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A GENERALIZED AVERAGING METHOD FOR LINEAR DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS

by Thomas J. Coakley

Ames Research Center
Moffett Field, Calif.

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SUMMARY

A new method for analyzing linear differential equations with almost periodic coefficients is described and discussed. The method is essentially a generalization of the classical Floquet theory developed for linear periodic systems. By use of the method, it is shown that an almost periodic system with rapidly varying coefficients can always be reduced or transformed to a corresponding system with either constant or slowly varying coefficients. An example is discussed which illustrates the method and indicates how the reduced system is frequently more easily analyzed than the original system.

INTRODUCTION

Ordinary differential equations constitute one of the principal tools for mathematically modeling physical systems. Within this general class, linear equations draw special attention because they are tractable and can be used frequently to approximate nonlinear systems. Most prominent within the class of linear differential equations are those with constant coefficients. These types of equations are particularly useful because their theory is quite simple and well developed. Unfortunately, the theory of linear differential equations with variable coefficients is not so well developed. The present investigation is a contribution to this theory and, in particular, is concerned with linear homogeneous equations with periodic and almost periodic coefficients. These equations have important applications in the dynamic stability of vehicles, structures and fluids, the theory of orbits, and the quantum theory of radiation.

Compared to the theory of linear differential equations with almost periodic coefficients, the theory of linear differential equations with periodic coefficients is more complete. For these systems, the classical Floquet theory (refs. 1 and 2) plays a central role. In this theory, it is shown that a linear system with periodic coefficients can always be transformed into a corresponding system with constant coefficients. Although Floquet theory is useful qualitatively, it is of limited practical application because no algorithm is provided for the actual construction of the transformation. For this purpose, additional and more specialized techniques must be used.

For linear systems with almost periodic coefficients, unfortunately, there is no general result such as Floquet theory to relate behavior and stability properties to systems with constant coefficients. As a consequence, methods for the analysis of these systems are much less developed and more highly specialized than those for linear periodic systems.

In this report, a new method is described which, in a certain sense, generalizes Floquet theory to linear equations with almost periodic coefficients. Our main result may be stated as follows: Under very broad conditions a bounded linear transformation can be found by means of which a linear system with rapidly varying, almost periodic coefficients may be transformed into a corresponding linear system with more slowly varying coefficients. Moreover, a useful algorithm is provided for constructing the transformation and the transformed system. The utility of the method rests on the fact that the transformed system is frequently more easily analyzed than the original one.

ANALYSIS

The method to be discussed is closely related to the so-called averaging methods previously developed for certain classes of periodic and almost periodic systems (refs. 2-6). However, it is more general and rigorous. This permits, on the one hand, application to wider classes of systems and, on the other, a more detailed discussion of existence and convergence.

Basic System

The basic system to be investigated is

$$\dot{x} = A(t)x \tag{1}$$

where $x(t)$ is an n-dimensional vector function, $A(t)$ is a bounded matrix function, and $\dot{x} = dx/dt$. For the current investigation, it is assumed that $A(t)$ can be represented as a generalized Fourier series, or almost periodic function, of the form

$$A(t) = \sum_n (A_n \cos \omega_n t + B_n \sin \omega_n t) = [a_{ij}(t)] \tag{2}$$

where the frequencies, ω_n, are arbitrary. The series is assumed to be absolutely convergent so that the L_∞ norm of A is bounded, that is,

$$||A|| = \sum_n (||A_n|| + ||B_n||) < \infty$$

where the matrix norms $||A_n||$ and $||B_n||$ are

$$||A_n|| = \max_i \sum_j |a_{nij}| \qquad \text{etc.}$$

Classical Floquet Theory

If $A(t)$ is periodic with period $T = 2\pi/\omega_o$ so that $\omega_n = n\omega_o$ in equation (2), the classical Floquet theory (refs. 1 and 2) states that a periodic transformation exists by means of which equation (1) can be reduced to a system with constant coefficients. This theory is outlined briefly here.

