NASA CR-66771

PROBABILISTIC SHOCK SPECTRA

By

Richard L. Barnoski

April 1969

Measurement Analysis Corporation
4818 Lincoln Boulevard
Marina del Rey, California 90291

NASA CR-66771

PROBABILISTIC SHOCK SPECTRA

By

Richard L. Barnoski

April 1969

Measurement Analysis Corporation
4818 Lincoln Boulevard
Marina del Rey, California 90291

NASA CR-66771

PROBABILISTIC SHOCK SPECTRA

By Richard L. Barnoski

Prepared under Contract No. NAS 1-8538 by

Measurement Analysis Corporation
Marina del Rey, California

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

ABSTRACT

This report concerns a procedure to characterize the single highest response of a single degree-of-freedom system to a burst of amplitude modulated random noise. The response maxima results are obtained by a digital simulation study and displayed as normalized families of curves in probability. Such curves are identified as probabilistic shock spectra; they have application in the design of structural systems in nonstationary random environments such as gusts, turbulence and earthquakes.

CONTENTS

1. INTRODUCTION

Some response characteristics of unimodal mechanical systems to finite duration deterministic inputs may be categorized by means of shock spectra. Shock spectra are plots which relate response maxima of a single degree of freedom system to parameters of both the system and the excitation. Such plots often are displayed as a family of curves whereby a normalized response is plotted as a function of the time duration of the input and the natural period of the system.

For a nonstationary random input such as an amplitude modulated burst of random noise (a type of shock pulse), it frequently is important in design to predict a response maximum. If we restrict our attention to a single degree of freedom mechanical system, it is consistent to identify as probabilistic shock spectra those plots which relate response maxima and exceedance probability with the system and excitation parameters. This study considers probabilistic shock spectra for amplitude modulated, broadband, Gaussian white noise of finite duration.

2. PROBLEM DEFINITION

The equation of motion for the system of Figure 1 is given by

$$m\ddot{y} + c\dot{y} + ky = f(t) \tag{1}$$

where m is the mass, c the viscous damping coefficient, k the linear spring constant and y the displacement from the static equilibrium position. Since

$$\omega_n^2 = \frac{k}{m}$$

$$2\zeta\omega_n = \frac{c}{m} \tag{2}$$

$$\zeta = \frac{c}{c_c} \quad ,$$

the system equation of motion may be written as

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = \frac{1}{m} f(t) \tag{3}$$

Let us assume the system is initially at rest.

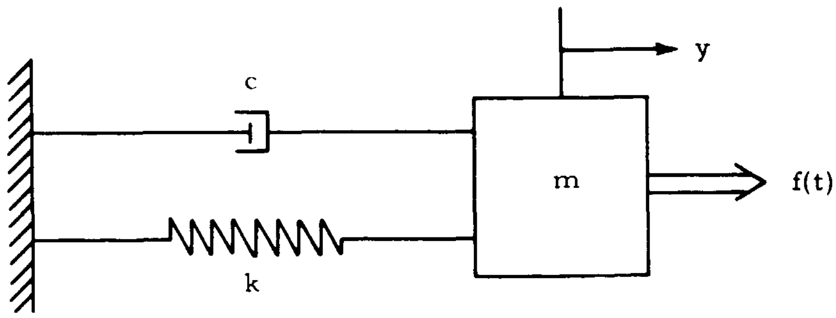


Figure 1. Single Degree of Freedom Mechanical System

The input force excitation is given by

$$f(t) = e(t)\, n(t) \quad , \quad 0 \le t \le t_0 \tag{4}$$

where $e(t)$ is a well-defined envelope function of time duration t_0 and $n(t)$ is stationary, Gaussian, broadband white noise with zero mean. This product defines a sample function of the process $f(t)$ as a burst of amplitude modulated white noise. Our problem is to predict the maximum response of the system to the nonstationary excitation $f(t)$.

3. GENERAL REMARKS

The problem of concern here is associated in the literature with random process problems dealing with barrier and threshold level crossings, first exceedances, and single highest peak (SHP) excursions. In theory, solutions can be determined when y(t) is a Markov process. In practice, solutions become tractable when y(t) is a one-dimensional Markov process functionally dependent upon a single random variable.

Our mechanical system is defined by a second-order differential equation in time with constant coefficients. The response y(t) to f(t) may be expressed in terms of a two-dimensional Markov process. Practically, however, in order to determine level crossing solutions, we seek methods which convert the response process to a Markov process of one dimension. Such methods generally provide approximate solutions since the conversion usually is not exact.

Rosenblueth and Bustamente [21] accomplished this conversion by using a variable substitution, and obtained bounding solutions by solving a boundary-value problem. Caughey and Gray [6] in addition to Mark [17] calculated similar results using Fokker-Planck equations. Gray [12] defined a variable substitution, used approximations for the transition probability, and subsequently made the conversion to a one-dimensional process. He then calculated bounding solutions as well as the mean and mean square for times to a first exceedance. Crandall, Chandiramani and Cook [9] obtained threshold value solutions by solving numerically the Smoluchowski integral equation.

Still other techniques applicable to random process level crossings are available. Shinozuka and Yao [22] developed bounds applicable to any first-passage problem. A renewal process approach was used by Rice and Beer [18]. A series solution approach was considered by Rice

[19], improved upon by Longet-Higgins [15], and developed further by Roberts [20]. The partial sums of the series comprise upper and lower bounds to the first passage density function. Both single and two-sided threshold levels were formulated. Other works that should be examined are those by Srinivasan, Subramanian, and Kumaraswamy [23] and Lin [14].

Yet another approach, very simple in concept and sufficiently accurate for engineering applications, is that of direct simulation. Crandall, Chandiramani and Cook [9] have considered a digital simulation, as has Barnoski [1], though to a more limited extent. Barnoski [3] also has obtained solutions by analog simulation methods. Principally due to the data processing convenience of a digital computer, a digital simulation is used in this study.

4. RESULTS FOR AN INPUT OF STATIONARY WHITE NOISE

Although our problem concerns a particular form of nonstationary excitation, it is worthwhile to examine first exceedance results and other properties of the response process when f(t) is bandlimited stationary white noise and the system response y(t) is stationary with zero mean. Accordingly, we wish to predict the single highest stationary (SHS) response the system will experience within the sampling time interval T.

4.1 RESPONSE PROPERTIES

Before we consider the prediction of this SHS response, let us first recall some pertinent properties associated with the system response y(t). When the damping is small and the bandwidth of the input excitation is wide compared with the half-power bandwidth of the system, the response appears as Figure 2. This is called narrowband noise. Since the zero crossings are very nearly equally spaced, such motion sometimes is described as a harmonic motion of frequency f_n with randomly varying amplitude and phase.

The mean square stationary response is given by

$$\sigma_y^2 = \int_0^{\omega_c} G_y(\omega)\, d\omega \tag{5}$$

where $G_y(\omega)$ is the response spectral density and ω_c is the upper cutoff frequency of the input excitation.

