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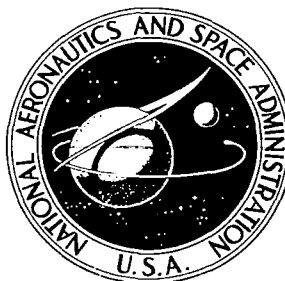
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HYDRODYNAMIC STABILITY OF INVISCID ROTATING COAXIAL JETS

by S. Leibovich

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FOREWORD

The research described herein, which was conducted at Cornell University, Department of Thermal Engineering, was performed under NASA Grant NGR-33-010-042 with Dr. John C. Evvard, NASA Lewis Research Center, as Technical Monitor.

SUMMARY

The inviscid hydrodynamic stability of a swirling coaxial jet to axisymmetric disturbances is investigated when the flow is incompressible but density stratified. In the first part of the paper, the axial and swirl velocity components and the density profiles are all allowed to be arbitrary functions of radius. Howard and Gupta's [1] stability results are then extended to this case with the results:

(i) If $\phi \equiv \frac{1}{r^3} \frac{d}{dr} [r^2 \rho_0(r) V^2(r)]$ with V the azimuthal velocity, then a sufficient condition for stability is $\phi \geqslant \frac{1}{4} \rho_0 \left(\frac{dW}{dr} \right)^2$

(ii) If the flow is unstable, then the growth rate of any unstable mode of wavenumber k cannot exceed $\frac{1}{2} k \, (W_{max} - W_{min})$ where W_{max} and W_{min} are the greatest and least axial velocities occurring in the flow.

In the second part, flow with a cylindrical vortex sheet is examined. The discontinuity sheet introduces an instability which cannot be stabilized by rotation as in the continuous case. Nevertheless, the presence of rotation reduces the growth rates of disturbances. The case of rapid rotation outside, and no rotation inside, is particularly simple to treat. For this case it is found that growth rates of waves of a given wavenumber are proportional to $\varepsilon \frac{\rho_i}{\rho_e}$ if there is a jump in swirl velocity as well as axial velocity at the vortex sheet, and to $\varepsilon^{1/2} (\rho_i | \rho_e)^{3/4}$ if the swirl is continuous (i.e., if it vanishes) at the interface. Here $\varepsilon = \frac{\Delta W}{\Omega a}$ (where W is the jump in axial velocity, a is the radius of the vortex sheet, and Ω is a typical angular velocity) is assumed to be small. The quantity $\rho_i | \rho_e$ is the ratio of density of fluid inside to that outside the vortex sheet.

INTRODUCTION

Prof. F. K. Moore has suggested that the nuclear fuel of a gas core nuclear rocket might be contained by a recirculating flow embedded in a propellant stream. The propellant would completely surround the nuclear core and no solid walls would be required. The sides of the rocket would therefore be cooled by the propellant, which would in the process absorb the energy required to generate thrust. A schematic of the situation appears in Figure 1.
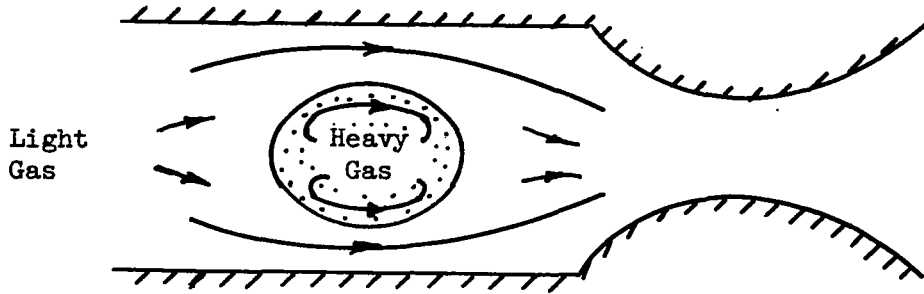
Light
Gas

Heavy
Gas

Figure 1

Recirculating eddies, almost completely self contained, have in fact been observed to form spontaneously in aerodynamic contexts, particularly in the vortical core of the rolled up vortex sheets above delta wings at an angle of attack. They have also been observed to occur in tubes, when the fluid introduced in the tube has a swirling component (cf. M. G. Hall's review article [2] for a summary and bibliography). The phenomenon has been called "vortex breakdown" or "vortex bursting" by aerodynamicists.

If the region inside the closed stream surface can be replaced by a second gas, then this phenomenon could be used as a containment mechanism.

Possible shapes and flow patterns for vortex bursts have been calculated by the author [3] under the following assumptions:

(a) the motion is steady,

(b) the fluids are incompressible and inviscid,

(c) the shapes of the bursts are long and slender,

(d) the motion of the fluid outside, if undisturbed by the presence of the burst, would be in solid body rotation superposed upon a uniform axial flow.

A sketch of the flow pattern found (for which we have coined the term "slender eddies") is shown in Figure 2. The surface of the eddy is a vortex sheet with the fluid inside moving more slowly than that outside. There is no velocity in the azimuthal direction inside the eddy. Outside the eddy the swirl component of velocity is zero on the eddy surface, and increases as the radial distance from the eddy increases, approaching solid body rotation at large distances.

There is no need for the gases inside and outside the eddy to be the same. In particular, a heavy gas could be caught in the eddy, and a light gas be blown past it. Figure 2 shows a sketch of the flow pattern uncovered.
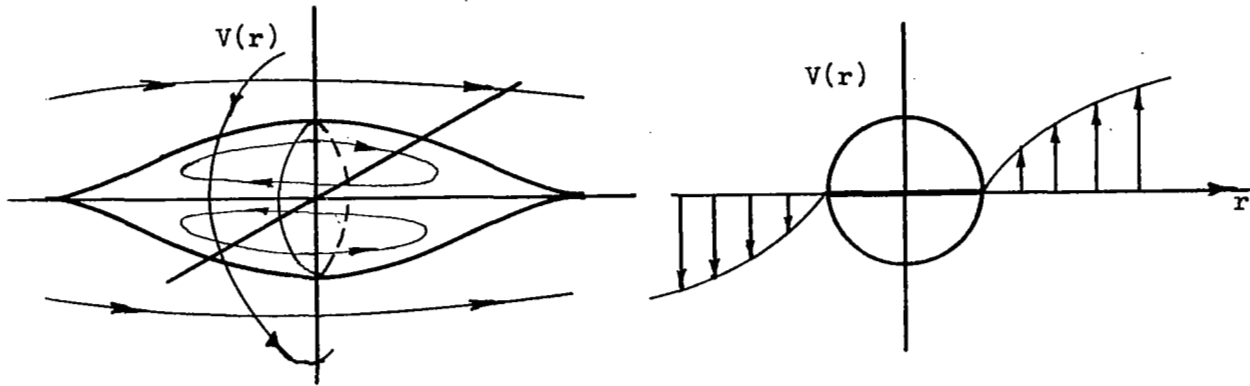

Figure 2

In this report, the stability of a simplified model of this flow pattern to axisymmetric disturbances is examined. No results are given for perturbations which are not axisymmetric.