Let $X(t)$ be a fundamental matrix of equation (1) so that

$$X = A(t)X \;, \qquad X(0) = I \tag{3}$$

where I is the identity matrix. It is easily shown that $X(t)C$, where C is an arbitrary nonsingular constant matrix and $X(t + T)$ are also fundamental matrices. The latter assertion follows from the periodicity of $A(t)$, that is,

$$\frac{d}{dt} X(t + T) = \frac{d}{d(t + T)} X(t + T) = A(t + T)X(t + T) = A(t)X(t + T) \tag{4}$$

By equating $X(t + T)$ and $X(t)C$, it follows that $X(T) = X(0)C = C$.

Now let the matrices B (constant) and $P(t)$ (periodic) be defined by

$$\left.\begin{aligned} e^{BT} &= C = X(T) \\ P(t) &= X(t)e^{-Bt} \end{aligned}\right\} \tag{5}$$

The matrix $P(t)$ is periodic (and nonsingular) since

$$P(t + T) = X(t + T)e^{-B(t + T)} = X(t)e^{BT}e^{-B(t + T)} = P(t) \tag{6}$$

By making the transformation

$$x = P(t)y \tag{7}$$

and using equations (3) and (5), it follows that the equation for y is

$$\dot{y} = By \tag{8}$$

In carrying out this transformation, several procedures are available. If the matrix $A(t)$ is arbitrary, then the reduction can generally only be made numerically. This can be done by first solving equation (3) over one period, that is, $0 \le t \le T$. The matrix B can then be obtained (often in diagonal form) from equations (5) by solving the matrix eigenvalue problem for $C = X(T)$ (refs. 7 and 8). If $A(t)$ contains a small parameter, for example, $A = A_o + \varepsilon A_1(t)$, where A_o is a constant matrix and ε is the small parameter, then perturbation methods can be used (refs. 2, 4-6).

Averaging and Integrating Operators

We propose to show that equation (1) can be transformed into a similar system but with only low-frequency components. For this purpose, we introduce two linear functional operators, M_p and N_p, called averaging and integrating operators, respectively, defined as follows:

$$\left.\begin{aligned} M_p e^{i\omega t} &= \begin{cases} e^{i\omega t}, & |\omega| < \omega_p \\ 0, & |\omega| \geq \omega_p \end{cases} \\ N_p e^{i\omega t} &= \begin{cases} 0, & |\omega| < \omega_p \\ (i\omega)^{-1} e^{i\omega t}, & |\omega| \geq \omega_p \end{cases} \end{aligned}\right\} \tag{9}$$

The number ω_p is called the cutoff frequency and is assumed to be greater than zero. It is easily verified that the operations M_pA and N_pA have the properties:

$$\left.\begin{aligned} M_pA &= \sum_{0 \leq \omega_n < \omega_p} (A_n \cos \omega_n t + B_n \sin \omega_n t) \\ N_pA &= \sum_{\omega_p \leq \omega_n} \omega_n^{-1} (A_n \sin \omega_n t - B_n \cos \omega_n t) \end{aligned}\right\} \tag{10}$$

$$\frac{d}{dt} N_pA = A - M_pA \tag{11}$$

From these relations, it is seen that M_pA, which we choose to call the generalized mean or average value of A, contains no frequencies greater than ω_p. Conversely, N_pA, which will be called the generalized integral of A, only contains frequencies equal to or greater than ω_p. It is easily shown that M_pA and N_pA are bounded, that is,

$$\|M_pA\| \leq \|A\| , \qquad \|N_pA\| \leq \omega_p^{-1} \|A\| \tag{12}$$

The operators defined above are closely related to those used by Krylov-Bogoliubov and Mitropolski (refs. 3 and 4). In fact, the operators they used may be obtained from equations (9) and (10) by letting the cutoff frequency go to zero, that is, $\omega_p \to 0$. In this case, the function M_pA is a constant equal to the mean value of A. However, the corresponding integrating operation, N_pA, is not bounded, and additional and rather severe restrictions must be placed on the class of almost periodic functions for this operation to be well defined (and bounded) (refs. 5 and 6).

