NASA TECHNICAL NOTE

NASA TN D-5215

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NASA TN D-5215

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by Richard E. Schwirian

Lewis Research Center
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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1969

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ABSTRACT

A new momentum-integral method has been devised for describing the incompressible flow of a conducting fluid in the entrance of an MHD channel. The method is distinctive in that an "edge" stress is permitted to exist at the point of intersection of the boundary layer and the free stream. A prudent choice of the edge stress variation forces the governing differential equation to yield an asymptotic solution with the correct wall stress. Accurate analytical solutions are obtained for laminar flow with and without a magnetic field. Nonmagnetic turbulent flow is also well described by this method.

A NEW MOMENTUM-INTEGRAL METHOD FOR TREATING MAGNETOHY-DRODYNAMIC AND SIMPLE HYDRODYNAMIC ENTRANCE FLOWS

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SUMMARY

A new momentum-integral method has been devised for describing the incompressible flow of a conducting fluid in the entrance of a magnetohydrodynamic (MHD) channel. The method is distinctive in that an "edge" stress is permitted to exit at the point of intersection of the boundary layer and the free stream. The edge stress exerts a drag force on the free stream and mathematically accounts for the fact that the "free" stream becomes less and less free as the fluid enters the channel. A prudent choice of the edge stress variation forces the governing differential equation to yield an asymptotic solution with the correct wall stress. The edge stress method allows accurate analytical solutions of the laminar flow momentum integral equations to be obtained for most reasonable initial conditions. Flows with and without magnetic fields can be treated. Nonmagnetic turbulent flow is also well described by this method.

INTRODUCTION

For the designer of liquid-metal magnetohdryodynamic (MHD) generators or electromagnetic pumps an accurate knowledge of wall shear stresses is imperative. Indeed, the practicality of the MHD generator concepts considered by Elliott (ref. 1) and Patrick and Lee (ref. 2) depends largely on keeping the viscous pressure drops across the device to a minimum. The most obvious way of meeting this need is to keep such devices as short as possible to minimize the time the fluid spends in the high velocity - high shear stress power generation section. It is quite possible, therefore, that fully developed flow may never be attained in a practical generator. If such is the case, the entrance flow problem is of prime importance.

A number of authors have considered the channel entrance flow problem. Dix (ref. 3), Rossow (ref. 4), and Moffatt (ref. 5) consider the laminar flow of a fluid over a flat plate perpendicular to which a magnetic field is impressed. Dix's (ref. 3) solutions are numerical, Rossow (ref. 4) takes a Blasius series approach, and Moffatt (ref. 5) uses a momentum integral method. Moffatt (ref. 5) also considers turbulent flow. These solutions are adequate descriptions of entrance flows only when the channel half-width a (see fig. 1) is very large relative to the boundary layer thickness δ and the free stream is not significantly accelerated. Bodoia and Osterle (ref. 6), Schlicting (ref. 7), Shohet (ref. 8), Roidt and Cess (ref. 9), Moffatt (ref. 10), Maciulaitis and Loeffler (ref. 11), and Dhanak (ref. 12) consider the more realistic laminar entrance

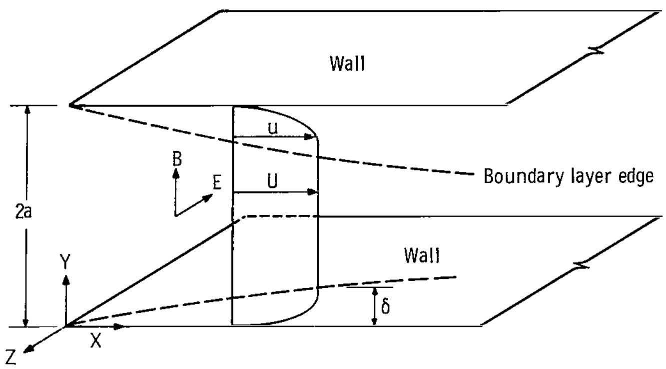


Figure 1. - Magnetohydrodynamic (MHD) entrance flow.

flow problem in which mass conservation demands that the free stream velocity change when the boundary layer thickness changes. Bodoia and Osterle (ref. 6) and Schlicting (ref. 7) consider the nonmagnetic case with a uniform velocity profile at the entrance. Bodoia and Osterle (ref. 6) present a finite-difference technique; Schlicting (ref. 7) matches an upstream Blasius series solution to a downstream series solution whose zeroth order term is the fully developed profile. Shohet (ref. 8) generalizes the work of Bodoia and Osterle (ref. 6) to include magnetic field effects. Similarly, Riodt and Cess (ref. 9) generalize Schlicting's method (ref. 7) to include magnetic field effects. Integral techniques are presented by Moffatt (ref. 10), Maciulaitis and Loeffler (ref. 11), and Dhanak (ref. 12). Moffatt (ref. 10) and Maciulaitis and Loeffler (ref. 11) assume similar profiles: Moffatt uses a "Hartmann" profile; Maciulaitis and Loeffler assume a parabolic profile. Dhanak (ref. 12) uses a fourth power polynomial to represent the velocity profile, applying Karman-Polhausen type conditions to relate the arbitrary constants. Finally, Maciulaitis and Loeffler (ref. 11) consider turbulent flow, using an integral method and a one-seventh power velocity profile. They also consider the case of nonmagnetically fully developed entrance conditions.

With the exception of the numerical techniques (refs. 3, 6, and 8), all of the previous methods have restricted spatial regions of validity. In most cases this means that the solutions obtained are valid only for points sufficiently close to the entrance or sufficiently far downstream. The purpose herein is to introduce an integral technique which (1) is valid throughout the channel, (2) gives accurate results with and without a magnetic field, (3) can be used with any reasonable entrance profile, including uniform entry and nonmagnetically fully developed entry, and (4) is good for laminar and restricted regions of turbulent flows. All this is made possible by the simple device of allowing a viscous edge stress term to exist in the free stream momentum equation. This term represents the drag that the boundary layer exerts on the free stream. If the boundary layer - free stream model of the flow is to be a valid approximation everywhere, such a stress must exist to satisfy the end conditions. If it is not included (as in refs. 4 to 7 and 9 to 12), the validity of the pressure distributions obtained is confined to a small region near the entrance.

The present investigation is concerned with determining the variation of the edge stress and obtaining analytical solutions for the friction factor and pressure defect. Both laminar and turbulent flows are considered.