The model chosen has a basic (unperturbed) motion which is cylindrical. Referring to Figure 3, which is a sketch of the model to be considered,
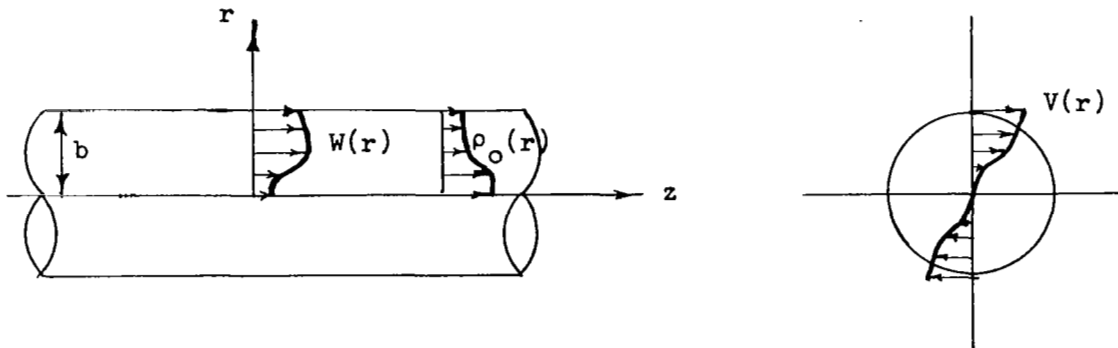

Figure 3

the model assumes motion confined to a tube of finite radius, with a base flow pattern of

$$w = W(r) \ , \quad v = V(r) \ , \quad u = 0 \ , \quad p = p(r) \ , \quad \rho = \rho_o(r).$$

Here p is pressure and ρ is density.

This model is at once more general and more particular than the slender eddy. The assumed velocity profiles may be arbitrary functions of r, but all base quantities are independent of the axial coordinate. It is not unreasonable to expect this cylindrical model to serve as a representation of the midsection of a slender eddy. Furthermore, this model can be used as an approximation of the coaxial jet gas core reactor scheme.

Stability results of a general nature are obtained for this strati-fied, swirling flow in sections 2 and 3 of Part I. The results are generalizations of two stability criteria discovered by Howard and Gupta [1] for the same problem but without density stratification. The present results may be summarized as follows.

(i)   The flow is stable if

$$r^{-3} \frac{d}{dr} \left[\rho_o(r) r^2 v^2(r) \right] \geq \frac{1}{4} \rho_o(r) \left(\frac{dW}{dr} \right)^2$$

For a flow with $W = 0$, this is just the Rayleigh stability criterion.

(ii)   If condition (i) is violated, and if the resulting flow proves to be unstable, then the growth rate is connected to the propagation speed of the mode by a "semicircle theorem". This states that the complex wave speed $c = c_r + i c_i$ of the unstable mode must lie in a semicircle in the upper half complex c-plane with center at $(c_r, c_i) = \left(\frac{1}{2}(W_{min} + W_{max}), 0 \right)$ and with radius $\frac{1}{2}(W_{max} - W_{min})$. W_{max} and W_{min} are the maximum and minimum values of axial speed which occur. Thus the maximum possible growth rate of an unstable mode of length $1/k$ is $\frac{1}{2}k(W_{max} - W_{min})$ and it would propagate at a speed $\frac{1}{2}(W_{max} + W_{min})$, (if it occurs at all).

Part II deals with the situation in which the density and velocity profiles are discontinuous. When a vortex sheet occurs, as in the slender eddy solution, the criteria of Part I, which require differentiable velocity and density profiles, cannot be applied. We therefore specialize our model even further. In doing so, we try to reproduce as closely as possible the salient features of the slender eddy. In the interior of the slender eddy, flow speeds are very low compared to the external flow. The model used in this part therefore is a cylindrical vortex sheet with zero (or constant, it does not matter) axial speed inside, and a different axial speed outside. There is no swirl component in the interior, and an arbitrary swirl component outside. The density ratio is arbitrary, and the heavy fluid may be either inside or outside the discontinuity sheet. The situation is sketched in Figure 4.
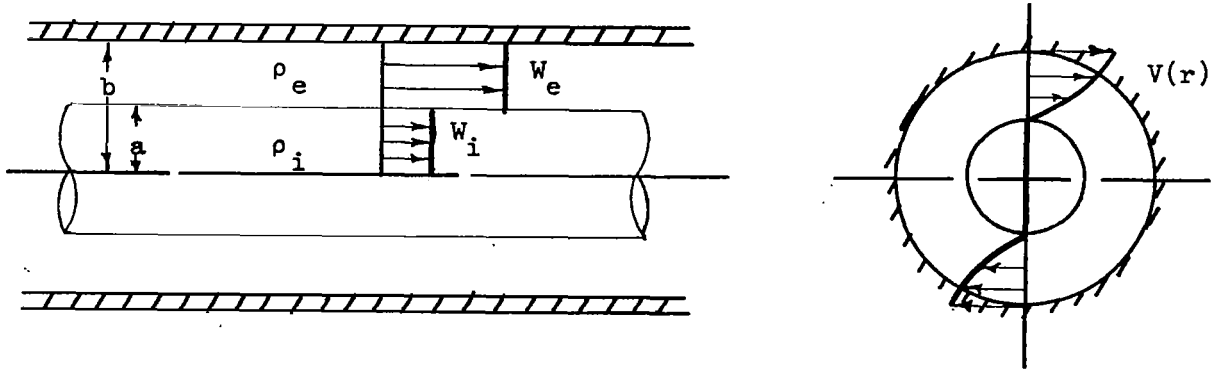
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Figure 4

Two sources of instability are present here. Because of the discontinuity of axial velocity, a Kelvin-Helmholtz instability mechanism is present. The second possibility is a Reyleigh instability [5], if $V^2(r)$ does not increase with radius. There is no possibility for stable flows if the Rayleigh criterion is violated (we show this in Part II), so we assume that $V^2(r)$ increases with r. It does so in the slender eddy flow. If this is the case, then rotation is a stabilizing influence and competes with the Kelvin-Helmholtz mechanism.

