NASA TECHNICAL REPORT

NASA TR R-310

c. 1

NASA TR R-310

A SEMIEMPIRICAL COLLISION MODEL FOR PLASMAS

by Willard E. Meador

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1969

A SEMIEMPIRICAL COLLISION MODEL FOR PLASMAS

By Willard E. Meador

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

CONTENTS

SUMMARY 1

INTRODUCTION 1

SYMBOLS 3

THE COLLISION MODEL 7

Derivation of Model 7

Determination of K and ξ 9

A NEW COLLISION TIME 10

THE MAGNETIC PROBLEM 14

TEMPERATURE EFFECTS 17

Thermal Conduction 17

Thermal Diffusion 18

ALTERNATE DETERMINATION OF ξ 19

CONCLUDING REMARKS 20

APPENDIX A – COLLISION INTEGRALS 21

APPENDIX B – THE KROOK COLLISION MODEL 23

APPENDIX C – CHAPMAN-ENSKOG SOLUTIONS 26

APPENDIX D – THE ONSAGER MAXIMIZATION PRINCIPLE 29

APPENDIX E – THE MEAN-FREE-PATH MODEL 30

REFERENCES 32

A SEMIEMPIRICAL COLLISION MODEL FOR PLASMAS

By Willard E. Meador
Langley Research Center

SUMMARY

A collision model similar in many respects to that of a Lorentz gas is developed for plasmas. Two assumptions are made: (1) the totality of interparticle force laws (including electron-electron) can be replaced with a single parameter determined by the unique ratio of generalized electric field to temperature gradient that can exist in a bounded, nonmagnetic, and currentless plasma of given composition; (2) the electron dynamics governed by such an average potential are Lorentz-like in the sense that the effective electron collision partners behave as heavy scattering centers. With the collision model thus specified and held fixed, the first-order kinetic equation is solved exactly for a succession of more complex plasma conditions; the corresponding distribution functions are used to predict such diverse properties as entropy, entropy and diffusion relaxation times, electromagnetic tensor conductivities, thermal conductivity, and thermal diffusion. Since the results are generally comparable to the third Sonine approximation and nearly always better than the second approximation (thermal conductivity excepted) for a fully ionized gas, the method apparently provides an adequate scheme for utilizing experimental data when microscopic collision parameters are scarcely known (e.g., electron—neutral-atom forces) and for directly extending previous calculations to more complex conditions. Interesting sidelights include the discovery that the elementary mean-free-path formulation of the Hall conductivity is left intact for small magnetic fields if the familiar collision time is replaced by a new definition associated with entropy relaxation.

INTRODUCTION

In the past few years, many authors have discussed statistical models as approximations to Boltzmann collision integrals for solving problems in rarefied gas dynamics (e.g., ref. 1). The philosophy is summarized briefly as follows: Whereas the exact collision integrals treat in detail the geometry of every binary encounter, the essential features can be described almost as well by first taking a statistical average over these encounters. Aside from providing validity tests for various approximate distribution functions, the models could prove useful in their own right for such purposes as adapting kinetic theory to experimental data in lieu of unknown microscopic parameters, directly extending previous calculations to more complex gas conditions (e.g., the extension of nonmagnetic results to electromagnetic fields), and solving problems that do not respond to classical perturbation methods. In addition, the flux expressions derived from simple models are often far more concise than those from polynomial expansions and are therefore easier to interpret.

Although some of the concepts can be generalized to neutral gases, the present investigation is concerned entirely with plasmas and particularly with those processes controlled by electrons. The development of a simple collision model and therefrom a reliable semiempirical electron distribution function is motivated by the following considerations:

(1) Electron-electron encounters prevent closed-form solutions of the Boltzmann equation and thus obscure important characteristics that might be better understood in a more empirical scheme.

(2) Most of the electron—neutral-atom interaction potentials are scarcely known.

However, one must recognize that the experimental data readily accessible from plasmas are very limited and, consequently, questions arise as to the type of data, the amount of data necessary for given purposes, and the theoretical framework in which the data are best represented. Information theory (ref. 1), for example, maximizes the observer's uncertainty about all plasma properties except those actually measured, but the entropy and distribution function, and so forth, so determined bear little resemblance to the true quantities if the experimental constraints are inadequate. Besides the usual conservation requirements, prior knowledge is ultimately assumed in reference 1 of all second-order velocity moments (including off-diagonal elements) in a multicomponent mixture and also the interparticle force laws for the specification, though somewhat arbitrarily, of certain other parameters. Such methods are of little value when a purpose of the semiempirical procedure is to avoid unknown potentials.

The problem then is to enhance the adequacy of a small number of practical constraints, at least one of which supplants theoretical cross sections, by somehow introducing a bias in the observer's attitude toward things not measured. Some indication of the proper bias can be deduced from the Boltzmann equation if a sufficiently simple collision model is adopted. The intent of the present research is to develop such a model based on the following assumptions: (1) the totality of interparticle force laws (including electron-electron) can be replaced with a single parameter determined by the unique ratio of generalized electric field to temperature gradient that can exist in a bounded, nonmagnetic, and currentless plasma of given composition; (2) the electron dynamics governed by this average potential are Lorentz-like in the sense that the effective electron collision partners behave as heavy scattering centers. With the collision model thus specified and held fixed, the first-order Chapman-Enskog equation is solved exactly for a succession of more complex plasma conditions; the corresponding distribution functions are used to predict such diverse quantities as entropy, entropy and diffusion relaxation times, electromagnetic tensor conductivities, thermal conductivity, and thermal diffusion.

Since the results are generally comparable to the third Sonine approximation and nearly always better than the second approximation (thermal conductivity excepted) for a fully ionized gas, the method apparently provides an adequate scheme for utilizing experimental data in multicomponent partially ionized plasmas and for directly extending previous calculations to more complex conditions. Interesting sidelights include the discovery that the elementary mean-free-path formulation of the Hall conductivity is left intact for small magnetic fields if the familiar collision time is replaced by a new definition associated with entropy relaxation.

For simplicity, it is assumed herein that heavy particles (e.g., ions and neutral atoms) are infinitely massive and are at rest relative to the laboratory.

SYMBOLS

a	parameter such that $a = 0$ refers to a Lorentz fully ionized plasma and $a = 2^{1/2}$ refers to a real fully ionized plasma
b	impact parameter
$\vec{B}$	magnetic field
$\hat{B}$	unit vector in direction of magnetic field
$\vec{c}$	particle velocity
D_1,D_2,D_3	functions defined in equation (C2)
e	negative of electron charge
$\vec{E}$	electric field
f	distribution function
$f^{(o)}$	Maxwellian distribution function
$\vec{g}$	function defined in equation (5)

$\vec{h}$	enthalpy flux
$\hat{i},\hat{j},\hat{k}$	unit vector in x-, y-, and z-direction, respectively, of Cartesian coordinate system
I_{ij}	integral defined by equation (C8)
$\vec{j}$	electron current density
k	Boltzmann's constant
K	parameter in the collision model defined by equation (C7)
m	particle mass
n	particle number density
p_e	electron partial pressure
$\vec{q}$	kinetic energy flux
R	combination $R_{13}^{-2}R_{04}R_{22}$ of R_{ij} integrals
R_{ij}	limit of I_{ij} integral as ω approaches zero
s	entropy density
$s^{(o)}$	equilibrium entropy density
$\dot{s}_c$	collisional production rate of entropy density
t	time
T_e	electron temperature
T_{ej}	measure of average energy with which electrons emerge from encounters with effective collision partners j

$\vec{u}_{ej}$ average velocity with which electrons emerge from encounters with effective collision partners j

$\vec{v}_e$ electron diffusion velocity

x integration variable

$\vec{\beta}$ reduced electron diffusion velocity

$\vec{\beta}_{ej}$ reduced average velocity with which electrons emerge from encounters with effective collision partners j

$\vec{\gamma}$ reduced electron particle velocity

γ_E Spitzer-Härm reduction factor for scalar electrical conductivity

γ_T Spitzer-Härm reduction factor for thermal diffusion

δ variational operator

δ_E Spitzer-Härm reduction factor for kinetic energy flux due to electric field and/or pressure gradient

δ_T Spitzer-Härm reduction factor for thermal conductivity

ϵ azimuthal angle in collision dynamics

$\vec{\epsilon}$ effective electron driving force (generalized electric field) defined by equation (C4)

$\vec{\epsilon}_{||}$ component of $\vec{\epsilon}$ parallel to magnetic field

$\vec{\epsilon}_{\perp}$ component of $\vec{\epsilon}$ perpendicular to magnetic field

ϵ_d electron diffusion energy density

ϵ_t total nonrandom electron energy density

θ polar angle in spherical coordinates

$\vec{\mu}$ Lagrangian multiplier

ξ effective interaction parameter for electron collisions

σ scalar electrical conductivity

$\sigma_{\|}$ parallel (to magnetic field) conductivity, $\sigma_{\|} = \sigma$

$\sigma_{\perp}$ perpendicular (to magnetic field) conductivity

σ_H Hall conductivity

$\sigma_L(\infty)$ exact first-order conductivity (infinite Sonine approximation) for a Lorentz gas

τ collision time equal to $\tau_\sigma(1)$, that is, first Sonine approximation to τ_σ

τ_s collision time based on entropy relaxation

τ_σ collision time deduced from scalar electrical conductivity

φ azimuthal angle in spherical coordinates

φ_e electron first-order perturbation function

χ scattering angle

ω cyclotron frequency for electrons

Subscripts:

c specifies effect due to collisions

e electrons

i index

j index; also particles of type j

L Lorentz gas, that is, no electron-electron collisions

Arguments 1, 2, 3, . . ., ∞ refer to level of Sonine approximation.

