2^2}{4} - \frac{5k_2^4}{64}\right)F_2 = \left(1 - \frac{k_2^2}{4} - \frac{5k_2^4}{64}\right)(f + \omega) \tag{7.13b}$$

These relations have already been anticipated in sections 5 and 6. We next introduce the above approximations into Eq. (4.11). If in Eq. (4.11a) we also substitute for δ from relation (4.12c), we obtain

$$\frac{1}{R} = \frac{1}{p}\left\{1 + e\left[\left(1 - \frac{k_1^2}{16} - \frac{9k_1^4}{256}\right)\cos G_1 + \frac{k_1^2}{16}\left(1 + \frac{k_1^2}{2}\right)\cos 3G_1\right.\right.$$
$$\left.\left. + \frac{k_1^4}{256}\cos 5G_1\right]\right\} \Big/ \left\{1 + \eta^2 de\left[\left(1 - \frac{k_1^2}{16}\right)\cos G_1 + \frac{k_1^2}{16}\cos 3G_1\right]\right\} \tag{7.14a}$$

and

$$\cos\sigma = \sqrt{1 - N^2}\left[\left(1 + \frac{k_2^2}{16} + \frac{7k_2^4}{256}\right)\sin G_2\right.$$
$$\left. + \frac{k_2^2}{16}\left(1 + \frac{k_2^2}{2}\right)\sin 3G_2 + \frac{k_2^4}{256}\sin 5G_2\right] \ . \tag{7.14b}$$

If we also introduce the approximation into Eq. (5.38) for Ω, we have:

$$\Omega = \Omega_o - \eta^2 m_s N(f+\omega) - \eta^2\left(1 + \frac{k_1^2}{16}\right)m_{p1}Ne$$
$$\cdot\left[b_1\left(\sin G_1 - \sin \frac{j_1\pi}{2K_1}\omega\right) + b_2\left(\sin 2G_2 - \sin \frac{j_1\pi}{K_1}\omega\right)\right]$$
$$+ \eta^4 N\left[m_{p1}e\left[\sum_{n=1}^{6} B_n\left(\sin nG_1 - \sin \frac{n\pi j_1}{2K_1}\omega\right)\right]\right.$$
$$\left. + m_{p2}\left(1 - N^2\right) \sin 2G_2\right] \qquad (7.15)$$

in which the coefficients B_n are given by

$$B_3 = \beta_3 - \frac{e^2\left(1 - N^2\right)b_1}{16}, \quad B_6 = \beta_6 - \frac{e^2\left(1 - N^2\right)b_2}{16}$$
$$B_n = \beta_n \quad \text{for} \quad n = 1, 2, 4, 5 \qquad . \quad (7.16)$$

The approximate formula for the third coordinate is obtained by introducing relation (7.12d) into Eq. (5.39). We get:

$$\tan\left[j_3(\phi - \Omega)\right] = N\left[\left(1 + \frac{k_2^2}{4} + \frac{11k_2^4}{64}\right) \tan G_2 - \frac{k_2^4}{64} \sin 2G_2\right]. \qquad (7.17)$$

With Ω as given by Eq. (7.15) and relations (7.16), then relations (7.14) and (7.17) are the second-order approximate formulae for the three coordinates in terms of the "true anomaly" f. We can get a corresponding form for the time-angle relationship by introducing the approximate forms into Eq. (6.50). Because of the unwieldy length of the resulting formula, we do not exhibit it at this time.

8. A GEOMETRICAL RESULT

In the case of the Kepler problem, the conservation of angular momentum implies that the motion of the particle is planar, and it can be immediately shown that for negative energy the particle moves between two concentric circles in the plane of motion. In this section, we seek the analog for the Vinti problem of the above geometrical result.

Here the motion is no longer planar and, in general, is quite complex. However, if we assume that the exact formulas (4.11) and the approximate formulas (5.38) and (5.39) for the three coordinates indeed give an exact discription of the motion, we can get some geometrical insight.

To describe this, it is again useful to think of the Vinti problem as a perturbation of the Kepler problem. If we give the plane of the corresponding Kepler problem an appropriate deformation and then allow this surface to rotate with the angular velocity of the instantaneous nodal line (Ω') derived from Eq. (5.38), we can show that the particle remains on this surface.