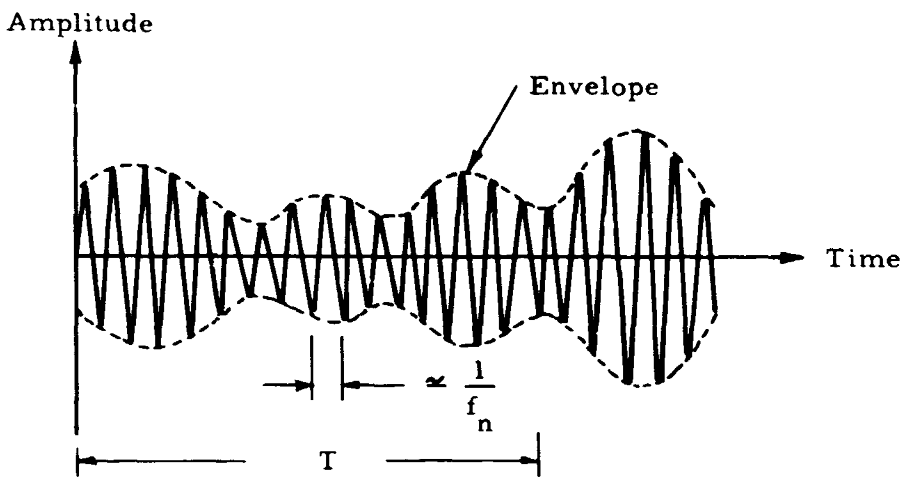


Figure 2. Stationary Response of a Lightly Damped Mechanical Oscillator to Broadband White Noise

Now

$$G_y(\omega) = \left|H(\omega)\right|^2 G_f(\omega) \tag{6}$$

where $H(\omega)$ is the system frequency response function

$$H(\omega) = \frac{1}{m\,\omega_n^2} \cdot \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i\,2\zeta\frac{\omega}{\omega_n}} \tag{7}$$

and $G_f(\omega)$ is the spectral density of the input excitation. For band-limited white noise, $G_f(\omega)$ is the constant G_0. By substituting Eq. (6) into (5) and carrying out the integration,

$$\sigma_y^2 = G_0 \int_0^{\omega_c} \left|H(\omega)\right|^2 d\omega = \frac{\pi Q G_0}{2m^2 \omega_n^3} I_n \tag{8}$$

where

$$Q = \frac{1}{2\zeta} \tag{9}$$

With $R = \omega_c/\omega_n$, the dimensionless term I_n is

$$I_n = \frac{1}{\pi} \tan^{-1} \frac{2\zeta R}{1 - R^2} + \frac{\zeta}{2\pi(1 - \zeta^2)^{1/2}} \ln\left[\frac{1 + R^2 + 2R(1 - \zeta^2)^{1/2}}{1 + R^2 - 2R(1 - \zeta^2)^{1/2}}\right] \tag{10}$$

and ranges in value between zero and one as shown in Figure 3. These curves point out that $H(\omega)$, for small damping values, acts as a highly selective bandpass filter and admits frequency components approximately equal to and centered about f_n. For broadband ncise, $\omega_c/\omega_n >> 1$ so that

$$\sigma_y^2 = \frac{\pi Q G_0}{2m^2 \omega_n^3} \tag{11}$$

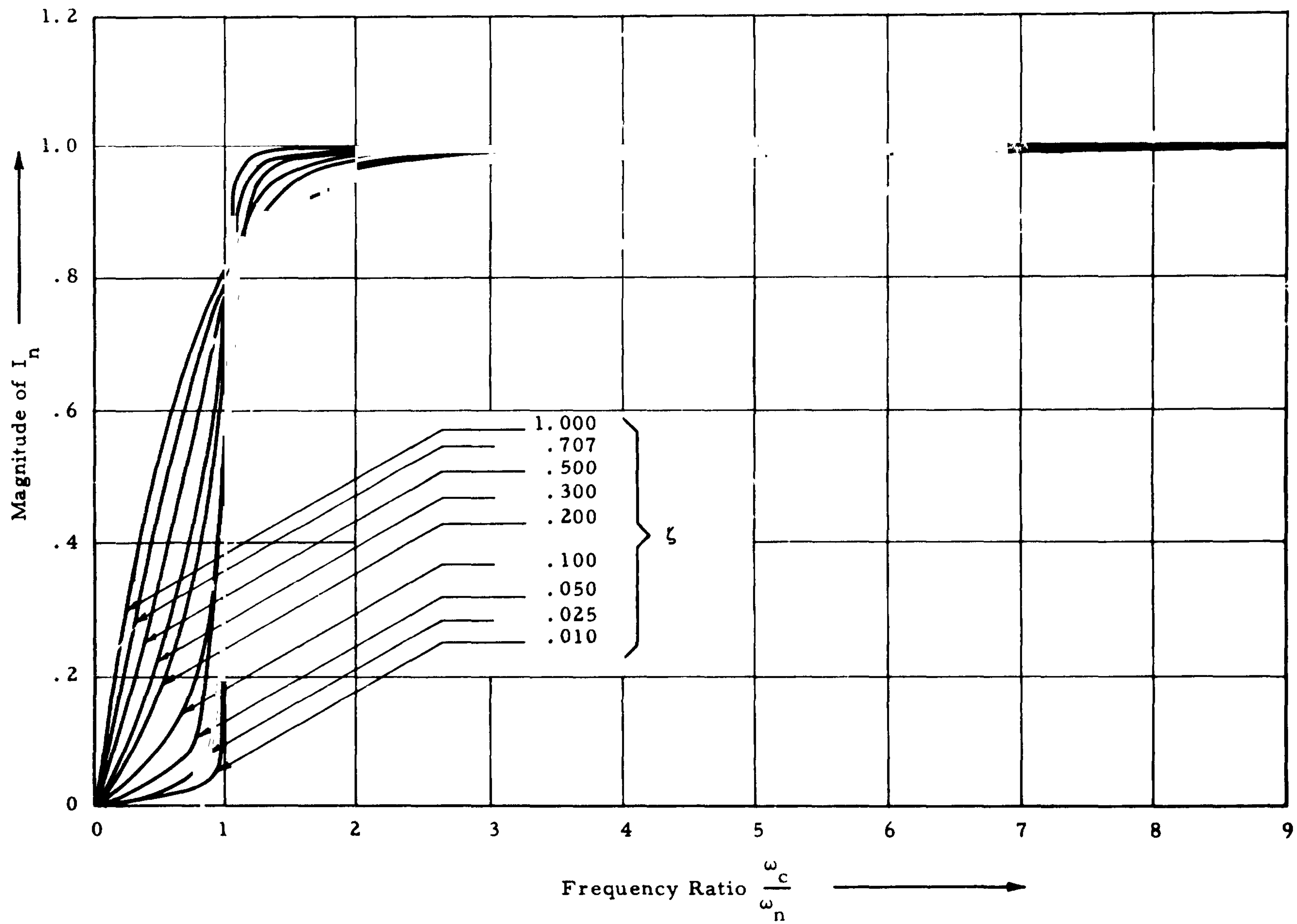


Figure 3. Numerical Value of I_n

For a stationary process, the expected number of crossings per unit time at the level $y = a$ with positive slope is given by [10]

$$n_a^+ = \int_0^{\infty} \dot{y}\, p(a, \dot{y})\, d\dot{y} \tag{12}$$

Since the joint density function $p(a, \dot{y})$ for a Gaussian process with zero mean is

$$p(a, \dot{y}) = \frac{1}{2\pi \sigma_y \sigma_{\dot{y}}} \exp\left[-\frac{1}{2}\left[\left(\frac{a}{\sigma_y}\right)^2 + \left(\frac{\dot{y}}{\sigma_{\dot{y}}}\right)^2\right]\right] , \tag{13}$$

then

$$n_a^+ = n_0^+ \exp\left[-\frac{1}{2}\beta_0^2\right] \tag{14}$$

where n_0^+ is the expected number of zero crossings per second with positive slope and β_0 is the ratio

$$\beta_0 = \frac{a}{\sigma_y} \tag{15}$$

Now n_0^+ is expressed by

$$n_0^+ = \frac{1}{2\pi}\left[\frac{\sigma_{\dot{y}}^2}{\sigma_y^2}\right]^{1/2} \tag{16}$$

where the mean square velocity response $\sigma_{\dot{y}}^2$ is

$$\sigma_{\dot{y}}^2 = G_0 \int_0^{\omega_c} \omega^2 \left|H(\omega)\right|^2 d\omega \tag{17}$$

