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ON ASYMPTOTIC STABILITY ACCORDING TO LYAPUNOV IN SOME CRITICAL CASES

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by

V. V. Kostin

Dopovidi Akademiyi Nauk URSR, 35, No. 1, pp. 21-25, 1967.

Translated from the Ukrainian

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ON ASYMPTOTIC STABILITY ACCORDING TO LYAPUNOV IN SOME CRITICAL CASES

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V. V. Kostin

Systems of the type (1), (2), and (3) are investigated in this paper. Some or all real parts of the eigenvalues of $A(t)$ approach zero as $t \to +\infty$. Sufficient conditions are obtained for asymptotic stability according to Lyapunov of the solutions of the systems at $t \to +\infty$.

Under consideration are systems of the type:

$$\frac{dx}{dt} = A(t)x \ , \tag{1}$$

$$\frac{dx}{dt} = A(t)x + f(t,x) \ , \tag{2}$$

$$\frac{dx}{dt} = A(t)x + \varphi(t) + F(t,x) \ , \tag{3}$$

where $x = \{x_1, \ldots, x_n\}$, $\varphi(t) = \{\varphi_1(t), \ldots, \varphi_n(t)\}$, $f(t,x) = \{f_1(t,x), \ldots, f_n(t,x)\}$, $F(t,x) = \{F_1(t,x), \ldots, F_n(t,x)\}$ are n-dimensional vectors. $A(t)$ is an n-dimensional matrix whose elements are complexly valued functions, continuous in the general case, of the real variable t. We will designate that $||A(t)|| = \sum_{i,j} |\alpha_{ij}(t)|$. It is required that $||A(t)|| \leq M$, $M > 0$ and M = constant for all cases where $t \geq t_0$.

In our work sufficient indications were obtained of the asymptotic stability of systems (1), (2), and (3), depending on the type of behavior of the given functions and the tending toward zero of the real parts of the roots of the characteristic equation

$$\det |A(t) - \lambda E| = 0 \ . \tag{4}$$

In that case, for systems of the type of (1) and (3) we used a method similar to that used in [1 - 3], where it was assumed that the roots of Eq. (4) have strictly negative real parts, which in the general case depend on $t \geq t_0$ [1,3]. In [4] an attempt was made to transfer the results of [1] to systems of types (1) and (2). The estimates obtained in that work are of little effectiveness. We partially eliminated that shortcoming and also obtained some new results.

Preliminarily, let us examine the system of differential equations

$$\frac{dV(t,t_0)}{dt} = AV(t,t_0) \quad , \tag{5}$$

where A is a constant n-dimensional matrix with the condition $||A|| \leq M$.

Let the roots of the characteristic equation $\det|A - pE| = 0$ be such that their real parts satisfy the condition

$$\mathrm{Re}_{\mathrm{I}}\, p \leq -\gamma < -\frac{\gamma}{4} < \mathrm{Re}_{\mathrm{II}}\, p \leq \Lambda \quad .$$

The constant $\gamma > 0$, $\Lambda = \max \mathrm{Re}\, p_i$ $(i = 1, 2, \ldots, n)$. For any two roots of the second group the condition of separation of the roots $|p_i - p_j| \geq c > 0$ and $c = \text{constant}$ $(i \neq j)$ is fulfilled. The multiplicity of the roots of the first group is sufficient.

Our further investigations are based on that lemma.

<u>Lemma.</u> Solution of system (5) $V(t,t_0)$ with the initial condition $V(t,t_0)|_{t=t_0} = E$ satisfies the condition

$$||V(t,t_0)|| \leq (K_1 + K_2)e^{\Lambda(t - t_0)} \quad ,$$

with the constants

$$0 \leq K_1 \leq 2^{2n}\gamma^{-n}M^{n}n^{2}(n - 1)^{\frac{n-1}{2}} = \alpha_1 \quad ,$$

$$0 < K_2 \leq 2^{n-1}c^{1-m}M^{n-1}n^{2}(n - 1)^{\frac{n-1}{2}}\left(\frac{\gamma}{2}\right)^{m-n} m = \alpha_2 \quad ,$$

where m is the number of roots of the first group, $m = 1, 2, \ldots, n$. Therefore, if $m = n$, then $K_1 = 0$. The method of I. Z. Shtokalo [5] has been used to demonstrate that lemma.

Now let us examine a system of the type of (1). It is assumed that the roots of the characteristic Eq. (4) are such that their real parts can be divided into two groups

$$\operatorname{Re}_{\mathrm{I}}\lambda(t) \leq -\gamma < -\frac{\gamma}{4} < \operatorname{Re}_{\mathrm{II}}\lambda(t) \leq -g\,\frac{1}{t^{1-\beta}} . \tag{6}$$

The constants γ, β, and $g > 0$; $0 < \beta < 1$. For any roots of the second group the condition of separation of the roots which figure in the lemma is fulfilled. The multiplicity of the roots of the first group is sufficient. It is also required that

$$\|A(t_2) - A(t_1)\| \leq \delta\left|\frac{1}{t_2^{1-\beta}} - \frac{1}{t_1^{1-\beta}}\right| \tag{7}$$

for all cases where t_2 and $t_1 \geq t_0 \geq T > 0$. The constant δ satisfies the condition

$$0 < \delta < \frac{\beta g}{(\alpha_1 + \alpha)(1 - \beta)} , \tag{8}$$

and so if $m = n$, then α_1 is not present in (8).