More precisely, we shall show that the motion takes place in a torsidal region defined by two ellipsoids of revolution and a hyperboloid of revolution. In this region the particle remains on a surface S rotating with the angular velocity of the instantaneous node, the surface S being a "small" deformation of a plane. The remainder of this section is devoted mainly to the explicit derivation of this result.

We first derive a relation between σ and ϕ. If we set:

$$\psi = j_3(\phi - \Omega), \tag{8.1}$$

then Eq. (5.39) takes the form:

$$\tan \psi = N \operatorname{sc}\left(F_2, k_2\right) . \tag{8.2}$$

It follows that

$$\tan^2 \psi = N^2 \operatorname{sc}^2\left(F_2, k_2\right) , \tag{8.3}$$

and so

$$\sec^2 \psi = 1 + N^2 \, sc^2\left(F_2, k_2\right)$$

$$= \frac{cn^2\left(F_2, k_2\right) + N \, sn^2\left(F_2, k_2\right)}{cn^2\left(F_2, k_2\right)} \quad . \tag{8.4}$$

Also, from Eq. (4.11b), it follows that

$$\sin^2 \sigma = 1 - \left(1 - N^2\right) sn^2\left(F_2, k_2\right)$$

$$= cn^2\left(F_2, k_2\right) + N^2 \, sn^2\left(F_2, k_2\right) \, . \tag{8.5}$$

Combining Eqs. (8.4) and (8.5), we see that

$$\sec^2 \psi = \frac{\sin^2 \sigma}{cn^2\left(F_2, k_2\right)} \quad , \tag{8.6}$$

and hence

$$\cos j_3(\phi - \Omega) = \cos \psi = \frac{cn\left(F_2, k_2\right)}{\sin \sigma} \, . \tag{8.7}$$

We introduce a system of rotating Cartesian axes by the relations:

$$\left.\begin{aligned} X &= \left(R^2 + c^2\right)^{1/2} \sin \sigma \cos (\phi - \Omega) \\ Y &= \left(R^2 + c^2\right)^{1/2} \sin \sigma \sin (\phi - \Omega) \\ Z &= R \cos \sigma \end{aligned}\right\} \tag{8.8}$$

where we note that X = 0 corresponds to $\phi = \Omega$. In relation to the fixed x-y-z system of Eq. (1.1), the Z-axis of Eq. (8.8) coincides with the z-axis of Eq. (1.1), while the X-Y system rotates at the angular velocity Ω' [obtainable from Eq. (5.38)] relative to the x-y system. Thus the X-axis always coincides with the instantaneous nodal line.

We further introduce an auxiliary system of variables (ξ_1, ξ_2, ξ_3) defined by

$$
\begin{aligned}
\xi_1 &= R \sin \sigma \cos j_3(\phi - \Omega) = R \sin \sigma \cos \psi \\
\xi_2 &= R \sin \sigma \sin j_3(\phi - \Omega) = R \sin \sigma \sin \psi \\
\xi_3 &= R \cos \sigma
\end{aligned}
\tag{8.9}
$$

and it immediately follows that

$$\frac{\xi_2}{\xi_1} = \tan \psi = N \operatorname{sc}\left(F_2, k_2\right) \tag{8.10}$$

where we have used Eq. (8.2). From Eq. (8.10), we have

$$\frac{\xi_2}{\xi_1} = \frac{N}{\sqrt{1 - N^2}} \cdot \frac{\sqrt{1 - N^2}\, \operatorname{sn}\left(F_2, k_2\right)}{\operatorname{cn}\left(F_2, k_2\right)}$$

which, if we use Eq. (4.11b) in the numerator and Eq. (8.7) in the denominator, gives:

$$
\begin{aligned}
\frac{\xi_2}{\xi_1} &= \frac{N}{\sqrt{1 - N^2}} \cdot \frac{\cos \sigma}{\sin \sigma \cos \psi} \\
&= \frac{N}{\sqrt{1 - N^2}} \cdot \frac{\xi_3}{R} \cdot \frac{R}{\xi_1} \quad ,
\end{aligned}
\tag{8.11}
$$

the latter following directly from Eq. (8.9). If we define an "asymptotic" angle of inclination I by the relation

$$\cos I = N \; , \tag{8.12}$$

then Eq. (8.11) takes the form:

$$\xi_3 = \xi_2 \cdot \tan I \; . \tag{8.13}$$

Using the above with Eqs. (8.8) and (8.9), we have that

$$\begin{aligned} Z = \xi_3 &= \xi_2 \cdot \tan I \\ &= R \cdot \sin \sigma \cdot \sin j_3 (\phi - \Omega) \cdot \tan I \\ &= \frac{R}{\left(R^2 + c^2\right)^{1/2}} \cdot \left(R^2 + c^2\right)^{1/2} \cdot \sin \sigma \cdot \sin (\phi - \Omega) \\ &\quad \cdot \frac{\sin j_3 (\phi - \Omega)}{\sin (\phi - \Omega)} \cdot \tan I \\ &= \frac{R}{\left(R^2 + c^2\right)^{1/2}} \cdot Y \cdot \frac{\sin j_3 (\phi - \Omega)}{\sin (\phi - \Omega)} \cdot \tan I \; . \end{aligned} \tag{8.14}$$

If we set $j_3 = 1 + \bar{\varepsilon}$, so that $\bar{\varepsilon} = O(\eta^2)$* and use the addition formula for the sine term in the numerator of Eq. (8.14), we get:

$$\begin{aligned} Z = Y \cdot \tan I \cdot & \frac{R}{\left(R^2 + c^2\right)^{1/2}} \\ & \cdot \left[\cos \bar{\varepsilon} (\phi - \Omega) + \cot (\phi - \Omega) \sin \bar{\varepsilon} (\phi - \Omega)\right] \; . \end{aligned} \tag{8.15}$$

---

*As remarked in the footnote on page 43, we have, to second order, $j_3 = 1$, i.e., $\bar{\varepsilon} = O(\eta^6)$.

Again using Eq. (8.8) to substitute for cot $(\phi - \Omega)$, we have:

$$Z = \tan I \cdot \frac{R}{\left(R^2 + c^2\right)^{1/2}} \; Y \cos \overline{\varepsilon}(\phi - \Omega) + X \sin \overline{\varepsilon}(\phi - \Omega) \; . \quad (8.16)$$

When $\eta = 0$, we also have $\Omega' = 0$ [see Eq. (5.38)] and the above equation takes the form

$$Z = Y \cdot \tan I \; , \qquad (8.16^*)$$

which is the equation of the plane of the Kepler motion. For $\eta \neq 0$, Eq. (8.16) represents a surface, clearly a deformed plane, moving with angular velocity Ω' relative to fixed x-y-z axes. We let S denote this surface.

As a description of S, Eq. (8.16) is incomplete since we have not expressed the quantities R and $(\phi - \Omega)$ in terms of X, Y and Z. This involves the solution of the following set of transcendental equations, obtained directly from Eq. (8.8), namely:

$$R^2 + c^2 \sin^2 \sigma = X^2 + Y^2 + Z^2 \qquad (8.17a)$$

$$R \cos \sigma = Z \qquad (8.17b)$$

and

$$\tan (\phi - \Omega) = Y/X \; . \qquad (8.18)$$

An adequate approximate solution of the pair of Eqs. (8.17) can be obtained by expanding in powers of $(c/R)^2$ to second order. However, since there is little to be gained, we do not include this calculation here. We only note that Eqs. (8.17) and (8.18) together complement Eq. (8.16) to give a complete description of the surface S.

It is clear from relation (3.17) of section 3 that the motion takes place between the two cenfocal ellipsoids of revolution defined by $R = R_1$ and $R = R_2$ where R_1 and R_2 are the roots of the first quadratic on the right of Eq. (3.17). We note furthur from

Eq. (4.11b) -- and using Eq. (8.12) -- that

$$|\cos \sigma| \leqq \sqrt{1 - N^2} = \sin I \tag{8.19}$$

so that the trajectory is bounded by the hyperboloid of revolution:

$$|\cos \sigma| = \sin I \ . \tag{8.20}$$

Hence the motion takes place in a toroidal region consisting of that portion of the "exterior" of the hyperboloid [Eq. (8.20)] lying between the ellipsoids of revolution defined by $R = R_1$ and $R = R_2$.