Upon integration,

$$\sigma_{\dot{y}}^2 = \frac{\pi Q G_0}{2m^2 \omega_n} II_n \tag{18}$$

where

$$II_n = \frac{1}{\pi} \tan^{-1} \frac{2\zeta R}{1 - R^2} - \frac{\zeta}{2\pi(1 - \zeta^2)^{1/2}} \ln\left[\frac{1 + R^2 + 2R(1 - \zeta^2)^{1/2}}{1 + R^2 - 2R(1 - \zeta^2)^{1/2}}\right] \tag{19}$$

with $a = \omega_c/\omega_n$. The term II_n ranges in value between zero and one as shown in Figure 4. For broadband noise, $II_n = 1$ and

$$\sigma_{\dot{y}}^2 = \frac{\pi Q G_0}{2m^2 \omega_n} \tag{20}$$

so that

$$n_0^+ \simeq f_n \tag{21}$$

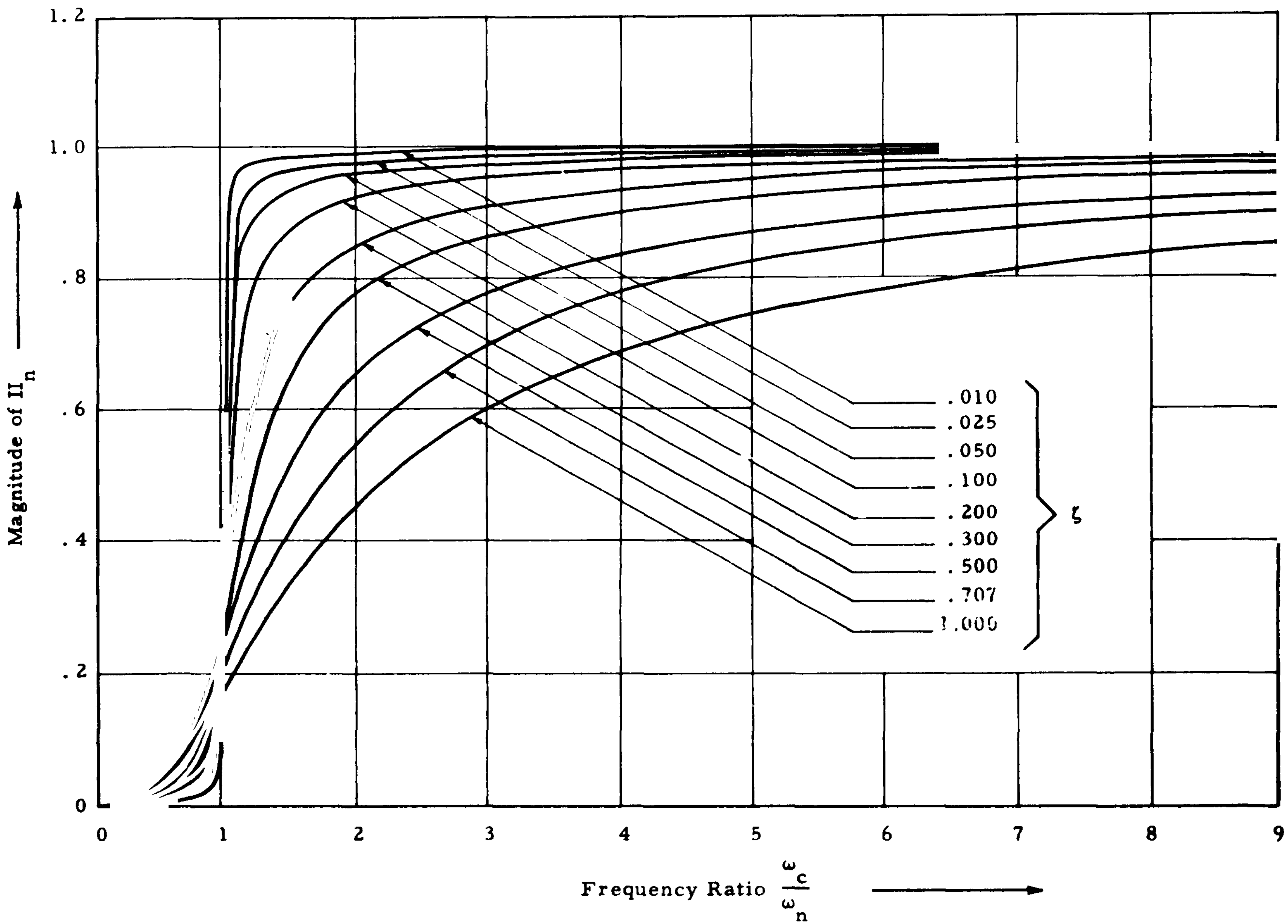


Figure 4. Numerical Value of II_n

Analogous in form to Eq. (16), the expected number of response maxima per unit time is given by [8]

$$N(y_{max}) = \frac{1}{2\pi}\left[\frac{\sigma_{\ddot{y}}^2}{\sigma_{\dot{y}}^2}\right]^{1/2} \tag{22}$$

where

$$\sigma_{\ddot{y}}^2 = G_0 \int_0^{\omega_c} \omega^4 \left|H(\omega)\right|^2 d\omega \tag{23}$$

Upon integration,

$$\sigma_{\ddot{y}}^2 = \frac{\pi Q G_0 \omega_n}{2m^2} \, III_n \tag{24}$$

where

$$III_n = \frac{4\zeta R}{\pi} + 2(1 - 2\zeta^2)\, II_n - I_n \tag{25}$$

Plots of III_n are shown as Figure 5. For increasing values of ω_c/ω_n, III_n increases in value to magnitudes much greater than one and becomes unbounded as $\omega_c/\omega_n \to \infty$. Discussions concerning this behavior are given by Crandall [8] and Lyon [16].

