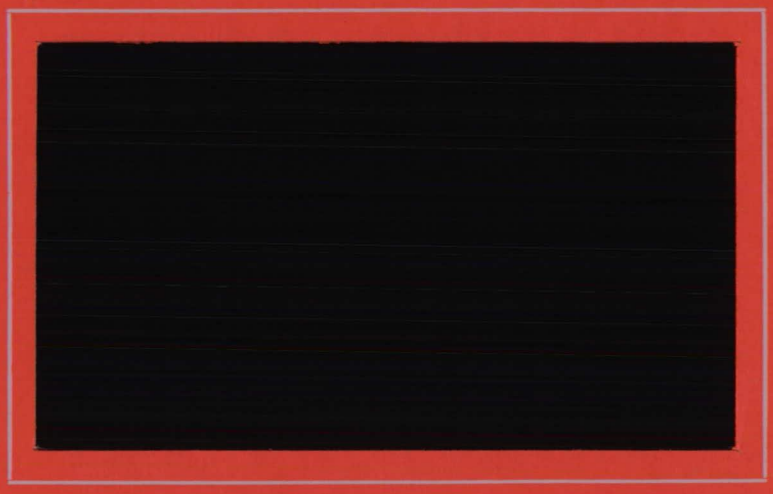

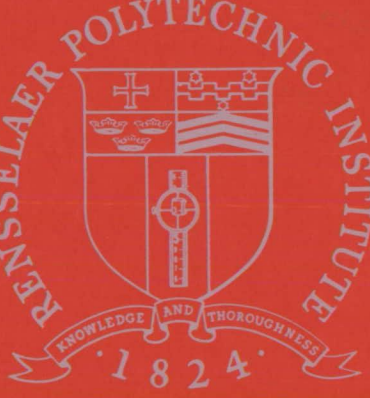


# Rensselaer Polytechnic Institute

Troy, New York

R.P.I. Technical Report - MP 2

# ON-LINE PARAMETER UPDATING OF THE MARTIAN ATMOSPHERE WITH MINIMUM STORAGE

Ronald E. Janosko

NASA Grant NGL 33-018-091

Analysis and Design of a Capsule Landing System
and Surface Vehicle Control System for Mars Exploration

JANUARY 1968

Rensselaer Polytechnic Institute
Troy, N.Y.

# ON-LINE PARAMETER UPDATING OF THE MARTIAN ATMOSPHERE WITH MINIMAL STORAGE

by

Ronald E. Janosko

## ABSTRACT

To relate the density of the Martian atmosphere to the height above the surface for a descending capsule, the use of an adaptive model-reference scheme has been investigated. It was assumed that the atmosphere is able to be structurally represented by the NASA adiabatic model which is good below the tropopause. Allowances have been made for the variation of parameters in this model to cover the whole atmosphere.

After this density equation was transformed to a suitable form, the parameters were identified and the independent variable changed so as to track in the direction of travel. Difference equations were formed based on the error equation applied at three different points. These difference equations were then solved for the change in model parameters so that they continuously follow the actual atmospheric parameters encountered in the Martian atmosphere.

A statistical approach was studied as a solution to the problem of minimizing the effect of noise on the system. The method of modified least squares was chosen as a solution to

this problem.

Computer results simulating the descent from the tropopause height of a design atmosphere to the surface indicate that the scheme does update the parameters properly. The parameters converge rapidly from their design values to those of the assumed actual atmosphere.

TABLE OF CONTENTS

# ON-LINE PARAMETER UPDATING OF THE MARTIAN ATMOSPHERE WITH MINIMAL STORAGE

## I. Introduction

In order to land a vehicle in the neighborhood of a particular point on the Martian surface various atmospheric parameters must be precisely known. These parameters are then used in the equations of motion and in the control equations for the vehicle. A nominal trajectory is designed on earth and the craft should then land in the near vicinity of the desired touchdown point. Unfortunately, there are at present at least ten models of the Martian atmosphere used in engineering design. These models are all of the same form and the only variation between them is in the parameter values used.[1,2]

The nominal trajectory will have to be designed on earth based upon one of these models. It would be desirable to develop a scheme to update the atmospheric parameters used in the equations of motion to correspond exactly to the conditions encountered in the Martian atmosphere when the craft arrives there. This would have to be done by an onboard facility that would be given up-to-date measurements of certain atmospheric and vehicle properties. This facility would then use these measurements to continuously update the atmospheric parameters.

These parameters should be available to the control system which would make use of this information in sizing the control variables.[3]

It is the aim of this report to present a possible scheme which would fulfill the design characteristics mentioned above.

The aerodynamic lift and drag of the capsule are affected by the atmospheric parameters. These are of the form

$$F_D = K_D \rho V^2 \tag{1}$$

and

$$F_L = K_L \rho V^2$$

where $K_D$ and $K_L$ are taken to be constant for fixed angle of attack, $\rho$ is the air density, a function of altitude, and V is the relative speed of the craft with respect to the atmosphere. We now see that two important atmospheric properties that should be corrected are the density model and the estimate of the wind velocity. This report will only be concerned with the density model corrections.

At present there are two density models for Mars being used by NASA. These models are[1]

$$\rho = \rho_{ref}\, e^{-\beta y} \quad \text{for } y > \text{Height of tropopause} \tag{2}$$

and

$$\rho = \rho_o \left(1 + \frac{\Gamma}{T_o} y\right)^{\frac{1}{\gamma - 1}} \quad \text{for } y < \text{Height of tropopause} \tag{3}$$

where

$\rho_o$ = surface density (slugs/ft)

$T_o$ = surface temperature ($^oR$)

$\beta$ = inverse scale height (1/ft)

$\Gamma$ = adiabatic lapse rate ($^oR$/ft)

$\gamma$ = ratio of specific heats

y = height above surface in feet

$$\rho_{ref} = \rho_o \left\{\frac{T_s}{T_o}\right\}^{\frac{1}{\gamma-1}} e^{\left\{\frac{\gamma}{\gamma-1}\ \frac{T_o}{T_s} -1\right\}}$$ (where $T_s$ is the temperature of the stratosphere)

Because the height of the tropopause varies from 56,000 ft. to 63,000 ft. depending upon atmospheric model used and because the lower portion of the atmosphere is where the aerodynamic characteristics become more pronounced we will use only the model given by equation (3). This model can also be used for the density variation in the stratosphere with some accuracy.

This can be seen by observing the density curves given in Figure 1. It is readily seen that the extension of the exponential model into the lower region is rather inaccurate, where as it appears that the adiabatic curve can be extended upward to fit the lower exponential region, i.e., from about 100,000 ft. to the tropopause.

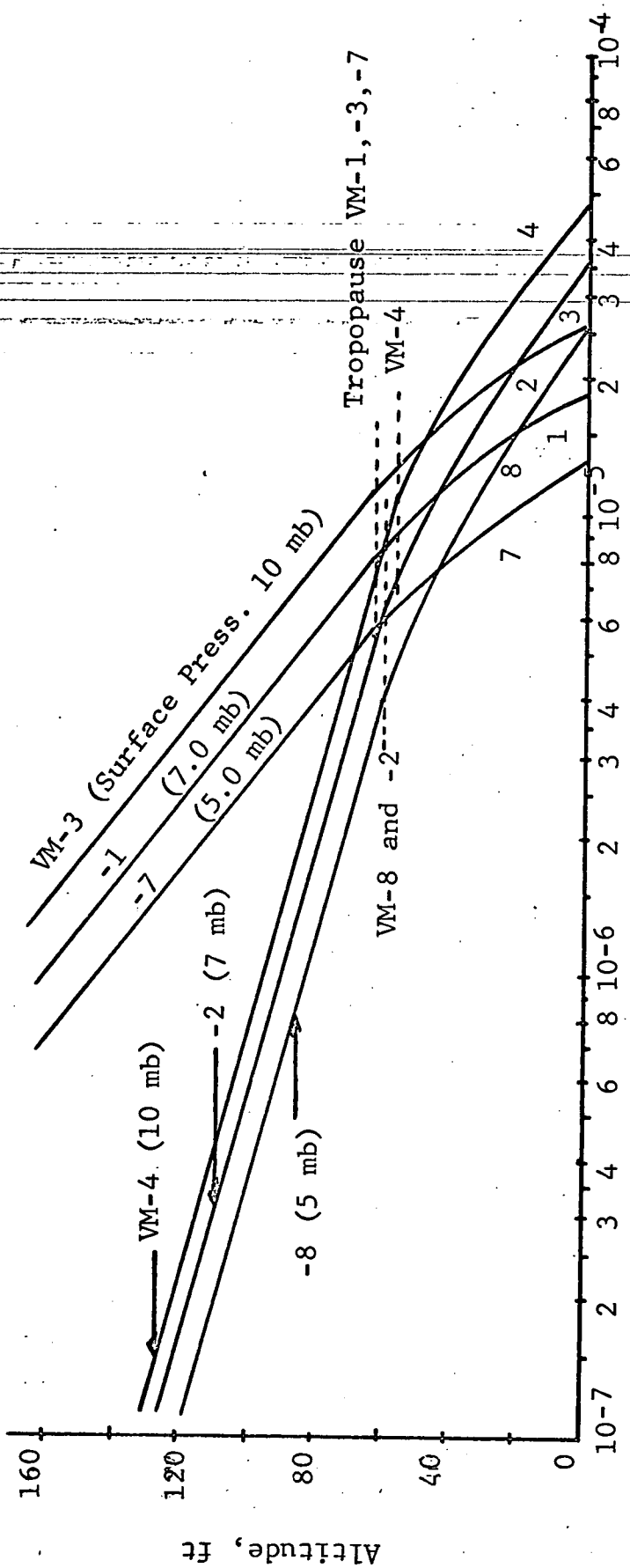
Altitude, ft
160
120
80
40
0
Density, slugs/ft³
10-7
2
3
4
6
8
10-6
2
3
4
6
8
10-5
2
3
4
6
8
10-4
VM-3 (Surface Press. 10 mb)
-1 (7.0 mb)
-7 (5.0 mb)
VM-4 (10 mb)
-2 (7 mb)
-8 (5 mb)
VM-8 and -2
Tropopause VM-1,-3,-7
VM-4
1
2
3
4
7
8