<u>Theorem 1.</u> Upon fulfillment of conditions (6), (7), and (8) the solutions of system (1) are asymptotically stable according to Lyapunov at $t \to +\infty$. More precisely, there exist constants $K > 0$ and $0 < \rho < 1$ such that if $x(t)$ is the solution of Eq. (1) then

$$\|x(t)\| \leq K \exp\left[-\frac{\rho g}{\beta}\left(t^{\beta} - t_0^{\beta}\right)\right]\|x(t_0)\| \tag{9}$$

for all cases where $t \geq t_0$.

For example, the system

$$x_1' = \left(-\frac{1}{t^{0.01}} + i\right)x_1 + 5ix_2 ,$$

$$x_2' = \left(3i - \frac{\ln t}{t}i - \frac{2}{5}\frac{1}{t^{0.01}}\right)x_1 + \left(-3\frac{1}{t^{0.01}} - i\right)x_2 , \tag{10}$$

starting at a certain $t \geq t_0 \geq T$ satisfies the conditions of Theorem 1 with the constants $g = 2$; $\beta = 0.99$, $m = n = 2$; $M < 11$; $\delta < 5$; $c > 6$. Now one can readily convince oneself that inequality (8) is satisfied. Therefore, the solutions of system (10) are asymptotically stable according to Lyapunov.

Note 1. Along with system (1) let us examine the system

$$\frac{dy}{dt} = B(t)y \quad , \tag{11}$$

where $B(t)$ is an n-dimensional matrix continuous in all cases where $t \geq t_0$.

It is required that

$$\int_{t_0}^{1} ||B(\tau) - A(\tau)||d\tau \leq Q|t^{\beta_1} - t_0^{\beta_1}| + R \quad ,$$

where β_1, Q, and R = constant, $Q > 0$, and $\beta_1 \leq \beta$. Therefore, Q is an arbitrary constant if $\beta_1 < \beta$ and $Q < \dfrac{g}{2(\alpha_1 + \alpha_2)}$ if $\beta_1 = \beta$.

Let inequality (8) be fulfilled for 2δ. Then upon fulfillment of those conditions the solutions of system (11) are asymptotically stable at $t \to +\infty$.

Note 2. Let in system (1) the roots of the characteristic Eq. (4) be such that $\mathrm{Re}\lambda(t) \leq -\gamma$; $||A(t_2) - A(t_1)|| \leq \delta|t_2 - t_1|$, where the constant δ satisfies the condition

$$\delta \leq \frac{\gamma}{4\alpha_1 T} \quad , \qquad T = \frac{8}{\gamma}\ln \alpha_1 \quad .$$

Then we get a previously known result — Theorem 2 from [1]. That estimate of the value of δ is more precise than the analogous estimate from [1]. {Note: In [3] the estimate for δ in Theorem 2 is given from [1].}

Let in system (2) $f(t,x)$ be a continuous vector function of the variables t and x; $f(t,0) = 0$. We will designate that $||f(t,x)|| \leq \omega(t)||x||$. The continuous function $\omega(t)$ satisfies the condition

$$\begin{cases} \text{either } \omega(t)t^{1-\beta} \to 0 \text{ at } t \to +\infty \quad , \\ \text{or } \omega(t) \leq \epsilon t^{-1} \quad , \quad \epsilon > 0 \quad , \quad \epsilon \text{ is a small parameter.} \end{cases} \tag{12}$$

Theorem 2. Let in system (2) the matrix $A(t)$ satisfy the conditions of Theorem 1. For the vector function $f(t,x)$, condition (12) is fulfilled. Then a zero solution of system (2) is asymptotically stable according to Lyapunov at $t \to +\infty$. More precisely, for solutions of system (2) an estimate of type (9) is made.

Now let us examine a system of the type of (3). There $\varphi(t)$ and $F(t,x)$ are continuous functions of the variables t and t, x respectively; $F(t,0) = 0$. We will designate that $||\varphi(t)|| \leq \omega_1(t)$; $||F(t,x) - F(t,y)|| \leq \omega_2(t)||x - y||$, where $\omega_i(t)$ $(i = 1, 2)$ are continuous functions of the variable $t \geq t_0$. We will also designate that $x(t) = x^{(1)}(t) - x^{(2)}(t)$, where $x^{(1)}(t)$ and $x^{(2)}(t)$ are any parts of the solution of system (3).

Theorem 3. Let for the matrix $A(t)$ the conditions of Theorem 1 be fulfilled and the vector-functions $\varphi(t)$ and $F(t,x)$ be such that for $\omega_i(t)$ $(i = 1, 2)$ condition (12) is fulfilled. Then for the vector function $x(t)$, starting from a certain $t \geq t_0$, the estimate is made

$$||x(t)|| \leq K \exp\left[-\frac{\rho^* g}{\beta}\left(t^\beta - t_0^\beta\right)\right]||x(t_0)||$$

with the constants $K > 0$ and $0 < \rho^{*\prime} < \rho$. Therefore, each partial solution of system (3) is asymptotically stable according to Lyapunov at $t \to +\infty$.

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