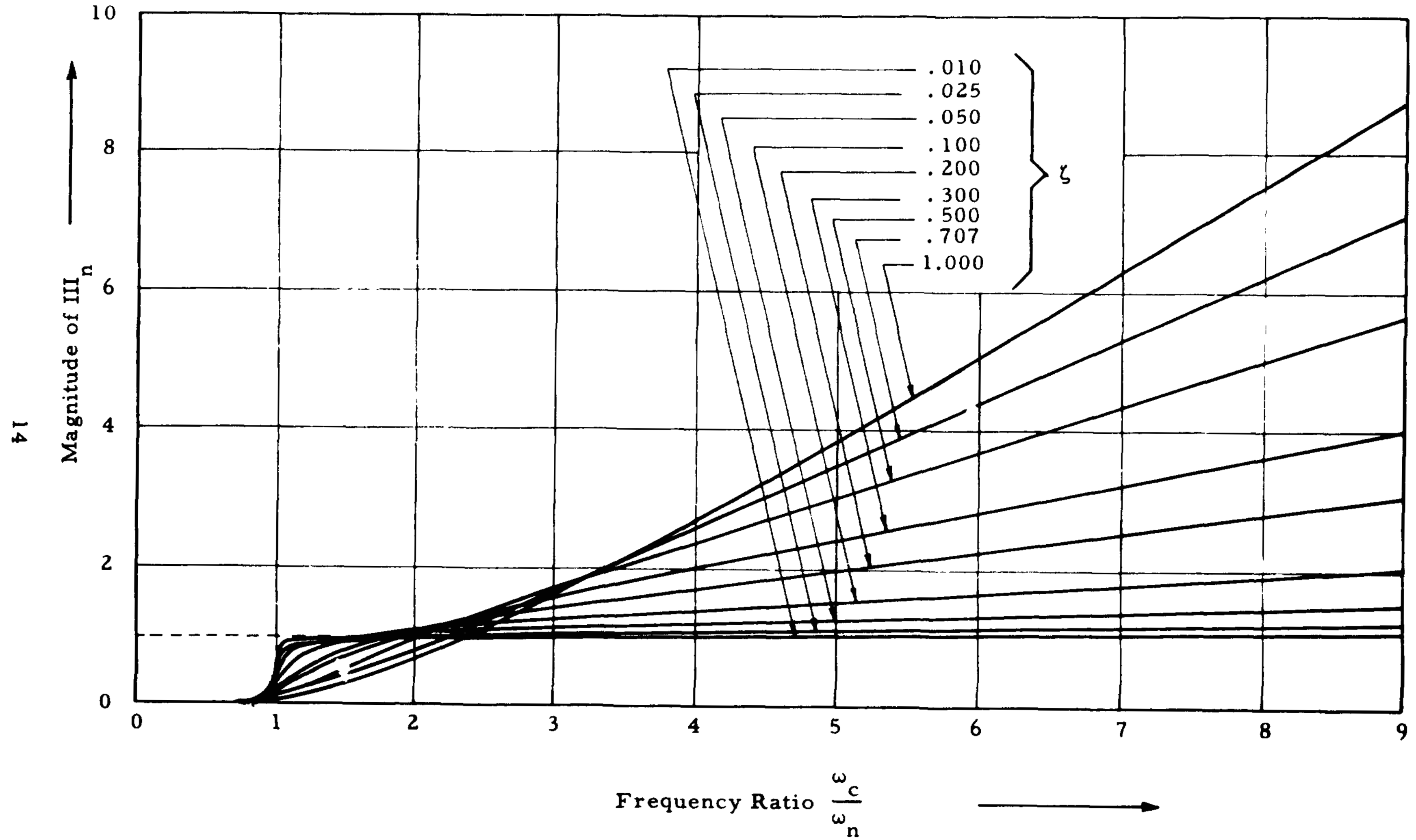


Figure 5. Numerical Value of III_n

Since the input is Gaussian and the system is linear, the probability density function for the process y(t) likewise is Gaussian. The density function for the envelope of the response is Rayleigh. The density function for the response maxima (that is, the number of individual peaks) is given by [8, 19]

$$p(\beta_0) = [1 - r^2]^{1/2} \; f\left(\frac{\beta_0}{[1 - r^2]^{1/2}}\right) + [2\pi r^2]^{1/2} \; \beta_0 f(\beta_0) \; F\left(\frac{\beta_0}{\left[\frac{1 - r^2}{r^2}\right]^{1/2}}\right) \tag{26}$$

where

$$r = \frac{n_0^+}{N(y_{max})} \tag{27}$$

The functions f( ) and F( ) are the tabulated normal functions

$$f(x) = \frac{1}{[2\pi]^{1/2}} \exp\left[-\frac{1}{2}x^2\right]$$
$$F(x) = \int_{-\infty}^{x} f(\eta)\, d\eta \tag{28}$$

For a narrowband process, $r \simeq 1$ and $p(\beta_0)$ reduces to the Rayleigh density function

$$p(\beta_0) = \beta_0 \exp\left[-\frac{1}{2}\beta_0^2\right], \quad a > 0 \tag{29}$$

This is noted to be the same as the density function for the envelope of the response process. As $r \to 0$, $p(\beta_0)$ reduces to the Gaussian density function

$$p(\beta_0) = \frac{1}{[2\pi]^{1/2}} \exp\left[-\frac{1}{2}\beta_0^2\right], \quad -\infty < a < \infty \tag{30}$$

which is that of the process $y(t)$.

4.2 SINGLE HIGHEST STATIONARY RESPONSE

To predict the SHS response we seek an expression for the probability that the maximum value of $|\beta|$ is less than or equal to the level β_0 within the time interval T. Such is noted symbolically by $P_T(|\beta| \leq \beta_0)$.

Let us restrict our attention to relatively high barrier levels, say $\beta_0 \geq 2.5$; we assume each exceedance at the level $y(t) = a$ is a statistically independent event and that the elapsed time to the first exceedance t_e is a random variable with the Poisson+ probability density.

+Although the Poisson assumption is imprecise, it simplifies enormously the mathematics associated with the problem as well as provides conservative bounds [10].

$$p(t_e) = n_a^+ \exp\left[- n_a^+ t_e\right] , \quad t_e \geq 0 \tag{31}$$

Now the probability of y(t) exceeding the level a within the interval $0 \leq t_e \leq T$ is

$$P_T(\beta > \beta_0) = \int_0^T p(t_e)\, d(t_e) \tag{32}$$

which, for the Poisson density and small probability values, reduces to

$$P_T(\beta > \beta_0) \simeq n_a^+ T \tag{33}$$

For the two-sided barrier, $P_T(|\beta| > \beta_0) \simeq 2\, P_T(\beta > \beta_0)$ so that

$$P_T(|\beta| > \beta_0) \simeq 2\, n_a^+ T \tag{34}$$

Upon substitution of Eq. (13),

$$P_T(|\beta| > \beta_0) \simeq 2\, f_n T \exp\left[- \frac{1}{2} \beta_0^2\right] \tag{35}$$

and

$$P_T(|\beta| \leq \beta_0) \leq 1 - P(|\beta| > \beta_0) \tag{36}$$

Alternatively, by means of either "equivalent" RC series circuits and Fokker-Planck solutions [17] or by a diffusion of joint probability in the phase plane [9],

$$\hat{P}_T(\beta \leq \beta_0) \simeq A_0 \exp\left[- \alpha_0 \omega_n T Q\right], \quad T > \tau_{corr} \tag{37}$$

where the "hat" over the P implies an estimate of $P_T(\beta \leq \beta_0)$. The term A_0 defines a constant dependent upon the initial conditions, α_0 is a parameter dependent upon both Q and the threshold level β_0, and τ_{corr} is the time lapse for the autocorrelation function to decay to a negligible value. For threshold levels where $\beta_0 \geq 2.5$, $T > \tau_{corr}$ can be ignored as a restriction and A_0 can be assumed equal to unity. Plots of α_0 versus Q are shown in Figure 6 as families of curves in β_0 for both $\beta = \beta^+$ and $\beta = |\beta|$.

Still another approach is by an active analog simulation [3]. The mechanical system is modeled by a network of operational amplifiers with the input supplied by a noise generator. The single highest system response from many trials are collected, processed, then used to establish estimates for $\hat{P}_T(|\beta| \leq \beta_0)$. Such empirical solutions are shown in Figures 7 and 8 as families of curves in Q for β_0 versus $f_n T/Q$. Solutions are displayed in Figure 7 for $\hat{P}_T(|\beta| \leq \beta_0) = 0.50$