SYMBOLS

a	channel half-width (Y-direction)
B	magnetic field strength
C_f	local friction factor
$\langle C_f \rangle$	average friction factor
C_{fB}	Blasius friction factor, $0.664(\mu/\rho UX)^{1/2}$
E	electric field strength
f	"edge" stress function
K	load parameter, $E/\langle U \rangle B$
M_H	Hartmann number, $Ba(\sigma/\mu)^{1/2}$
$\mathscr{P}$	pressure defect, $(p - p_0)/\rho\langle U \rangle^2$
$\mathscr{P}'$	hydromagnetic pressure defect, $\mathscr{P} + \left(4M_H^2/R_e\right)x(1 - K)$
Re	Reynolds number, $4\rho\langle U \rangle a/\mu$
Re_m	magnetic Reynolds number, $\mu_o\sigma a\langle U \rangle \lvert K - 1 \rvert$

U	free stream velocity
$\langle U \rangle$	average velocity
u	velocity in boundary layer (X-direction)
u^*	dimensionless velocity, $u/\langle U \rangle$
v	transverse velocity (Y-direction)
X	coordinate parallel to flow
X_e	entrance length
x	X/a
x_1	$4\,x/Re$
x_2	$4\,M_H^2 x/Re$
Y	coordinate parallel to magnetic field
y	Y/a
Z	coordinate perpendicular to X and Y
α_L, α'_L	see eq. (21)
α_T	see eq. (34)
δ	boundary layer thickness
δ^*	displacement thickness
ϵ	dimensionless boundary layer thickness, δ/a
ϵ^*	ϵ/ϵ_∞
η	Y/δ
θ	momentum thickness
μ	viscosity
μ_o	permeability of free space
ρ	fluid density
τ_e	"edge" stress
τ_w	wall stress

Subscripts:

L	laminar
o	entrance value
T	turbulent

w wall

∞ asymptotic value

ANALYSIS

The problem concerns the flow of an incompressible, viscous, electrically conducting fluid in the entrance region of a channel (see fig. 1). The fluid is assumed to have scalar conductivity and viscosity. The width of the duct in the Z-direction is taken to be infinite so that the problem is two-dimensional in the independent variables X and Y. Constant electric and magnetic fields are impressed in the Z and Y directions, respectively. The slip magnetic Reynolds number

$$Re_m = \mu_o a \left| \frac{E}{B} - \langle U \rangle \right|$$

is taken to be small so that the induced magnetic field is also small. Finally, the fluid is assumed to be electrically neutral and Hall effect-free.

For the uniform entry case it is common to use the following boundary layer and free stream momentum integral equations (e.g., refs. 10 to 12):

Boundary layer:

$$\frac{d}{dX}(\rho U^2 \theta) + \rho U \delta^* \frac{dU}{dX} = \tau_w - \sigma B^2 U \delta^* \tag{1}$$

Free stream:

$$\frac{dp}{dX} = \sigma B(E - UB) - \rho U \frac{dU}{dX} \tag{2}$$

Maciulaitis and Loeffler use equations (1) and (2) even for the nonmagnetically fully developed inlet condition. This is clearly a questionable procedure since, in this case, there is no reason to suspect that the pressure gradient is accurately given by equation (2) near the inlet. Schlicting (ref. 7) and Roidt and Cess (ref. 9) also use equation (2), but since theirs are Blasius series solutions, the unintegrated form of equation (1) is used.

Clearly, equations (1) and (2) cannot hold for large X because (1) the asymptotic solution $\tau_w = \sigma B^2 U \delta^*$ is not, in general, compatible with the correct asymptotic solution and (2) there is no shear stress term in equation (2). Both of these difficulties stem from the same source. Equations (1) and (2) are only valid in regions where a true boundary layer exists (i.e., a region where the Y-extent of the constant-velocity free stream is very large in comparison with the boundary layer thickness). The region near the inlet of the uniform entry case is an example of such a region. Far from the inlet the boundary layer thickness and free stream width can be of the same order of magnitude. The result is that a velocity profile and shear stresses develop across the entire channel in the real flow. In this way the downstream pressure gradient becomes dependent on the wall stress. If the boundary layer - free stream model is to produce a similar dependence, allowance must be made for a viscous coupling between the boundary layer and the free stream. This is done in the present work by permitting an edge stress to exist at the boundary layer edge $Y = \delta$.

The manner in which the edge stress enters can be seen by integrating the MHD version of Prandtl's boundary layer equations:

$$\rho u \frac{\partial u}{\partial X} + \rho v \frac{\partial u}{\partial Y} = \frac{\partial \tau}{\partial Y} - \frac{dp}{dX} + \sigma B[E - uB]$$

$$\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} = 0$$

The integration is performed over the variable Y for both the boundary layer and the free stream. To compute the integrals of the inertia and Lorentz force terms in the free stream, the approximations $u = U(X)$ and $v = 0$ are used. Also, a shear stress τ_e is allowed to act at the boundary layer edge. The integrated equations of motion are the following ($\epsilon = \delta/a$):

Boundary layer:

$$\frac{d}{dX}(\rho U^2 \theta) + \rho U \delta^* \frac{dU}{dX} = \tau_w - \frac{\tau_e}{(1 - \epsilon)} - \sigma B^2 U \delta^* \tag{3}$$

Free stream:

$$\frac{dp}{dX} = \sigma B(E - UB) - \rho U \frac{dU}{dX} - \frac{\tau_e}{a(1 - \epsilon)} \tag{4}$$

As noted, the derivation of equation (4) assumes the velocity U is constant over a cross section of the free stream flow. This may be bothersome to the reader in that the existence of shear stresses (in particular, an edge stress) in a flow demands that the velocity profile be nonconstant. The velocity U in equation (4) is, therefore, more aptly thought of as an average value of velocity over the cross section. However, for computational purposes, it is taken to be invariant with Y.

The asymptotic condition is now

$$\left[\tau_w - \frac{\tau_e}{(1 - \epsilon)} - \sigma B^2 U \delta^*\right]_{X=\infty} = 0 \tag{5}$$

Substituting equation (5) in equation (4) and noting that dU/dX approaches zero as X becomes large gives

$$\left(\frac{dp}{dX}\right)_{X=\infty} = \sigma B\left(E - \frac{B}{a}\int_0^{\delta} u\, dY - \frac{B}{a} U \int_0^{a} dY\right)_{\infty} - \frac{\tau_{w,\infty}}{a} \tag{6}$$

Equation (6) is the correct form for the balance of pressure, shear stress, and electromagnetic forces in the fully developed flow. If, further, boundary layer - free stream model is chosen so that (1) the fully developed wall stress $\tau_{w,\infty}$ is the correct value and (2) the mean velocity

$$\langle U\rangle = \frac{1}{a}\left(\int_0^{\delta} u\, dY + U \int_0^{a} dY\right)$$

is the correct value, then the fully developed pressure gradient is automatically correct.

It is clear that, in order that equations (3) and (4) be valid for all X, the edge stress τ_e must vary. Since, at present, the only knowledge of τ_e concerns its end values, assume momentarily that τ_e can be put in the form

$$\frac{\tau_e}{1 - \epsilon} = \tau_w f\left(\frac{\epsilon}{\epsilon_\infty}, \frac{\epsilon_o}{\epsilon_\infty}, \epsilon_\infty\right) \equiv \tau_w f\left(\epsilon^*, \epsilon_o^*, \epsilon_\infty\right) \tag{7}$$

Equations (3) and (4) then become the following:

Boundary layer:

$$\frac{d}{dX}\left(\rho U^2 \theta\right) + \rho U \delta^* \frac{dU}{dX} = \tau_w\left(1 - f - \frac{\sigma B^2 U \delta^*}{\tau_w}\right) \tag{8}$$

Free stream:

$$\frac{dp}{dX} = \sigma B(E - UB) - \rho U \frac{dU}{dX} - \frac{\tau_w f}{a} \tag{9}$$

The use of equations (8) and (9) now awaits only the selection of the boundary layer profiles and the function f. This will be done for laminar and turbulent flow separately.

Laminar Flow

The laminar flow profile is assumed to be parabolic:

$$\frac{u}{U} = 2\eta - \eta^2 \tag{10}$$

where $\eta = Y/\delta$.

This profile is chosen because it is simple and, for zero Hartmann number, exactly describes the true, downstream Poiseuille flow.