Primes with symbols refer to quantities after a collision.

The notation $\langle A \rangle$ refers to average of quantity A over velocity space.

THE COLLISION MODEL

Derivation of Model

The Boltzmann collision integral (ref. 2) for a two-component Lorentz plasma (no electron-electron encounters) is written to first order and for fixed scattering centers j as

$$\left(\frac{\partial f_e}{\partial t}\right)_{ej} = \int \left(f_e{}' f_j{}' - f_e f_j\right)\left|\vec{c}_e - \vec{c}_j\right| b \, db \, d\epsilon \, d\vec{c}_j$$

$$= \int \left(f_e{}' - f_e\right) f_j c_e b \, db \, d\epsilon \, d\vec{c}_j$$

$$= n_j \gamma f_e^{(o)} \left(\frac{2kT_e}{m_e}\right)^{1/2} \int \left(\varphi_e{}' - \varphi_e\right) b \, db \, d\epsilon \qquad (1)$$

where primed symbols refer to quantities after a collision (as opposed to unprimed symbols for quantities before a collision), $f_e^{(o)}$ is the Maxwellian distribution function for electrons given by

$$f_e^{(o)} = n_e \left(\frac{m_e}{2\pi k T_e}\right)^{3/2} e^{-\gamma^2} \qquad (2)$$

$\vec{\gamma}$ is the reduced electron particle velocity expressed as

$$\vec{\gamma} = \left(\frac{m_e}{2kT_e}\right)^{1/2} \vec{c}_e \qquad (3)$$

and φ_e is the electron perturbation function defined by

$$f_e = f_e^{(o)} \left(1 + \varphi_e\right) \qquad (4)$$

Equation (1) is further simplified by observing that φ_e is always dependent upon $\vec{\gamma}$ in the following manner (ref. 3) for the assumptions stated in the last paragraph of the Introduction:

$$\varphi_e = \vec{g}(\gamma,\text{fields and gradients}) \cdot \vec{\gamma} \tag{5}$$

Accordingly,

$$\left(\frac{\partial f_e}{\partial t}\right)_{ej} = n_j \gamma f_e^{(o)} \left(\frac{2kT_e}{m_e}\right)^{1/2} \vec{g} \cdot \int \left(\vec{\gamma}' - \vec{\gamma}\right) b \, db \, d\epsilon = -K\gamma^{1-4/\xi} f_e^{(o)} \varphi_e \tag{6}$$

if ξ is the power in the electron—heavy-particle inverse power interaction potential. (See appendix A.) The parameter K embraces a collection of thermodynamic parameters and also may be a function of ξ.

No matter how many heavy components the plasma contains, the assumption that ξ can be so adjusted as to make the far right side of equation (6) a fair representation of the sum of all electron—heavy-particle collision integrals appears justified. It seems less likely, on the other hand, because of the more complex integrals, that electron-electron encounters can be thus included. Nevertheless,

$$\left(\frac{\partial f_e}{\partial t}\right)_c = \sum_j \left(\frac{\partial f_e}{\partial t}\right)_{ej} = -K\gamma^{1-4/\xi} f_e^{(o)} \varphi_e \tag{7}$$

is adopted as the complete (j = e included) real-gas collision model, the validity of which will be tested by macroscopic comparisons. This collision model definitely establishes K and ξ as empirical parameters and has an advantage over other simple models in that ξ is immediately identifiable and $\left(\partial f_e/\partial t\right)_c$ is the exact first-order expression if the plasma is truly Lorentz with particles of only a single heavy species serving as active electron scattering centers. This condition holds true either for a slightly ionized pure gas (negligible electron-electron and electron-ion effects) or for full ionization with large ionic charge.

It is further evident that equation (7) reduces to the well-known Krook model (cf., ref. 4 and appendix B) only when $\xi = 4$. The present method is thus a generalization of Krook's work and shows his model to be restricted to Maxwellian molecules ($\xi = 4$) for a Lorentz plasma and to the first Sonine approximation to the Chapman-Enskog solution (ref. 2 and, also, appendix B) for a real gas. Since the first Sonine approximation is completely inadequate in its description of many plasma features, the additional γ-dependent factor in equation (7) should constitute a major improvement.

Determination of K and ξ

Appendix C provides the exact solution of the Chapman-Enskog first-order perturbation equation corresponding to the collision model of equation (7) and to the application of a pressure gradient and an electromagnetic field to a zero temperature gradient plasma. Of more immediate interest, however, is the nonmagnetic plasma, the perturbation function for which is found by setting ω equal to zero in equation (C9) and using Ohm's law $\vec{j} = -en_e\vec{v}_e = \sigma\vec{\epsilon}$ to write

$$\varphi_e = 2R_{04}R_{13}^{-1}\gamma^{(4/\xi)-1}\vec{\beta}\cdot\vec{\gamma} \tag{8}$$

The R_{ij} integrals are defined in appendix C and

$$\vec{\beta} = \left(\frac{m_e}{2kT_e}\right)^{1/2}\vec{v}_e \tag{9}$$

is the reduced electron diffusion velocity.

In addition, K is determined for equation (C7) by the requirement that the velocity moment of φ_e, which originally depended on K, must yield the correct electron diffusion. Therefore, from equation (7)

$$\left(\frac{\partial f_e}{\partial t}\right)_c = -\frac{R_{13}}{\tau_\sigma R_{04}}\gamma^{1-4/\xi}f_e^{(o)}\varphi_e \tag{10}$$

where τ_σ is the familiar collision time defined in terms of the electrical conductivity

$$\sigma = e^2 n_e\tau_\sigma / m_e \tag{11}$$

Only ξ remains unspecified and, therefore, the entire burden of accounting for the average interparticle collision properties falls on that parameter.

It seems reasonable to propose that an adequate value of ξ is obtained by requiring the energy moment of equation (8) to duplicate the experimental kinetic energy flux $\vec{q}$. One therefore computes

$$\vec{q} \propto \left\langle\gamma^2\vec{\gamma}\right\rangle = n_e^{-1}\int\gamma^2\vec{\gamma}f_e^{(o)}\varphi_e\, d\vec{c}_e = 2\xi^{-1}(1+\xi)\vec{\beta} \tag{12}$$

the energy-flux—enthalpy-flux difference

$$\left\langle\gamma^2\vec{\gamma}\right\rangle - \frac{5}{2}\left\langle\vec{\gamma}\right\rangle = \left\langle\gamma^2\vec{\gamma}\right\rangle - \frac{5}{2}\vec{\beta} = \frac{(4-\xi)\vec{\beta}}{2\xi} \tag{13}$$

for future reference, and finally

$$2\xi^{-1}\vec{\beta} = \left\langle \gamma^2 \vec{\gamma} \right\rangle - 2\vec{\beta} \tag{14}$$

from either equation (12) or equation (13). Each expression refers explicitly to the nonmagnetic, zero-temperature-gradient gas.

Equations (10) and (14) with n_e, T_e, and measurements of $\vec{\beta}$ and $\left\langle \gamma^2 \vec{\gamma} \right\rangle$ represent the complete semiempirical collision model. Although equation (14) obviously does not refer to the bounded, nonmagnetic, and currentless plasma postulated in the Introduction for the determination of ξ, the two sets of conditions are subsequently shown to be exactly equivalent in that respect – that is, the values found for ξ are identical in the two situations. Also, ξ may not be the power in any particular inverse power interaction potential, nor is it necessary to suppose that the constituent force laws are of that type. Macroscopic data have replaced the microscopic dynamics.