Our results for Part II may be summarized as follows:

(i) Although rotation has a stabilizing effect, a Kelvin-Helmholtz instability prevails, so that the flow (which has a jump without structure) is unstable,

(ii) The growth rates decrease as the rotation rate increases, being proportional to

$$\omega_i = \frac{\rho_i}{\rho_e} \frac{k(W)^2}{\Omega a} \frac{ka I_0(ka)}{I_1(ka)}$$

if the swirl at the vortex sheet (located at r = a) is not zero, and being proportional to

$$Wk \sqrt{\frac{W}{\Omega a}} \left[\frac{\rho_i}{\rho_e} \frac{ka I_0(ka)}{I_1(ka)} \right]^{3/4}$$

when the swirl vanishes at r = a.

Here ρ_i and ρ_e are, respectively, interior, and exterior densities, Ω is a typical angular velocity in the outer flow, and I_0 and I_1 are modified Bessel functions. The results hold for Ω large (this is made more precise later).

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The relationship between results of Parts I and II is discussed in the conclusion.

PART I

EQUATIONS AND GENERAL RESULTS FOR INVISCID STABILITY TO AXISYMMETRIC DISTURBANCES

Basic Flow and Small Disturbance Equations

The basic flow field is taken to be,

$$p = P(r)$$

$$\rho = \rho_o(r)$$

$$\Gamma = \Gamma(r)$$

$$u_z = W(r)$$

$$u_r = 0$$

where $\Gamma = ru_\theta$ is the circulation, $\rho(r)$ density, and p pressure. Some general features of its stability to axisymmetric disturbances will now be derived. The results obtained serve to generalize the work of Howard and Gupta [1] (1962) to which the results of this section reduce if the fluid is homogeneous (ρ_o = constant).

The equations of motion governing an axisymmetric flow of an inviscid and incompressible (but density stratified) flow are

$$\bar{\rho}[\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} - \frac{\Gamma^2}{r^3}] = - \frac{\partial p}{\partial r}$$

$$\frac{\partial ru_r}{\partial r} + \frac{\partial ru_z}{\partial z} = 0$$

$$\bar{\rho}[\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z}] = - \frac{\partial p}{\partial z} \tag{1}$$

$$\frac{\partial \Gamma}{\partial t} + u_r \frac{\partial \Gamma}{r\partial r} + u_z \frac{\partial \Gamma}{\partial z} = 0$$

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$$\frac{\partial \bar{\rho}}{\partial t} + u_r \frac{\partial \bar{\rho}}{\partial r} + u_z \frac{\partial \bar{\rho}}{\partial z} = 0$$

$$\Gamma = r u_\theta \quad .$$

We assume that the basic flow is subjected to a small perturbation, so that

$$\bar{\rho} = \rho_o(r) + \rho(r,z,t)$$

$$u_z = W(r) + w(r,z,t)$$

$$u_r = u(r,z,t) \tag{2}$$

$$p = P(r) + \pi(r,z,t)$$

$$\Gamma = \Gamma(r) + \gamma(r,z,t)$$

Thus, the perturbations $\underline{Q} = (u,w,\gamma,\rho,\pi)$ are governed by the linear set of equations

$$u_t + W u_z - \frac{2\Gamma}{r^3}\gamma = -\frac{1}{\rho_o}\pi_r + \frac{\rho}{\rho_o}\frac{\Gamma^2}{r^3}$$

$$w_t + W w_z + uDW = -\frac{1}{\rho_o}\pi_z$$

$$\gamma_t + W\gamma_z + uD\Gamma = 0 \tag{3}$$

$$\rho_t + W\rho_z + uD\rho_o = 0$$

$$(ru)_r + (rw)_z = 0 \quad , \quad D = \frac{d}{dr} \quad .$$

Letting $\underline{Q} = \underline{q}(r)\exp[i(kz - \omega t)] = \underline{q}(r)\exp[ik(z-ct)]$ we may replace $\frac{\partial}{\partial t}(\ )$ by $-i\omega(\ )$ and $\frac{\partial(\ )}{\partial z}$ by $ik(\ )$. The equations for the Fourier components are then

(a) $\qquad ik(W-c)u - \dfrac{2\Gamma}{r^3}\gamma = -\dfrac{1}{\rho_0}D\pi = \dfrac{\Gamma^2}{\rho_0 r^3}\rho$

(b) $\qquad ik(W-c)w + uDW = -\dfrac{1}{\rho_0}ik\pi$

(c) $\qquad ik(W-c)\gamma + uD\Gamma = 0$ $\hfill$ (4)

(d) $\qquad ik(W-c)\rho + uD\rho_0 = 0$

(e) $\qquad ikw = -\dfrac{1}{r}D(ru) = -D_*u$

where $\qquad D_*(\ ) = \dfrac{1}{r}\dfrac{d}{dr}[r(\ )].$

Now $c = c_r + ic_i$ and we assume that c is not real (if it is, the motion is stable). Then we may let

$$u = ik(W-c)F(r)\ .$$

Then from the continuity equation (4e)

$$w = -D_*[(W-c)F(r)]$$

while, from (4c,d),

$$\gamma = -FD\Gamma$$

$$\rho = -FD\rho_0\ .$$

Eliminating π between (4a) and (4b), we find that

$$D[\rho_0(W-c)^2D_*F] - k^2\rho_0(W-c)^2F = -\Phi(r)F \qquad (5)$$

where

$$\Phi(r) = \dfrac{1}{r^3}D(\rho_0 r^2)\ . \qquad (6)$$

In the case $W = 0$, we can see from (5) that $\Phi \geqslant 0$ is a sufficient condition for stability, and this condition is Reyleigh's criterion.