Transformation of the Basic System

The averaging and integrating operators defined above will now be used to transform equation (1). For this purpose, we assume that the vector, x, in equation (1) is transformed into a new vector, y, by a matrix $I + V(t)$, that is,

$$x = [I + V(t)]y \tag{13}$$

where I is the identity matrix and $V(t)$ is to be determined. Assume further that the transformed differential equation for y is of the form

$$\dot{y} = B(t)y \tag{14}$$

where $B(t)$, like $V(t)$, is to be determined. Substituting equations (13) and (14) into equation (1) and rearranging, we obtain the following equation for V:

$$\dot{V} = U - B \tag{15}$$

where

$$U = A + AV - VB \tag{16}$$

Suppose for the moment that the matrix function $U(t)$ in equation (15) is known and of the same form as $A(t)$ (i.e., with an arbitrary frequency spectrum). We desire to choose $B(t)$ in such a way that solutions to equation (15) for V are bounded for all t regardless of the form, or frequency spectrum, of $U(t)$. It is readily verified that this can be done by taking

$$B = M_pU \tag{17}$$

The solution for V then becomes

$$V = N_pU + V_o \tag{18}$$

where V_o is an arbitrary constant matrix that will be taken as zero in the current development. By means of equation (12), one may verify that V is bounded if U is and if $\omega_p > 0$.

In order that B and V, as defined above, yield the transformation given by equations (13) and (14), U(t) must satisfy equation (16). By setting $B = M_pU$ and $V = N_pU$, this equation is reduced to a single equation for U, that is,

$$U = A + A(N_pU) - (N_pU)(M_pU) \tag{19}$$

The function U defined by this equation will be called the generating function of the transformation. It would be very useful to know that this function exists for arbitrarily small cutoff frequencies including the value

$\omega_p = 0$. In this case, the reduced system, equation (14), would have constant coefficients, and one would have the direct generalization of Floquet theory to almost periodic systems. Unfortunately, because of the unboundedness of N_p as $\omega_p \to 0$, this is not generally possible. However, solutions to equation (19) can be shown to exist for sufficiently large values of the cutoff frequency, ω_p, or for sufficiently small matrices $A(t)$. The explicit condition on A and ω_p which guarantees the existence of solutions to equation (19) is

$$\|A\| < (3 - 2\sqrt{2})\omega_p \tag{20}$$

If this condition is satisfied, it can also be shown that solutions to equation (19) can be constructed by successive approximations, starting with the initial approximation $U = U_1 = A$. Further, the matrix N_pU is bounded by unity, that is, $\|N_pU\| < 1$, so that the transformation matrix $I + V$ of equation (13) is bounded and nonsingular. It then follows that the ultimate stability and behavior of solutions to equation (1) are controlled by the solutions to equation (14). These results can be proven by means of the contraction mapping principle, or fixed point theorem in Banach space (refs. 5 and 9). However, the details of the proof are beyond the scope of the present discussion and will be omitted.

In addition to solving equation (19) by successive approximations, one can also use series expansions in terms of a parameter, say, ε. In this event, one writes

$$\left.\begin{aligned} A(t) &= \varepsilon\tilde{A}(t, \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n A_n(t) \\ U(t) &= \varepsilon\tilde{U}(t, \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n U_n(t) \end{aligned}\right\} \tag{21}$$

where the series for $A(t)$ is assumed to converge in some neighborhood, $0 \le \varepsilon \le \varepsilon_o$, and $\|A(t, \varepsilon)\| \le 1$. The series for $U(t)$ can be constructed by substituting equations (21) into (19) and equating coefficients of ε^n. It can be shown that this series converges if $\varepsilon < \varepsilon_o < k\omega_p$, where k is a constant of the same order of magnitude as the parenthetical constant appearing in equation (20), that is, $3 - 2\sqrt{2}$.