Figure 1

That only one model need be used is seen by noting that both models have a series expansion in powers of y about some reference height $y_o$ in the form

$$\rho = \rho' \sum_{i=0}^{\infty} a_i (y-y_o)^i \tag{4}$$

where the $a_i$ will differ with each model. This leads us to believe that if we use only one model throughout the entire atmosphere we will get a different set of parameters for each portion of the atmosphere. However, if the scheme is developed to update these parameters and if we are concerned only with predicting the density a short distance ahead of the craft we can see that only one model need be used throughout the atmosphere. It should be noted that the parameters of the adiabatic model may vary somewhat in the upper portion of the atmosphere but that this model will still be able to predict the density short distances ahead of the craft.

We can now see that the problem is to develop a scheme to correct the parameters in the equation

$$\rho = \rho_o \left\{1 + \frac{\Gamma}{T_o} y\right\}^{\frac{1}{\gamma-1}} \tag{5}$$

which relates density to altitude. The current NASA models of the Martian atmosphere are such so as to give ranges for each of these parameters as shown below.

$$1.32 \times 10^{-5} \text{ (slugs/ft}^3) < \rho_o < 4.98 \times 10^{-5} \text{ (slugs/ft}^3)$$

$$-3.21 \ (^{o}R/1000 \text{ ft.}) < \Gamma < -2.13 \ (^{o}R/1000 \text{ ft.})$$

$$360 \ ^{o}R < T_o < 495 \ ^{o}R$$

$$1.37 < \gamma < 1.43$$

$$-.0043 \ (1/\text{ft.}) < \frac{\Gamma}{T_o} < -.0089 \ (1/\text{ft.})$$

Presently NASA is using the values below for their design calculations (VM-8 Model).

$$\rho_o = 2.56 \times 10^{-5} \quad \text{slugs/ft}^3$$

$$\Gamma = -2.96 \ ^{o}R/1000 \text{ ft.}$$

$$T_o = 360 \ ^{o}R$$

$$\gamma = 1.37$$

We can therefore assume that it is parameters of this magnitude that will have to be tracked.

## II. Summary

It is shown that the problem of predicting the parameters in the density equation can be solved by a discrete system of equations for the case of no measurement errors. Computer results show that the system will converge rapidly to the desired final values.

A statistical scheme gives convergent results for the case of no measurement errors as expected. If random errors are introduced in the measurement of the scheme also converges. This represents statistically independent measurements.

It was found that the parameter weights used in the modified least squares solution greatly affected the performance of the system. The behavior of the system cam be controlled by proper manipulation of these weights.

## III. Derivation of Modified Density Equation

We can assume that the density variation with height is given by the NASA adiabatic model which is valid below the tropopause. This model is

$$\rho = \rho_o \left\{1 + \frac{\Gamma}{T_o} y\right\}^{\frac{1}{\gamma - 1}} \tag{6}$$

where the parameters are

$\rho_o$ = surface density (slugs/ft$^3$)

$T_o$ = surface temperature ($^oR$)

$\Gamma$ = adiabatic lapse rate ($^oR$/ft)

$\gamma$ = ratio of specific heats

$y$ = height above the surface (ft)

Instead of using y as the independent variable, let us define the quantity X as

$$X = H - y \tag{7}$$

where H is a reference height above the surface of Mars. With y less than H we can now get the density as a function of X as shown below

$$\rho = \rho_o \left\{1 + \frac{\Gamma}{T_o} y \ (H - X)\right\}^{\frac{1}{\gamma - 1}} \tag{8}$$

Introducing a scale factor of the form $(N)^{\frac{1}{\gamma-1}}$ into the first factor and $\frac{1}{(N)^{\frac{1}{\gamma-1}}}$ into the second factor of equation (8) gives us with some algebraic manipulation

$$\rho = \rho_o (N)^{\frac{1}{\gamma-1}} \left\{ \frac{1}{N} + \frac{\Gamma H}{T_o N} \left(1 - \frac{X}{H}\right) \right\}^{\frac{1}{\gamma-1}} \tag{9}$$

Now letting

$$a = \ln \rho_o (N)^{\frac{1}{\gamma-1}}$$

$$b = \frac{1}{\gamma-1}$$

$$c = \frac{\Gamma H}{T_o N}$$

$$x = \frac{X}{H}$$

equation (9) can be written as

$$\rho = e^a \left\{ \frac{1}{N} + c(1 - x) \right\}^b \tag{10}$$

or using natural logarithms as

$$\ln \rho = a + b \ln \left\{ \frac{1}{N} + c(1 - x) \right\} \tag{11}$$

with a,b, and c dimensionless parameters to by tracked.

We notice that we are now tracking the parameters with respect to an increasing variable x, i.e. we are now tracking the parameters in the direction of travel.

## IV. Development of Plant and Model Equations Scheme

### A. Introduction to Scheme

From the statement of the problem we can see that

an adaptive control system that will track the parameters of the density equation is a solution to the problem. It is this type of scheme which will be developed.

Knowing that we wish to adjust parameters of a model suggests that we use a model-reference method of adaptive control. From the diagram of such a system given in Figure 2 we

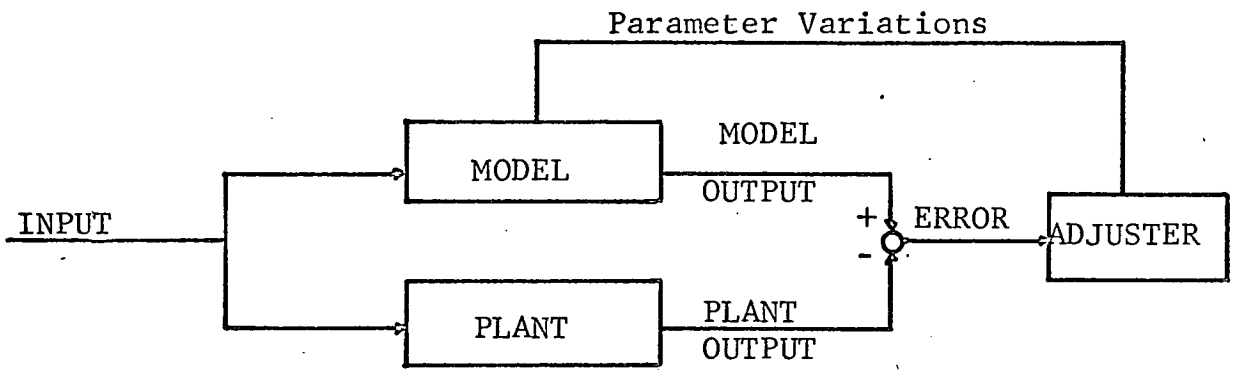


Figure 2

can see that what happens with this method is that an input, in this case altitude, is fed into both the model of the density equation and into the plant or the actual physical process relating the density to the height above the surface. The outputs of both blocks are compared and an error signal is generated. This error signal is then used to correct or to update the parameters in the model of the system until the model parameters approach those of the plant and the error signal is minimized toward zero.

B. Equations for Plant and Model

Using equation (10) as the reference or plant in the model-reference scheme referred to above suggests that we use a model of the density as below

$$P = e^{A} \left(\frac{1}{N} + C(1-x)\right)^{B} \tag{12}$$

where A, B, and C are to be corrected until they approach a, b, and c respectively. We modify the parameters of the model because we have direct information about them, whereas we cannot obtain the parameters of the plant directly.