Semi-Circle Theorem

We now follow Howard and Gupta. Multiply (5) by rF^*, where the star superscript stands for complex conjugation, and integrate from $r = 0$ to $r = R$. Remembering that the boundary conditions require that

$$u(0) = 0 \ , \quad \text{so} \quad F(0) = 0$$

$$u(R) = 0 \ , \quad \text{so} \quad F(R) = 0$$

we find that

$$\int_0^R \rho_0 (W-c)^2 [\,|D_* F|^2 + k^2 |F|^2]r\,dr = \int_0^R r\phi(r)|F|^2\,dr \ .$$

Separate this equation into its real and imaginary parts, let

$$S = \rho_0 r[\,|D_* F|^2 + k^2 |F|^2]$$

and assume that $c_i \neq 0$. The imaginary part is

$$c_i \int_0^R (W - c_r)S\,dr = 0$$

or

$$\int_0^R WS\,dr = c_r \int_0^R S\,dr \ .$$

The real part yields

$$\int_0^R [(W-c_r)^2 - c_i^2]\,S\,dr = \int_0^R r\phi|F|^2\,dr$$

Consider a statically stable configuration,

$$\phi \geqslant 0$$

then

$$\int_0^R W^2 S\,dr \geqslant (c_r^2 + c_i^2) \int_0^R S\,dr$$

If a and b are any two real numbers, the above equation implies that

$$\int_0^R (W-a)(W-b)Sdr \geq \{[c_r - \frac{1}{2}(a+b)]^2 + c_i^2 - \frac{1}{4}(a-b)^2\} \int_0^R Sdr .$$

But if

$$a \leq W \leq b$$

then the left hand side is less than or equal to zero, in which case

$$[c_r - \frac{1}{2}(a+b)]^2 + c_i^2 \leq \frac{1}{4}(a-b)^2$$

Since a positive c_i implies instability, all unstable modes lie in the semi-circle in the upper half complex c-plane with center at $(c_r, c_i) = (\frac{1}{2}(a+b), 0)$ and of radius $\frac{1}{2}(b-a)$. A disturbance with a wave speed not in the semi-circle either is stable, with $c_i = 0$, or not a possible solution to the equation of motion.

Howard and Gupta's Semicircle theorem therefore carries over to the stratified case without change.

Sufficient Conditions for Stability

In this section we show that the general stability criterion found by Howard and Gupta for the case of a homogeneous fluid also holds if the fluid is stratified, provided the criterion is suitably modified.

We return to equation (5) and write

$$M = W - c,$$

and

$$F = M^{-1/2}G.$$

We note that

$$D_* F = DF + \frac{F}{r} = M^{-1/2}D_* G - \frac{1}{2}M^{-3/2}(DM)G$$

then, in terms of G equation (5) is (upon division by $M^{1/2}$)

$$D[\rho_o M D_* G] - k^2 \rho_o M G + \frac{1}{2} G \{\rho_o \frac{DM}{r} - \rho_o D^2 M - DMD\rho_o\} = -\frac{G}{M}[\phi - \frac{1}{4} \rho_o (DM)^2] \, .$$

Multiplying this equation by rG^* and integrating from 0 to R, we get

$$-\int_o^R \rho_o M[|D_* G|^2 + k^2|G|^2]rdr + \frac{1}{2}\int_o^R r\rho_o|G|^2[\frac{DM}{r} - D^2 M - DMD\rho_o]dr$$

$$= -\int_o^R \frac{|G|^2}{M} r[\phi - \frac{1}{4}\rho_o(DW)^2]dr \, .$$

Collecting the imaginary parts of the complex equation, we get

$$c_i\{\int_o^R \{\rho_o[|D_* G|^2 + k^2|G|^2] + \frac{|G|^2}{|W-c|^2}[\phi - \frac{1}{4}\rho_o(W')^2]\}rdr\} = 0 \, .$$

Therefore, either $c_i = 0$ (stable) or else

$$\int_o^R \rho_o[|D_* G|^2 + k^2|G|^2]rdr = \int_o^R \frac{r|G|^2}{|W-c|^2}\left[\frac{\rho_o}{4}(W')^2 - \phi\right]dr$$

which is a modified form of the Howard-Gupta result.

Since the left hand side must be positive, this equation cannot be met when

$$\phi \geq \frac{1}{4}\rho_o(W')^2$$

which is therefore a sufficient condition for stability of the flow.

PART II

THE CYLINDRICAL VORTEX SHEET

When the density, shear and swirl distributions have very sharp
gradients at some cylindrical surface, the stability criterion of Part I
becomes difficult to apply. In particular, these distributions can, on
the inviscid scale, appear to be discontinuous. Although the continuously
stratified criterion may be piecewise satisfied, the discontinuity inter-
face may be unstable.

This situation, where the instability arises purely from the presence
of a discontinuity sheet, is familiar. For example, even though the
inflection point criteria may be piecewise satisfied in plane parallel
flow, a vortex sheet is unstable.

In this section, we examine this possibility. To do so, we must
actually solve for the perturbed motion. Since this is intractable for
arbitrary profiles, we must abandon the general viewpoint taken in Part I
and consider a special, and simple, basic motion.

The model we choose is a cylindrical vortex sheet, closely resembling
the middle section of a long, slender eddy found previously by the author.
Sketch 4 illustrates the geometry considered.

The discontinuity surface $r = a$ divides the inner (i) region $r < a$
from the outer (e) region $r > a$. The base flow in each region is taken
to be as follows:

$$r < a; \qquad \rho = \rho_i = \text{constant}$$

$$W = W_i = \text{constant}$$

$$V = V_i = 0 :$$

$$\text{and } r > a; \qquad \rho = \rho_e = \text{constant}$$

$$W = W_e = \text{constant}$$

$$V = V_e(r)$$

and without loss of generality, we may take $W_e = 0$. Thus in what follows,
we drop the subscript of i on W_i and assume it to be the only non-zero
axial velocity.

This particular flow features a general external swirl velocity,
but no swirl inside in accordance with the slender eddy solution. As in
that work, the density is taken as piecewise constant. For simplicity,
however, the shear associated with the slender eddy is suppressed and

the axial velocity is taken to be piecewise constant. It is thought that this model will provide useful qualitative information about the stability of the slender eddy.

Although we are principally interested in the gas core application, with $\rho_i \gg \rho_e$, we can carry out the analysis for ρ_i/ρ_e arbitrary.

Eigenvalue Problem

Equation (5) holds in each region. Therefore,

$$DD_*F_i - k^2F_i = 0 \tag{7a}$$

and

$$DD_*F_e - k^2F_e = -\frac{\Phi e(r)}{\rho_e c^2} F_e \tag{7b}$$

If the deviation of the interface from $r = a$ is $\delta(z,t)$, so that

$$r - a = \delta(z,t)$$

describes the radial position of the interface, then

$$\frac{\partial\delta}{\partial t} + \underline{u} \cdot \nabla\delta = u_r$$

If

$$\delta = \zeta e^{ik(z-ct)}$$

where ζ = constant, then

$$ik(W_{e,i}-c)\zeta = ik(W_{e,i}-c)F_{e,i}(a) \ .$$

Thus the kinematical condition at the interface is

$$F_e(a) = F_i(a) \ .$$

Also, with $W_e = 0$, $W_i = W$,

$$u_i = ik(W-c)F_i$$

$$u_e = -ikcF_e$$

and, since $u_i(0) = 0$, $u_e(R) = 0$, we have

$$F_i(0) = F_e(R) = 0 \quad . \tag{8}$$

Furthermore, the dynamical condition at the interface requires that the pressure match there, or

$$\pi_e(a) = \pi_i(a) \quad .$$

Referring to (4b,e)

$$\pi = +\rho(W-c)^2 D_* F$$

so that at $r = a$,

$$\rho_i(W-c)^2(D_* F_i) = \rho_e c^2(D_* F_e) \quad . \tag{9}$$

Equation (7) must now be solved subject to these boundary conditions, where the object is to find the eigenvalues c. If c is real, the flow is stable, if complex, it is unstable.