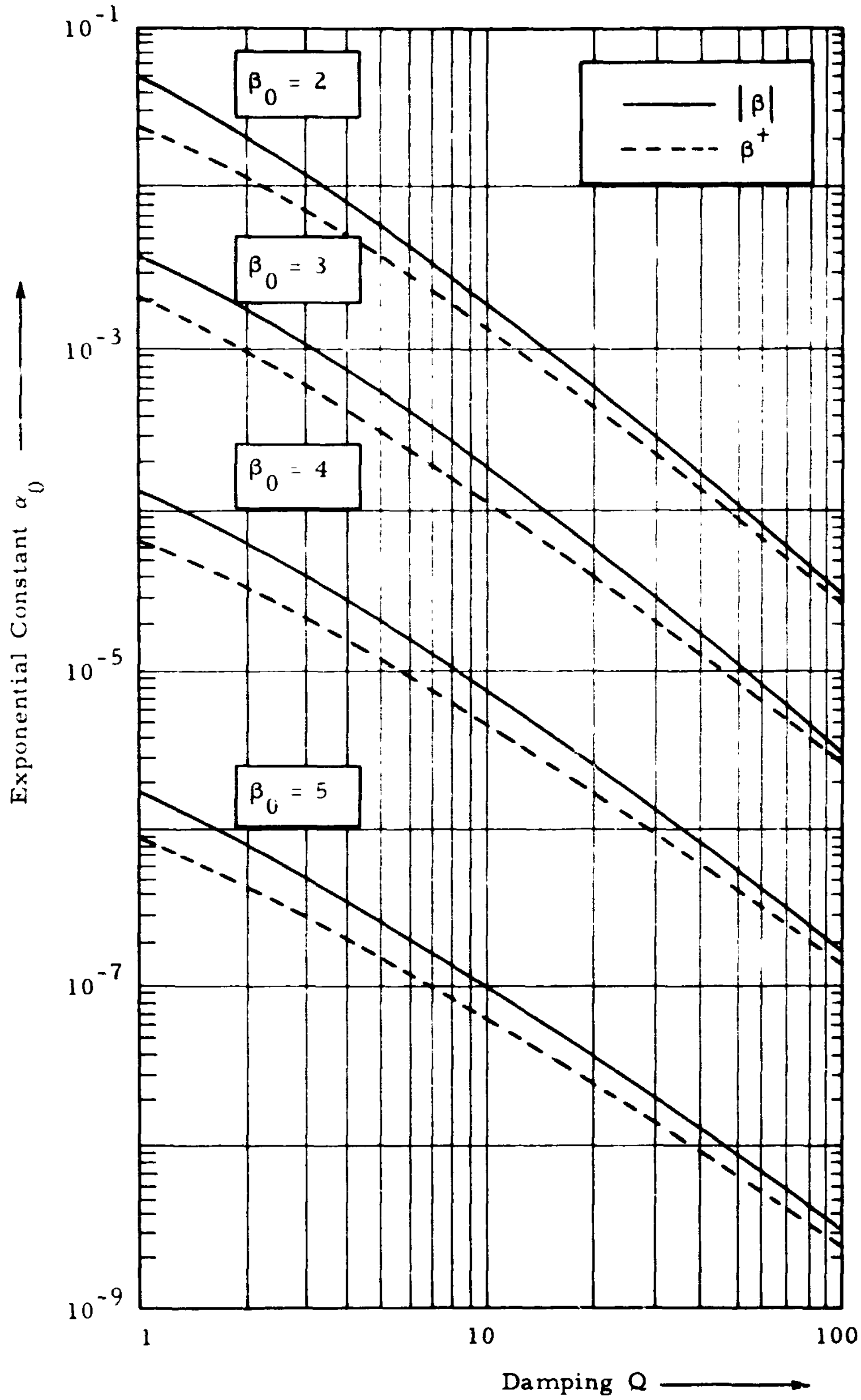


Figure 6. Plot of Exponential Constant α_0

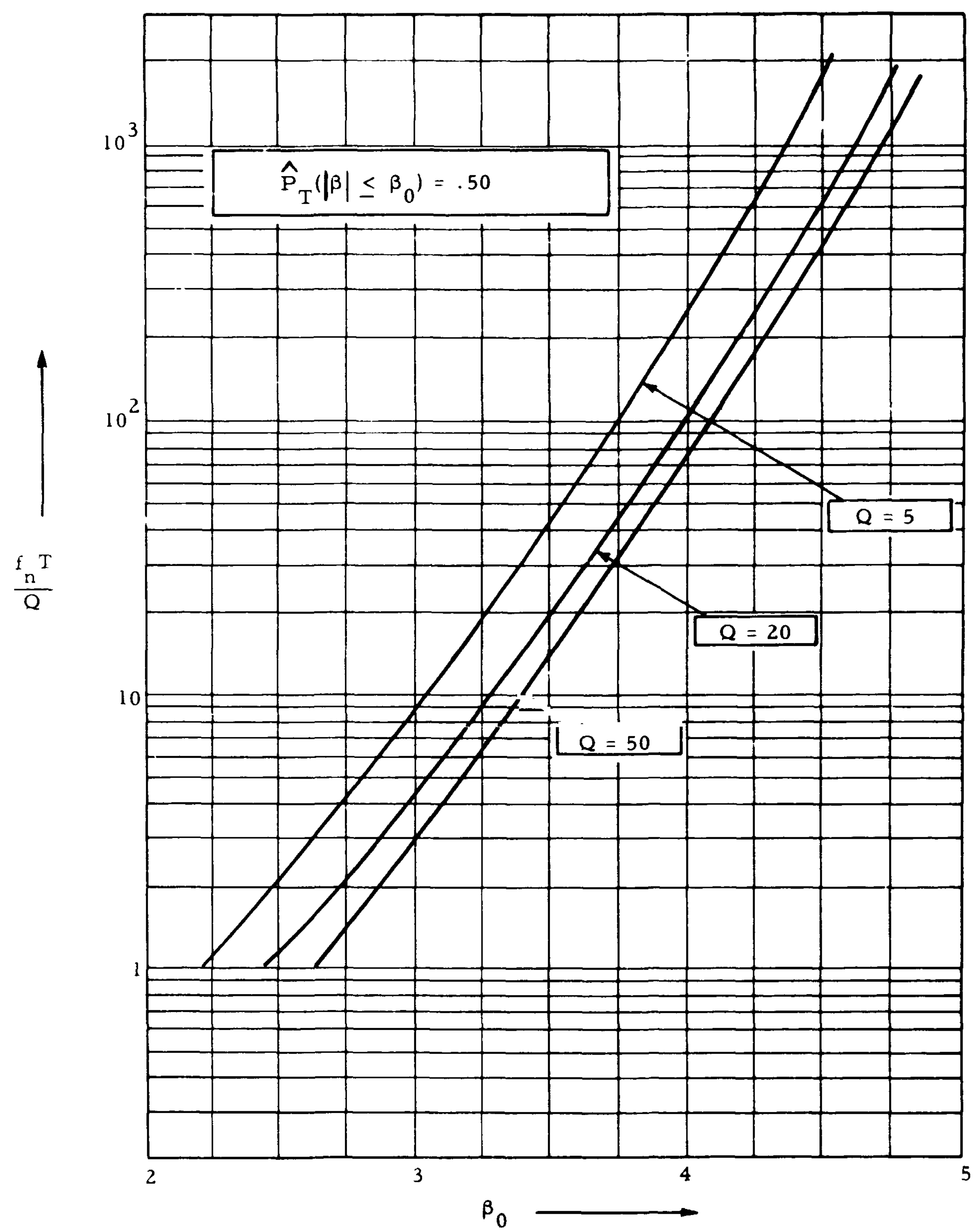


Figure 7. Response Maxima of a Single Degree-of-Freedom System to Stationary White Noise