The above equations (10) and (12) are somewhat cumbersome to work with in the exponential form. They can be made easier to handle by taking the natural log of both giving

$$z = \ell n\, \rho = a + b\ \ell n\ \left(\frac{1}{N} + c(1-x)\right) \text{ Plant} \tag{13}$$

and

$$Z = \ell n\ P = A + B\ \ell n\ \left(\frac{1}{N} + C(1-x)\right) \text{ Model} \tag{14}$$

We now see that the model will be a true representation of the plant if A = a, B = b, and C = c. The error is defined as the difference between Z and z or

$$\varepsilon = Z - z \tag{15}$$

This can be represented by Figure 3.

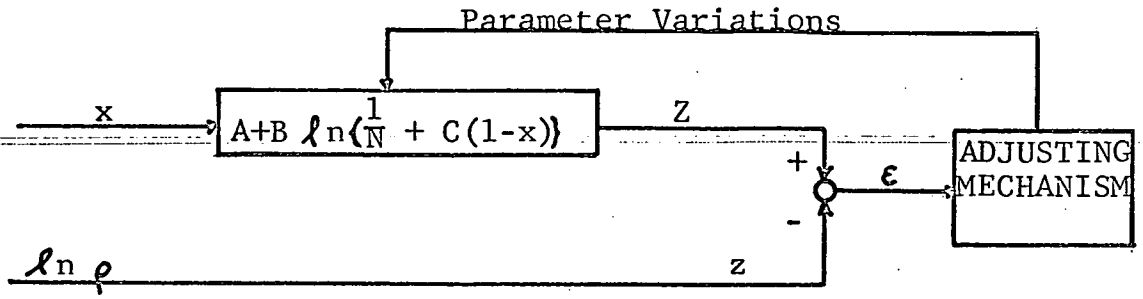


Figure 3

In this system we want to correct A, B, and C based on the error generated. By making this error as small as possible we can use the values A, B, and C from the model to predict the density continuously until touch-down. The adjusting mechanism must adjust A, B, and C to track a, b, and c based on the error between the output of the model and the physical relationship between the density and altitude.

In the physical system onboard, the input x will be fed into a model which gives the structure or form of the physical relationship between density and height above the Mars surface. The output of this model will then be compared with the output of the actual plant or the real atmosphere. That is $\ell n\ P$ will be compared with $\ell n\ \rho$. We cannot obtain directly the relationship between density and height above the surface but we can

get a density for each height we feed into the model.

The two outputs are then compared and an error signal is generated. This signal is then fed into an adjusting mechanism that can track the parameters A, B, and C and the error as functions of x. This mechanism should be able to find discrete changes in A, B, C, and $\varepsilon$ for discrete changes in x and correct A, B, and C in the model. This would give us

$$\begin{aligned} A_{new} &= A_{old} + \Delta A \\ B_{new} &= B_{old} + \Delta B \\ C_{new} &= C_{old} + \Delta C \end{aligned} \tag{16}$$

where $\Delta A$, $\Delta B$, and $\Delta C$ are found from the adjusting mechanism equations.

## V. Discrete Case Adjusting Mechanism Equations

The above stated that the parameters A, B, and C would have to be adjusted until a zero error was generated.

Recall that the error was defined as

$$\varepsilon = \left\{A = B \ln \left(\frac{1}{N} + C(1-x)\right)\right\} - \left\{a + b \ln\left(\frac{1}{N} + c\ (1-x)\right)\right\} \tag{17}$$

or based on the measured density

$$\varepsilon = A + B \ln \left(\frac{1}{N} + C(1-x)\right) - \ln \rho \tag{18}$$

Since we are looking for a zero error operating

state, we want parameters A, B, and C such that

$$0 = A + B \ln \left(\frac{1}{N} + C(1-x)\right) - \ln \rho \tag{19}$$

Equation (19) is clearly a function of A, B, and C only for fixed x. Taking three such x's, i.e. $x_n$, $x_{n+1}$, and $x_{n+2}$, and writing three equations similar to equation (19) we now get

$$0 = A + B \ln \left(\frac{1}{N} + C\,(1-x_n)\right) - \ln \rho_n \tag{20}$$

$$0 = A + B \ln \left(\frac{1}{N} + C\,(1-x_{n+1})\right) - \ln \rho_{n+1} \tag{21}$$

$$0 = A + B \ln \left(\frac{1}{N} + C\,(1-x_{n+2})\right) - \ln \rho_{n+2} \tag{22}$$

where the $\rho_n$, $\rho_{n+1}$, and $\rho_{n+2}$ correspond to the density at $x_n$, $x_{n+1}$, and $x_{n+2}$ respectively.

Equations (20), (21), and (22) are non-linear in A, B, and C. Using a standard technique for the solution of simultaneous non-linear equations we expand these three equations about some initial values, $A_{n-1}$, $B_{n-1}$, and $C_{n-1}$, by a Taylor Series expansion and keep only first order terms.[5,6] Doing this, and noticing that

$$A_{n-1} + B_{n-1} \ln \left(\frac{1}{N} + C_{n-1}(1-x_n)\right) - \ln \rho_n = \varepsilon_n$$

$$A_{n-1} + B_{n-1} \ln \left(\frac{1}{N} + C_{n-1}(1-x_{n+1})\right) - \ln \rho_{n+1} = \varepsilon_{n+1}$$

$$A_{n-1} + B_{n-1} \ln \left(\frac{1}{N} + C_{n+1}(1-x_{n+2})\right) - \ln \rho_{n+1} = \varepsilon_{n+2}$$

where $\varepsilon_n$, $\varepsilon_{n+1}$, and $\varepsilon_{n+2}$ are the errors at $x_n$, $x_{n+1}$, and $x_{n+2}$ due to the differences in the assumed values of A, B, and C and the actual parameters needed for the zero error operating state. Also defining

$\Delta A = A - A_{n-1}$ ($A_{n-1}$= an approximation to A when $x = x_{n-1}$)

$\Delta B = B - B_{n-1}$ ($B_{n-1}$= an approximation to B when $x = x_{n-1}$)

$\Delta C = C - C_{n-1}$ ($C_{n-1}$= an approximation to C when $x = x_{n-1}$)

We can get

$$-\varepsilon_n = \Delta A + \Delta B\left(\frac{1}{N} + C_{n-1}(1-x_n)\right) + \Delta C\left[\frac{B_{n-1}(1-x_n)}{\frac{1}{N} + C_{n-1}(1-x_n)}\right] \quad (23)$$

$$-\varepsilon_{n+1} = \Delta A + \Delta B\left(\frac{1}{N} + C_{n-1}(1-x_{n+1})\right) + \Delta C\left[\frac{B_{n-1}(1-x_{n+1})}{\frac{1}{N} + C_{n-1}(1-x_{n+1})}\right] \quad (24)$$

$$-\varepsilon_{n+2} = \Delta A + \Delta B\left(\frac{1}{N} + C_{n-1}(1-x_{n+2})\right) + \Delta C\left[\frac{B_{n-1}(1-x_{n+2})}{\frac{1}{N} + C_{n-1}(1-x_{n+2})}\right] \quad (25)$$

These equations are now linear in $\Delta A$, $\Delta B$, and $\Delta C$ and can be solved by Cramer's Rule for $\Delta A$, $\Delta B$, and $\Delta C$. Before doing this however, we examine the determinate of this system of equations. We see that the determinate is nearly singular for small differences between the x's.

In the original problem of updating the parameters of the density equation, we wish to have new, corrected values of the parameters early in the descent through the atmosphere. To do this we need to take readings as rapidly as possible which of course calls for a small

difference in the x's. This places the x's in the determinate very close together and the determinate becomes nearly singular and the system of equations cannot be solved.

If we go back to equations (23), (27), and (25) and now take the difference between (23) and (27), we get

$$\Delta B \ln\left[\frac{\frac{1}{N}+C_{n-1}(1-X_{n+1})}{\frac{1}{N}+C_{n-1}(1-X_n)}\right] + \Delta C\, B_{n-1}\left[\frac{1-X_{n+1}}{\frac{1}{N}+C_{n-1}(1-X_{n+1})} - \frac{1-X_n}{\frac{1}{N}+C_{n-1}(1-X_n)}\right] = \varepsilon_n - \varepsilon_{n+1} \quad (26)$$

and working likewise with equations (24) and (25), we get

$$\Delta B \ln\left[\frac{\frac{1}{N}+C_{n-1}(1-X_{n+2})}{\frac{1}{N}+C_{n-1}(1-X_{n-1})}\right] + \Delta C\, B_{n-1}\left[\frac{1-X_{n+2}}{\frac{1}{N}+C_{n-1}(1-X_{n+2})} - \frac{1-X_{n+1}}{\frac{1}{N}+C_{n-1}(1-X_{n+1})}\right] = \varepsilon_{n+1} - \varepsilon_{n+2} \quad (27)$$