The solution for F_i which satisfies the boundary condition on the axis is

$$F_i = AI_1(kr)$$

Since the equation for F_e is considerably more complicated, it is worthwhile to pause to see whether anything of a general nature can be deduced about the problem short of actually solving it in detail.

We can now show that the flow is always unstable to waves of large enough wavenumber providing that there is an axial velocity difference, i.e., $W \neq 0$. A byproduct of this is a demonstration that the flow is stable for $W = 0$ if $\phi > 0$ (Rayleigh criterion) and unstable if $\phi < 0$ for

$W = 0$. If ϕ changes sign for $W = 0$, then this approach gives no information as to stability.

To show this we begin by making the following change of variables:

$$G = rF_e$$

$$\eta = \frac{r^2}{a^2}$$

$$b = \frac{R^2}{a^2}$$

$$\phi_e(r) = \phi(\eta)$$

thus,

$$\frac{1}{r}\frac{d}{dr} = \frac{2}{a^2}\frac{d}{d\eta}$$

and equation (7b) becomes

$$\frac{d^2G}{d\eta^2} - \frac{1}{4}\frac{k^2a^2}{\eta}G = -\frac{a^2}{4\eta}\frac{\phi(\eta)}{\rho_e c^2}G \tag{10}$$

and G is subject to the boundary conditions:

$$G(1) = aA\, I_1(ka)$$

$$\frac{dG}{d\eta}(1) = \frac{\rho_i}{\rho_e}\frac{(W-c)^2}{c^2}aA\frac{ka}{2}I_0(ka) \tag{11}$$

$$G(b) = 0 \quad .$$

Multiplying equation (10) by G^* and integrating from $\eta = 1$ to $\eta = b$, we have

$$-G^*(1)\frac{dG}{d\eta}(1) - \int_1^b\left|\frac{dG}{d\eta}\right|^2 d\eta = \frac{a^2k^2}{4}\int_1^b\frac{1}{\eta}\left[1 - \frac{\phi}{k^2\rho_e c^2}\right]|G|^2 d\eta \quad .$$

But from the boundary conditions (11),

$$G^*(1)\frac{dG}{d\eta}(1) = \frac{\rho_i}{\rho_e}|aA|^2 I_1(ka)I_0(ka)\frac{ka}{2}\frac{(W-c)^2}{c^2} .$$

so that, on putting $G = aAg$

$$-\int_1^b \left\{ \left|\frac{dg}{d\eta}\right|^2 + \frac{a^2k^2}{4\eta}|g|^2 \right\}d\eta = \frac{\rho_i}{\rho_e}\frac{ka}{2}I_0(ka)I_1(ka)\frac{(W-c)^2}{c^2} - \frac{a^2}{4\rho_e c^2}\int_1^b \frac{\phi}{\eta}|g|^2 d\eta .$$

With obvious notation, this equation may be written

$$Q\frac{(W-c)^2}{c^2} - \frac{1}{\rho_e}\frac{R}{c^2} + S = 0$$

where Q, R, S, are all real numbers with $Q > 0$, and $S > 0$.

Rearranging, we have

$$(Q + S)c^2 - 2QWc - \left(\frac{R}{\rho_e} - QW^2\right) = 0$$

If the discriminant of this equation

$$W^2Q^2 + \left(\frac{1}{\rho_e}R - QW^2\right)(Q + S) \geqslant 0$$

then c is real, and the flow stable, whereas if this is not so, the flow is unstable. Rewriting this condition,

$$R(Q + S) - \rho_e W^2 SQ \geqslant 0 .$$

Clearly, if $R < 0$, then the inequality is violated since $Q > 0$, $S > 0$, so the flow is unstable. Note that

$$R = \frac{a^2}{4}\int_1^b \frac{1}{\eta}\phi|g|^2 d\eta$$

is negative if ϕ is negative. Thus we assume that $\phi > 0$ to proceed.

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With R > 0, the flow will be stable if either

$$R \geqslant \rho_e W^2 S$$

$$R \geqslant \rho_e W^2 Q$$

The Rayleigh criterion is recovered as a sufficient condition for stability from either of these by putting W = 0.

From the first criterion,

$$\int_1^b \frac{1}{\eta}[\frac{a^2}{4}\phi - \rho_e W^2]|g|^2 d\eta \geqslant \rho_e W^2 \int_1^b |\frac{dg}{d\eta}|^2 d\eta \geqslant 0$$

From the second criterion,

$$\int_1^b \frac{a^2}{4\eta}\phi|g|^2 d\eta \geqslant \rho_i W^2 \frac{ka}{2}I_0(ka)I_1(ka)$$

Both of these are violated for ϕ fixed if k is large enough. Thus the flow must be unstable at least to short waves (k large) and it is possible that no wavelengths are stable.

Certainly it appears that the larger $\frac{a^2\phi}{\rho_e W^2}$ is, the greater are the chances for stability. This would lead one to hope that ϕ could be raised high enough so that the instability would arise only at very high axial wave numbers, where viscous damping enters as a stabilizing factor. Also, it appears that, for any given ϕ, low frequency waves would be more likely to be stable.

Therefore, one is lead to examine what seems to be the most favorable situation for stability, i.e., $\frac{a^2\phi}{\rho_e W^2} \gg 1$, to see if it is in fact stable. This limiting condition will be made more precise in what follows.

We now non-dimensionalize ϕ by introducing a typical angular velocity Ω, and letting

$$\Gamma = a^2 \Omega f(\eta)$$

so that

$$\phi = \frac{2}{a^4 \eta} \frac{d}{d\eta} \rho \Gamma^2 = 4 \frac{\rho_e \Omega^2}{\eta} ff'(\eta) \ .$$

Large $\frac{\phi a^2}{\rho_e W^2}$ thus will mean large values of the frequency ratio

$$\frac{\Omega a}{W} = \frac{1}{\varepsilon}$$

where ε is a Rossby number based upon a.