The wall stress is then given by

$$\tau_w = \frac{2\mu U}{\delta} \tag{11}$$

The derivative du/dY is zero at the boundary layer edge $Y = \delta$. Although the idea of an edge stress τ_e at a point where the velocity derivative is zero may seem contradictory, it should be remembered that the boundary layer - free stream model is an approximation to the true flow, which contains shear stresses throughout. Because of this approximation the relation $\tau = \mu(du/dY)$ cannot be used to calculate the viscous stresses at all points in the flow. In particular, it cannot be used at the boundary layer edge or in the free stream. The requirement of a nonzero stress at the boundary layer edge is not, therefore, a contradiction. It is a necessary condition if the boundary layer - free stream model is to yield accurate results both near and far from the entrance.

When equation (10) is used, the quantities θ, δ^*, and U can be computed in terms of ϵ, a, and $\langle U \rangle$:

$$\theta = \frac{2}{15}\delta \equiv \frac{2}{15} a\epsilon$$

$$\delta^* = \frac{1}{3}\delta \equiv \frac{1}{3}a\epsilon$$

$$U = \frac{\langle U\rangle}{1 - \frac{\epsilon}{3}}$$

Inserting these equations together with equation (11) into equation (8) yields

$$\left[\frac{2 + \frac{7}{3}\epsilon}{\left(1 - \frac{\epsilon}{3}\right)^2}\right]\frac{d(\epsilon^*)^2}{dx} = \frac{120}{Re}\left[\frac{2(1 - f)}{\epsilon_\infty^2} - \frac{M_H^2}{3}(\epsilon^*)^2\right] \tag{12}$$

where

$$\epsilon^* = \frac{\delta}{\delta_\infty} = \frac{\epsilon}{\epsilon_\infty}$$

$$M_H = Ba\left(\frac{\sigma}{\mu}\right)^{1/2}$$

$$Re = \frac{4\rho\langle U\rangle a}{\mu}$$

$$x = \frac{X}{a}$$

At this point it is necessary to decide the manner is which the asymptotic boundary layer - free stream solution shall approximate the true Hartmann flow. Above all, the asymptotic wall stress and Hartmann wall stress should be equal. The latter is given by (ref. 13)

$$\left(\tau_w\right)_{Hartmann} = \frac{\mu M_H\langle U\rangle \tanh M_H}{a\left(1 - \frac{\tanh M_H}{M_H}\right)}$$

The asymptotic boundary layer - free stream wall stress is given by equation (11). Noting that $\delta_\infty = \epsilon_\infty a$ and $U = \langle U \rangle/(1 - \epsilon/3)$ results in

$$\tau_{w,\infty} = \frac{2\mu\langle U \rangle}{a\epsilon_\infty\left(1 - \frac{\epsilon_\infty}{3}\right)}$$

Equating the previous quantities results in the quadratic equation

$$\epsilon_\infty^2 - 3\epsilon_\infty + \frac{6\left(1 - \frac{\tanh M_H}{M_H}\right)}{M_H \tanh M_H} = 0 \tag{13}$$

The appropriate root of equation (13) for the present problem is

$$\epsilon_\infty = \frac{3}{2}\left[1 - \sqrt{1 - \frac{8}{3}\,\frac{\left(1 - \frac{\tanh M_H}{M_H}\right)}{M_H \tanh M_H}}\right] \tag{14}$$

The selection of the negative root is based on the fact that ϵ must equal one when $M_H = 0$. The value of f at $X = \infty$ is obtained by equating the right-hand side of equation (12) to zero:

$$f\left(1, \epsilon_0^*, \epsilon_\infty\right) \equiv f_\infty = 1 - \frac{M_H^2}{6}\,\epsilon_\infty^2 \tag{15}$$

The initial values of f depend on the inlet conditions. The two obvious cases are the (1) uniform velocity and (2) fully developed (Poiseuille flow) profiles. The values of ϵ and f at the inlet are, therefore, as follows:

Uniform velocity inlet:

$$\left.\begin{aligned} \epsilon_0 &= 0 \\ f\left(\epsilon_0^*, \epsilon_0^*, \epsilon_\infty\right) &\equiv f_0 = 0 \end{aligned}\right\} \tag{16}$$

Fully developed inlet:

$$\left.\begin{aligned} \epsilon_o &= 1 \\ f\left(\epsilon_o^*, \epsilon_o^*, \epsilon_\infty\right) \equiv f_o &= 1 \end{aligned}\right\} \tag{17}$$

In the uniform velocity case, $f_o = 0$ so that the flow near the inlet is Blasius-like. In the fully developed case, f_o is one for zero magnetic field strength (eq. (15) for $M_H = 0$).

Equations (15) to (17) summarize all that is known of the function f. This author has not been able to deduce the functional relation of f to the other variables in the problem although this may, indeed, be possible. Instead, an inductive approach which uses certain known limiting solutions is used. The function f is required to be such that the solutions of equation (12) are consistent with these limiting solutions. The requirements are now listed:

(a) The function f should be such that ϵ varies monotonically with x. That is, if $\epsilon_\infty > \epsilon_o$, ϵ should monotonically increase, and, if $\epsilon_\infty < \epsilon_o$, ϵ should monotonically decrease.

(b) For large x-values, ϵ^* should be expandable in the form (refs. 7 and 9)

$$\epsilon^* = 1 - a_1 \exp(-\lambda_1 x) + \ldots \tag{18}$$

(c) For the uniform inlet case at low x-values, ϵ^* should be expandable as (refs. 7 and 9)

$$\epsilon^* = (x)^{1/2}\left(b_1 + b_2 x^{1/2} + b_3 x + \ldots\right) \tag{19}$$

when ϵ_∞ is of order 1.

(d) For the uniform inlet case at low x-values, ϵ^* should be expandable as (ref. 4)

$$\epsilon^* = (x)^{1/2}\left(c_1 + c_2 x + c_3 x^2 + \ldots\right) \tag{20}$$

when ϵ_∞ is very small compared to unity.

(e) As $x \to \infty$, the value of ϵ^* should approach unity.

For the case $f \equiv 0$, the solutions of equation (12) satisfy conditions (a) to (d). It is, therefore, logical to choose f so that the right-hand side of equation (12) has the same kind of dependence on ϵ^* as it does when $f \equiv 0$. However, when $f \equiv 0$, the resulting solutions do not satisfy condition (e). One choice of f which has the appropriate dependence on ϵ^*, but which also satisfies conditions (e), is obtained by demanding that the right-hand side of equation (12) be directly proportional to $(1 - \epsilon^{*2})$ or

$$\frac{120}{Re}\left[\frac{2(1 - f)}{\epsilon_\infty^2} - \frac{M_H^2}{3}\epsilon^{*2}\right] = \alpha_L(1 - \epsilon^{*2}) \tag{21}$$

With this choice, the solutions of equation (12) asymptotically approach the condition $\epsilon^* = 1$ when X becomes large. The proportionality constant α_L is chosen by insisting that equation (21) be compatible with the entrance conditions (e.g., eq. (16) or (17)). The result is

$$\alpha_L = \frac{120}{Re}\left[\frac{\dfrac{2(1 - f_o)}{\epsilon_\infty^2} - \dfrac{M_H^2}{3}\epsilon_o^{*2}}{1 - \epsilon_o^{*2}}\right]$$

If one uses this definition of α_L, equation (21) can be put in the form

$$f = f_o + \frac{\Delta f}{\Delta\epsilon^{*2}}\left(\epsilon^{*2} - \epsilon_o^{*2}\right) \tag{22}$$

where

$$\Delta f = f_\infty - f_o$$

$$\Delta\epsilon^{*2} = 1 - \epsilon_o^{*2}$$

It is clear that a more arbitrary choice of f would not satisfy all of conditions (a) to (e). For instance, a choice which makes the right-hand side of equation (21) equal to $\alpha'_L(1 - \epsilon^*)$ would satisfy every condition but (d). This does not preclude other possibilities, of course. However, the one given by equation (21) is reasonable, simple, and provides accurate analytical solutions.