As mentioned previously, the special plasma conditions underlying equation (14) serve only to fix experimentally the numerical value of ξ, which is constant for a given set of plasma constituents. The electron first-order perturbation function φ_e therefore remains the lone condition-dependent function in equation (10), and it must be determined separately for each problem considered. The procedure is always the same:

Replace the Boltzmann collision integrals with equation (10), substitute the appropriate fields and gradients into the first-order kinetic equation, and solve the resulting expression exactly.

Illustrations and tests of this theory are provided herein.

A NEW COLLISION TIME

As a preliminary to the first application and test of the above semiempirical theory, it is instructive to compute the rate at which collisions produce entropy density $\dot{s}_c$; accordingly, from equations (8) and (10)

$$\dot{s}_c = -k\int \ln\left(1 + \varphi_e\right)\left(\frac{\partial f_e}{\partial t}\right)_c d\vec{c}_e \approx \frac{kR_{13}}{\tau_\sigma R_{04}}\int \gamma^{1-4/\xi} f_e^{(o)} \varphi_e^2 \, d\vec{c}_e = \frac{n_e m_e v_e^2}{\tau_\sigma T_e} \tag{15}$$

The rate of dissipation of the diffusion energy density ϵ_d is then related to that energy by

$$\frac{1}{2}\tau_\sigma T_e \dot{s}_c = \epsilon_d = \frac{1}{2} n_e m_e v_e^2 \tag{16}$$

It thus appears that τ_σ measures quite adequately the relaxation time scale for entropy, at least to the extent that ϵ_d represents the complete nonrandom electron energy density. But it will now be shown that this condition does not hold. The best way of defining the total nonrandom electron energy density ϵ_t is through the following relation involving the difference between the entropy density s and its equilibrium value $s^{(o)}$:

$$\epsilon_t = T_e\left(s^{(o)} - s\right) = kT_e\int f_e \ln\left(f_e/f_e^{(o)}\right) d\vec{c}_e \approx \frac{kT_e}{2}\int f_e^{(o)}\varphi_e^2 \, d\vec{c}_e = R\epsilon_d \tag{17}$$

The far right side of equation (17) is obtained from the substitution of equation (8) for φ_e and R for $R_{13}^{-2}R_{04}R_{22}$. (See appendix C.)

A convenient connection between equations (16) and (17) and the concepts of irreversible thermodynamics is provided by Onsager's fundamental principle, which states that the collisional production rate of entropy density is a maximum (ref. 5, pp. 116 and 145-148). See also appendix D where it is shown that the present distribution function can be obtained by the maximization of $\dot{s}_c$ subject to the constraint that $\vec{\beta}$ is known. This principle suggests a new collision time in equation (16) of the form

$$\tau_s = \epsilon_d^{-1}\epsilon_t\tau_\sigma = R\tau_\sigma \tag{18}$$

so that the entire ϵ_t is subject to dissipation in the manner

$$\frac{1}{2}\tau_s T_e \dot{s}_c = \epsilon_t \tag{19}$$

Since τ_s/τ_σ exceeds unity (fig. 1) for all but Maxwellian molecules $(\xi = 4)$, entropy equilibrization generally must lag that of electron diffusion. Such behavior is not surprising in view of the many collisions normally required to produce randomization. E. A. Mason of Brown University has suggested the possibility of $\tau_s - \tau_\sigma$ being a correction for the so-called persistence of velocities in free-flight or free-path theory. If this suggestion is valid, the nonrandom energy $\epsilon_t - \epsilon_d$ might be explained in terms of the "memory" of an electron, that is, the time required for an electron to forget its original speed and direction. In any event, the method for calculating τ_s is well prescribed by the relation

$$\frac{\tau_s}{\tau_\sigma} = \frac{2T_e\left(s^{(o)} - s\right)}{n_e m_e v_e^2} \approx \frac{kT_e}{n_e m_e v_e^2}\int f_e^{(o)}\varphi_e^2 \, d\vec{c}_e \tag{20}$$

and the parameter is shown to occur naturally in the Hall conductivity discussed subsequently.

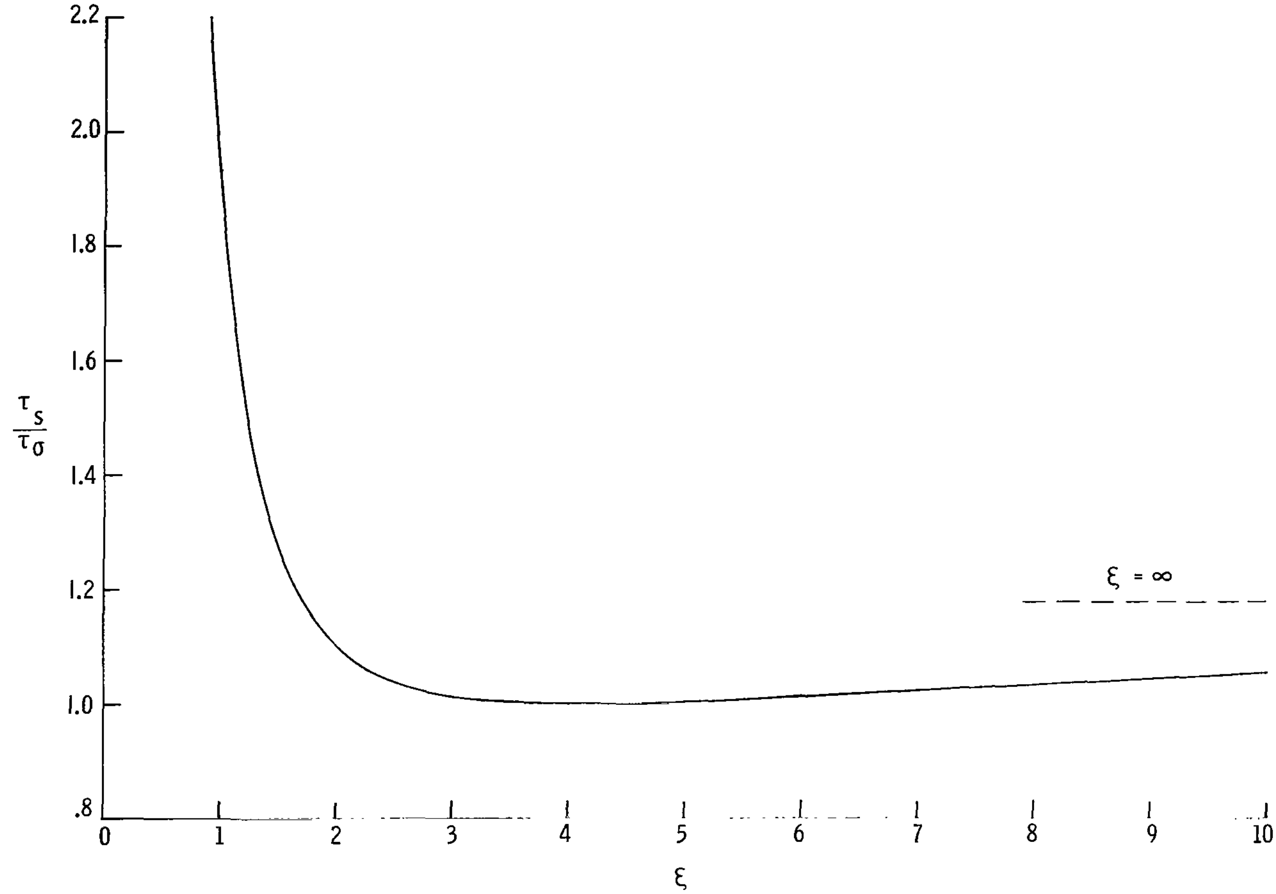

Figure 1.- Ratio of collision times as a function of effective collision parameter.