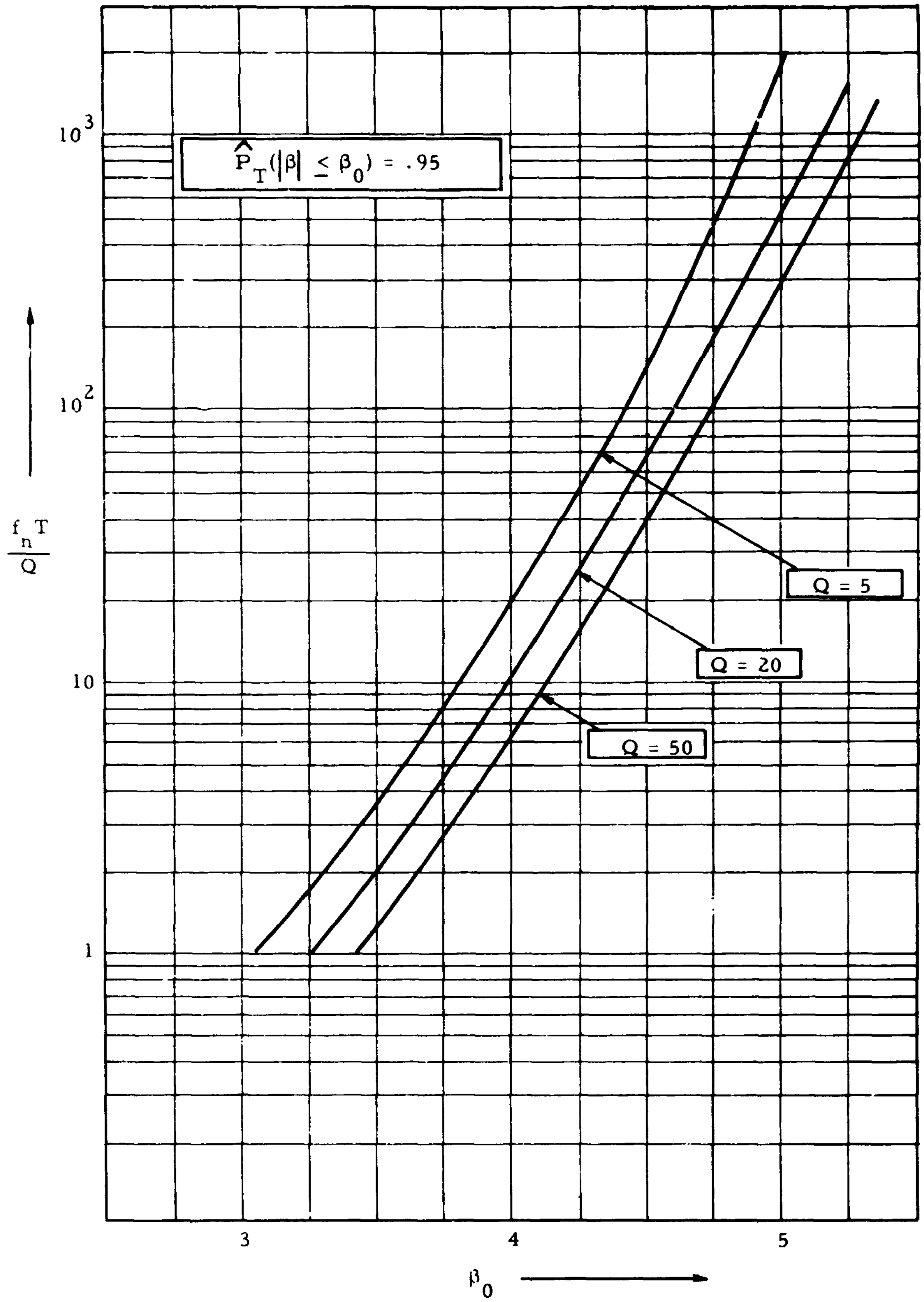


Figure 8. Response Maxima of a Single Degree-of-Freedom System to Stationary White Noise

(this percentile happens to correspond to the average peak value) and in Figure 8 for $P_T(|\beta| \leq \beta_0) = 0.95$.

Now Eqs. (36) and (37) and Figures 7 and 8 all constitute solutions to the SHS problem for Gaussian, stationary, narrowband noise. The analog results can be compared more conveniently with the analytical expressions by rewriting Eqs. (36) and (37) in the form

$$2 f_n T = \hat{P}_T(|\beta| \leq \beta_0) \ \exp\left[\frac{1}{2} \ \beta_0^2\right]$$

$$\alpha_0 = -\frac{1}{2\pi} \ \frac{\ln\left[\hat{P}_T(|\beta| \leq \beta_0)\right]}{Q^2\left[\frac{f_n T}{Q}\right]} \tag{38}$$

Such a comparison shows close agreement between the analog data and Eq. (37). The analog data also support Eq. (36), but show close agreement only for the larger values of $f_n T$. Let us consider the practical ramifications of this behavior.

Equation (36) is a bounding solution, and it departs somewhat from the other solutions for a lesser number of response cycles, say $f_n T < 1000$. Since the SHS response is influenced by Q for the smaller values of $f_n T$, this departure logically could be predicted since Eq. (36) is independent of Q. Now for the larger values of $f_n T$, say 10,000, Eq. (36) yields results approximately the same as the values obtained from the other solutions. This behavior suggests the SHS response becomes independent of Q for a large number of response cycles.

In summary, these solutions markedly point out the dependency of the single highest stationary response upon both the sampling time and an acceptable value of probability. For example, Figure 8 shows that $\beta_0 = 3$ (i. e., a peak value which is three times the stationary rms response) corresponds to $\hat{P}_T(|\beta| \leq \beta_0)$ of approximately 0.95 at $f_n T/Q \simeq 0.2$ and to $\hat{P}_T(|\beta| \leq \beta_0)$ of approximately 0.50 at $f_n T/Q \simeq 5.0$. Expressed in another way, such statements point out there is a 5% probability that a peak value will exceed $3\sigma_y$ for $T \simeq 0.2\, Q/f_n$ and a 50% probability that a peak value will exceed 3σ for $T \simeq 5Q/f_n$. It is clear from these results that the design of structure and or equipment based upon a single exceedance criteria in a random environment entails knowledge of the system, statistical assumptions of the environment, an estimate of the time duration in the environment, and an acceptable probability level of design. The extension of such results to include multi-degree of freedom structures, although examined partially [2], remains as a future task.

5. NOISE BURST STUDY

5.1 PRELIMINARY CONSIDERATIONS

Unfortunately, many random environments cannot be characterized as an input excitation of stationary, Gaussian white noise. Environments such as those for gusts, turbulence, and earthquakes can be represented more realistically by

$$f(t) = e(t)\, n(t)\ , \qquad 0 \leq t \leq t_0$$

where, as mentioned previously, e(t) is a well defined envelope or modulation function and n(t) is bandlimited, Gaussian white noise.

This excitation is a sample function from a nonstationary random process and can be formed by shaping the output of a random noise generator with a time limited envelope function. A typical input for broadband white noise modulated by a rectangular step and the subsequent system response are shown in Figure 9. With the system initially at rest, our task is to predict the maximum value of the response y(t). This solution is noted by $P_{t_0}(|\beta| \leq \beta_0)$.

Consider the five envelope functions shown in Figure 10. They are classified as rectangular, half-sine and sawtooth. The notation ΔT_1 refers to the time duration of a rectangular noise burst and LRMP denotes an initial peak sawtooth, RRMP a terminal peak sawtooth, and SYMRMP a symmetrical sawtooth. Let us elect to formu late solutions for the single highest response by means of a digital

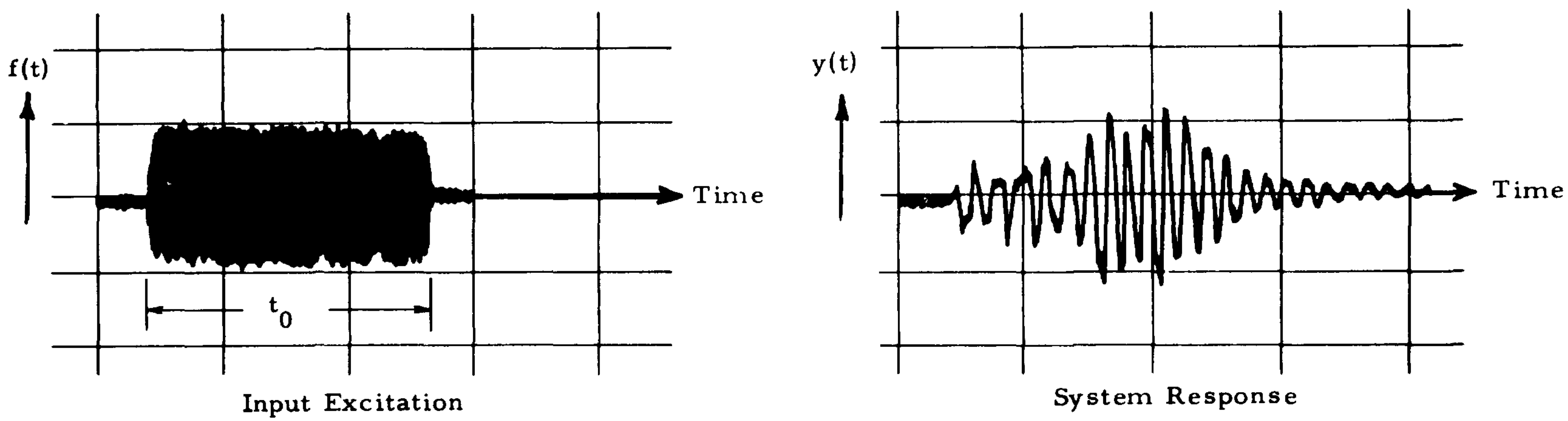


Figure 9. Response of a Single Degree-of-Freedom System to a Rectangular Burst of White Noise

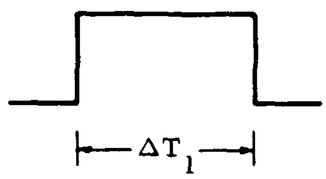
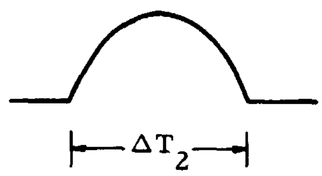
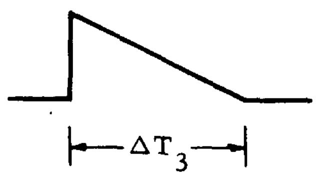
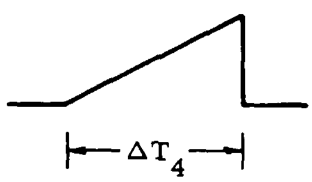
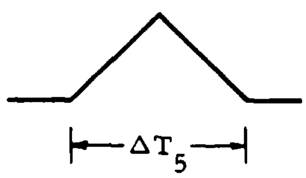
Modulation Function		K
Rectangular		1.00
Half-Sine		2.00
Sawtooth	LRMP 	3.00
	RRMP 	3.00
	SYMRMP 	3.00

Figure 10. Modulation Functions for the Input Noise

simulation and to represent the results by plots similar to Figures 7 and 8. We call such single highest response solutions probabilistic shock spectra in as much as they represent (estimates of) the maximum response of a single degree of freedom system to bursts of random excitation.