The criteria (a) to (e) say nothing of the near entrance behavior of the fully developed entrance case. This is because, to this author's knowledge, no such information exists. Therefore, the behavior of f according to equation (22) does not necessarily apply to this case. However, since the chosen f is not necessarily inapplicable and since no other solutions are available, calculations are performed for this case also.

Substituting equation (21) gives equation (12) as

$$\left[\frac{2 + \frac{7}{3}\epsilon}{\left(1 - \frac{\epsilon}{3}\right)^2}\right]\frac{d(\epsilon^{*2})}{dx} = \alpha_L(1 - \epsilon^{*2}) \tag{23}$$

which can be rearranged to give

$$x = \frac{2}{\alpha_L}\int_{\epsilon_o^*}^{\epsilon^*}\frac{\epsilon^*\left(2 + \frac{7}{3}\epsilon_\infty\epsilon^*\right)d\epsilon^*}{\left(1 - \frac{\epsilon_\infty\epsilon^*}{3}\right)(1 - \epsilon^{*2})} \tag{24}$$

Equation (24) can be integrated by partial fractions to yield

$$x = \frac{18}{\epsilon_\infty^2\alpha_L}\left\{\varphi_1 \ln\left(\frac{\epsilon^* - 1}{\epsilon_o^* - 1}\right) + \varphi_2 \ln\left(\frac{\epsilon^* + 1}{\epsilon_o^* + 1}\right) + \varphi_3 \ln\left(\frac{\epsilon^* - \frac{3}{\epsilon_\infty}}{\epsilon_o^* - \frac{3}{\epsilon_\infty}}\right) - \varphi_4\left[\frac{1}{\left(\epsilon^* - \frac{3}{\epsilon_\infty}\right)} - \frac{1}{\left(\epsilon_o^* - \frac{3}{\epsilon_\infty}\right)}\right]\right\} \tag{25}$$

where

$$\varphi_1 = \frac{-\left(2 + \frac{7}{3}\epsilon_\infty\right)}{2\left(\frac{3}{\epsilon_\infty} - 1\right)^2}$$

$$\varphi_2 = \frac{-\left(2 - \frac{7}{3}\epsilon_\infty\right)}{2\left(\frac{3}{\epsilon_\infty} + 1\right)^2}$$

$$\varphi_3 = -\varphi_1 - \varphi_2$$

$$\varphi_4 = \frac{27}{\epsilon_\infty\left(1 - \frac{9}{\epsilon_\infty^2}\right)}$$

Using equation (11) and the continuity relation $U = \langle U\rangle/(1 - \epsilon_\infty\epsilon^*/3)$ gives the local friction factor C_{fL} in the form

$$C_{fL} = \frac{2\tau_w}{\rho\langle U\rangle^2} = \frac{16}{Re\left(1 - \frac{\epsilon_\infty\epsilon^*}{3}\right)\epsilon_\infty\epsilon^*} \tag{26}$$

The average friction factor $\langle C_{fL}\rangle$ is given by

$$\langle C_{fL}\rangle \equiv \frac{1}{x}\int_0^x C_{fL}\,dx = \frac{864}{\epsilon_\infty^4\alpha_L}\left(\frac{1}{Re\,x}\right)\left\{-\psi_1\ln\left(\frac{\epsilon^* - 1}{\epsilon_o^* - 1}\right) + \psi_2\ln\left(\frac{1 + \epsilon^*}{1 + \epsilon_o^*}\right) - \psi_3\ln\left(\frac{\epsilon^* - \frac{3}{\epsilon_\infty}}{\epsilon_o^* - \frac{3}{\epsilon_\infty}}\right)\right.$$

$$\left. -\psi_4\left[\frac{1}{\left(\epsilon^* - \frac{3}{\epsilon_\infty}\right)} - \frac{1}{\left(\epsilon_o^* - \frac{3}{\epsilon_\infty}\right)}\right] - \frac{\psi_5}{2}\left[\frac{1}{\left(\epsilon^* - \frac{3}{\epsilon_\infty}\right)^2} - \frac{1}{\left(\epsilon_o^* - \frac{3}{\epsilon_\infty}\right)^2}\right]\right\} \tag{27}$$

where

$$\psi_1 = -\frac{\varphi_1}{\left(\frac{3}{\epsilon_\infty} - 1\right)}$$

$$\psi_2 = -\frac{\varphi_2}{\left(\frac{3}{\epsilon_\infty} + 1\right)}$$

$$\psi_3 = \psi_2 - \psi_1$$

$$\psi_4 = \left(1 - \frac{9}{\epsilon_\infty}\right)\psi_1 + \left(1 + \frac{9}{\epsilon_\infty}\right)\psi_2 - \frac{6}{\epsilon_\infty}\psi_3$$

$$\psi_5 = \frac{9}{\left(1 - \frac{9}{\epsilon_\infty^2}\right)}$$

Finally, the dimensionless pressure defect $\mathscr{P}_L$ is obtained by solving for $\tau_e/(1 - \epsilon)$ in equation (3), substituting into equation (4) using the relations $\delta = \epsilon a$ and $U = \langle U \rangle/(1 - \epsilon/3)$, and integrating the resulting equation. This gives

$$\mathscr{P}_L = \frac{p - p_o}{\rho\langle U \rangle^2} = -\frac{x}{2}\langle C_{fL}\rangle - \frac{4M_H^2}{Re}x(1 - K) + \frac{\left(1 - \frac{7}{15}\epsilon_o\right)}{\left(1 - \frac{\epsilon_o}{3}\right)^2} - \frac{\left(1 - \frac{7}{15}\epsilon_\infty\epsilon^*\right)}{\left(1 - \frac{\epsilon_\infty\epsilon^*}{3}\right)^2} \tag{28}$$

where $K = E/\langle U \rangle B$.

Turbulent Flow

The entrance length problem for turbulent flow is complicated by the fact that no simple formula for the fully developed flow is known for arbitrary magnetic field strengths. Murgatroyd (ref. 14) provides experimental results in the Reynolds number range $10^4 < Re < 10^5$. His experiments show that the wall shear stress varies from the Blasius formula value at zero Hartmann number to the Hartmann value at high Hartmann numbers. The behavior is such that, except for a small region near $M_H = 0$, the wall stress monotonically increases with Hartmann number. Harris (ref. 15) has done an excellent dimensional analysis of the fully developed flow and shows that the fully developed wall stress depends specifically on the parameters Re, M_H/Re, and M_H^2/Re. Unfortunately, he is unable to evaluate certain unknown functions of these parameters for all magnetic field strengths. Therefore, rather than attempt an analytical description of the downstream flow, the results of Murgatroyd (ref. 14) are used to obtain $\tau_{w,\infty}$.

For present purposes it is assumed that the velocity profile and wall stress are given by

$$\frac{u}{U} = \eta^{1/7} \tag{29}$$

and

$$\tau_w = \frac{\beta \rho U^2}{2^{1/2}} \left(\frac{\mu}{\rho U \delta} \right)^{1/4} \tag{30}$$

where $\beta = (0.0225) 2^{1/2} = 0.0318$.