Some Properties of the Reduced System

In certain cases, the reduced system, equation (14), may have properties that considerably simplify its analysis. This is certainly true when the system matrix, $B = M_pU$, is constant. An important example is the case when $A(t)$ is <u>periodic</u> and expressible in the form

$$A(t) = \sum_{n=0}^{\infty} (A_n \cos n\omega_o t + B_n \sin n\omega_o t) \tag{22}$$

By choosing the cutoff frequency equal to the base frequency, that is, $\omega_p = \omega_0$, and assuming the existence criterion, equation (20), to be satisfied, one may show that M_pA and thus $B = M_pU$ are constant matrices. Although this is simply a restatement of Floquet theory, there is the additional feature that solutions can be constructed by the rather straightforward and convergent processes of successive approximations or series expansions. In this aspect the method is similar to methods developed earlier by Cesari (ref. 2), Hale (ref. 5), and Golomb (ref. 6). However, in those methods the matrix B (or, equivalently, its eigenvalues) is defined implicitly by a nonlinear matrix equation that must be solved. In our method, it is defined explicitly, and this feature generally simplifies theoretical and numerical analyses.

In the more general situation in which A(t) is almost periodic, that is, where some or all of the frequencies ω_n are incommensurable, there is no assurance that the matrix B will be constant. In fact, B will almost always be time dependent. However, it frequently can be made constant to an acceptable order of approximation. An important example is when A(t) can be expressed in the series form of equations (21) and each term, $A_1(t)$, $A_2(t)$, etc., is a trigonometric polynomial containing only a finite number of frequencies. In this case, it is possible to insure that the first N terms in the series for B are constant by choosing the cutoff frequency, ω_p, sufficiently small. When only the first term is to be made constant, that is, N = 1, then ω_p is chosen equal to the smallest frequency in $A_1(t)$, say, ω_1. The first approximation to $B = M_pU$ is then εM_pA_1 = const, and the sufficient condition on the size of ε for the series to converge is estimated by the condition $\varepsilon < k\omega_1$. One may show that as the level of approximation for which the matrix B is to be made constant, that is, N, is increased, the required cutoff frequency (and thus the sufficient interval of convergence in ε) decreases so that ω_p and $\varepsilon \to 0$ in the limit as $N \to \infty$. The series expansion obtained in this limit, although constant, is generally not convergent for any finite value of ε. In fact, it is only a formal or asymptotic series with the property that beyond some level of approximation the series begins to diverge without bound (refs. 10-12). This divergence is due to small divisors that appear in the expansion for N_pU, and occur as a result of the fact that N_p is an unbounded operator on the space of almost periodic functions. The asymptotic series derived in this manner is essentially the one obtained by using the averaging and perturbation techniques developed in references 3 and 4. According to the above arguments, one may show that if the first N terms of a convergent series expansion for B are constant (with $\omega_p \neq 0$), then these terms are equal to the corresponding first N terms in the asymptotic expansion of B (with $\omega_p = 0$).

If the procedure described above for obtaining the matrix B(t) = const up to some level of approximation is inapplicable or impractical (e.g., if A(t) contains dominant low-frequency terms), it may still be possible to put B in a form that makes equation (14) easier to analyze than the original system. An important example occurs when B(t) or any of its related approximate systems have very slowly varying coefficients. In this case, it is assumed that A(t) has certain properties that permit B (or any approximation) to be expressed in the typical form

$$B = \varepsilon \tilde{B}(\varepsilon^h t, \varepsilon)$$

where $h \geq 2$ is an integer. For systems of this type, singular perturbation methods, applicable to systems containing a large parameter, are available (refs. 1 and 12).

In those cases where it is not possible or practical to choose the cutoff frequency in such a way that B is constant or a very slowly varying matrix to an acceptable order of approximation, then it is generally not possible to find a theoretical solution of the reduced system, and numerical or analog integration methods must be used. If the original system matrix contains only low-frequency elements, then there is usually no advantage in using the averaging method. On the other hand, if $A(t)$ contains both high- and low-frequency elements, the numerical analysis may be greatly facilitated (i.e., speeded up) by first filtering out the high-frequency elements through a suitable averaging transformation.

If, for the case described above, a complete solution is not sought but only information on stability is required, then the second method of Liapunov (ref. 2) can be used to advantage. In its most simple form, this theory states that the system $\dot{y} = B(t)y$ will be asymptotically stable if the eigenvalues of the symmetric matrix $S(t) = (1/2)(B' + B)$ (where B' is the transpose of B) are negative for all t. One should note that this is only a sufficient condition for stability and that the system may be stable even if the condition is violated.