Meanwhile, the ratio τ_s/τ_σ can be computed and compared by several methods as a first test of the semiempirical distribution function. It is convenient for this purpose to express equation (14) in terms of the Spitzer-Härm relations (ref. 6)

$$\vec{\beta} = -\frac{1}{en_e}\left(\frac{m_e}{2kT_e}\right)^{1/2}\vec{j} = -\frac{1}{en_e}\left(\frac{m_e}{2kT_e}\right)^{1/2}\sigma\vec{\epsilon} = -\frac{\gamma_E}{en_e}\left(\frac{m_e}{2kT_e}\right)^{1/2}\sigma_L\vec{\epsilon} = \gamma_E\vec{\beta}_L \tag{21}$$

$$\left\langle\gamma^2\vec{\gamma}\right\rangle = \delta_E\left\langle\gamma^2\vec{\gamma}\right\rangle_L \tag{22}$$

between real and Lorentz plasmas operating under identical electric field and thermodynamic conditions. Hence,

$$2\xi^{-1}\gamma_E\vec{\beta}_L = \delta_E\left\langle\gamma^2\vec{\gamma}\right\rangle_L - 2\gamma_E\vec{\beta}_L \tag{23}$$

Although the data necessary for experimental verification of equations (10) and (23) are not available, one can still check their accuracy on the basis of the Spitzer-Härm exact first-order solution for a fully ionized gas. The parameters $\gamma_E = 0.5816$ and $\delta_E = 0.4652$ are known for this plasma, as is the Lorentz quantity

$$\left\langle \gamma^2 \vec{\gamma} \right\rangle_L = 4\vec{\beta}_L \tag{24}$$

from equation (12) with $\xi = 1$; consequently, ξ has the value

$$\xi = \left(2\gamma_E^{-1}\delta_E - 1\right)^{-1} = 1.6674 \tag{25}$$

Electron-electron effects are therefore quite substantial in their adjustment of the original ($\xi = 1$) electron-ion force law. This result implies that the subsequent tests of equations (10) and (25) are indeed meaningful, even though the plasma is without neutral components.

The integrals comprising R in equation (18) were computed numerically for the value of ξ of equation (25) to yield $R_{04} = 0.66467$, $R_{13} = 1.2113$, and $R_{22} = 2.6461$; accordingly,

$$\tau_\sigma^{-1}\tau_s = 1.199 \tag{26}$$

In like manner, substitution into equation (20) of equation (C11) of appendix C with $\omega = 0$ gives the second Sonine approximation

$$\frac{\tau_s(2)}{\tau_\sigma(2)} = \frac{16a^2 + 104a + 259}{16a^2 + 104a + 169} = 1.259 \tag{27}$$

when a equals $2^{1/2}$ which corresponds to the real plasma. Although the agreement is already reasonable, it is much improved by a similar comparison with the third Sonine approximation

$$\frac{\tau_s(3)}{\tau_\sigma(3)} = \frac{20736a^4 + 173952a^3 + 518392a^2 + 370856a + 93169}{20736a^4 + 173952a^3 + 427312a^2 + 262136a + 47089} = 1.207 \tag{28}$$

obtained from the matrix elements of Schweitzer and Mitchner (ref. 3).

The advantage of the present technique is even more pronounced for the Lorentz gas ($\xi = 1$) for which equation (18) yields the exact first-order result

$$\tau_\sigma^{-1}\tau_s = \frac{315\pi}{512} = 1.933 \tag{29}$$

as compared with the values

$$\frac{\tau_s(2)}{\tau_\sigma(2)} = 1.533 \qquad \frac{\tau_s(3)}{\tau_\sigma(3)} = 1.979 \tag{30}$$

which correspond to $a = 0$ in equations (27) and (28). Equations (26) to (30) also illustrate the large sensitivity of τ_s/τ_σ to the effective interaction potential ($\xi = 1.6674$ and 1), as do the following ratios from equation (18) representing Maxwellian ($\xi = 4$), Coulombic ($\xi = 1$), and rigid-sphere ($\xi = \infty$) forces, respectively:

$$\tau_s/\tau_\sigma = 1.000 \qquad \tau_s/\tau_\sigma = 1.933 \qquad \tau_s/\tau_\sigma = 1.178 \tag{31}$$

Similar conclusions apply as well to other plasma properties (e.g., thermal and magnetic), so that the previously stated purpose of using experimental data to avoid scarcely known collision parameters indeed has merit.

It is further observed that the first Sonine approximation always gives unity for τ_s/τ_σ, regardless of ξ, and so do equations (27) and (28) when a is very large. Since large values of a absolve the ions of their scattering responsibilities, such results support the well-known claim that the first Sonine approximation is adequate for pure gases and for mixtures with small mass disparities.

THE MAGNETIC PROBLEM

A more exacting test of equations (10) and (14) or (25) is obtained by the application of an electromagnetic field. As explained earlier, the method will be useful for directly extending nonmagnetic measurements and previous nonmagnetic calculations to the full field if the specification of ξ in terms of the nonmagnetic energy-flux—diffusion-velocity ratio is adequate. Again the fully ionized gas is considered, although one should recognize that other types of plasmas merely involve different empirical or theoretical values of γ_E and δ_E in equation (25).

The complete first-order solution of the electronic Lorentz-like Boltzmann equation for a plasma in an electromagnetic field is derived in appendix C, as is the current density

$$\vec{j} = \sigma_{||}\vec{\epsilon}_{||} + \sigma_\perp\vec{\epsilon}_\perp + \sigma_H\hat{B}\times\vec{\epsilon} = \sigma\left(\vec{\epsilon}_{||} + R_{13}{}^{-1}I_{13}\vec{\epsilon}_\perp + \omega\tau_\sigma R_{04}R_{13}{}^{-2}I_{22}\hat{B}\times\vec{\epsilon}\right) \tag{32}$$

Additional notation includes the parallel (to $\hat{B}$), perpendicular, and Hall conductivities $\left(\sigma_{||},\ \sigma_{\perp},\right.$ and $\left.\sigma_H,\right.$ respectively$\left.\right)$, the cyclotron frequency $\omega = eB/m_e$, and the integral

$$I_{ij} = \int_0^{\infty} x^{(4i/\xi)+j}\left(1 + \omega^2\tau_\sigma{}^2 R_{04}{}^2 R_{13}{}^{-2} x^{(8/\xi)-2}\right)^{-1} e^{-x^2}\, dx \tag{33}$$

The limit of this integral, as ω approaches zero, is R_{ij}. One can also write the following combination of equations (18) and (32):

$$\vec{j} = \sigma\left(\vec{\epsilon}_{||} + R_{13}{}^{-1} I_{13}\vec{\epsilon}_{\perp} + \omega\tau_S R_{22}{}^{-1} I_{22}\hat{B} \times \vec{\epsilon}\right) \tag{34}$$

Since I_{ij} and R_{ij} are identical when quadratic and higher powers of $\omega\tau_\sigma$ are neglected, equation (34) becomes

$$\vec{j} = \sigma\left(\vec{\epsilon} + \omega\tau_S\hat{B} \times \vec{\epsilon}\right) \tag{35}$$

for small magnetic fields; likewise, the second Sonine approximation of equations (27) and (C12) yields the same formal result. These results are interesting for at least two reasons: (1) the appearance of τ_S rather than τ_σ in the Hall conductivity indicates a preference for the relaxation time of entropy over that of diffusion; (2) except for the new collision time, equation (35) is exactly Spitzer's formula for the simple mean-free-path current density (ref. 7; see also appendix E). In addition, a comparison of the numerical results of equations (26) to (30) suggests that the present technique for computing σ_H to linear $\omega\tau_S$ might be better than the third Sonine approximation.

Although equation (34) still provides the most concise description, the quadratic and higher effects of large magnetic fields are difficult to interpret. Even so, the conductivity ratios are found quite easily from equation (32) by using the nonmagnetic value of ξ from equation (25) and $\gamma_E(\infty) = 0.5816$ in the following formulas:

$$\frac{\sigma_\perp}{\sigma_L(\infty)} = \gamma_E(\infty) R_{13}(\xi)^{-1} I_{13}(\xi,\omega\tau_\sigma) \tag{36}$$

and

$$\frac{\sigma_H}{\sigma_L(\infty)} = \gamma_E(\infty)\omega\tau_\sigma R_{04} R_{13}(\xi)^{-2} I_{22}(\xi,\omega\tau_\sigma) \tag{37}$$

where $\sigma_L(\infty)$ is the exact Lorentz conductivity $\gamma_E{}^{-1}\sigma = \gamma_E{}^{-1}\sigma_{||}$. The results obtained with equations (36) and (37) are listed in the last column in tables I and II, respectively, for the extension of Spitzer and Härm's scalar conductivity (ref. 6) to the electromagnetic tensor conductivity at several values of $\omega\tau$.

Also listed in tables I and II are calculations from the first Sonine expressions (see appendix E)

$$\frac{\sigma_{\perp}(1)}{\sigma_L(\infty)} = \frac{\gamma_E(1)}{1 + \omega^2\tau_\sigma(1)^2} \tag{38a}$$

$$\frac{\sigma_H(1)}{\sigma_L(\infty)} = \frac{\gamma_E(1)\omega\tau_\sigma(1)}{1 + \omega^2\tau_\sigma(1)^2} \tag{38b}$$

and the second, third, and sixth approximations of Kaneko (ref. 8). Besides $\gamma_E(1) = 3\pi/32$ from reference 6, a few conversion factors are necessary: the $\tau_\sigma(1)$ of equation (38) is equal to the τ of tables I and II and each entry in table I of reference 8 should be multiplied by $\pm\gamma_E(1)$, where the minus sign is used in the Hall conductivity.