By using equations (29) and (30) in equation (8), one obtains, in a manner similar to that already outlined for laminar flow, the governing equation for turbulent flow:

$$\frac{d(\epsilon^*)^{5/4}}{dx} = \frac{90}{Re} \left[\frac{\left(1 - \frac{\epsilon}{8}\right)^{5/4}}{7 + 2\epsilon} \right] \left[\frac{\beta(1 - f)}{\epsilon_\infty^{5/4}} - \frac{4M_H^2}{Re^{3/4}} \left(1 - \frac{\epsilon}{8}\right)^{3/4} \epsilon^{*5/4} \right] \tag{31}$$

Letting

$$(\epsilon')^{5/4} = \left(1 - \frac{\epsilon}{8}\right)^{3/4} \epsilon^{*5/4} \tag{32}$$

using the approximations

$$d(\epsilon')^{5/4} \cong \left(1 - \frac{\epsilon}{8}\right)^{3/4} d\epsilon^{*5/4}$$

$$\frac{\left(1 - \frac{\epsilon}{8}\right)^2}{7 + 2\epsilon} \cong \frac{1}{2} \left[\frac{\left(1 - \frac{\epsilon_o}{8}\right)^2}{7 + 2\epsilon_o} + \frac{\left(1 - \frac{\epsilon_\infty}{8}\right)^2}{7 + 2\epsilon_\infty} \right]$$

and assuming that

$$f = f_o + \frac{\Delta f}{\Delta\epsilon'^{5/4}}\left(\epsilon'^{5/4} - \epsilon_o'^{5/4}\right) \tag{33}$$

gives equation (31) as

$$\frac{d\epsilon'^{5/4}}{dx} = \alpha_T\left(1 - \epsilon'^{5/4}\right) \tag{34}$$

where

$$\alpha_T = \frac{45\beta}{Re^{1/4}\left(1 - \epsilon_o'^{5/4}\right)}\left[\frac{(1 - f_o)}{\epsilon_\infty'^{5/4}} - \frac{4M_H^2}{\beta Re^{3/4}}\epsilon_o'^{5/4}\right]\left[\frac{\left(1 - \frac{\epsilon_o}{8}\right)^2}{7 + 2\epsilon_o} + \frac{\left(1 - \frac{\epsilon_\infty}{8}\right)^2}{7 + 2\epsilon_\infty}\right] \tag{35}$$

The solution to equation (34) is

$$\epsilon'^{5/4} = 1 + \left(\epsilon_o'^{5/4} - 1\right)\exp(-\alpha_T x) \tag{36}$$

Unlike the laminar flow case the variation of the function f (eq. (33)) is not easily justified. The only justifications offered here are that the variation given strongly resembles equation (22) for the laminar flow and the simple form of the solution given by equation (36) would be impossible otherwise. Also, any variation of f that satisfies both end conditions is better than simply omitting f entirely, which is what is inadvertently done in references (5) and (11).

As in laminar flow the two entrance conditions given by equations (16) and (17) apply for turbulent flow as well. The value of ϵ_∞ is obtained by equating the known value of τ_w to the value given by equation (30). The local friction factor C_{fT} is approximated by

$$C_{fT} = \frac{2\tau_w}{\rho\langle U\rangle^2} = \frac{2\beta}{\left(1 - \frac{\epsilon}{8}\right)^{7/4}}\left(\frac{1}{Re\epsilon}\right)^{1/4} = \frac{2\beta}{\epsilon_\infty^{1/4}\left(1 - \frac{\epsilon}{8}\right)^{8/5}}\left(\frac{1}{Re\epsilon'}\right)^{1/4} \tag{37}$$

with ϵ approximated from equation (32) as

$$\epsilon \cong \frac{\epsilon_\infty \epsilon'}{\left(1 - \frac{\epsilon_\infty}{8}\right)^{3/5}} \tag{38}$$

Equation (38) is slightly inaccurate at small x-values but approaches the value given by equation (32) as x increases.

The average value $\langle C_{fT} \rangle$ is obtained from

$$\begin{aligned}
\langle C_{fT} \rangle &= \frac{1}{x} \int_0^x C_{fT}\, dx \\
&= \frac{1}{\alpha_T x} \int_{\epsilon_o'^{5/4}}^{\epsilon'^{5/4}} C_{fT} \frac{d\epsilon'^{5/4}}{1 - \epsilon'^{5/4}} \\
&\cong \frac{2\beta}{(Re\epsilon_\infty)^{1/4} \alpha_T x \left(1 - \frac{\epsilon}{8}\right)^{8/5}} \int_{\epsilon_o'^{5/4}}^{\epsilon'^{5/4}} \frac{d\epsilon'^{5/4}}{\epsilon'^{1/4}\left(1 - \epsilon'^{5/4}\right)} \\
&= \frac{2\beta Q}{(Re\epsilon_\infty)^{1/4} \alpha_T x \left(1 - \frac{\epsilon}{8}\right)^{8/5}}
\end{aligned} \tag{39}$$

where

$$Q = \ln\left(\frac{1 - \gamma_o}{1 - \gamma}\right) + \frac{5}{2} \sum_{j=1}^{2} A_j \ln\left(\frac{\gamma^2 + 2\omega_j\gamma + 1}{\gamma_o^2 + 2\omega_j\gamma_o + 1}\right)$$

$$+ 5 \sum_{j=1}^{2} \left[\frac{B_j - A_j\omega_j}{\left(1 - \omega_j^2\right)^{1/2}}\right]\left\{\tan^{-1}\left[\frac{\gamma + \omega_j}{\left(1 - \omega_j^2\right)^{1/2}}\right] - \tan^{-1}\left[\frac{\gamma_o + \omega_j}{\left(1 - \omega_j^2\right)^{1/2}}\right]\right\} \tag{40}$$

$$\omega_1 = - \cos \frac{2\pi}{5} = - 0.30902$$

$$\omega_2 = \cos \frac{\pi}{5} = 0.80902$$

$$\gamma = (\epsilon')^{1/4}$$

$$A_1 = - \frac{\frac{1}{5}(1 + \omega_1)}{\omega_2 - \omega_1}$$

$$A_2 = \frac{\frac{1}{5}(1 + \omega_2)}{\omega_2 - \omega_1}$$

$$B_1 = - \frac{1}{10}\left(\frac{3 - 2\omega_1}{\omega_2 - \omega_1}\right)$$

$$B_2 = \frac{1}{10}\left(\frac{3 - 2\omega_2}{\omega_2 - \omega_1}\right)$$

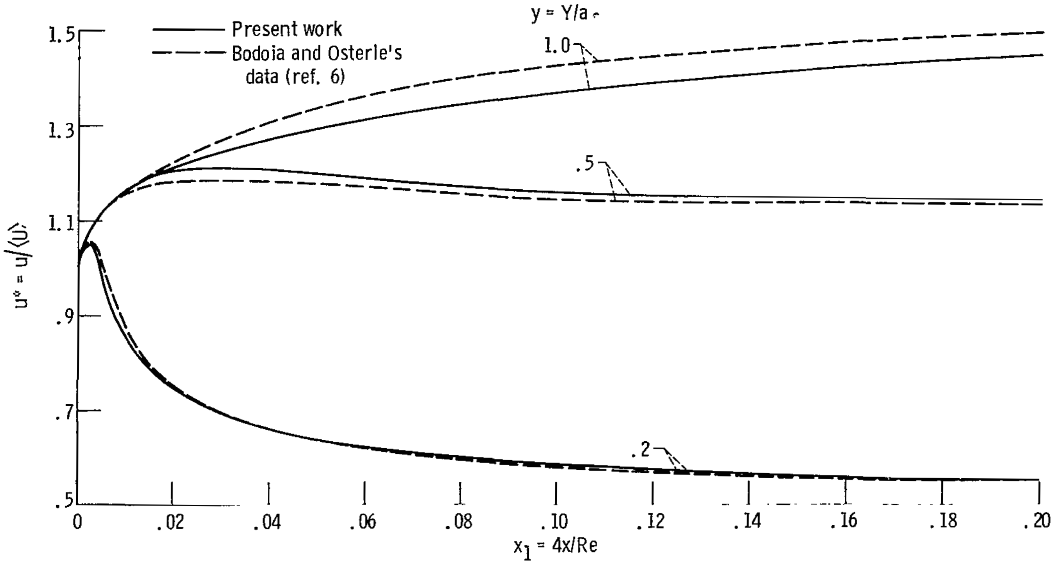
Finally, the pressure defect $\mathscr{P}_T$ is given by