It can be shown that if we use equations (23), (26), and (27) we still get a determinate that is very nearly singular for small $\Delta x$. We therefore try taking the difference between equations (26) and (27) getting

$$\Delta B \ln\left[\frac{\left\{\frac{1}{N}+C_{n-1}(1-X_{n+2})\right\}\left\{\frac{1}{N}+C_{n-1}(1-X_n)\right\}}{\left\{\frac{1}{N}+C_{n-1}(1-X_{n+1})\right\}^2}\right] +$$

$$\Delta C\, B_{n-1}\left[\frac{1-X_{n+2}}{\frac{1}{N}+C_{n-1}(1-X_{n+2})} - \frac{2(1-X_{n+1})}{\frac{1}{N}+C_{n-1}(1-X_{n+1})} - \frac{1-X_n}{\frac{1}{N}+C_{n-1}(1-X_n)}\right] = 2\varepsilon_{n+2} - \varepsilon_{n+1} - \varepsilon_n \quad (28)$$

Let us define:

$$x_n \equiv n\Delta x$$

$$x_{n+1} \equiv (n+1)\Delta x$$

$$x_{n+2} \equiv (n+2)\Delta x$$

we can get from equations (23), (26), and (28) the following three equations

$$\Delta A + \Delta B \ln\left[\frac{1}{N} + C_{n-1}(1-n\Delta x)\right] + \Delta C \frac{B_{n-1}(1-n\Delta x)}{\frac{1}{N} + C_{n-1}(1-n\Delta x)} = -\varepsilon_n \qquad (29)$$

$$\Delta B \ln\left[1 - \frac{C_{n-1}\Delta x}{\frac{1}{N} + C_{n-1}(1-n\Delta x)}\right] + \Delta C B_{n-1}\left[\frac{-\frac{\Delta x}{N}}{\left\{\frac{1}{N} + C_{n-1}(1-n\Delta x)\right\}\left\{\frac{1}{N} + C_{n-1}(1-(n+1)\Delta x)\right\}}\right] = \varepsilon_n - \varepsilon_{n+1} \qquad (30)$$

$$\Delta B \ln\left[1 - \frac{C_{n-1}^2 \Delta x^2}{\left\{\frac{1}{N} + C_{n-1}(1-(n+1)\Delta x)\right\}^2}\right] + \Delta C B_{n-1}\left[\frac{-2C_{n-1}\frac{\Delta x^2}{N}}{\left\{\frac{1}{N} + C_{n-1}(1-n\Delta x)\right\}\left\{\frac{1}{N} + C_{n-1}(1-(n+1)\Delta x)\right\}\left\{\frac{1}{N} + C_{n-1}(1-(n+2)\Delta x)\right\}}\right]$$

$$= 2\varepsilon_{n+1} - \varepsilon_n - \varepsilon_{n+2} \qquad (31)$$

With equations (29), (30), and (31) the determinate of the system is no longer singular and thus one can use Cramer's Rule to solve those three equations for $\Delta A$, $\Delta B$, $\Delta C$.

The scheme that has been introduced above to update the atmospheric parameters is deterministic, i.e. it uses three equations to solve for three unknowns. Unfortunately this type of scheme is very sensitive to measurement errors. It is therefore thought desirable to

modify that scheme so as to minimize the effect of measurement errors on it. The most common way to attack a problem of this sort is through a statistical approach. This section concerns itself with the application of statistical principles to the problem of minimizing the effect of measurement errors on our prediction of the model parameters.

## VI. Statistical Adjusting Mechanism Equations

Commonly there are three major approaches to this problem. The method of least squares fitting of data is quite common and straight forward however it's accuracy decreases with decreasing number of data points. In this problem we desire a solution rapidly and hence cannot afford to take many data readings between updatings of the equation parameters.[8]

Another common technique is that of Linear Filtering and Prediction theory[8,9]. This solution is by no means straight forward and becomes even more complex than usual if the equation relating the parameters to the measured quantity is dependent on an independent variable as is in our system of equation. This type of solution would require more computation time between updatings as well as a larger computational facility. Neither of these requirements is advantageous to our problem solution. It however has the advantage that the weighting matrix is self correcting between iterations. Also this is more

adaptable to differential equations.

The third method is the method of maximum likelihood. There are two approaches to the use of this method. One approach utilizes redudant measurements of measured quantities to get estimates of the parameters.[10,11] The other approach uses the measurement of an additional measurable quantity that is a function of the known parameters.[12]

It can be shown that if the measured quantities are assumed to come from normal populations and if further assumed statistically independent this method reduces to the problem of finding a least squares fit for $\Delta A$, $\Delta B$, and $\Delta C$.

Initial testing showed that the maximum likelihood schemes failed once measurement errors were introduced into the system.

After testing other schemes that don't seem to work we go back and take another look at the least-squares method. We notice that the problem is in the need for a large number of data points between updatings. This problem can be eliminated by fitting a least-squares curve to all the available data points at the time of the fit. That is we fit the curve to say 12 data points, then using the next data point fit the curve to 13 data points, etc. until

all data points have been fit adding a data point continually until touchdown. This way we can have an on-line fitting yet still have the large number of data points at each fitting.

A. Derivation of Modified On-line Least Square Equation

In this section we will assume that we have a function of the form

$$y_i = f_i(x_i, a, b, c) \tag{32}$$

were for our case this is

$$\ell n\ \rho_i = a + b\ \ell n\ (\tfrac{1}{N} + C(1-x_i)) \tag{33}$$

The form given in Equation (32) is used to simplify the derivation. In Equation (32) it is desired to determine the parameters a, b, and c from a set of data of the form:

$$Y_i = F_i\ (x_i) \tag{34}$$

That is, for every height we can get a measurement of the actual density, which may or may not include noise. In this derivation we assume that the measurement $Y_i$ includes gaussian noise, hence we now desire the "best" estimate of the parameters a, b and c.

A least squares approach is used to solve the problem applying modifications originally introduced by Levenberg[13] and Hartley[14].

We first define

$$s(a,b,c) = \sum_{i=1}^{n} \omega_i \left[-Y_i + f_i\ (x_i,a,b,c)\right]^2 \quad (35)$$

where the a,b and c are the desired parameters, and $w_i$ are statistical weights which take into account the relative accuracy of the instrument used to measure the $Y_i$ at each $x_i$. Unfortunately the function $f_i\ (x_i,a,b,c)$ is non-linear and hence is difficult to solve for the true a, b and c.

To overcome this difficulty we take a Taylor series expansion of $f(x_i,a,b,c)$ about some initial estimated values A, B and C. Doing this and letting S(a,b,c) be the approximation to s(a,b,c) we get

$$S(a,b,c) = \sum_{i=1}^{n} \omega_i \left[-Y_i + f_i(x_i,a,b,c) + \frac{\partial f_i}{\partial a}\Delta a + \frac{\partial f_i}{\partial b}\Delta b + \frac{\partial f_i}{\partial c}\Delta c\right]^2 \quad (36)$$

where the $\Delta a$, $\Delta b$ and $\Delta c$ represent the difference between the true and estimated value of the parameters. The partial derivatives are evaluated using the estimated parameter values. The upper limit on the summation in Equations (35) and (36) represents the number of data points to be fit. Equation (36) is now linear in $\Delta a$, $\Delta b$ and $\Delta c$ and can be solved for the change in parameters needed to minimize the sum of the squares.

The standard least squares approach attempts to minimize Equation (36) by taking the partial derivations of Equation (36) with respect to the parameters a, b, and c. These derivatives are then set equal to zero and then

solved for the incremental changes $\Delta a$, $\Delta b$ and $\Delta c$ needed to correct the estimated values A, B and C towards the true a, b, and c.

However as often happens these incremental changes are too large in magnitude and "overshoot" the desired solution values thus delaying the convergence of A, B and C to the true values sought. These large changes also affect the linearity assumption used in the Taylor series expansion in the formulation of Equation (36).

We can eliminate this problem of large parameter changes if instead of minimizing Equation (36) we minimize

$$J(a,b,c) = S(a,b,c) + \alpha(\Delta a)^2 + \beta(\Delta b)^2 + \gamma(\Delta c)^2 \tag{37}$$

where the S(a,b,c) has previously been defined. The $\alpha$, $\beta$ and $\gamma$ are positive weights expressing the relative value of minimizing each parameter change as well as indicating the relative importance of minimizing the sum of the squares versus the incremental change in the parameters. Minimizing Equation (37) forces the parameter changes to be small.