The comparisons are generally favorable and show equations (36) and (37) to be nearly always superior to the second Sonine approximation. One can thus proceed with some assurance to the extensions and semiempirical applications originally intended for the present method.

TABLE I.- PERPENDICULAR CONDUCTIVITIES

$\omega\tau$[†]	$\sigma_\perp(1)/\sigma_L(\infty)$	$\sigma_\perp(2)/\sigma_L(\infty)$	$\sigma_\perp(3)/\sigma_L(\infty)$	$\sigma_\perp(6)/\sigma_L(\infty)$	$\sigma_\perp/\sigma_L(\infty)$ (eq. (36))
0.0	0.295	0.569	0.574	0.581	0.582
.2	.283	.458	.468	.471	.471
.5	.236	.236	.253	.256	.266
1.0	.147	.100	.106	.110	.118
2.0	.0589	.0406	.0367	.0381	.0393
4.0	.0173	.0147	.0126	.0117	.0111
6.0	.0080	.0073	.0066	.0057	.0051

[†]The first-order collision time τ is related to τ_σ by the ratio $\tau_\sigma/\tau = 32(3\pi)^{-1}\gamma_E$, where γ_E is given in reference 6; in particular, $\gamma_E(\infty)$ yields $\tau_\sigma/\tau = 1.975$ for the conversion of the independent variable in equation (36).

TABLE II.- HALL CONDUCTIVITIES

$\omega\tau$[†]	$\sigma_H(1)/\sigma_L(\infty)$	$\sigma_H(2)/\sigma_L(\infty)$	$\sigma_H(3)/\sigma_L(\infty)$	$\sigma_H(6)/\sigma_L(\infty)$	$\sigma_H/\sigma_L(\infty)$ (eq. (37))
0.0	0.000	0.000	0.000	0.000	0.000
.2	.057	.220	.215	.217	.212
.5	.118	.266	.271	.271	.268
1.0	.147	.193	.207	.208	.214
2.0	.118	.117	.123	.127	.132
4.0	.0693	.0672	.0673	.0689	.0713
6.0	.0478	.0469	.0466	.0471	.0483

[†]The first-order collision time τ is related to τ_σ by the ratio $\tau_\sigma/\tau = 32(3\pi)^{-1}\gamma_E$, where γ_E is given in reference 6; in particular, $\gamma_E(\infty)$ yields $\tau_\sigma/\tau = 1.975$ for the conversion of the independent variable in equation (37).

TEMPERATURE EFFECTS

Thermal Conduction

Temperature gradients provide third and fourth checks of the collision model of equations (10) and (14) or (25) through comparisons with Spitzer and Härm's thermal conductivity and thermal diffusion.

By a procedure similar to that used in appendix C, the Lorentz-like first-order Boltzmann equation for a plasma subject to a temperature gradient (but no pressure gradient) can be written

$$\vec{c}_e \cdot \nabla f_e^{(o)} = f_e^{(o)}\left(\gamma^2 - \frac{5}{2}\right)\vec{c}_e \cdot \nabla \ln T_e = -\frac{R_{13}}{\tau_\sigma R_{04}} \gamma^{1-4/\xi} f_e^{(o)} \varphi_e \tag{39}$$

The solution of equation (39) is

$$\varphi_e = -\frac{\tau_\sigma R_{04}}{R_{13}}\left(\frac{2kT_e}{m_e}\right)^{1/2} \gamma^{(4/\xi)-1}\left(\gamma^2 - \frac{5}{2}\right)\vec{\gamma} \cdot \nabla \ln T_e \tag{40}$$

which yields, respectively, the real and Lorentz ($\xi = 1$) kinetic-energy fluxes

$$\vec{q} = \frac{1}{2} n_e m_e \left\langle c_e{}^2 \vec{c}_e \right\rangle = -k^2 T_e \sigma (e\xi)^{-2}\left(\xi^2 + 5\xi + 4\right)\nabla T_e \tag{41}$$

and

$$\vec{q}_L = -10k^2 T_e e^{-2} \gamma_E(\infty)^{-1} \sigma \nabla T_e \tag{42}$$

and, hence, the Spitzer-Härm thermal reduction factor

$$\delta_T = \left(\vec{q}_L\right)^{-1}\vec{q} = \gamma_E(\infty)\left(10\xi^2\right)^{-1}\left(\xi^2 + 5\xi + 4\right) \tag{43}$$

If the same value of ξ is substituted into equation (43) which was derived previously in equation (25) for the nonmagnetic zero-temperature-gradient problem, there results

$$\delta_T(\xi = 1.6674) = 0.3162 \tag{44}$$

This time, however, the comparisons with Spitzer and Härm's exact value of 0.2252 and Kaneko's second Sonine approximation of 0.1906 are fairly poor. Even so, the value in equation (44) is better than the corresponding Krook number of 0.0736 obtained from the substitution (see appendix B) of $\gamma_E(1)$ and $\xi = 4$ into equation (43).

Thermal Diffusion

Spitzer and Härm's thermal diffusion reduction factor γ_T also can be derived from equation (40) by first taking the velocity moment and then comparing real and Lorentz plasmas in the manner of equations (41) to (43). The pertinent expressions are

$$\vec{j} = k\sigma(2e\xi)^{-1}(4 - \xi)\nabla T_e \tag{45}$$

$$\gamma_T = \left(\vec{j}_L\right)^{-1}\vec{j} = \gamma_E(\infty)(3\xi)^{-1}(4 - \xi) \tag{46}$$

and

$$\gamma_T(\xi = 1.6674) = 0.2712 \tag{47}$$

Despite the excellent agreement between the value in equation (47) and the exact value (ref. 6) of 0.2727, the preceding calculation is a valid test of equations (10) and (14) only in that it demonstrates the automatic incorporation by the present model of Onsager's reciprocal relations (ref. 9, pp. 704-717). This fact is illustrated quite directly by the following manipulation of equation (13) for no temperature gradient and of equations (41) and (45) for only a temperature gradient:

$$\begin{aligned}\vec{q} - \vec{h} &= \frac{n_e m_e}{2}\left(\frac{2kT_e}{m_e}\right)^{3/2}\left(\left\langle \gamma^2 \vec{\gamma} \right\rangle - \frac{5}{2}\vec{\beta}\right) = \frac{n_e m_e(4 - \xi)}{4\xi}\left(\frac{2kT_e}{m_e}\right)^{3/2}\vec{\beta} \\ &= -\frac{(4 - \xi)kT_e}{2e\xi}\vec{j} = -\frac{(4 - \xi)\sigma k T_e}{2e\xi}\vec{\epsilon}\end{aligned} \tag{48}$$

$$\begin{aligned}\vec{q} - \vec{h} &= -\frac{\sigma k^2 T_e\left(\xi^2 + 5\xi + 4\right)}{e^2\xi^2}\nabla T_\epsilon - \frac{5}{2}n_e k T_e \vec{v}_e \\ &= -\frac{\sigma k^2 T_e\left(\xi^2 + 5\xi + 4\right)}{e^2\xi^2}\nabla T_e + \frac{5kT_e}{2e}\vec{j} \\ &= -\frac{\sigma k^2 T_e^2\left(9\xi^2 + 16\right)}{4e^2\xi^2}\nabla \ln T_e\end{aligned} \tag{49}$$

The linear combination

$$\vec{q} - \vec{h} = -\frac{\sigma k T_e(4 - \xi)}{2e\xi}\vec{\epsilon} - \frac{\sigma k^2 T_e^2\left(9\xi^2 + 16\right)}{4e^2\xi^2}\nabla \ln T_e \tag{50}$$

is appropriate when the field and temperature gradients are present simultaneously.

Likewise, the addition of equation (45) and Ohm's law for no temperature gradient gives the current density

$$\vec{j} = \sigma\vec{\epsilon} + \frac{\sigma k T_e(4 - \xi)}{2e\xi}\nabla \ln T_e \tag{51}$$

so that the matrix representation of equations (50) and (51)

$$\begin{pmatrix} \vec{j} \\ \vec{q} - \vec{h} \end{pmatrix} = \sigma \begin{pmatrix} 1 & -\dfrac{kT_e(4 - \xi)}{2e\xi} \\ -\dfrac{kT_e(4 - \xi)}{2e\xi} & \dfrac{k^2T_e^2\left(9\xi^2 + 16\right)}{4e^2\xi^2} \end{pmatrix} \begin{pmatrix} \vec{\epsilon} \\ -\nabla \ln T_e \end{pmatrix} \tag{52}$$

clearly reveals the symmetry between the off-diagonal elements.