We can now give a geometrical interpretation to what we termed the asymptotic angle of inclination I. The bounding hyperboloid [Eq. (8.20)] has an asymptotic cone: the angle I is the angle of inclination of the tangent plane to this asymptotic cone.

9. THE POTENTIAL

By an elegant device involving the introduction of complex variables and the use of the generating function for Legendre polynomials, Vinti (ref. 1) has shown that when the potential [Eq. (1.5)] is expressed in terms of spherical coordinates r, θ, and ϕ, we get:

$$V = -\frac{\mu R}{R^2 + c^2 \cos^2 \sigma} = -\frac{\mu}{r} \sum_{n=0}^{\infty} (-)^n \left(\frac{c}{r}\right)^{2n} P_{2n}(\cos \theta) \tag{9.1}$$

where the P_{2n} term denotes the Legendre polynomials of even order. If r_o denotes a length scale, which we shall later identify with the radius of the Earth, we can write Eq. (9.1) as

$$V = -\frac{\mu}{r} \sum_{n=0}^{\infty} (-)^n \left(\frac{c}{r_o}\right)^{2n} \left(\frac{r_o}{r}\right)^{2n} P_{2n}(\cos \theta) \ . \tag{9.2}$$

If we now make the identification

$$J_2 = \left(\frac{c}{r_o}\right)^2 , \tag{9.3}$$

then Eq. (9.3) becomes:

$$V = -\frac{\mu}{r} \sum_{n=0}^{\infty} \left(-J_2\right)^n \left(\frac{r_o}{r}\right)^{2n} P_{2n}(\cos\,\theta) \tag{9.4a}$$

$$= -\frac{\mu}{r} \left\{1 - \sum_{n=1}^{\infty} (-)^{n+1}\, J_2^n \left(\frac{r_o}{r}\right)^{2n} P_{2n}(\cos\,\theta)\right\} . \tag{9.4b}$$

The standard representation of the geopotential in spherical coordinates--referred to an origin at the center of mass--is

$$V_G = -\frac{\mu}{r} \left\{1 - \sum_{n=2}^{\infty} J_n \left(\frac{r_o}{r}\right)^n P_n(\cos\,\theta) - \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} J_{nm} \left(\frac{r_o}{r}\right)^n P_n^m(\cos\,\theta)\, \cos\,(\phi - \alpha_{mn})\right\} \tag{9.5}$$

In Eq. (9.5) r_o denotes the mean Earth radius and the J_{nm}'s are called the geopotential coefficients.

Apart from the Kepler term the dominant term in the representation [Eq. (9.5)] is that with coefficient J_2, which exceeds by an order of magnitude the effect of any other term. If, for purposes of comparison, we restrict our attention to that part of the geopotential which is both rotationally symmetric and symmetric about the equatorial plane, then we include this dominant term. We denote this part of the geopotential by V_{GS} and the residual $(V_G - V_{GS})$ can be considered a perturbation which must be taken into account later. Then:

$$V_{GS} = -\frac{\mu}{r}\left\{1 - \sum_{m=1}^{\infty} J_{2n}\left(\frac{r_o}{r}\right)^{2n} P_{2n}(\cos\theta)\right\} . \tag{9.6}$$

We now see that when c is chosen in accordance with the identification [Eq. (9.3)], then the Vinti potential [Eq. (9.4)] agrees with the symmetric geopotential [Eq. (9.6)] up to the second (zonal) harmonic. Further identification would require that

$$J_{2n} = (-)^{n+1} J_2^n \quad \text{for} \quad n \geqslant 2.$$

This is not true for any $n > 1$ and is not even true in an order of magnitude sense for any $n > 2$. In fact:

$$J_4 \approx -\frac{3}{2} J_2^2 , \quad |J_{2n}| \gg |J_2|^n \quad \text{for } n \geqslant 3, \tag{9.7}$$

so that if the Vinti problem is taken as the base solution, we must add as a perturbation the residual ($V_{GS} - V$) which starts with a term in the fourth harmonic with coefficient of order J_2^2.

As far as we know, there has not been proposed a real physical situation giving rise to the Vinti potential field. Although this question has no relevance for the dynamical problem and its relation to satellite orbit prediction, nevertheless it is of interest when we consider how closely the Vinti potential approximates the geopotential. The latter feature suggests that there is a hypothetical geoid whose potential matches the geopotential exactly. This is indeed the case. We shall show that a solid sphere with an appropriate interior mass distribution induces an external potential of the Vinti type.