Relationships Between Exact and Approximate Solutions

Since it is generally not possible to obtain the exact solution for U (and B and V), it is important to know the relationships that exist between the exact solutions of the reduced system and the approximate ones obtained by terminating the successive approximations or series expansions for B at some level. We note that if A and U (and B) are expressed in the series form of equations (21) then the error involved in terminating the series at some level, say, ε^N, is of order ε^{N+1}. One can also show that a similar statement is true for the process of successive approximations. It is thus clear that under appropriate conditions the behavior and stability of the exact and approximate solutions will be essentially the same. These conditions depend on the properties of the matrix $A(t)$, the level of the approximation N, and the types of behavior and stability under investigation. Because of the complicated nature of these conditions, it is impossible to give a detailed account of them here. Only a brief and qualitative discussion will be given.

The most important principle in relating the behavior and stability of the exact and approximate systems may be stated as follows: If some or all of the solutions to an approximate system can be shown to grow or decay exponentially (over an appropriate time interval) at an absolute rate greater than a quantity proportional to the error or remainder terms omitted in the approximation (e.g., ε^{N+1}), then there will be corresponding exact solutions with the same behavior. If the Nth approximation to B, say, B_N, is a constant matrix, then the rates of growth or decay of the approximate solutions for Y are given by the real parts of the eigenvalues of B_N. If the approximation, B_N, is a very slowly varying function, for example, of the form $\varepsilon B_N(\varepsilon^h t, \varepsilon)$, $h \geq 2$, then the rates of growth or decay of the approximate solutions are given, to order ε^h, again by the real parts of the eigenvalues of B_N. In this case, however, solutions may grow over some intervals and decay over others.

In the event some solutions of an approximate system neither grow nor decay (which may occur when the approximation $M_pU \sim B_N$ is constant with some zero or imaginary eigenvalues), the stability of the exact solutions cannot be inferred except in special cases. All that can be said generally is that the exact solutions cannot grow or decay at a rate that is (usually) greater than a quantity proportional to the remainder terms in the approximation for B. For a more complete discussion of these questions, including special cases, see references 2, 5, and 13.

Example

In applying the results discussed in previous paragraphs, we consider the following second-order system

$$\ddot{u} + 2\omega_o\zeta\dot{u} + \omega_o^2[1 + 2\phi(t)]u = 0 \tag{23}$$

which describes the motion of a damped harmonic oscillator parametrically excited by a forcing function $\phi(t)$. It is assumed that ω_o and ζ are constant and ζ and $\phi(t)$ are small. In addition, it is assumed that $\phi(t)$ can be represented as a generalized Fourier series of the form

$$\phi(t) = \sum_{\omega_n} \phi_n \cos(\omega_n t - \theta_n) \tag{24}$$

By means of the transformation

$$\left.\begin{aligned} u &= x_1 \cos \omega_o t + x_2 \cos \omega_o t \\ \dot{u} &= \omega_o[-x_1 \sin \omega_o t + x_2 \cos \omega_o t] \end{aligned}\right\} \tag{25}$$

equation (23) can be put into the form of equation (1):

$$\dot{x} = A(t)x \tag{26}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{27}$$

and

$$a_{11} = \omega_o[\phi \sin 2\omega_o t - \zeta(1 - \cos 2\omega_o t)]$$

$$a_{12} = \omega_o[\phi(1 - \cos 2\omega_o t) + \zeta \sin 2\omega_o t]$$

$$a_{21} = -\omega_o[\phi(1 + \cos 2\omega_o t) - \zeta \sin 2\omega_o t]$$

$$a_{22} = -\omega_o[\phi \sin 2\omega_o t + \zeta(1 + \cos 2\omega_o t)]$$

To analyze equation (26), only those properties of the solution that can be determined by a study of the first approximation to U, that is, U = A, will be investigated. The approximate system for y is then