The fact that Onsager's reciprocal relations are guaranteed in the present procedure is, of course, another important advantage over several previous models. In particular, the Krook model (appendix B), even though it technically satisfies the reciprocal requirements, does so in a trivial fashion because it predicts zero values for the off-diagonal matrix elements by forcing ξ to equal 4.

ALTERNATE DETERMINATION OF ξ

Equation (51) immediately suggests a more convenient experimental method for determining the value of ξ in equation (10) than the method offered by equation (14). If a temperature gradient alone is initially applied to an enclosed plasma, a secondary generalized electric field $\vec{\epsilon}$ will build up until the current produced by the temperature gradient is canceled (ref. 7, p. 144). The final stationary state so obtained corresponds to $\vec{j} = 0$ in equation (51) and hence to the following expression for ξ in terms of the balancing $\vec{\epsilon}$ and ∇T_e:

$$\xi = 4\left(1 - \frac{2e\vec{\epsilon}}{k\nabla T_e}\right)^{-1} \tag{53}$$

The values of ξ obtained from equation (53) are, of course, numerically equivalent to those obtained from equation (14) because of the guaranteed satisfaction in the present technique of Onsager's reciprocal relations.

As in equation (14), the conditions underlying equation (53) are assumed only for the purpose of establishing ξ, which remains fixed irrespective of subsequent conditions and forms of φ_e if the plasma constituents do not change. It is difficult to imagine a simpler experimental procedure for completely defining the collision model than that of equation (53) and a corresponding one for the determination of τ_σ in equation (10). The latter involves measurements of a field-induced electric current and the subsequent use of Ohm's law and equation (11).

CONCLUDING REMARKS

The collision model derived herein has been shown to predict quite adequately such diverse plasma properties as entropy, entropy and diffusion relaxation times, electromagnetic tensor conductivities, and either thermal diffusion or off-diagonal kinetic-energy flux. The model thus provides a satisfactory scheme for utilizing macroscopic experimental data when microscopic parameters (e.g., electron—neutral-atom force laws) are scarcely known and for directly extending previous calculations to more complex conditions (e.g., to the full electromagnetic field). Only the simplest measurements, that is, electron diffusion and static field and temperature conditions, are required for the first application. In addition, the closed-form distribution functions are easier to interpret than the functions from polynomial expansions and reveal, for example, the natural appearance of entropy relaxation in Hall conductivities.

Generalizations of this technique can apparently be made to higher perturbation orders. Certain modifications to the equations defining the collision model and to the kinetic equations are necessary, of course, but the manipulations should be much easier in a Lorentz-like framework than with the more exact Boltzmann collision integrals. Perhaps one or two second-order constraints will suffice to overcome Chapman-Enskog convergence difficulties just as the introduction of an effective collision parameter ξ (empirical) overcame electron-electron difficulties in the present report.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., February 3, 1969,
129-02-01-01-23.

APPENDIX A

COLLISION INTEGRALS

It is shown in this appendix that the last part of equation (6) follows directly from the second part when the electron—heavy-particle interaction potential can be written as the inverse ξ-power of the separation distance.

The first step in such a development is the formulation of the reduced electron particle velocity after a collision $\vec{\gamma}'$ in terms of the reduced electron particle velocity before the collision $\vec{\gamma}$ and the corresponding scattering or deflection angle χ. This straightforward exercise in geometry yields

$$\vec{\gamma} - \vec{\gamma}' = (1 - \cos\chi)\vec{\gamma} - \gamma \sin\chi\Big[\hat{i}(\sin\varphi\cos\epsilon + \cos\theta\cos\varphi\sin\epsilon) - \hat{j}(\cos\varphi\cos\epsilon - \cos\theta\sin\varphi\sin\epsilon) - \hat{k}\sin\theta\sin\epsilon\Big] \tag{A1}$$

where θ and φ are the polar and azimuthal angles, respectively, of $\vec{\gamma}$ in spherical coordinates and ϵ is the azimuthal angle of $\vec{\gamma}'$ with respect to the direction of $\vec{\gamma}$.

Accordingly, equation (6) may be derived as follows with the aid of equation (5):

$$\left(\frac{\partial f_e}{\partial t}\right)_{ej} = n_j \gamma f_e^{(o)}\left(\frac{2kT_e}{m_e}\right)^{1/2} \vec{g} \cdot \int\left(\vec{\gamma}' - \vec{\gamma}\right) b\, db\, d\epsilon$$

$$= -2\pi n_j \gamma f_e^{(o)} \varphi_e \left(\frac{2kT_e}{m_e}\right)^{1/2} \int_0^\infty (1 - \cos\chi) b\, db \tag{A2}$$

The remaining integral involves, of course, the detailed collision dynamics, but that calculation has already been performed for the inverse power potential by Hirschfelder, Curtiss, and Bird (ref. 9, pp. 546-549). Their results can be written

$$\int_0^\infty (1 - \cos\chi) b\, db = \frac{K(\xi)}{2\pi n_j}\left(\frac{m_e}{2kT_e}\right)^{1/2} \gamma^{-4/\xi} \tag{A3}$$

so that

$$\left(\frac{\partial f_e}{\partial t}\right)_{ej} = -K\gamma^{1-4/\xi} f_e^{(o)} \varphi_e \tag{A4}$$

APPENDIX A

It must be mentioned, however, that the Coulomb potential ($\xi = 1$) satisfies equation (A3) only if a cut-off (usually the Debye length) replaces the upper limit of the integral and if the γ^2 appearing in the argument of the logarithm in equation (8.3-8) of reference 9 is replaced by its average value over all collisions. Both practices are rather standard in the current literature.

APPENDIX B

THE KROOK COLLISION MODEL

Holway (ref. 1) has used some concepts from information theory to derive the Krook collision model defined by

$$\left(\frac{\partial f_e}{\partial t}\right)_c = -K\left\{f_e - n_e\left(\frac{m_e}{2\pi kT_{ej}}\right)^{3/2} \exp\left[-\frac{m_e\left(\vec{c}_e - \vec{u}_{ej}\right)^2}{2kT_{ej}}\right]\right\} \tag{B1}$$

More specifically, he obtained this result by means of a variational procedure based on the maximization of entropy subject to mass conservation and to experimental knowledge of the average velocity $\vec{u}_{ej}$ and the average energy $3kT_{ej}/2$ with which electrons emerge from encounters with their effective collision partners j. The pertinent relations are equations (46) to (49) of reference 1, from which it is evident that T_{ej} equals the electron temperature T_e in the first Chapman-Enskog perturbation order.

Other first-order expressions include

$$f_e = f_e^{(o)}\left(1 + \varphi_e\right) \tag{B2}$$

and the following linear expansion of equation (B1):

$$\left(\frac{\partial f_e}{\partial t}\right)_c = -Kf_e^{(o)}\left(\varphi_e - 2\vec{\beta}_{ej} \cdot \vec{\gamma}\right) \tag{B3}$$

where $\vec{\beta}_{ej}$ and $\vec{\gamma}$ are the reduced velocities

$$\vec{\beta}_{ej} = \left(\frac{m_e}{2kT_e}\right)^{1/2} \vec{u}_{ej} \tag{B4}$$

and

$$\vec{\gamma} = \left(\frac{m_e}{2kT_e}\right)^{1/2} \vec{c}_e \tag{B5}$$

The present objective is to determine φ_e from the corresponding perturbation equation

$$-\frac{e}{m_e}\vec{E} \cdot \frac{\partial f_e^{(o)}}{\partial \vec{c}_e} = \left(\frac{\partial f_e}{\partial t}\right)_c = -Kf_e^{(o)}\left(\varphi_e - 2\vec{\beta}_{ej} \cdot \vec{\gamma}\right) \tag{B6}$$

for an applied electric field. If $\vec{E}$ is related to the reduced electron diffusion velocity

$$\vec{\beta} = \left(\frac{m_e}{2kT_e}\right)^{1/2} \vec{v}_e \tag{B7}$$

through Ohm's law

$$\vec{E} = \frac{\vec{j}}{\sigma} = \frac{m_e}{e^2 n_e \tau_\sigma} \vec{j} = -\frac{m_e}{e\tau_\sigma} \vec{v}_e = -\frac{m_e}{e\tau_\sigma}\left(\frac{2kT_e}{m_e}\right)^{1/2} \vec{\beta} \tag{B8}$$

there results

$$\frac{1}{\tau_\sigma} \vec{\beta} \cdot \frac{\partial f_e^{(o)}}{\partial \vec{\gamma}} = -\frac{2f_e^{(o)}}{\tau_\sigma} \vec{\beta} \cdot \vec{\gamma} = -Kf_e^{(o)}\left(\varphi_e - 2\vec{\beta}_{ej} \cdot \vec{\gamma}\right) \tag{B9}$$

and

$$\varphi_e = 2\left(\frac{\vec{\beta}}{\tau_\sigma K} + \vec{\beta}_{ej}\right) \cdot \vec{\gamma} \tag{B10}$$