We could start by posing the larger question, namely, what mass distribution in the geoid gives rise to the geopotential V_G? Although the restriction is not necessary, we shall here restrict our consideration of this question to the simpler case of the symmetric geopotential V_{GS}. Our problem then is to determine the mass distribution inside a sphere consistent with an arbitrary external field of the form V_{GS}.

Referred to a spherical coordinate system we let $P(r,\theta,\phi)$ represent an arbitrary exterior point and $Q(r',\theta',\phi')$ an arbitrary interior point at which is situated the mass element dm. We denote the position vector of P and Q by $\underset{\sim}{r}$ and $\underset{\sim}{r}'$, respectively, and let χ denote the angle between $\underset{\sim}{r}$ and $\underset{\sim}{r}'$. If we let $\tau(r',\theta',\phi')$ denote the mass density at any interior point, we see that

$$dm = \tau \cdot r'^2 \sin\theta' \, dr' \cdot d\theta' \cdot d\phi' \quad . \tag{9.7*}$$

If we let γ denote the gravitational constant, then the potential at P due to the mass element dm at Q is given by

$$dU_Q = -\gamma \frac{dm}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \tag{9.8a}$$

$$= -\gamma \frac{dm}{r} \cdot \frac{1}{\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\chi\right]^{1/2}} \tag{9.8b}$$

where, in deriving Eq. (9.8b), we have used the cosine law. Noting that on the right of Eq. (9.8b) we have the generating function for Legendre polynomials, it follows that

$$dU_Q = -\gamma \frac{dm}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\chi)$$

$$= -\gamma \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \cdot r'^n P_n(\cos\chi) \, dm \quad . \tag{9.9}$$

We now use the addition theorem for Legendre functions (ref. 9), namely, when

$$\cos\chi = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi') \tag{9.10}$$

(as is the case here), we have the following relation for the Legendre polynomials:

$$P_n(\cos\chi) = P_n(\cos\theta)\, P_n(\cos\theta')$$
$$+ 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi - \phi'). \tag{9.11}$$

Introducing Eq. (9.11) into Eq. (9.9) and substituting for dm from Eq. (9.7), we get:

$$dU_Q = -\gamma \left\{ \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \left[r'^n P_n(\cos\theta) P_n(\cos\theta') \right] \right.$$
$$\cdot \tau r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi' + 2\sum_{n=1}^{\infty} \frac{1}{r^{n+1}}$$
$$\cdot \left[r'^n \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi - \phi') \right]$$
$$\left. \cdot \tau r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi' \right\}. \tag{9.12}$$

To determine the potential at P due to the sphere we must perform the integration

$$U = \int dU_Q$$

when the integration is taken over the sphere, namely, r' ranges from 0 to r_O, θ' from 0 to π, and ϕ' from 0 to 2π. If we take τ to be rotationally symmetric, i.e., independent of ϕ', then the ϕ'-integration in the first summation integrates to 2π, while the ϕ'-dependence in the second summation integrates to zero. After performing the ϕ-integration, we therefore have:

$$U = -2\pi\gamma \sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{r^{n+1}} \int_0^{r_0}\int_0^{\pi} r'^{\,n+2} P_n(\cos\theta') \sin\theta' \cdot \tau \cdot dr'\, d\theta'. \tag{9.13}$$

In all the above manipulations we have interchanged the integration and summation operations.

If we further assume that τ is not dependent on r', then the r'-integration in Eq. (9.13) is immediate and we get

$$U = -2\pi\gamma r_0^2 \sum_{n=0}^{\infty} \left(\frac{r_o}{r}\right)^{n+1} \frac{P_n(\cos\theta)}{(n+3)} \int_0^{\pi} \tau \cdot P_n(\cos\theta') \sin\theta'\, d\theta'. \tag{9.14}$$