To adhere to the format of Figures 7 and 8, two questions must be answered:

- What time duration should be used in lieu of the sampling time interval T?
- What rms response should be used in order to form the ratio $|\beta|$?

An answer to the first question is provided by replacing T with a measure of the time duration of the input burst. Let ΔT_1 be the time duration of a rectangular noise burst. On the assumption of equal energy inputs (governed by the envelope functions), other envelope functions can be equated to this time base according to

$$t_0 = K \Delta T_1 \qquad (39)$$

where

$$K = \frac{\pi E_m^2}{\int_0^{\pi} e^2(t)\, dt} \qquad (40)$$

For a rectangular envelope, K = 1 so that $t_0 = \Delta T_1$. Values of K for the other envelope functions are included in Figure 10.

An answer to the second question is provided by first examining the factors which influence the system mean square response when e(t) is a rectangular step and n(t) is correlated according to the function

$$R_n(\tau) = R_0 \, e^{-\alpha|\tau|} \cos \rho\tau \tag{41}$$

This correlation function is suitable for the output of white noise generators used in the laboratory. The quantity α defines a correlation decay constant and ρ a harmonic frequency. The system mean square response may be expressed by [4]

$$E[y^2(t)] = \frac{R_0}{m^2}\left[R_1T_1 - X_1T_2 + R_3T_3 - X_3T_4\right] \tag{42}$$

where

$$\begin{aligned}
T_1 &= \frac{a}{2b}\left[1 - e^{-2bt}\left(1 + \frac{b}{a}\sin 2at\right)\right] \\
T_2 &= -e^{-2bt}\sin^2 at \\
T_3 &= 1 + e^{-2bt}\left(1 + \frac{b}{a}\sin 2at + \frac{b^2 - a^2 + \rho^2 - \alpha^2}{a^2}\sin^2 at\right) \\
&\quad - 2e^{-(\alpha+b)t}\left(\cos at \cos \rho t + \frac{b+\alpha}{a}\sin at \cos \rho t + \frac{\rho}{a}\sin at \sin \rho t\right) \\
T_4 &= 2\left[e^{-2bt}\left(\frac{\rho\alpha}{a^2}\sin^2 at\right) + e^{-(\alpha+b)t}\left(\frac{\rho}{a}\sin at \cos \rho t\right.\right. \\
&\quad \left.\left. - \cos at \sin \rho t - \frac{b+\alpha}{a}\sin at \sin \rho t\right)\right]
\end{aligned} \tag{43}$$

The remaining terms are written as

$$R_1 = \mathrm{Re}\left[\frac{\left(\rho^2 + \alpha^2 + s_1^2\right)}{s_1\left(s_1^2 - s_3^2\right)\left(s_1^2 - s_4^2\right)}\right] \cdot \frac{\alpha}{a^2}$$

$$R_3 = \mathrm{Re}\left[\frac{1}{\left(s_3^2 - s_1^2\right)\left(s_3^2 - s_2^2\right)}\right]$$

(44)

$$X_1 = \mathrm{Im}\left[\frac{\left(\rho^2 + \alpha^2 + s_1^2\right)}{s_1\left(s_1^2 - s_3^2\right)\left(s_1^2 - s_4^2\right)}\right] \cdot \frac{\alpha}{a^2}$$

$$X_3 = \mathrm{Im}\left[\frac{1}{\left(s_3^2 - s_1^2\right)\left(s_3^2 - s_2^2\right)}\right]$$

and the complex numbers as

$$s_1 = a + ib \qquad s_3 = \rho + i\alpha$$

(45)

$$s_2 = -a + ib \qquad s_4 = -\rho + i\alpha$$

with the system properties

$$a = \omega_n \left[1 - \zeta^2 \right]^{1/2}$$

$$b = \zeta \omega_n \tag{46}$$

From the plots in [4], it is noted that the mean square response may exceed its stationary value for various combinations of α, ρ, τ, ω_n and Q. Such is not convenient for the $|\beta|$ normalization.

For a white noise input, the mean square response is determined from Eq. (42) by

$$\lim_{\alpha \to \infty} E[y^2(t)] \tag{47}$$

which reduces to+

$$E[y^2(t)] = \frac{\pi G_0}{4\zeta m^2 \omega_n^3} \left[1 - e^{-2bt} \left(1 + \frac{b}{a} \sin 2at + \frac{2b^2}{a^2} \sin^2 at \right) \right] \tag{48}$$

+ This expression agrees identically with that of Caughey and Stumpf [7]. A relevant discussion is by Janssen and Lambert [13].