The definition

$$\vec{\beta} = \langle\vec{\gamma}\rangle = \frac{1}{n_e} \int \vec{\gamma} f_e^{(o)} \varphi_e \, d\vec{c}_e \tag{B11}$$

is used with equation (B10) to yield

$$\vec{\beta} = \frac{\vec{\beta}}{\tau_\sigma K} + \vec{\beta}_{ej} \tag{B12}$$

and

$$\varphi_e = 2\vec{\beta} \cdot \vec{\gamma} \tag{B13}$$

and with equations (B3) and (B13) to yield

$$\left(\frac{\partial f_e}{\partial t}\right)_c = -\frac{1}{\tau_\sigma} f_e^{(o)} \varphi_e \tag{B14}$$

The physical interpretation of equations (B13) and (B14), and therefore the first-order Krook model, is clear: They correspond either to the first Sonine approximation to a real plasma or to the exact first-order solution for a Lorentz gas (no electron-electron collisions) composed of Maxwellian molecules. Neither is adequate. Also in line with these conclusions is the observation from equation (10) that $\xi = 4$ and $\tau_\sigma = \tau_\sigma(1)$ must be employed in the collision model of this paper if Krook model type of results are desired.

APPENDIX C

CHAPMAN-ENSKOG SOLUTIONS

The standard Chapman-Enskog first-order perturbation equation (ref. 2) for electrons can be written as follows for pressure gradients and an applied electromagnetic field $\vec{E}$ and $\vec{B}$, no temperature gradients, infinitely massive heavy particles at rest relative to the laboratory, and the collision model of equation (7):

$$\left(\frac{2kT_e}{m_e}\right)^{1/2}\vec{\gamma}\cdot\nabla f_e^{(o)} - \frac{e}{m_e}\left(\frac{m_e}{2kT_e}\right)^{1/2}\vec{E}\cdot\frac{\partial f_e^{(o)}}{\partial\vec{\gamma}} - \omega f_e^{(o)}\left(\vec{\gamma}\times\hat{B}\right)\cdot\frac{\partial\varphi_e}{\partial\vec{\gamma}} = -K\gamma^{1-4/\xi}f_e^{(o)}\varphi_e \tag{C1}$$

Since

$$\varphi_e = -\left[D_1(\gamma)\vec{\gamma} + D_2(\gamma)\vec{\gamma}\times\hat{B} + D_3(\gamma)\vec{\gamma}\cdot\hat{B}\hat{B}\right]\cdot\vec{\epsilon} \tag{C2}$$

is a standard solution (ref. 3) of equation (C1), there results

$$\frac{2e}{m_e}\left(\frac{m_e}{2kT_e}\right)^{1/2}\vec{\gamma} + \omega\left(D_2\vec{\gamma}\cdot\hat{B}\hat{B} - D_2\vec{\gamma} + D_1\vec{\gamma}\times\hat{B}\right) = K\gamma^{1-4/\xi}\left(D_1\vec{\gamma} + D_2\vec{\gamma}\times\hat{B} + D_3\vec{\gamma}\cdot\hat{B}\hat{B}\right) \tag{C3}$$

upon taking the spatial and velocity derivatives of $f_e^{(o)}$ and defining the effective driving force as

$$\vec{\epsilon} = \vec{E} + (en_e)^{-1}\nabla p_e \tag{C4}$$

It is readily confirmed by equating coefficients of $\vec{\gamma}$, $\vec{\gamma}\times\hat{B}$, and $\vec{\gamma}\cdot\hat{B}\hat{B}$ in equation (C3) that the exact closed-form solution is

$$\varphi_e = -\frac{2e}{m_eK}\left(\frac{m_e}{2kT_e}\right)^{1/2}\gamma^{(4/\xi)-1}\left(1 + \frac{\omega^2}{K^2}\gamma^{(8/\xi)-2}\right)^{-1}\left[\left(1 + \frac{\omega^2}{K^2}\gamma^{(8/\xi)-2}\right)\vec{\epsilon}_{||} + \vec{\epsilon}_{\perp} + \frac{\omega}{K}\gamma^{(4/\xi)-1}\hat{B}\times\vec{\epsilon}\right]\cdot\vec{\gamma} \tag{C5}$$

where $\vec{\epsilon}_{||}$ and $\vec{\epsilon}_{\perp}$ are the components of $\vec{\epsilon}$ which are parallel and perpendicular, respectively, to $\hat{B}$. The current density then becomes

$$\vec{j} = -en_e\left\langle\vec{c}_e\right\rangle = -\frac{en_e}{\pi^{3/2}}\left(\frac{2kT_e}{m_e}\right)^{1/2}\int e^{-\gamma^2}\vec{\gamma}\varphi_e\, d\vec{\gamma}$$

$$= \frac{e^2 n_e R_{13}}{m_e K R_{04}}\left(\vec{\epsilon}_{\parallel} + \frac{I_{13}}{R_{13}}\vec{\epsilon}_{\perp} + \frac{\omega I_{22}}{KR_{13}}\hat{B}\times\vec{\epsilon}\right) \tag{C6}$$

so that

$$K = \frac{e^2 n_e R_{13}}{m_e \sigma R_{04}} = \frac{R_{13}}{\tau_\sigma R_{04}} \tag{C7}$$

Other definitions include the electrical conductivity $\sigma = e^2 n_e \tau_\sigma / m_e$ and the integrals

$$I_{ij} = \int_0^\infty x^{(4i/\xi)+j}\left(1 + \omega^2\tau_\sigma^2 R_{04}^2 R_{13}^{-2} x^{(8/\xi)-2}\right)^{-1} e^{-x^2}\, dx \tag{C8}$$

and $R_{ij} = I_{ij}(\omega = 0)$.

Except for the determination of ξ, the following expressions thus represent the complete specification of the zero-temperature-gradient plasma:

$$\varphi_e = -\frac{2\sigma R_{04}}{en_e R_{13}}\left(\frac{m_e}{2kT_e}\right)^{1/2}\gamma^{(4/\xi)-1}\left(1 + \frac{\omega^2\tau_\sigma^2 R_{04}^2}{R_{13}^2}\gamma^{(8/\xi)-2}\right)^{-1}\left[\left(1 + \frac{\omega^2\tau_\sigma^2 R_{04}^2}{R_{13}^2}\gamma^{(8/\xi)-2}\right)\vec{\epsilon}_{\parallel} + \vec{\epsilon}_{\perp} + \frac{\omega\tau_\sigma R_{04}}{R_{13}}\gamma^{(4/\xi)-1}\hat{B}\times\vec{\epsilon}\right]\cdot\vec{\gamma} \tag{C9}$$

and

$$\vec{j} = \sigma\left(\vec{\epsilon}_{\parallel} + R_{13}^{-1} I_{13}\vec{\epsilon}_{\perp} + \omega\tau_\sigma R_{04} R_{13}^{-2} I_{22}\hat{B}\times\vec{\epsilon}\right) \tag{C10}$$

Also discussed in the main body of the report are the corresponding second Sonine approximation formulas derived from the matrix elements of Schweitzer and Mitchner (ref. 3). If the parameter a is introduced in these expressions for the purpose of describing the Lorentz (a = 0) and real $\left(a = 2^{1/2}\right)$ plasmas simultaneously, there results

$$\varphi_e = -\frac{4\sigma}{en_e(4a + 13)}\left(\frac{m_e}{2kT_e}\right)^{1/2}\left\{\left(3\gamma^2 + 2a - 1\right)\vec{\epsilon}\right.$$

$$\left. + \frac{\omega\tau_\sigma}{4a + 13}\left[3(4a + 23)\gamma^2 + 8a^2 + 22a - 43\right]\hat{B}\times\vec{\epsilon}\right\}\cdot\vec{\gamma} \tag{C11}$$

and

$$\vec{j} = \sigma\left[\vec{\epsilon} + \omega\tau_{\sigma}\left(\frac{16a^2 + 104a + 259}{16a^2 + 104a + 169}\right)\hat{B} \times \vec{\epsilon}\right] \tag{C12}$$

Only the linear terms have been retained in an $\omega\tau_{\sigma}$ expansion.