We can now use the previously mentioned technique to find the minimum of Equation (37). That is we take the partial derivatives of Equation (37) with respect to each of the parameters a, b, and c and set them equal to zero. Doing this gives us

$$\frac{\partial J}{\partial a} = \frac{\partial S}{\partial a} + 2\alpha\,\Delta a = 0 \tag{38}$$

$$\frac{\partial J}{\partial b} = \frac{\partial S}{\partial b} + 2\beta\,\Delta b = 0 \tag{39}$$

$$\frac{\partial J}{\partial c} = \frac{\partial S}{\partial c} + 2\gamma\,\Delta c = 0 \tag{40}$$

where

$$\frac{\partial S}{\partial a} = 2\sum_{i=1}^{n}\left[-Y_i + f_i(x_i,a,b,c) + \frac{\partial f_i}{\partial a}\Delta a + \frac{\partial f_i}{\partial b}\Delta b + \frac{\partial f_i}{\partial c}\Delta c\right]\frac{\partial f_i}{\partial a} \tag{41}$$

$$\frac{\partial S}{\partial b} = 2\sum_{i=1}^{n}\left[-Y_i + f_i(x_i,a,b,c) + \frac{\partial f_i}{\partial a}\Delta a + \frac{\partial f_i}{\partial b}\Delta b + \frac{\partial f_i}{\partial c}\Delta c\right]\frac{\partial f_i}{\partial b} \tag{42}$$

$$\frac{\partial S}{\partial c} = 2\sum_{i=1}^{n}\left[-Y_i + f_i(x_i,a,b,c) + \frac{\partial f_i}{\partial a}\Delta a + \frac{\partial f_i}{\partial b}\Delta b + \frac{\partial f_i}{\partial c}\Delta c\right]\frac{\partial f_i}{\partial c} \tag{43}$$

with $w_i$ set equal to 1. In Equation (41), (42) and (43) it should be noted that in our case

$$\frac{\partial f_i}{\partial a} = 1 \tag{44}$$

$$\frac{\partial f_i}{\partial b} = \ell n\ \left(\frac{1}{N} + C(1-x_i)\right) \tag{45}$$

$$\frac{\partial f_i}{\partial c} = \frac{B(1-x_i)}{\frac{1}{N} + C(1-x_i)} \tag{46}$$

The above partial derivatives are valuated for each $x_i$ using the (n-1) value of A, B, C for n data points. This is equivalent to updating the partial derivative for each interval.

Substituting equations (41), (42) and (43) into Equation (38), (39), and (34) respectively we can get with some rearrangement a system of equations similiar to what Levenberg refers to as the "damped normal

equation".

These equations are shown below with $y_i$ representing $f_i(x_i, A,B,C)$

$$\left[\sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial a}\right)^2+\alpha\right]\Delta a+\left[\sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial a}\right)\left(\frac{\partial f_i}{\partial b}\right)\right]\Delta b+\left[\sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial a}\right)\left(\frac{\partial f_i}{\partial c}\right)\right]\Delta c=\sum_{i=1}^{n}(Y_i-y_i)\frac{\partial f_i}{\partial a} \quad (47)$$

$$\left[\sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial a}\right)\left(\frac{\partial f_i}{\partial b}\right)\right]\Delta a+\left[\sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial b}\right)^2+\beta\right]\Delta b+\left[\sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial b}\right)\left(\frac{\partial f_i}{\partial c}\right)\right]\Delta c=\sum_{i=1}^{n}(Y_i-y_i)\frac{\partial f_i}{\partial b} \quad (48)$$

$$\left[\sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial a}\right)\left(\frac{\partial f_i}{\partial c}\right)\right]\Delta a+\left[\sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial b}\right)\left(\frac{\partial f_i}{\partial c}\right)\right]\Delta b+\left[\sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial c}\right)^2+\gamma\right]\Delta c=\sum_{i=1}^{n}(Y_i-y_i)\frac{\partial f_i}{\partial c} \quad (49)$$

where it should be noted that all the partial derivatives are evaluated for the estimated parameter values. For our case they are as given in Equations (44), (45) and (46). Mention is also made at this time of the fact that at each data point they are reevaluated using the latest estimates of the true parameter values, i.e. the values of A, B and C found at the (n-1)th interval are used to find the nth value of the parameters. In this way the adjusting mechanism equations are adaptive in that they use the latest parameter estimates in obtaining the new values for the parameter changes. This set of equations requires storage of only the $\ell n\ \rho$ in an ordered sequence. All other values are recalculated and take minimum storage requirements in the onboard computer.

Equations (47), (48) and (49) can be written in a

more compact form if we let

$$a_1 = a \qquad \alpha_1 = \alpha$$

$$a_2 = b \qquad \alpha_2 = \beta$$

$$a_3 = c \qquad \alpha_3 = \gamma$$

and then define an element of a matrix $\Psi$ by the following.

$$\{\Psi_{jk}\} = \begin{cases} \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial a_j}\right)\left(\frac{\partial f_i}{\partial a_k}\right) & \text{(for } j \neq k\text{)} \\ \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial a_j}\right)^2 + \alpha_j & \text{(for } j = k\text{)} \end{cases}$$

with the elements of column vectors $\underline{\Delta a}$ and $\underline{\Theta}$ given by

$$\{\Delta a_j\} = \Delta a_j \qquad j = 1,2,3$$

$$\{\Theta_j\} = \sum_{i=1}^{n} (Y_i - y_i)\frac{\partial f_i}{\partial a_j} \qquad j = 1,2,3$$

Using the above definitions Equations (47), (48) and (49) can be combined and written as

$$\Psi \underline{\Delta a} = \underline{\Theta} \tag{50}$$

Equation (50) can now be solved by straightforward matrix algebra once the $\alpha$, $\beta$ and $\gamma$ are known. That is to say that the parameter changes are a function of the weights and also because S(a,b,c) is a function of the parameter changes it is an implicit function of the weights.

The determination of the optimal weights is left as a task for future research. Examples showing the effect of varying the weights are discussed in the section on

numerical results.

B. Improvement of Parameter Changes by Weighted Step Sizes

Once the step size is determined for all of the parameters it is possible that the new estimates of the parameter values will not minimize Equation (35). It can be shown[14] that the sum of the squares is a function of the step size taken to update the parameters, that is to say if

$$A_i = A_{i-1} + \nu \Delta a_i \tag{51}$$

where $A_i$ is the updated parameter value, $A_{i-1}$ is the previous value $\Delta a_i$ is the parameter step obtained from the solution to Equation (50), and $\nu$ is a number from 0 to 1, then the sum of the squares is a function of $\nu$.

It is now clear that the $\nu$ should be chosen so as to minimize the sum of the squares for each fixed set of parameter changes. One way of obtaining this $\nu$ is to choose three trial values of $\nu$ from 0 to 1, solve Equation (35) for the sum of the squares at these three points, fit a parabula to these three points and then solve for the $\nu$ which minimizes the parabolic function[14] as shown in Figure 4.

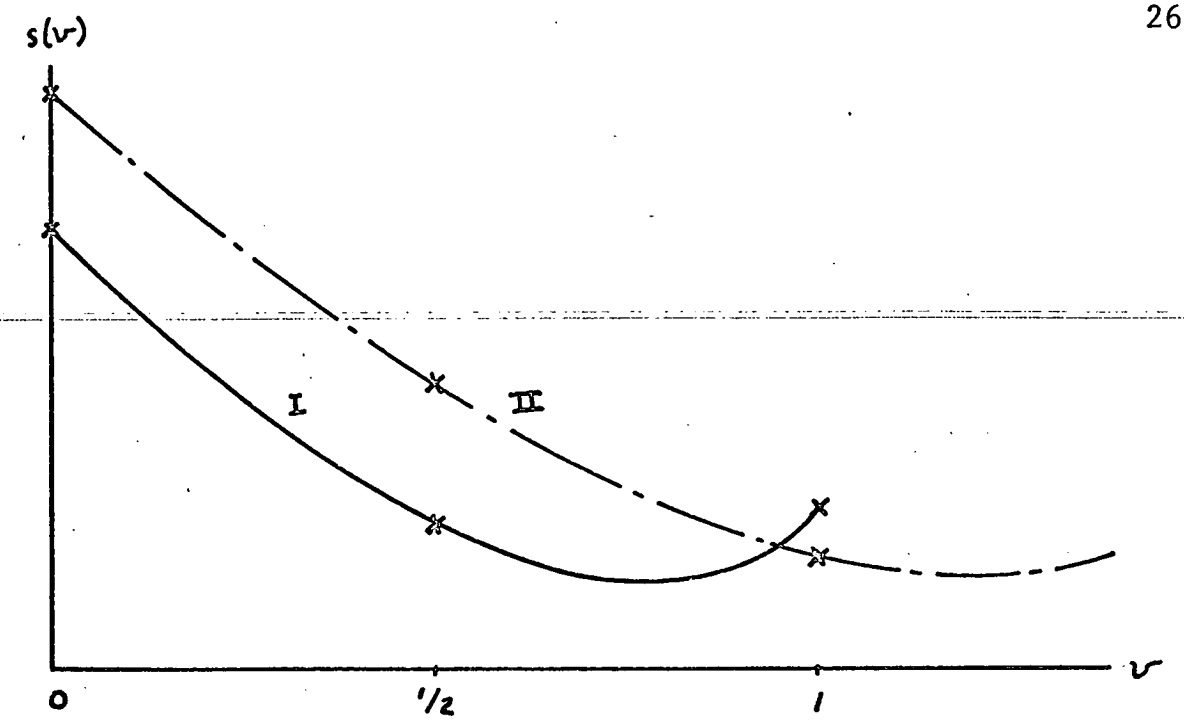


Fig. 4

If the three points are chosen as shown in the above Figure, i.e. 0, 1/2 and 1, and the sum of the squares at each respectively called s(o), s(1/2) and s(1), then it is easily found through ordinary calculus that

$$v_{min} = \frac{1}{2} + \frac{1}{4}\left[\frac{s(0) - s(1)}{s(0) - 2s(1/2) + s(1)}\right] \tag{52}$$

with the restraint that $0 \leq v_{min} \leq 1$.