$$\mathscr{P}_T = \frac{p - p_o}{\rho\langle U\rangle^2} = -\frac{x}{2}\langle C_{fT}\rangle - \frac{4M_H^2}{Re}x(1 - K) + \frac{\left(1 - \frac{2}{9}\epsilon_o\right)}{\left(1 - \frac{\epsilon_o}{8}\right)^2} - \frac{\left(1 - \frac{2}{9}\epsilon\right)}{\left(1 - \frac{\epsilon}{8}\right)^2} \tag{41}$$

DISCUSSION

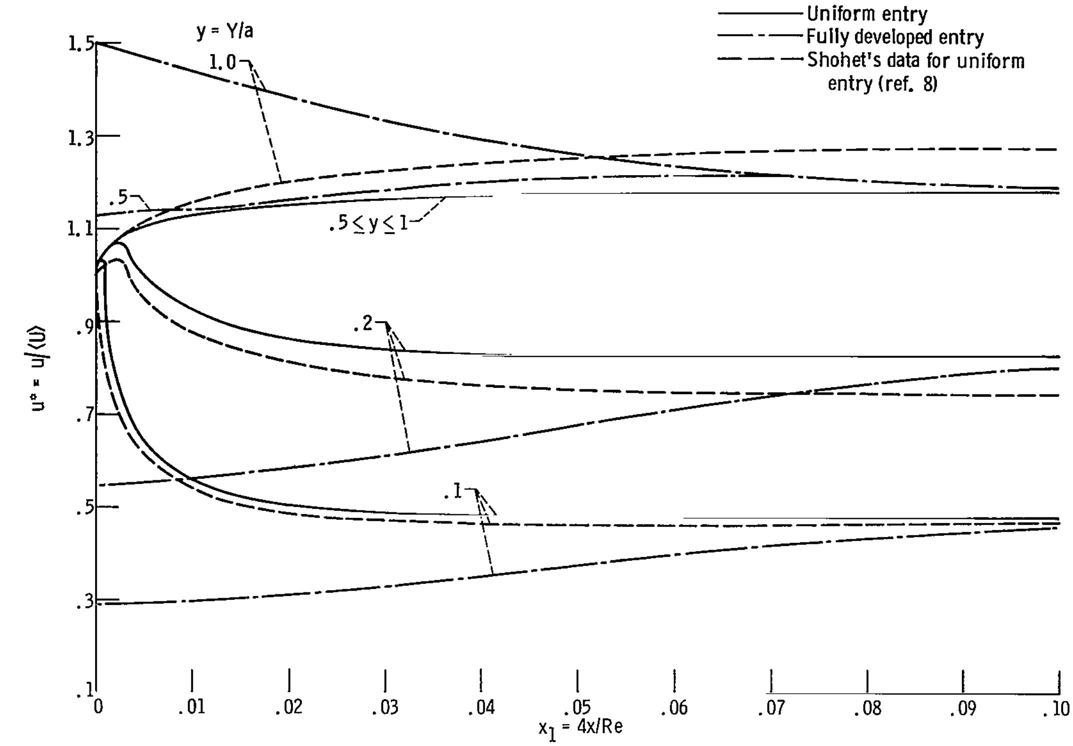
Laminar Flow

Figures 2(a) and (b) show the laminar flow velocity developments for $M_H = 0$ and $M_H = 4$ with uniform entrance conditions. These are compared with the numerical solutions of Bodoia and Osterle (ref. 6) and Shohet (ref. 8). Even though the free stream velocities (large y-values) are somewhat in error, the velocities near the wall (small y-values) follow the numerical solutions quite closely. For friction factor calculations



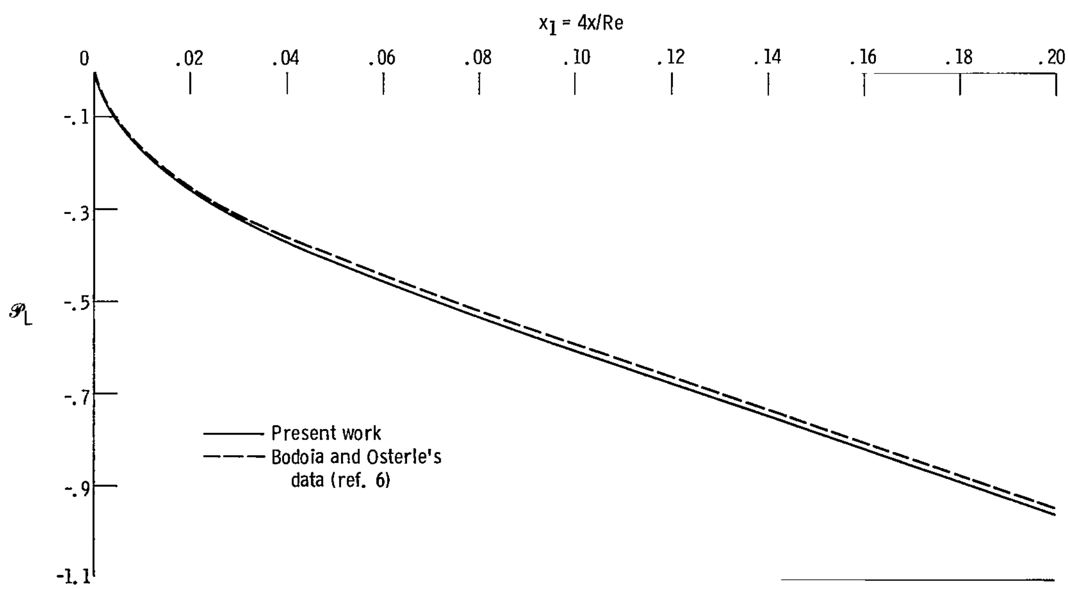
(a) Hartmann number, $M_H = 0$.

Figure 2. - Laminar flow velocity development.