$$\dot{y} = (M_p A)y \tag{28}$$

We choose the cutoff frequency of the operator M_p to be some fraction of $2\omega_o$, that is, $\omega_p = 2\ell\omega_o$, $\ell \le 1$. Now solutions to equation (19) for U exist if equation (20) is satisfied. This is equivalent to the following condition on ϕ and ζ:

$$|\phi| + |\zeta| < \frac{2}{3}(3 - 2\sqrt{2})\ell \approx 0.114\ell$$

By means of equations (11) and (27), we may write the matrix $M_p A$ of equation (28) as

$$M_p A = B_1 = -\omega_o \begin{bmatrix} \zeta - a_1 & -a_o + a_2 \\ a_o + a_2 & \zeta + a_1 \end{bmatrix} \tag{29}$$

where

$$a_o = M_p\phi , \qquad a_1 = M_p(\phi \sin 2\omega_o t) , \qquad a_2 = M_p(\phi \cos 2\omega_o t) \tag{30}$$

Using equation (24), the explicit representation of these functions is

$$\left.\begin{aligned}
a_o(t) &= \sum_{\omega_n<\omega_p} \phi_n \cos(\omega_n t - \theta_n) \\
a_1(t) &= -\frac{1}{2} \sum_{|\omega_n-2\omega_o|<\omega_p} \phi_n \sin[(\omega_n - 2\omega_o)t - \theta_n] \\
a_2(t) &= \frac{1}{2} \sum_{|\omega_n-2\omega_o|<\omega_p} \phi_n \cos[(\omega_n - 2\omega_o)t - \theta_n]
\end{aligned}\right\} \tag{31}$$

Unfortunately, it is not possible to solve equation (28) for completely arbitrary perturbation functions $\phi(t)$. For this purpose, $\phi(t)$ must be further restricted. We will consider two cases in which a_o, a_1, and a_2 are (1) constants and (2) very slowly varying functions.

Case 1: Constant coefficients.- Here we assume $\phi(t)$ has no frequency components in the bandwidths $0 < \omega_n < \omega_p = 2\ell\omega_o$ and $|\omega_n - 2\omega_o| < \omega_p = 2\ell\omega_o$ with the exception of one at $\omega_n = 0$ and one at $\omega_n = \omega_r = 2\omega_o$. In this case, choosing $\theta_o = 0$, we have

$$a_o = \phi_o\ , \qquad a_1 = \frac{1}{2}\phi_r \sin\theta_r\ , \qquad a_2 = \frac{1}{2}\phi_r \cos\theta_r \tag{32}$$

The frequency $\omega_r = 2\omega_o$ is customarily called the principal resonance frequency.

The behavior and stability of the (approximate) solutions for y can be determined from the eigenvalues of M_pA. These eigenvalues are

$$\begin{aligned}
\lambda_i &= \omega_o[-\zeta \pm \sqrt{a_1^2 + a_2^2 - a_o^2}\,] \qquad i = 1, 2 \\
&= \omega_o\left[-\zeta \pm \sqrt{\left(\frac{\phi_r}{2}\right)^2 - \phi_o^2}\right]
\end{aligned} \tag{33}$$

The approximate solutions for $y(t)$ behave like $\exp(\lambda_i t)$.

Applying the arguments made in the section concerning the relationship between exact and approximate solutions, one can state that the exact solutions for y will be asymptotically stable (or decay exponentially with increasing time) if

$$\mathrm{Re}(\lambda_i) = \omega_o \mathrm{Re}\left[-\zeta \pm \sqrt{\left(\frac{\phi_r}{2}\right)^2 - \phi_o^2}\right] < -\mu \tag{34}$$

where μ is a small, positive, second-order quantity (proportional to the squares ζ^2, ϕ_n^2, etc.) and Re denotes the real part of the indicated expression. The condition that the solutions be asymptotically unstable (or that they grow exponentially) may be obtained from equation (34) by reversing the inequality and sign of μ.