Therefore the ideal updated parameters are obtained from

$$A_i = A_{i-1} + v_{min} \Delta a_i \tag{53}$$

where the $\Delta a_i$ is obtained from Equation (50). The values obtained from Equation (53) limit the overshoot of the updated parameters while minimizing the standard sum of the squares.

Curve I in Figure 4 shows the minimum value occurring in the 0 to 1 interval and hence no problem is discovered in applying Equation (53). On the other hand, Curve II has a minimum outside the 0 to 1 interval. In this case the value of $v$ is set equal to 1. If the minimizing $v$ had been negative, $v_{min}$ would have been set equal to zero.

With these special cases taken care of Equation (53) can now be applied for all $v$.

## VII. Numerical Testing of Adjusting Mechanism Equations

Before developing this adjusting mechanism it would be best to mathematically investigate this scheme to see if it will do what we claim of it. To perform this test we will have to solve the adjusting mechanism equations. They can be solved numerically on a digital computer where the a, b, and c will be some assumed actual atmospheric parameters and the initial values for A, B and C will be the design parameters and used by NASA. This program should also be able to be used to test the stability of the system as a function of the scale factor N to be used in the solution.

If this scheme is able to adjust the values of A, B and C from the initial values to the values of a, b, and c we will say that the scheme works.

Presently, NASA is using a VM-8 atmospheric model (corresponding to a surface pressure of 5mb.) for reference trajectory computations. We will therefore use the values of the VM-8 atmosphere to calculate the initial values of the model parameters. The values of the atmospheric properties used for this model are[4]

$$\rho_o = 2.56 \times 10^{-5} \text{ slugs/ft}^3$$
$$\Gamma = -2.96 \ ^oR/1000 \text{ ft.}$$
$$T_o = 360 \ ^oR$$
$$\gamma = 1.37$$

We will assume for this example that when the vehicle reaches Mars it encounters an actual atmosphere which is represented by the VM-4 atmospheric properties (corresponding to a 10 mb. surface pressure) given below[4]

$$\rho_o = 4.98 \times 10^{-5} \text{ slugs/ft}$$
$$\Gamma = -3.21 \ ^oR/1000 \text{ ft.}$$
$$T_o = 360 \ ^oR$$
$$\gamma = 1.43$$

A reference height of 61,000 ft. was chosen because this is the height of the tropopause height of the design atmosphere.

Using these values, and a scale factor of 1 , we get the initial model parameters $A_o$, $B_o$, and $C_o$ as

$$A_o = -10.572918$$
$$B_o = 2.702702$$

$C_o = -0.50155$

and the actual atmospheric parameters are then

$a = -9.907495$

$b = 2.325581$

$c = -0.543916$

We wish to allow for one hundred readings between the reference height and the Mars surface. Therefore we will use a dimensional step-size of 610 ft. Using this value as well as the scale factor of 1 and the 61,000 ft. reference height we get that the scaled $\Delta x$ will be equal to .01 as seen directly from the derivation of the density model.

A. Discrete System

Instead of solving equations (29), (30), and (31), by Cramer's Rule, we apply it first to equations (30), and (31). We then solve for $\Delta B$ and $\Delta C$ first and then substitute these values into equation (29) to solve for $\Delta A$. This is a somewhat easier operation and requires less computational time.

Also because

$$\ell n(1+\delta) \simeq \delta$$

for small $\delta$ we can write

$$\ell n\left[1 - \frac{C_{n-1}\Delta x}{\frac{1}{N} + C_{n-1}(1-n\Delta x)}\right] \simeq - \frac{C_{n-1}\Delta x}{\frac{1}{N} + C_{n-1}(1-n\Delta x)} \qquad (54)$$

and

$$\ln\left[1-\frac{\Delta x^2 C_{n-1}^2}{\left\{\frac{1}{N}+C_{n-1}(1-(n+1)\Delta x)\right\}^2}\right] \simeq \frac{\Delta x^2 C_{n-1}^2}{\left\{\frac{1}{N}+C_{n-1}(1-(n+1)\Delta x)\right\}^2} \tag{55}$$

Now using equations (53) and (54) in equations (30) and (31) and at the same time dividing equation (30) by $\Delta x$ and (31) by $\Delta x^2$ we can get

$$\Delta B\left[\frac{-C_{n-1}}{\frac{1}{N}+C_{n-1}(1-n\Delta x)}\right]+\Delta C\left[\frac{-\frac{B_{n-1}}{N}}{\left\{\frac{1}{N}+C_{n-1}(1-n\Delta x)\right\}\left\{\frac{1}{N}+C_{n-1}(1-(n+1)\Delta x)\right\}}\right] = \left[\frac{\varepsilon_n-\varepsilon_{n+1}}{\Delta x}\right] \tag{56}$$

$$\Delta B\left[\frac{-C_{n-1}^2}{\left\{\frac{1}{N}+C_{n-1}(1-(n+1)\Delta x)\right\}^2}\right]+\Delta C\ \frac{-\frac{2B_{n-1}C_{n-1}}{N}}{\left\{\frac{1}{N}+C_{n-1}(1-n\Delta x)\right\}\left\{\frac{1}{N}+C_{n-1}(1-(n+1)\Delta x)\right\}\left\{\frac{1}{N}+C_{n-1}(1-(n+2)\Delta x)\right\}} = \left[\frac{2\varepsilon_{n+1}-\varepsilon_n-\varepsilon_{n+2}}{\Delta x^2}\right] \tag{57}$$

for which the determinate is clearly not singular.

It is noted that the right hand side of Equations (56) and (57) represent the first and second derivatives of $\varepsilon$ with respect to x. Thus the problem mentioned earlier that if $\varepsilon$ is noisy the scheme fails because taking the derivative of a noisy variably magnifies the noise and leads to unstable situations.

The $\Delta B$ and $\Delta C$ obtained from equations (56) and (57) will not be the actual $\Delta B$ and $\Delta C$ needed to correct the $B_{n-1}$ and $C_{n-1}$ to the values which are the solutions of equations (20), (21), and (22). They will only be approximations to the actual $\Delta B$ and $\Delta C$ needed. The values obtained will have to be used to generate new

approximations to the solution and a repetitive scheme will be used to solve for the A, B, and C needed to operate with a zero error. The scheme will be repeated, each time increasing the index n by one. The approximations for $\Delta B$ and $\Delta C$ will be used to get new approximations to $\Delta A$ from equation (20) which can be written in the form

$$\Delta A = -\varepsilon_n - \Delta B \ln\left[\frac{1}{N} + C_{n-1}(1-n\Delta x)\right] - \Delta C\left[\frac{B_{n-1}(1-n\Delta x)}{\frac{1}{N} + C_{n-1}(1-n\Delta x)}\right] \quad (58)$$

New approximations to A, B, and C will be found each time from equation (16) given in a more accurate form as

$$\begin{aligned} A_n &= A_{n-1} + \Delta A \\ B_n &= B_{n-1} + \Delta B \\ C_n &= C_{n-1} + \Delta C \end{aligned} \quad (59)$$

It is now possible to compute the values for $\Delta A$, $\Delta B$, and $\Delta C$ from equations (56), (57), and (58). New values for the design parameters A, B, and C are then determined by equation (59).

The results obtained with this scheme are plotted in Figure 5.

B. Statistical Case

Testing of the statistical equations used the same scale factor, reference height, and step-size as the

discrete case. The numerical results of the last scheme discussed in the statistical section are plotted in Figures 6,7,8 and 9.

Noise is introduced into the system by a Monte Carlo simulation of errors on the measured quantities. Random normal deviates from a standard normal population are read in as data. These are then corrected to correspond to random deviates of the assumed populations of each measured quantity. (See Appendix A)

Errors are independently introduced into each measurement of $\rho$. This corresponds to a statistically independent set of measurements.

## VIII. Results and Discussion of Results

### A. Discrete Case

The $A_o$, $B_o$, and $C_o$ in Figure 5 are the initial design values given previously. The a, b, and c in this figure are the values corresponding to the parameters of the actual atmosphere. From the plots of A, B, and C versus the index n, it is seen that the values of the model are essentially equal to those of the plant after three iterations. Numerical results showed that the model parameters are exactly equal to the plant parameters after six iterations. The error at the next point caused by

the difference between the plant and the updated model parameters was also calculated. It was found that $|\varepsilon_{MAX}| = 1.06 \times 10^{-14}$.