In terms of $\alpha = \frac{1}{\varepsilon} \frac{Wk}{\omega}$, the governing differential equation is

$$\frac{d^2 G}{d\eta^2} + \frac{G}{\eta} \left\{ \frac{\alpha^2 ff'}{\eta} - \frac{a^2 k^2}{4} \right\} = 0 \qquad (12)$$

We now obtain the first term of the asymptotic expansion of G for large $|\alpha|$*, by the WBKJ technique (Jeffreys [4], "Asymptotic Approximations").

Put $G = A(\eta) \exp \alpha B(\eta)$, upon separation of powers of α, we find

$$(B')^2 = - \frac{ff'}{\eta^2} \ , \qquad \frac{A'}{A} = - \frac{1}{2} \frac{B''}{B'} \ .$$

Therefore

$$B = \pm i \int_1^\eta \frac{1}{\eta} (ff')^{1/2} d\eta$$

and

$$A \propto (B')^{-1/2} \ .$$

Let

$$\frac{1}{\eta} (ff')^{1/2} \equiv \chi(\eta)$$

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then the first approximation to G is

$$G = \chi^{-1/2} \{C \exp (i\alpha \int_1^\eta \chi d\eta) + D \exp (-i\alpha \int_1^\eta \chi d\eta)\} \ . \tag{13}$$

There are two important special cases to consider

(i) χ is always non-zero

(ii) the slender eddy case where $\chi(1) = 0$, i.e., at the interface the angular velocity is continuous, and therefore vanishes there.

We consider each case in turn.

Case i

Since χ does not vanish in the interval $1 < \eta < b$, the expression (13) for G is a uniformly valid approximation. Applying boundary conditions, we have

$$G'(1) = \frac{\rho_i}{\rho_e} \frac{(W-c)^2}{c^2} \frac{ka^2}{2} AI_0(ka) = -\frac{1}{2}\chi^{-3/2}(1)(C + D) + i\alpha(C - D)\chi^{-1/2}(1)$$

$$G(1) = \chi^{-1/2}(1)(C + D) = aAI_1(ka)$$

$$G(b) = \chi^{-1/2}(b)\{C \exp (i\alpha \int_1^b \chi d\eta) + D \exp (-i\alpha \int_1^b \chi d\eta)\} = 0 \ .$$

This leads to the following eigenvalue problem (after some manipulation)

$$\begin{vmatrix} \frac{1}{2} \frac{\rho_i}{\rho_e} (\frac{W-c}{c})^2 kaI_0(ka) + (\frac{\chi'}{\chi})(1)I_1(ka) & - i\alpha\chi(1) & i\alpha\chi(1) \\ \\ I_1(a) & - 1 & - 1 \\ \\ 0 & \exp(i\alpha\int_1^b \chi d\eta) & \exp(-i\alpha\int_1^b \chi d\eta) \end{vmatrix} = 0$$

or, when the determinant is expanded,

$$[\nu(\epsilon\alpha - 1)^2 + \mu]\tan(\alpha\Delta) + 2\alpha = 0 \qquad (14)$$

where

$$\mu = \frac{\chi'(1)}{\chi^2(1)} \ , \qquad \Delta = \int_1^b \chi d\eta \ , \qquad \nu = \frac{\rho_i ka I_o(ka)}{\rho_e I_1(ka)\chi(1)} \ .$$

We are primarily interested in values of ka ranging from small to moderate. It would not be appropriate to attempt to infer much about the behavior of very short waves (ka >> 1) from our inviscid theory since viscosity assumes an increasingly important role as ka increases (the Reynolds number based upon wavelength decreases as k increases, so that ka large corresponds to low wave Reynolds numbers).

With ka finite, then ϵ is small (since $\Omega >> 1$), but $\epsilon\alpha$ may be moderate or even large. In fact

$$\epsilon\alpha = \frac{W}{c}$$

which can assume any value. In particular, if there are any unstable modes, then for that mode the semicircle theorem (which still applies, by virtue of the boundary condition (9)) assures us that

$$1 < \frac{W}{c} = \epsilon\alpha < \infty$$

There are clearly an infinite number of real solutions, α_n, to the eigenvalue equation (14) and these represent stable oscillations.

On the other hand, there are complex solutions as well, showing the motion to be unstable. These modes and their amplification rates are now uncovered.

Let

$$\alpha = \alpha_1 + i\alpha_2$$

and look for solutions with α_2 large. In terms of $\omega = \omega_1 + i\omega_2$,

$$\omega_2 = -\frac{Wk}{\epsilon}\frac{\alpha_2}{|\alpha|^2}$$

Thus the larger is α_2, the greater the amplification rate. In this way we find the greatest possible amplification rates for given Ω (but large).

$$\text{As } \alpha_2 \to \infty, \ \tan(\alpha\Delta) \sim i\,\frac{\alpha_2}{|\alpha_2|} = \begin{cases} i, & \text{if } \alpha_2 > 0 \\ -i, & \text{if } \alpha_2 < 0 \end{cases}.$$

Assuming amplification, $\alpha_2 < 0$ (to be checked a postiori for consistency, so that

$$(\epsilon\alpha - 1)^2 + \frac{2i}{\epsilon\nu}(\epsilon\alpha - 1) + \frac{\mu}{\nu} + \frac{2i}{\epsilon\nu} = 0$$

(15)

$$\epsilon\alpha - 1 = -\frac{i}{\epsilon\nu}(1 \pm \sqrt{1 + \epsilon^2\mu\nu + 2i\epsilon\nu})$$

or

$$\epsilon\alpha - 1 = -\frac{i}{\epsilon\nu}\Big(1 + \{\tfrac{1}{2}(1 + \epsilon^2\mu\nu) + \tfrac{1}{2}\big((1 + \epsilon^2\mu\nu)^2$$

$$+ 4\epsilon^2\nu^2\}^{1/2}\}^{1/2}$$

$$+ i\{\tfrac{1}{2}\Big[(1 + \epsilon^2\mu\nu)^2 + 4\epsilon^2\nu^2\Big]^{1/2} - \tfrac{1}{2}(1 + \epsilon^2\mu\nu)\}^{1/2}\Big)$$

where the sign has been chosen to ensure that $\alpha_2 = \text{Im}(\alpha)$ is negative, since we have assumed this to be the case.

Now from its definition ν depends on wavenumber, but for all finite wavenumber, it is finite. (It always exceeds $\frac{1}{2}\rho_i/\rho_e$, however, so that it is large in the containment problem.)

Fixing ν, we have the asymptotic value for α_2 for ϵ small

$$\alpha_2 = -\frac{2+\dots}{\epsilon^2\nu}$$

where the dots represent additional small corrections of order ϵ^2.