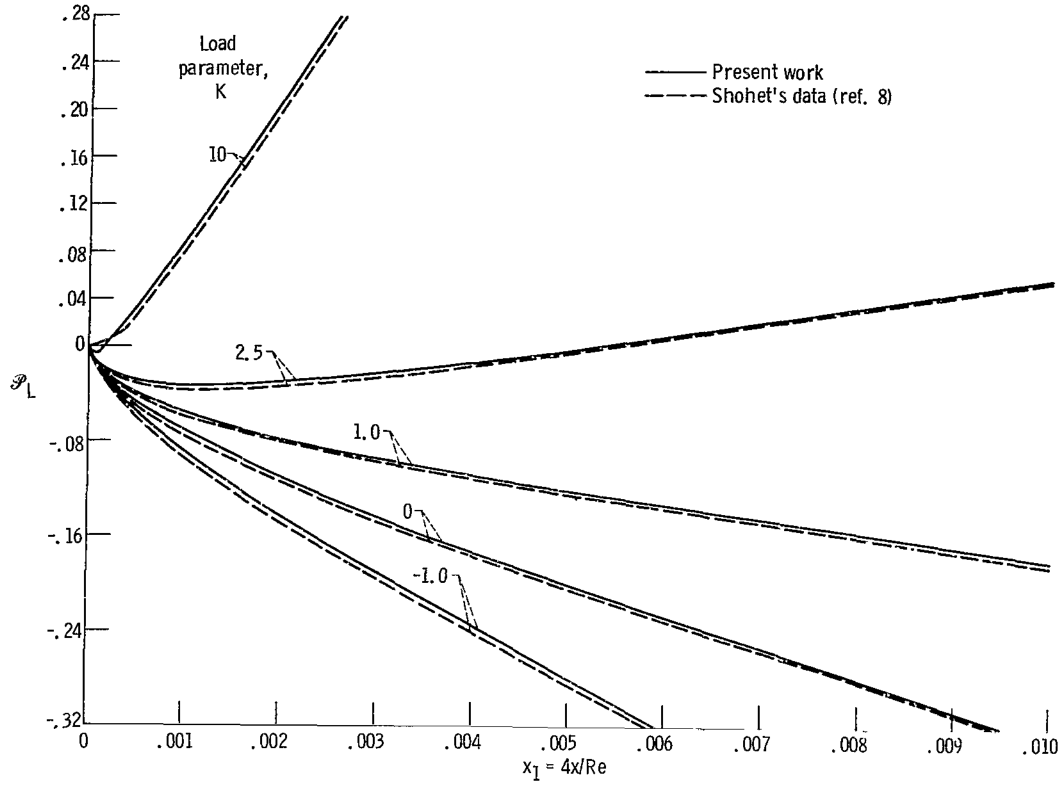
(b) Hartmann number, $M_H = 4.0$.

Figure 2. - Concluded.



(a) Hartmann number, $M_H = 0$.

Figure 3. - Laminar flow pressure defect development.



(b) Hartmann number, $M_H = 4$; uniform entry only.

Figure 3. - Concluded.

only the accuracy of the velocities near the wall is relevant. Figure 2(b) also shows the development for the fully developed entrance case (Poiseuille flow) for $M_H = 4$. The flow is as one might expect - deceleration at high y-values, acceleration at low y-values. No comparisons with numerical solutions are possible here since, to this author's knowledge, no such solutions have been obtained.

Figures 3(a) and (b) illustrate the pressure defect development corresponding to figures 2(a) and (b). In figure 3(b), the load parameter was also varied and its effect is evident. The agreement is excellent.

Figure 4 shows the hydromagnetic pressure defect $\mathscr{P}'_L$ development for $M_H = 4$ and both entrance conditions. The hydromagnetic pressure defect is defined as

$$\mathscr{P}'_L = \mathscr{P}_L + \frac{4M_H^2}{Re} x(1 - K) \tag{42}$$

From equation (28) it can be seen that $\mathscr{P}'_L$ is independent of the electric field. Figure 4 indicates a less severe entrance pressure drop for the fully developed entrance condi-

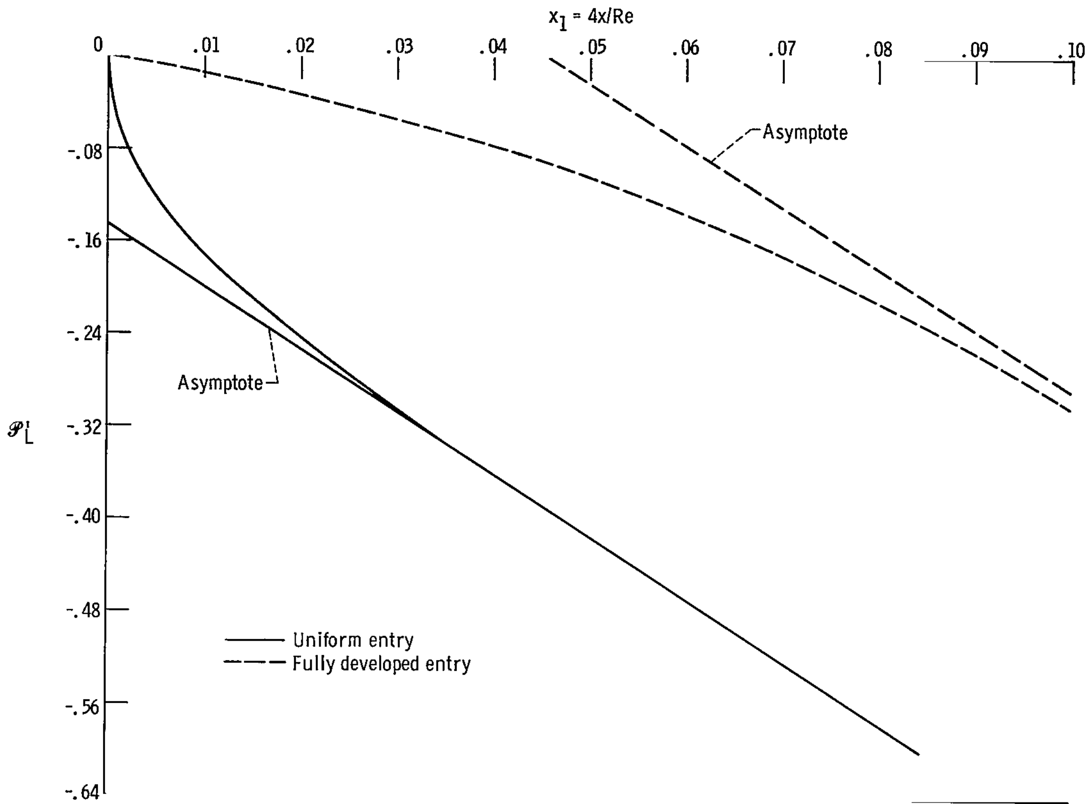


Figure 4. - Laminar flow hydromagnetic pressure defect for Hartmann number $M_H = 4$.

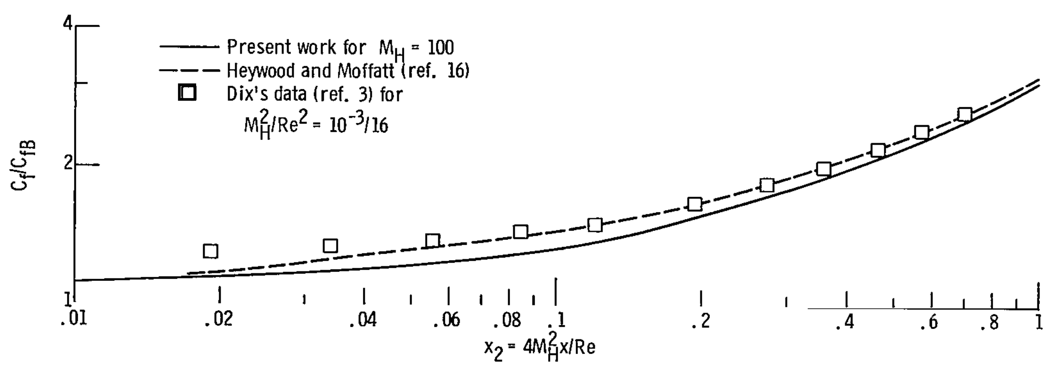


Figure 5. - Ratio of local friction factor to Blasius friction factor as function of distance. $(C_{fB} = 0.664(\mu/\rho UX)^{1/2})$.

tion. It also shows that, for this case, a greater distance from the entrance is required for the pressure gradient to approach its asymptotic value.