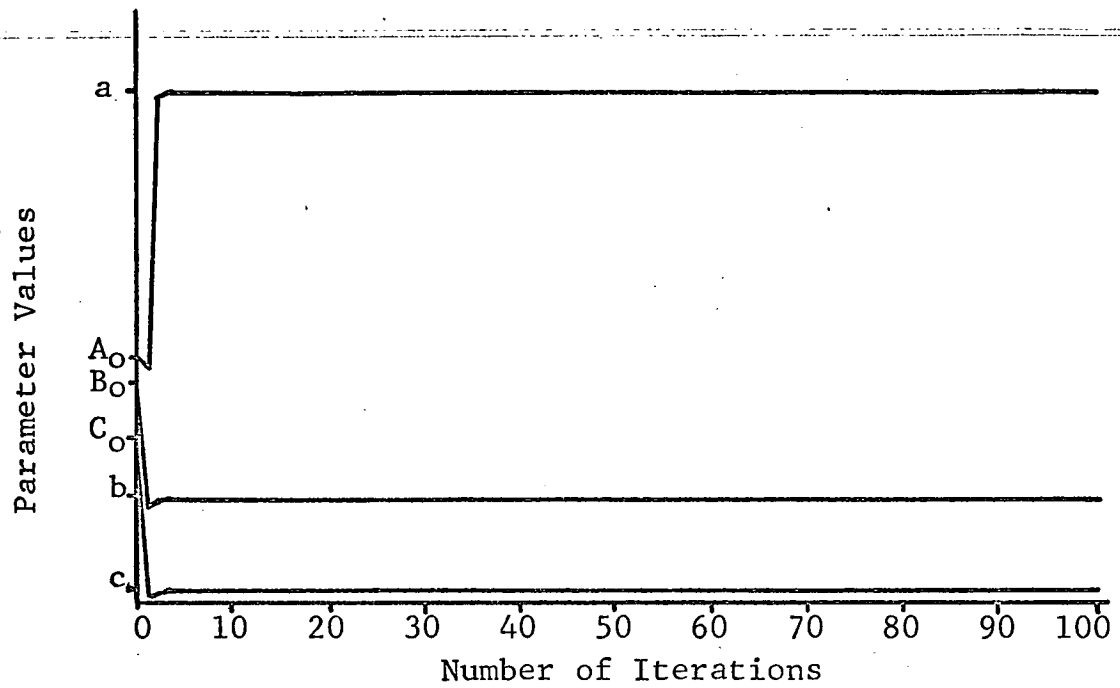


Fig. 5

The plots show that the scheme works very well. It can be seen that the parameters of the model converge rapidly to the desired true values and then stay at these values.

## VIII.B. Statistical Case

Results plotted in Figure 6 show that the scheme does converge to the desired final values rapidly if no measurement errors are present. Introduction of these errors in the manner previously mentioned causes the system to converge slightly slower as seen in that Figure.

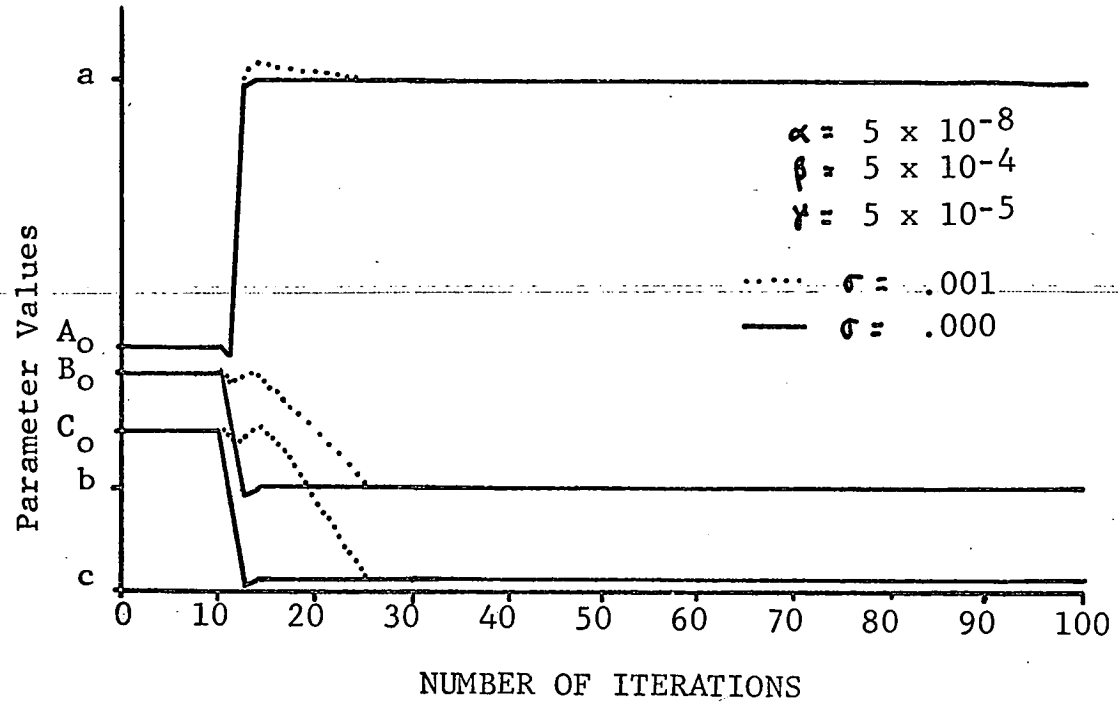


Fig. 6

Another result of this research is the conclusion that the relative value of minimizing the sum of the squares versus minimizing the parameter changes is not as suggested by Levenberg. This is seen in Figure 7.

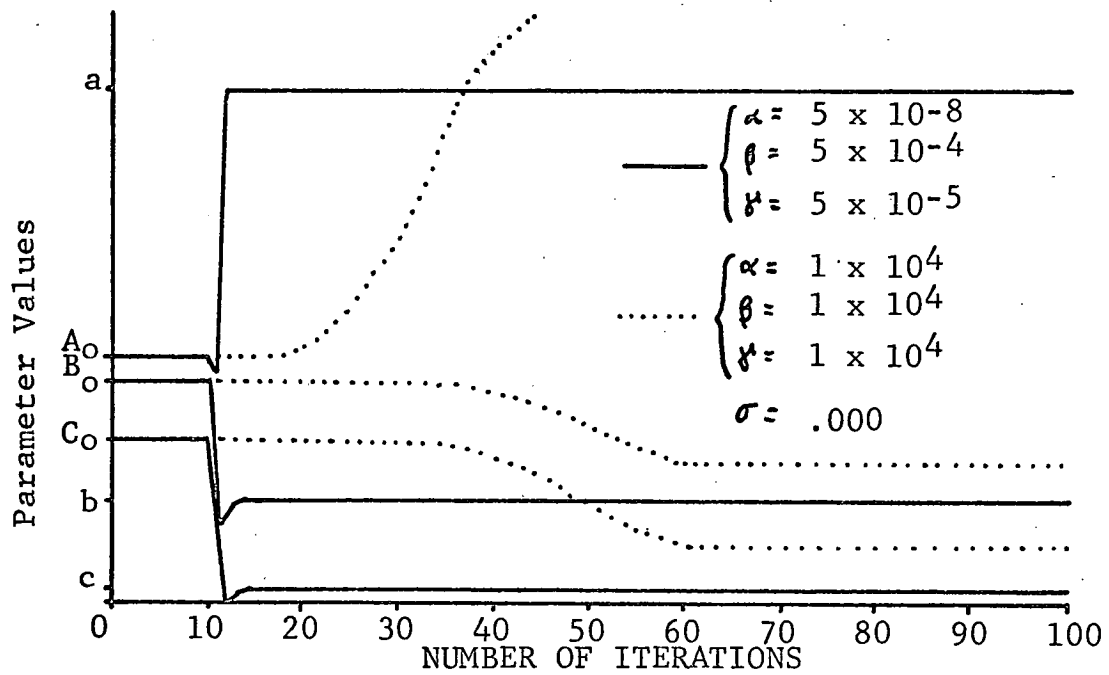


Fig. 7

Levenberg assumes large weights on the parameter changes, whereas the author assumes small weights. Small weights indicate that the prime concern is the minimization of the sum of the squares with only a minor constraint on the size of the parameter changes.

Figure 8 shows the advantage of not using equal parameter weights as suggested by Levenberg. By wisely choosing the weights we can in effect dampen the response of each parameter separately.