Furthermore, the real part of α has the approximate value

$$\alpha_1 \sim \frac{1}{\epsilon} + \frac{1}{2\epsilon}(\frac{\mu}{\nu} + 4)^{1/2}.$$

Amplification rates

Since

$$\omega = \frac{\Omega ka}{\alpha} = \frac{\Omega ka}{|\alpha|^2}\alpha^*$$

the growth rate for an unstable mode is

$$\omega_2 = -\frac{ka\Omega}{|\alpha|^2}\alpha_2$$

and to lowest order this is,

$$\omega_2 = \frac{ka}{2}\Omega\nu\epsilon^2 = \frac{1}{2}kW\nu\left(\frac{W}{\omega\alpha}\right) = \frac{\epsilon}{2}kW\frac{\rho_i}{\rho_e}\frac{kaI_o(ka)}{I_1(ka)[f(1)f'(1)]^{1/2}} \quad . \tag{16}$$

The growth rate for waves of moderate length is thus inversely proportional to the angular velocity of swirl, and directly proportional to the density ratio.

Although the effect of rotation is stabilizing, no amount of rotation can completely stabilize the flow to a disturbance of any wavelength.

Case ii

Now we allow $\chi(1) = 0$, which is a feature of the slender eddy theory. Thus the solution (13) is not uniformly valid, and in particular it fails at $\eta = 1$, where a boundary condition must be applied. We render the solution uniformly valid by the standard WBKJ treatment.

First, however, we apply the boundary condition at $\eta = b$ to (13) which is valid there. This yields

$$G = K\chi^{-1/2}\left\{\exp\left(i\alpha\int_1^\eta\chi d\eta\right) - e^{2i\alpha\Delta}\exp\left(-i\alpha\int_1^\eta\chi d\eta\right)\right\} \quad . \tag{17}$$

Near $\eta = 1$,

$$\chi^2(\eta) \sim h^2(\eta - 1) + \ldots$$

where h is constant = f'(1). The turning point occurs at some point $\eta > 1$ where

$$\alpha^2 \eta \chi^2 = \frac{k^2 a^2}{4}$$

or approximately at

$$\eta_c = 1 + \frac{k^2 a^2}{4h^2 \alpha^2}$$

as $\alpha \to \infty$.

If we now introduce the stretched coordinate

$$\xi = (h\alpha)^{2/3}(\eta - 1)$$

equation (12) becomes

$$\frac{d^2 G}{d\xi^2} + \xi G = 0 \tag{18}$$

if error terms of relative error $0 (\alpha^{-2})$ are ignored. Thus the turning point η_c lies within the ξ "boundary layer", and is indistinguishable from $\xi = 0$. The solution of this equation links the point $\eta = 1$ with the solution (17), when properly matched. The matching involves $\xi \to \infty$ in this "inner" region, and $\eta \to 1$ in the "outer" region.

The general solution of (18) is

$$G = \xi^{1/2}[CJ_{1/3}(\frac{2}{3}\xi^{3/2}) + DJ_{-1/3}(\frac{2}{3}\xi^{3/2})] \ .$$

Boundary conditions at $\xi = 0$ ($\eta = 1$) are

$$G = aAI_1(ka)$$

$$\frac{dG}{d\eta} = \frac{1}{2} \bar{\nu}(\frac{W-c}{c})^2 aAI_1(ka) = (h\alpha)^{2/3} \frac{dG}{d\xi}$$

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where

$$\bar{\nu} = \frac{\rho_i}{\rho_e} \frac{kaI_o(ka)}{I_1(ka)} \quad ,$$

and since

$$J_{1/3}(\tfrac{2}{3}\xi^{3/2}) \sim \frac{\xi^{1/2}}{3^{1/3}\Gamma(\tfrac{4}{3})} \quad , \qquad J_{-1/3}(\tfrac{2}{3}\xi^{3/2}) \sim \frac{3^{1/3}}{\xi^{1/2}\Gamma(\tfrac{2}{3})}$$

as $\xi \to 0$,

$$\frac{3^{1/3}}{\Gamma(\tfrac{2}{3})} D - aAI_1(ka) = 0$$

and

$$\frac{(h\alpha)^{2/3}}{3^{1/3}\Gamma(\tfrac{4}{3})} C - \frac{1}{2}\bar{\nu}\left(\frac{W-c}{c}\right)^2 aAI_1(ka) = 0 \quad .$$

Upon eliminating A,

$$\frac{(h\alpha)^{2/3}}{3^{1/3}\Gamma(\tfrac{4}{3})} C = \frac{1}{2}\bar{\nu}\left(\frac{W-c}{c}\right)^2 \frac{3^{1/3}}{\Gamma(\tfrac{2}{3})} D \quad . \tag{19}$$

As $\xi \to \infty$ and $\eta \to 1$, the inner and outer solutions for G must match. As $\eta \to 1$,

$$\alpha \int_1^\eta \chi d\eta \sim \int_0^\xi \xi^{1/2} d\xi = \frac{2}{3}\xi^{3/2} \quad .$$

Thus, as $\eta \to 1$, the outer solution tends to

$$K \frac{1}{\xi^{1/4}}\left(\frac{\alpha}{h^2}\right)^{1/6}\{\exp(\tfrac{2}{3}i\xi^{3/2}) - \exp(2i\alpha\Lambda)\exp(-\tfrac{2}{3}i\xi^{3/2})\} \quad .$$

As $\xi \to \infty$, the inner solution is asymptotic to

$$\frac{1}{\xi^{1/4}} \sqrt{\frac{3}{\pi}} \{C \cos (\frac{2}{3} \xi^{3/2} - \frac{5\pi}{12}) + D \cos (\frac{2}{3} \xi^{3/2} - \frac{\pi}{12})\} \; .$$

Comparing the two, it is seen that a match requires

$$C = \frac{4}{3} iK \sqrt{\pi} (\frac{\alpha}{h^2})^{1/6} e^{i\alpha\Delta} \cos (\alpha\Delta - \frac{\pi}{12})$$

$$\tag{20}$$

$$D = - \frac{4}{3} iK \sqrt{\pi} (\frac{\alpha}{h^2})^{1/6} e^{i\alpha\Delta} \cos (\alpha\Delta - \frac{5\pi}{12}) \; .$$

Combining (19) and (20), we arrive at the eigenvalue equation

$$\frac{\cos(\alpha\Delta - \frac{\pi}{12})}{\cos(\alpha\Delta - \frac{5\pi}{12})} (h\alpha)^{2/3} + \frac{1}{2} \bar{\nu} (\epsilon\alpha - 1)^2 \frac{3^{2/3}\Gamma(\frac{4}{3})}{\Gamma(\frac{2}{3})} = 0 \; . \tag{21}$$