If we now represent the dependence of τ on θ' in the form:

$$\tau = \tau_o \left\{ 1 + \sum_{\ell=0}^{\infty} \tau_\ell P_\ell(\cos\theta') \right\}, \tag{9.15}$$

so that τ_o denotes the (constant) mean density and the τ_ℓ's ($\ell \geqslant 1$) are the dimensionless coefficients for the higher moments, then using the orthogonality of the Legendre polynomials we have:

$$\int_0^{\pi} P_n(\cos\theta')\sin\theta' \cdot \tau \cdot d\theta' = \tau_0 \int_0^{\pi} P_n(\cos\theta') \cdot \left[1 + \sum_{\ell=1}^{\infty} \tau_\ell P_\ell(\cos\theta') \right] \sin\theta'\, d\theta'$$

$$= \begin{cases} 2\tau_o & \text{for } n=0 \\ 2\tau_o\left(\dfrac{\tau_n}{2n+1}\right) & \text{for } n \geqslant 1 \end{cases} \tag{9.16}$$

and substituting Eq. (9.16) into Eq. (9.14), we get:

$$U = -4\pi\gamma r_o^2\tau_o \left\{ \frac{r_o}{r}\,\frac{1}{3} + \sum_{n=1}^{\infty} \frac{\tau_n}{(n+3)(2n+1)} \left(\frac{r_o}{r}\right)^{n+1} P_n(\cos\,\theta) \right\}$$

$$= -\frac{\gamma}{r}\cdot\frac{4\pi r_o^3}{3}\cdot\tau_o \left\{ 1 + \sum_{n=1}^{\infty} \frac{3\tau_n}{(n+3)(2n+1)} \left(\frac{r_o}{r}\right)^{n} P_n(\cos\,\theta) \right\}. \quad (9.17)$$

If we now call the sphere with constant density τ_o the mean sphere and denote its mass by μ_o, then clearly

$$\mu_o = \frac{4\pi r_o^3}{3}\,\tau_o\;, \quad (9.18)$$

and setting $\mu = \gamma\mu_o$ (the normalized gravitational constant), we see that Eq. (9.17) may be written:

$$U = -\frac{\mu}{r} \left\{ 1 + \sum_{n=1}^{\infty} \frac{3\tau_n}{(n+3)(2n+1)} \left(\frac{r_o}{r}\right)^{n} P_n(\cos\,\theta) \right\} . \quad (9.19)$$

It is clear from Eq. (9.19) that the τ_n can now be chosen to fit an arbitrary axisymmetric potential. We can make U symmetric about the equatorial plane by requiring all odd coefficients to vanish, that is, by setting:

$$\tau_{2k+1} = 0 \quad \text{for all k.} \quad (9.20)$$

We then get the symmetric part of U which we call U_s in the form:

$$U_s = -\frac{\mu}{r} \left\{ 1 + \sum_{n=1}^{\infty} \frac{3\tau_{2n}}{(2n+3)(4n+1)} \left(\frac{r_o}{r}\right)^{2n} P_{2n}(\cos\,\theta) \right\} . \quad (9.21)$$

Then, comparing Eq. (9.6) with Eq. (9.21), we see that U_s becomes identical with V_{GS} if we take:

$$\tau_{2n} = \frac{(2n + 3)(4n + 1)}{3} J_{2n} \tag{9.22}$$

and all odd coefficients zero. In particular, we can identify U_s with the Vinti potential V given by Eq. (9.4b) if again we have the odd coefficients in the density distribution identically zero and the even ones as given by

$$\tau_{2n} = \frac{(2n + 3)(4n + 1)}{3} (-J_2)^n \ . \tag{9.23}$$

We have thus produced a physical realization of the Vinti potential field.

We have also produced a density distribution whose potential can be matched to the symmetric part of the geopotential. This suggests a procedure which, by successive refinement, may lead to an approximation for the density distribution of the Earth, which, except for layers of sharp discontinuity, should give some insight into the actual distribution. The above analysis assumes continuity--and, in fact, analyticity--in the variables for the density distribution.

10. CONCLUSION

We have derived a solution of the Vinti dynamical problem in the relatively compact form given by relations (4.11), (5.38), and (5.39). These relations are clear generalizations of the solution of the Kepler problem in terms of true anomaly. As an alternative we have the more elementary expanded form in relations (7.14) to (7.17). Each form is complemented by the time-angle relationship (6.50).

We have also derived some qualitative results on the motion (section 8) and a physical interpretation of the Vinti potential in terms of mass distribution (section 9).

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National Aeronautics and Space Administration
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