Figure 5 is a comparison of the present work with the numerical flat plate results of Dix (ref. 3) and the momentum integral - Hartmann profile method of Heywood and Moffatt (ref. 16). The solution given by equation (26) is representative of flat plate results only when the Hartmann number M_H is very large. Otherwise, the free stream velocity varies with X. The value $M_H = 100$ was chosen as being representative of a large Hartmann number. It can be seen that the agreement is again quite good.

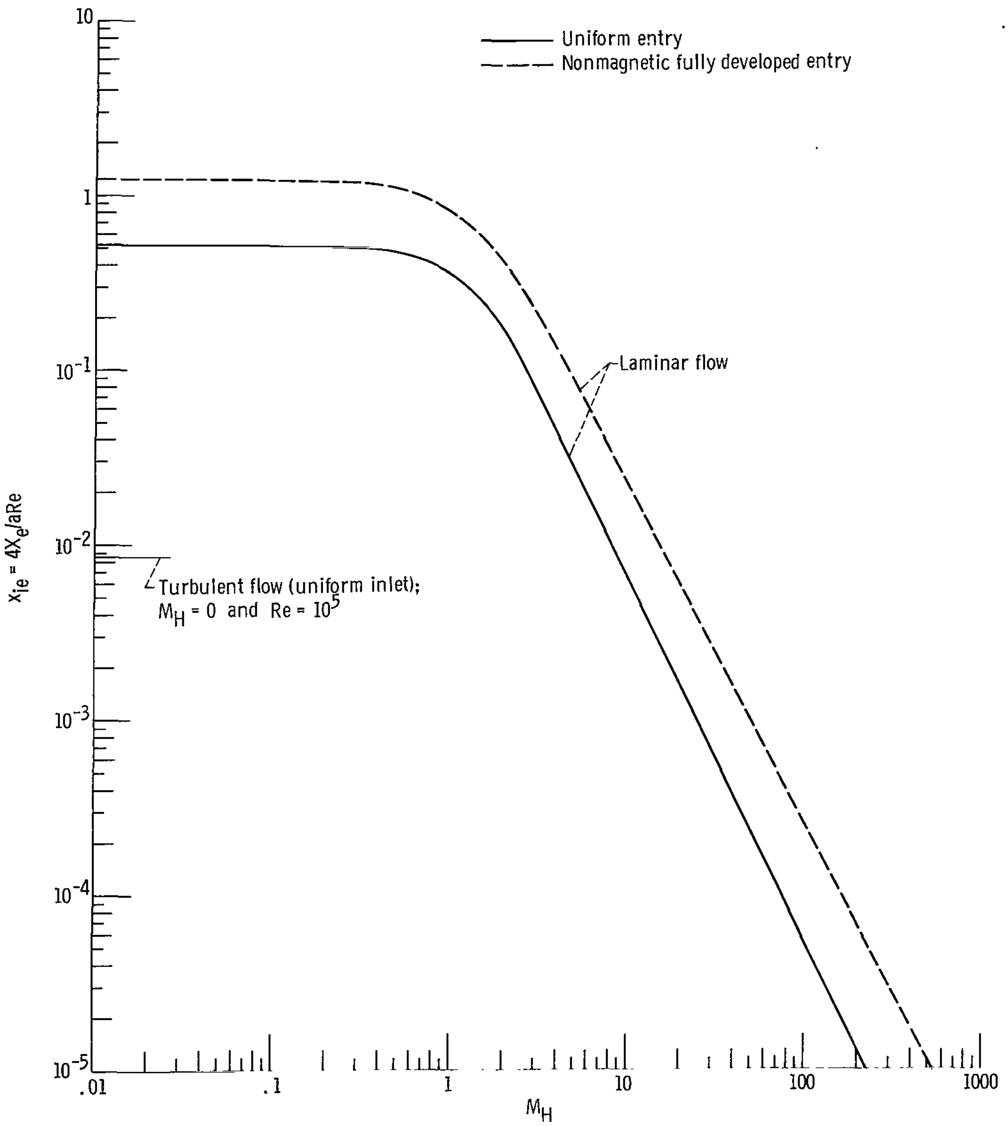


Figure 6. - Entrance lengths for laminar and turbulent flows.

Entrance lengths for both entrance conditions are depicted in figure 6. The entrance length X_e is defined here as the distance from the entrance at which the corresponding $\epsilon(x)$ satisfies

$$\epsilon - \epsilon_o = 0.99(\epsilon_\infty - \epsilon_o) \tag{43}$$

It is evident that the entrance lengths for the fully developed entrance case are sizably larger than for the uniform entrance case. The behavior is similar, however; both approach constant values as M_H becomes small.

Turbulent Flow

When the Hartmann number is zero, equations (29) and (30) should be adequate to describe the turbulent flow entrance problem. This problem was examined (for the uniform inlet case only, of course) and the following quantities were obtained:

$$X_e = 12.1\ \mathrm{Re}^{1/4} a \tag{44}$$

$$\Delta \mathscr{P}_T = 0.088 \tag{45}$$

where X_e is the entrance length, and $\Delta \mathscr{P}_T$ is defined as

$$\Delta \mathscr{P}_T = \lim_{x \to \infty} \left[\left(\frac{8}{7}\right)^{7/4} \beta \left(\frac{1}{\mathrm{Re}\,\epsilon}\right)^{1/4} - \mathscr{P}_T \right] \tag{46}$$

The quantity $\Delta \mathscr{P}_T$ is therefore the difference between the dimensionless pressure drop in a very long section of channel where the flow is fully developed (first term in brackets) and the drop in an equal-length channel with entrance effects $\mathscr{P}_T$.

Equations (29) and (30) were found to be inadequate for describing the flow at Hartmann numbers other than zero. Asymptotic wall shear stresses computed from Murgatroyd's (ref. 14) data frequently yielded negative edge stresses and, for some Hartmann numbers, boundary layer thicknesses greater than the channel half width. Both of these results are physically meaningless.

The only reason for trying equations (29) and (30) at $M_H \neq 0$ is simply expedience; no better models are available. However, the erroneous results mentioned in the preceding paragraph indicate than any computations based on these results would also be erroneous. Hence, no calculations were performed. Clearly, if a meaningful description of the asymptotic flow cannot be found, there is no point in making entrance flow calculations. It should be apparent, however, that this is not a shortcoming of the edge stress method.

CONCLUDING REMARKS

A momentum integral method has been developed for describing the incompressible flow of a conducting fluid in the entrance of an MHD channel. The method is amenable to various initial conditions and the wall shear stress asymptotically approaches the correct downstream value. All this is possible by introducing a boundary layer edge stress, the correct choice of which forces the governing differential equation to the correct asymptotic solution. The technique is accurate for describing laminar entrance flows, with or without a magnetic field. Turbulent flow without a magnetic field can also be analyzed.

The analysis of turbulent entrance flow in a magnetic field is dependent on a knowledgable choice of boundary layer velocity profiles and wall stresses. In the present work one-seventh power velocity profiles and Blasius formula wall stresses are used. This description is found to be good for a Hartmann number of zero but inadequate otherwise.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, February 7, 1969,
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