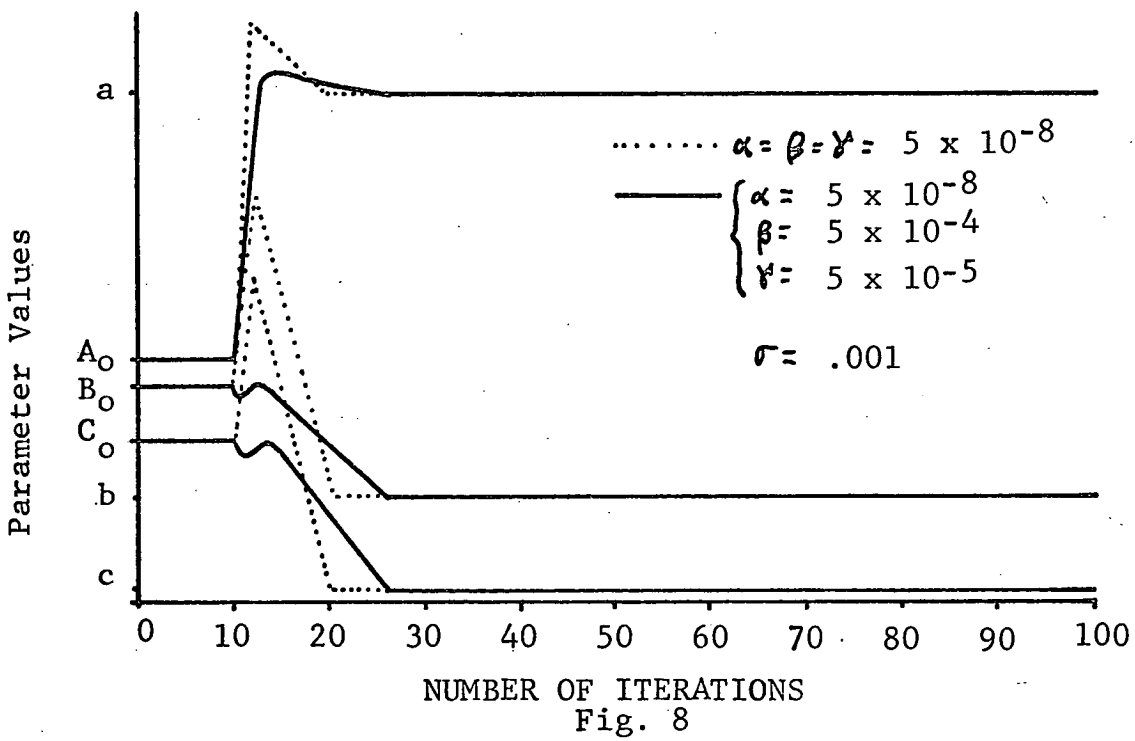

Fig. 8

The final result in this section is the observation that increasing the standard deviation of the sampled density causes sharper peaks in the response of the parameter changes as shown in Figure 9.

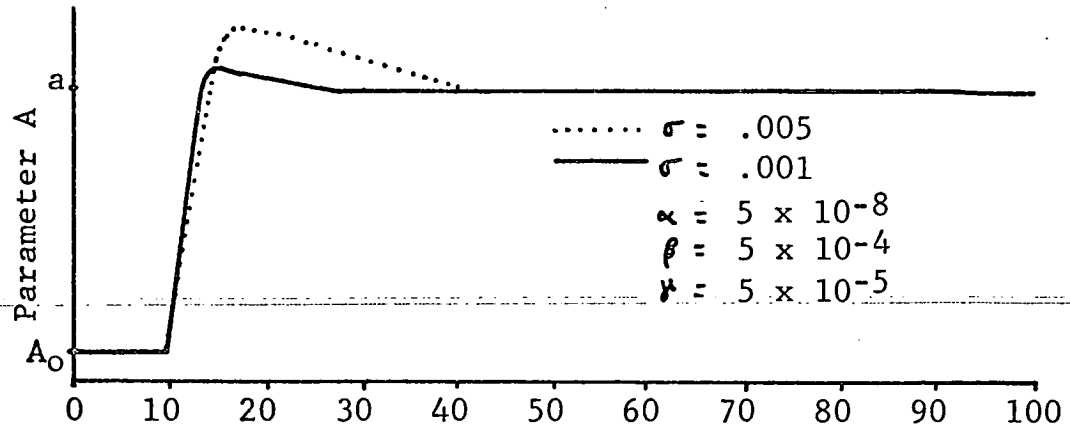


Fig. 9

This indicates that we can eliminate this problem by proper choice of weights $\alpha$, $\beta$, and $\gamma$. We would again dampen the response.

This result indicates that it might be possible to develop a scheme to continuously update the parameter weights to give optimal response of the parameters.

References

1. Levine, G., Evens, D. and Stevens, V. "NASA Models of the Mars Atmosphere for Entry Vehicle Design", NASA TN-D-2525, 1964.

2. Brooks, Edward M. "Comprehensive Summary of Available Knowledge of the Meteorology of Mars and Venus", NASA CR-786, 1967.

3. Shen, C.N., and Cefola, P.J. "Adaptive Control for Mars Entry based on Sensitivity Analysis", AIAA 68-835, presented at AIAA Guidance, Control and Flight Dynamics Conference, Pasadena, Calif., August 1968.

4. Stone, I. "Atmospheric Data to Alter Voyager Design", Aviation Week & Space Technology, Nov. 22, 1965.

5. Conte, S.D. Elementary Numerical Analysis, New York: McGraw-Hill Book Company, 1965.

6. Jennings, Walter. First Course in Numerical Methods, New York: The Macmillian Company, 1964.

7. Miller, I. and Freund, J. Probability and Statistics for Engineers, New Jersey: Prentice-Hall, Inc., 1965.

8. Ho, Yu Chi. "On the Stochastic Approximation Method and Optimal Filtering Theory", Journal of Mathematical Analysis and Applications, No. 6, 1962.

9. Kalman, R.E. and Bucy, R.S. "New Results in Linear Filtering and Prediction Theory", Transactions of the ASME, Journal of Basic Engineering, March 1961.

10. Hald, A. Statistical Theory with Engineering Applications, New York: John Wiley & Sons, Inc., 1952.

11. Cramer, Harold. Mathematical Methods of Statistics, Princeton: Princeton University press, 1945.

12. Battin, Richard H. Astronautical Guidance, New York: McGraw-Hill Book Company, 1964.

13. Levenberg, K., "A Method for the Solution of Certain Non-Linear Problems in Least Squares", Qu. Appl. Mathe., 2, 164-8 (1944).

14. Hartley, H.O. "The Modified Gauss-Newton Method for the Fitting of Non-linear Regression Functions by Least Squares", Technometrics, Vol. 3, No. 2, May 1961.

# APPENDIX A

## Notes on the Monte Carlo Simulation of Measurement Errors.[7]

The probability density function of a gausion population is

$$p(\tilde{\varepsilon}) = \frac{1}{\sqrt{2\pi}\,\sigma_{\tilde{\varepsilon}}} \int_{-\infty}^{\tilde{\varepsilon}} e^{-(z-\mu)^2/2\sigma_{\tilde{\varepsilon}}^2}\, dz \qquad (1A)$$

where $\mu$ is the mean of the population, z a dummy variable of integration, $\sigma_{\tilde{\varepsilon}}$ the population standard deviation, and $\tilde{\varepsilon}$ is the random variable.

For a normal population this reduces to

$$p(\hat{\varepsilon}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\hat{\varepsilon}} e^{-z^2/2}\, dz \qquad (2A)$$

because for a normal population $\mu = 0$, and the standard deviation is defined to be 1. In Equation (2A) $\hat{\varepsilon}$ is the normalized Random variable.

In a Monte Carlo simulation random numbers from 0 to 1 are set equal to $p(\hat{\varepsilon})$ and then equation (2A) is solved for the $\hat{\varepsilon}$ for which the following holds

$$RN = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\hat{\varepsilon}} e^{-z^2/2}\, dz \qquad (3A)$$

where RN designates the random number. The $\hat{\varepsilon}$ thus obtained are called random normal deviates. These numbers then can be used to simulate the measurement of a gaussion population of known mean and standard deviation.

The random variable for a normalized population can

be written as

$$\hat{\varepsilon} = \frac{\tilde{\varepsilon} - \mu}{\sigma_{\tilde{\varepsilon}}} \tag{4A}$$

where $\hat{\varepsilon}$, $\tilde{\varepsilon}$, $\mu$, and $\sigma_{\tilde{\varepsilon}}$ are as previously defined. From Equation (4A) we can get that the random deviate for the true population is

$$\tilde{\varepsilon} = \sigma_{\tilde{\varepsilon}} \hat{\varepsilon} + \mu \tag{5A}$$

Now if the standard deviation is normalized, i.e., given as a percentage. It can be dimensionalized by multiplying that dimensionless number by the population mean giving

$$\sigma_{\tilde{\varepsilon}} = \sigma \mu \tag{6A}$$

where $\sigma$ is a normalized standard deviation.

Therefore using (6A) in (5A) we can get

$$\tilde{\varepsilon} = (\sigma + 1)\mu \tag{7A}$$

where for our example for each data point

$\mu_i = \ln \rho_i$ = true density

and $\tilde{\varepsilon}_i = \ln \tilde{\rho}_i$ = measured density.

It should be noted that if the measurement is noiseless, i.e. $\sigma = 0$ then the measured value equals the true value.