APPENDIX D

THE ONSAGER MAXIMIZATION PRINCIPLE

It is shown in this appendix that the perturbation function of equation (8) can be deduced from Onsager's principle on the maximization of the collisional production rate of entropy density (ref. 5). The mathematical statement of this principle is as follows:

$$\delta\dot{s}_c = -k\delta\int\left[1 + \ln\left(1 + \varphi_e\right)\right]\left(\frac{\partial f_e}{\partial t}\right)_c d\vec{c}_e = -k\delta\int \ln\left(1 + \varphi_e\right)\left(\frac{\partial f_e}{\partial t}\right)_c d\vec{c}_e$$

$$\approx kK\delta\int\gamma^{1-4/\xi} f_e^{(o)}\varphi_e^{2}\, d\vec{c}_e = 2kK\int\gamma^{1-4/\xi} f_e^{(o)}\varphi_e\delta\varphi_e\, d\vec{c}_e = 0 \qquad \text{(D1)}$$

in which equation (7) is substituted for the collision integrals.

If the diffusion velocity $\vec{\beta}$ is taken to be the only nonequilibrium parameter on which φ_e depends, the perturbation function can be varied arbitrarily in equation (D1) subject to the constraint

$$\delta\vec{\beta} = \delta\left\langle\vec{\gamma}\right\rangle = \delta n_e^{-1}\int\vec{\gamma} f_e^{(o)}\left(1 + \varphi_e\right) d\vec{c}_e = n_e^{-1}\int\vec{\gamma} f_e^{(o)}\delta\varphi_e\, d\vec{c}_e = 0 \qquad \text{(D2)}$$

Accordingly, the application of the method of Lagrange multipliers to equations (D1) and (D2) yields

$$\int\gamma^{1-4/\xi} f_e^{(o)}\delta\varphi_e\left(\varphi_e - \vec{\mu}\cdot\vec{\gamma}\gamma^{(4/\xi)-1}\right) d\vec{c}_e = 0 \qquad \text{(D3)}$$

and

$$\varphi_e = \gamma^{(4/\xi)-1}\vec{\mu}\cdot\vec{\gamma} = 2R_{04}R_{13}^{-1}\gamma^{(4/\xi)-1}\vec{\beta}\cdot\vec{\gamma} \qquad \text{(D4)}$$

Since equation (D4) is identical with equation (8), Onsager's maximization principle is embodied in the Boltzmann equation. It is further suggested that the present variational procedure may be more efficient than the maximization of entropy used in information theory (ref. 1). At least equation (D1) is based upon an objective physical theory, whereas the alternate technique refers to the uncertainty of the observer and provides only an upper limit to the true entropy unless the constraints form a complete set for describing the macroscopic state of the system.

APPENDIX E

THE MEAN-FREE-PATH MODEL

The electron equation of motion (ref. 7, p. 28) in the mean-free-path statistical model can be written as follows for infinite ionic mass, an ionic charge number of unity, and zero-plasma-flow velocity:

$$\vec{j} = \sigma\left[\vec{\epsilon} - (en_e)^{-1}\vec{j} \times \vec{B} - m_e\left(e^2 n_e\right)^{-1} \frac{\partial \vec{j}}{\partial t}\right] \tag{E1}$$

It is observed, however, from the macroscopic equations of continuity and energy transfer that the time derivatives of both particle density and temperature are second order in that they involve products of gradients and squares and divergences of the electron diffusion velocity; consequently, since the partial time derivative of equation (2) involves only $\partial n_e/\partial t$ and $\partial T_e/\partial t$, there can be no time derivative on the left side of the first-order Chapman-Enskog perturbation equation (see appendix C). This argument means that the $\partial \vec{j}/\partial t$ appearing in reference 7 must be set to zero for comparisons with the present report.

The cross product of equation (E1) and $\vec{B}$ and the subsequent combination with equations (11) and (E1) yield

$$\begin{aligned}\vec{j} \times \vec{B} &= -\sigma B\hat{B} \times \vec{\epsilon} + \sigma B^2(en_e)^{-1}\left(\hat{B} \times \vec{j}\right) \times \hat{B} \\ &= -en_e\omega\tau_\sigma\hat{B} \times \vec{\epsilon} + en_e\omega^2\tau_\sigma^2\left[\vec{\epsilon}_\perp - (en_e)^{-1}\vec{j} \times \vec{B}\right]\end{aligned} \tag{E2}$$

the solution of which is

$$\vec{j} \times \vec{B} = en_e\omega\tau_\sigma\left(1 + \omega^2\tau_\sigma^2\right)^{-1}\left(\omega\tau_\sigma\vec{\epsilon}_\perp - \hat{B} \times \vec{\epsilon}\right) \tag{E3}$$

A further combination of equation (E3) with equation (E1) then gives

$$\vec{j} = \sigma\left[\vec{\epsilon}_{||} + \left(1 + \omega^2\tau_\sigma^2\right)^{-1}\left(\vec{\epsilon}_\perp + \omega\tau_\sigma\hat{B} \times \vec{\epsilon}\right)\right] \tag{E4}$$

and the tensor conductivities

$$\sigma_{||} = \sigma \tag{E5a}$$

$$\sigma_{\perp} = \sigma\left(1 + \omega^2\tau_\sigma^2\right)^{-1} \tag{E5b}$$

$$\sigma_H = \omega\tau_\sigma\sigma\left(1 + \omega^2\tau_\sigma^2\right)^{-1} \tag{E5c}$$

Except for the substitution of τ_s for τ_σ, equation (E4) is the same as equation (35) to linear $\omega\tau_\sigma$. Equation (E4) also gives the first Sonine approximation to $\vec{j}$ when σ assumes the corresponding value $\sigma(1)$.

REFERENCES

1. Holway, Lowell H., Jr.: New Statistical Models for Kinetic Theory: Methods of Construction. Phys. Fluids, vol. 9, no. 9, Sept. 1966, pp. 1658-1673.
2. Chapman, Sydney; and Cowling, T. G.: The Mathematical Theory of Non-Uniform Gases. Second ed., Cambridge Univ. Press, 1952. (Reprinted 1960.)
3. Schweitzer, S.; and Mitchner, M.: Electrical Conductivity of Partially Ionized Gases. AIAA J., vol. 4, no. 6, June 1966, pp. 1012-1019; Errata, AIAA J., vol. 4, no. 10, Oct. 1966, p. 1888.
4. Bhatnagar, P. L.; Gross, E. P.; and Krook, M.: A Model for Collision Processes in Gases. I. Small Amplitude Processes in Charged and Neutral One-Component Systems. Phys. Rev., Second ser., vol. 94, no. 3, May 1, 1954, pp. 511-525.
5. Robinson, B. B.; and Bernstein, I. B.: A Variational Description of Transport Phenomena in a Plasma. Ann. Phys. (N.Y.), vol. 18, no. 1, Apr. 1962, pp. 110-169.
6. Spitzer, Lyman, Jr.; and Härm, Richard: Transport Phenomena in a Completely Ionized Gas. Phys. Rev., Second ser., vol. 89, no. 5, Mar. 1, 1953, pp. 977-981.
7. Spitzer, Lyman, Jr.: Physics of Fully Ionized Gases. Second rev. ed., Interscience Publ., 1962.
8. Kaneko, Shobu: Transport Coefficients of Plasmas in a Magnetic Field. J. Phys. Soc. Jap., vol. 15, no. 9, Sept. 1960, pp. 1685-1696.
9. Hirschfelder, Joseph O.; Curtiss, Charles F.; and Bird, R. Byron: Molecular Theory of Gases and Liquids. John Wiley & Sons, Inc., c.1954. (Reprinted with corrections 1964.)

NASA-Langley, 1969 —— 25 L-6301

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D. C. 20546

OFFICIAL BUSINESS

FIRST CLASS MAIL

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS AND
SPACE ADMINISTRATION

020 001 50 51 30S 69100 00903
AIR FORCE WEAPONS LABORATORY/AFWL/
KIRTLAND AIR FORCE BASE, NEW MEXICO 87117

ATT E. LOU BOWMAN, ACTING CHIEF TECH. LIB

POSTMASTER: If Undeliverable (Section 158 Postal Manual) Do Not Return

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546