As in case (i), there are clearly an infinite number of real solutions to this equation, but we look for complex roots, with $\alpha = \alpha_1 + i\alpha_2$ and α_2 large and negative. In that event

$$\frac{\cos(\alpha\Delta - \frac{\pi}{12})}{\cos(\alpha\Delta - \frac{5\pi}{12})} \sim \frac{e^{-\frac{\pi i}{12}}}{e^{-\frac{5\pi i}{12}}} = e^{\frac{\pi i}{3}}$$

if exponentially small errors are ignored. Making this substitution in (21) and rearranging, the equation assumes the following form,

$$\lambda^2 (\epsilon\alpha - 1)^6 = (h\alpha)^2$$

where

$$\lambda^2 = \frac{9(\bar{\nu})^3}{8} \left[\frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{2}{3})}\right]^3 \; .$$

Putting $\varepsilon\alpha = x$ and taking the square root of both sides,

$$\lambda(x - 1)^3 \pm \frac{h}{\varepsilon} x = 0 .$$

As $\varepsilon \to 0$, with h and λ fixed, the six solutions of this equation are asymptotically

$$x_{1,2} = \pm \left(\frac{h}{\varepsilon\lambda}\right)^{1/2}$$

$$x_{3,4} = \pm i\left(\frac{h}{\varepsilon\lambda}\right)^{1/2}$$

$$x_{5,6} = \pm \frac{\lambda\varepsilon}{h} .$$

Since the equation was derived to hold for $\mathrm{Im}(\frac{x}{\varepsilon})$ large and negative, the only significant root of this equation is

$$x_4 = - i\left(\frac{h}{\varepsilon\lambda}\right)^{1/2} ,$$

corresponding to the amplification rate

$$\omega_2 = Wk\left(\frac{\varepsilon\lambda}{h}\right)^{1/2} = (Wk)\left[\frac{\bar{\nu}}{2} \frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{2}{3})}\right]^{3/4} \sqrt{3} \left(\frac{W}{a\Omega f'(1)}\right)^{1/2} .$$

Again rotation appears as a stabilizing influence, but is unable to completely suppress the Kelvin-Helmholtz instability. As might be expected, when the angular velocity starts from zero at the interface, the stabilizing effect of rotation is weaker than in case (i).

DISCUSSION

In light of Part I, some instabilities uncovered in Part II are anomalous. All shear layers have a real structure, and replacing a thin structure by a discontinuity is done only to achieve mathematical simplifications. However, in this case it may lead to erroneous results if a thin but stable structure (by Part I) is replaced by a vortex sheet, since Part II sees the discontinuity sheet as unstable.

This is truly a peculiar situation, and demands that the results of Part II be used with an appropriate understanding of their limitations.

All flows with jumps in axial velocity appear unstable in Part II, although, at least, those with stable structure are probably stable. However, the results of Part II have the following utility: if a flow (with a thin layer connecting regions where the motion fits the model of Part II) is unstable, then the amplification rates found in Part II apply.

It should be emphasized that the instabilities found in this paper are local, and do not discredit the existence of slender eddies as a global phenomenon. Such eddies have axially varying shapes and flow quantities which must crucially affect their overall stability. In fact, it is known experimentally that these flows have a stable existence. Instead, an instability uncovered here should be regarded as an indication of the extent of mixing which can be expected between the eddy interior and the outer stream.

The results of Part II (which may also apply to the coaxial jet reactor) suggest that the instability, and hence the vigor of mixing is controllable by rotation of the outer flow. Without rotation, the Kelvin-Helmholtz instability is much stronger, with a growth rate proportional to kW.

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LIST OF SYMBOLS FOR PART II

A,B $A \propto (B')^{-1/2}$, $B = \pm\, i\int_1^\eta \frac{1}{\eta}(ff')^{1/2}d\eta$

        Functions used in WBKJ asymptotic expansion

C,D,K Constants used in WBKJ expansion

F Perturbation stream function

G $G = rF_e$

$I_{o,1}$ Modified Bessel Functions

Q $Q = \frac{\rho_i}{\rho_e}\frac{ka}{2}I_o(ka)I_1(ka)$

R $R = \frac{a^2}{4}\int_1^b \frac{\Phi}{\eta}|g|^2 d\eta$ (Also used as [dimensional] radius of tube wall.)

S $S = \int_1^b \left[|g'|^2 + \frac{a^2 k^2}{4\eta}|g|^2\right] d\eta$

V Azimuthal velocity

W Axial velocity

a Radius of vortex sheet

b R^2/a^2, where R = tube radius

c Complex wave speed $c = c_r + ic_i$

e Subscript referring to region external to vortex sheet

$f(\eta)$ $rV = a^2\Omega f(\eta)$

$g(\eta)$ $g = G/aA$

h $h = \frac{df}{d\eta}(1)$

i	Subscript referring to region inside vortex sheet (also $\sqrt{-1}$)
k	Wavenumber
u,v,w	Radial, azimuthal and axial velocity perturbations
x	$x = \varepsilon\alpha$
α	$\alpha = \dfrac{1}{\varepsilon}\dfrac{Wk}{\omega} = ak\dfrac{\Omega}{\omega} = \alpha_1 + i\alpha_2$
Γ	$\Gamma = rV$ is the circulation (also the gamma function)
δ	Displacement of discontinuity interface from $r = a$, i.e. equation for interface is $r = a + \delta(z,t)$
Δ	$\Delta = \int_1^b \chi d\eta$
ε	$\varepsilon = \dfrac{W}{\Omega a}$, Rossby number
ζ	$\zeta = |\delta|$
η	$\eta = \dfrac{r^2}{a^2}$
λ	$\lambda^2 = \dfrac{9}{8}(\bar{\nu})^3 \left[\dfrac{\Gamma(\frac{4}{3})}{\Gamma(\frac{2}{3})}\right]^3$
μ	$\mu = \dfrac{\chi'(1)}{\chi^2(1)}$
ν	$\nu = \dfrac{\rho_i}{\rho_e}\dfrac{kaI_o(ka)}{I_1(ka)\chi(1)}$
$\bar{\nu}$	$\bar{\nu} = \chi(1)\,\nu$
ξ	Stretched coordinate, $\xi = (h\alpha)^{2/3}(\eta - 1)$
ρ	Density
Φ	$\Phi \equiv \dfrac{1}{r^3}\dfrac{d}{dr}(\rho_o \Gamma^2)$

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$\chi(\eta)$ $\chi = \frac{1}{\eta}(ff')^{1/2}$

ω Wave frequency