When $\zeta = 0$, $\omega_o = 1$, and $\phi(t)$ is the periodic function given by

$$\begin{aligned} 2\phi(t) &= \delta - 1 + \varepsilon \cos 2t \\ &= 2(\phi_o + \phi_r \cos 2t) \end{aligned} \tag{35}$$

equation (23) reduces to the Mathieu equation. For sufficiently small values of ϕ_o and ϕ_r, or $\delta - 1$ and ε, the matrix M_pU of the reduced system, equation (28), is constant. The eigenvalues of the first approximation system are then given by

$$\lambda_i = \pm \frac{1}{2} \sqrt{\left(\frac{\varepsilon}{2}\right)^2 - (\delta - 1)^2} \tag{36}$$

The curves in the $\delta - \varepsilon$ plane separating zones of stability and instability are the transition curves. To a first approximation, these may be obtained by setting $\lambda_i = 0$ in the above expression. This gives

$$\delta = 1 \pm \frac{\varepsilon}{2} \tag{37}$$

which has been obtained previously by other methods (refs. 4, 5, 10, 14, and 15).

Case 2: Very slowly varying coefficients.- As remarked previously, solutions of equation (28) cannot be found if $\phi(t)$ or the functions a_o, a_1, and a_2 of equations (31) are completely arbitrary. An exception occurs when these functions are very slowly varying. To be specific, we assume that all the frequencies appearing in these functions are small and of second order, that is, of order ε^2 where ε is of the same order as ζ and ϕ. In this case, equation (28) can be analyzed by singular perturbation methods (refs. 1 and 12). The approximate solutions for y have the form

$$y \sim \exp\left\{\int [\lambda_i(t) + O(\varepsilon^2)]dt\right\} \tag{38}$$

where $\lambda_i(t)$ are the (time dependent) eigenvalues of the matrix M_pA and are given, as before, by equation (33). From this it can be seen that, depending on whether the real parts of each $\lambda_i(t)$ are positive or negative, solutions may grow over certain intervals and decay over others.

It is of interest to estimate the conditions on the coefficients ϕ_n of equation (31), for which solutions will decay monotonically (i.e., will be asymptotically stable). This is the same as requiring that the real parts of each $\lambda_i(t)$ be negative for all t. A sufficient condition to insure this type of behavior may be obtained from equations (31) and (33) and is given by

$$-\zeta \pm \frac{1}{2}\sum{}' |\phi_n| < -\mu \tag{39}$$

where μ is a small second-order quantity and where the summation is over all coefficients in the series for $a_1(t)$ or $a_2(t)$ (i.e., eqs. (31)).

It is interesting to compare the above stability criterion with one obtained by using the second method of Liapunov (ref. 2). In this method, no restrictions on the coefficents a_o, a_1, and a_2 in equation (28) need be made. In its simplest form, this theory states that the solutions to equation (28) will be asymptotically stable if the eigenvalues of the symmetric matrix $S = (1/2)(B' + B)$ are negative for all t. These eigenvalues are

$$\lambda = \omega_o \left[-\zeta \pm \sqrt{a_1{}^2 + a_2{}^2}\right]$$

Taking into account the explicit form of the functions $a_1(t)$ and $a_2(t)$ given by equations (31), it is easily shown that a sufficient condition for these eigenvalues to be negative is simply

$$\zeta > \frac{1}{2}\sum{}' |\phi_n|$$

This is essentially the same result as that given by equation (39), which was based on the assumption that $a_1(t)$ and $a_2(t)$ are very slowly varying functions.

CONCLUDING REMARKS

In the preceding paragraphs, we have described a method of averaging applicable to linear, homogeneous, almost periodic differential equations, which is a generalization of the well-known methods developed in references 3 and 4. Using the procedure, we have shown that it is always possible to transform an almost periodic system with rapidly varying coefficients into one with more slowly varying coefficients (i.e., with only low-frequency components). We have indicated some of the conditions under which the reduced system may be more easily analyzed than the original system and have illustrated these by a simple example. We have also indicated a theoretical and practical advantage of the new method over the previous methods. This advantage stems from the fact that the processes used in constructing the transformation and the reduced system are convergent. Thus, estimates of the error incurred in terminating the approximations at some level are obtained. In the previous methods, the processes are nonconvergent and error estimates cannot be obtained. Finally, we have shown how the error estimates are useful in relating the stability and behavior of the exact solutions to the approximate solutions.

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