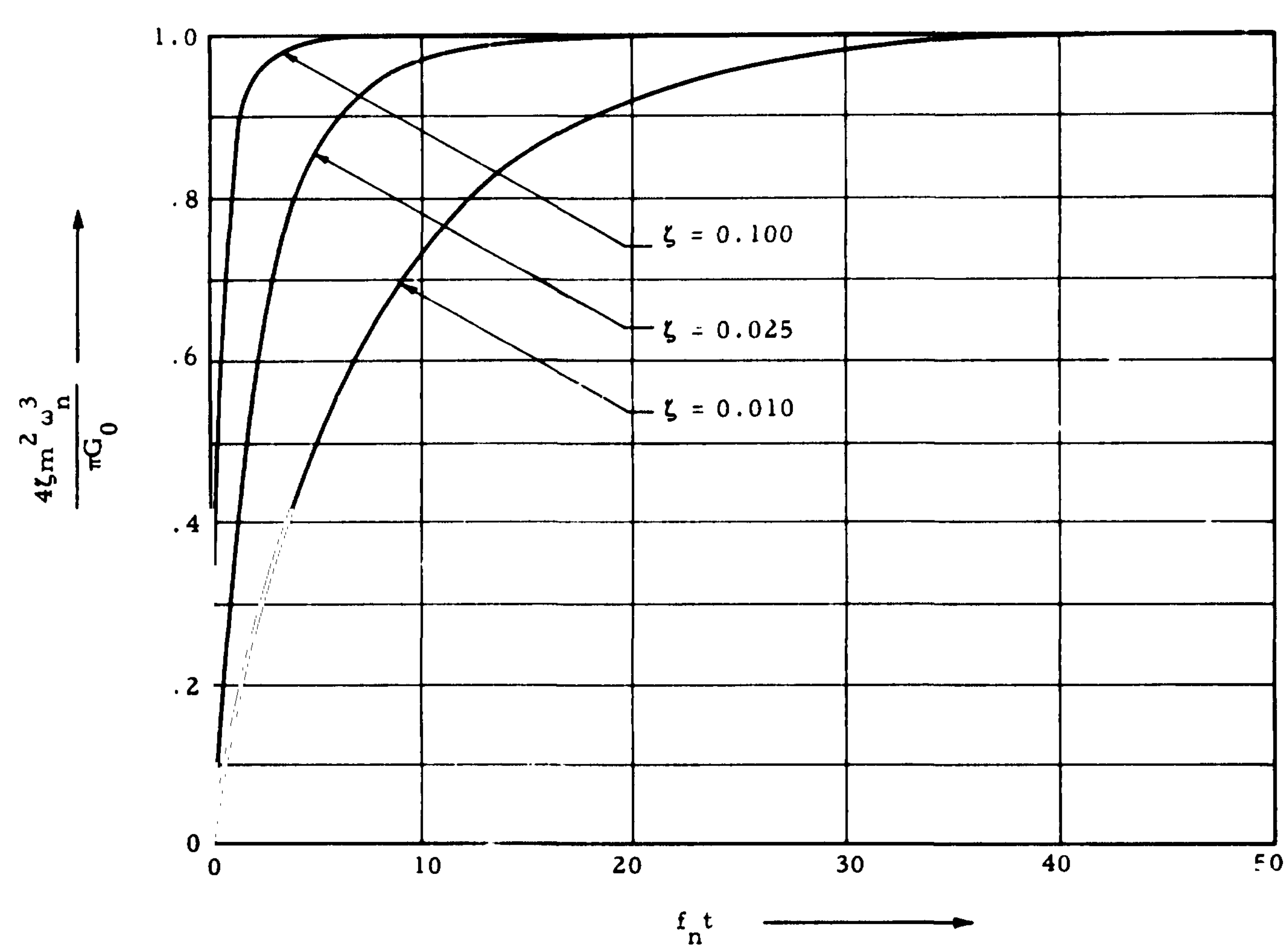


Figure 11. Normalized Mean Square Response to White Noise Modulated by a Unit Step Function

Figure 11 depicts the behavior of this equation. A family of normalized curves in Q is shown where $f_n t$ is interpreted as the number of response cycles. For damped systems, the mean square response is seen to approach asymptotically (but does not exceed) the limiting value

$$\lim_{t \to \infty} E[y^2(t)] = \sigma_y^2 = \frac{\pi G_0}{4\zeta m^2 \omega_n^3}$$

which is the stationary mean square response of the system to an input of broadband white noise. Since the σ_y^2 values are not exceeded, their use as normalization constants for $|\beta|$ is suitable. In addition, the plots of Figure 11 may be used to assess the time lapse (or number of response cycles) for unimodal mechanical systems to attain near stationarity in their response.

5.2 COMPUTER SIMULATION

The block diagram for the digital simulation on the 1108 Univac computer is shown as Figure 12. Digital filtering techniques [11] were used to represent the mechanical system. The random noise generator consisted of a computer subroutine which produced Gaussian white noise with zero mean and unit variance. The envelope functions were those of Figure 10 where

$$\begin{aligned} &\text{sawtooth:} && \Delta T_3 = \Delta T_4 = \Delta T_5 = 3\Delta T_1 = t_0 \\ &\text{half-sine:} && \Delta T_2 = 2\Delta T_1 = t_0 \end{aligned} \tag{49}$$

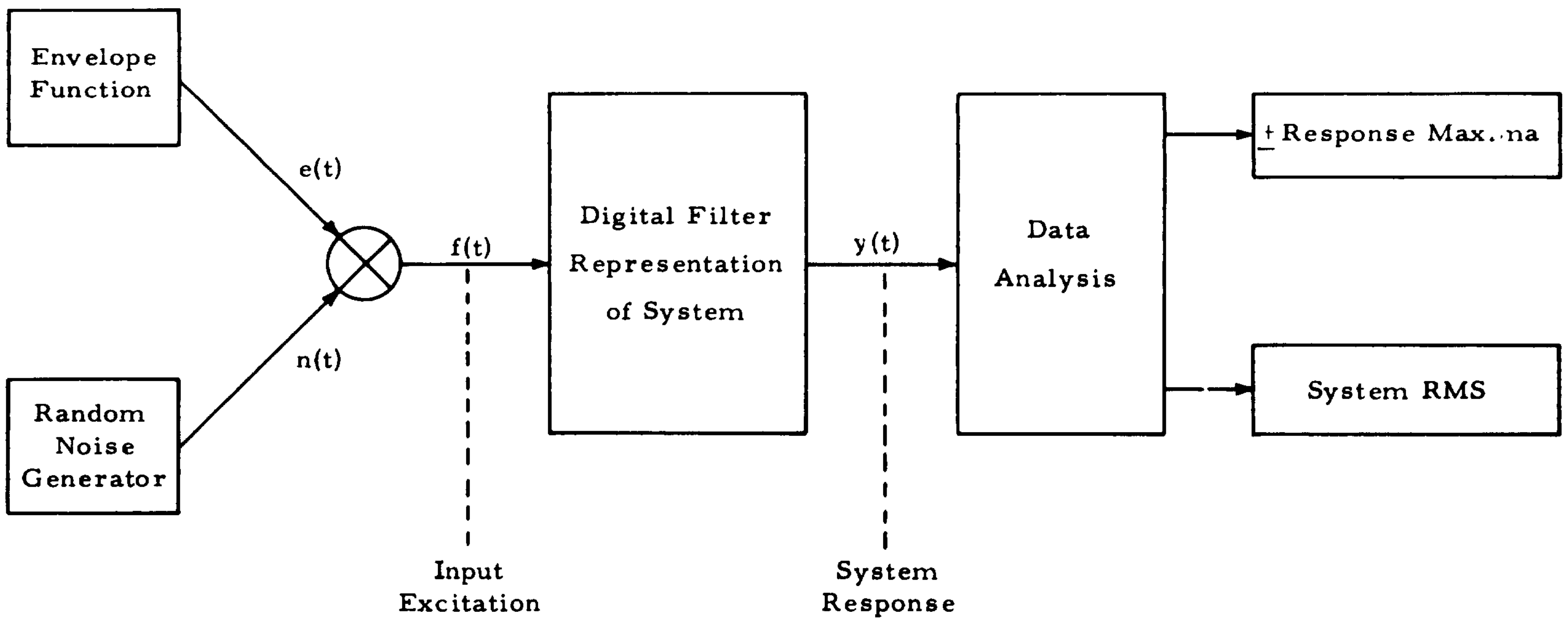


Figure 12. Digital Simulation Procedure

By discretizing the time variable in the form

$$t_0 = \lambda \Delta t \quad , \tag{50}$$

the input excitation to the single tuned digital filter for each envelope function is

$$f(\lambda\Delta t) = e(\lambda\Delta t)\ n(\lambda\Delta t) \tag{51}$$

The filter output is defined by

$$y(\lambda\Delta t) = C\, f(\lambda\Delta t) + h_1\, f([\lambda - 1]\ \Delta t) + h_2\, f([\lambda - 2]\ \Delta t) \tag{52}$$

where

$$\begin{aligned} h_1 &= 2 \exp[-\zeta\omega_n \Delta t] \cos(\omega_n [1 - \zeta^2]^{1/2}\ \Delta t) \\ h_2 &= -\exp[-2\zeta\omega_n\ \Delta t] \\ \omega_n &= 2\pi f_n \quad , \quad f_n = 100 \text{ Hz} \end{aligned} \tag{53}$$

For each combination of $f_n t_0/Q$, three hundred runs were made. For each run, the data were sampled and the positive and negative response maxima ($\pm$ peaks) were noted. After each set of three hundred runs, the data were normalized by the white noise stationary rms response of the system and arranged sequentially according to order of magnitude. The sample mean and variance were calculated by

$$\overline{y^+_{max}} = \frac{1}{N} \sum_{k=1}^{N} y^+_k \, max$$

$$\overline{y^-_{max}} = \frac{1}{N} \sum_{k=1}^{N} y^-_k \, max$$

$$S^2_+ = \frac{1}{N-1} \sum_{k=1}^{N} \left[y^+_{max} - \overline{y^+_{max}} \right]^2 \tag{54}$$

$$S^2_- = \frac{1}{N-1} \sum_{k=1}^{N} \left[y^-_{max} - \overline{y^-_{max}} \right]^2$$

where the sample size N = 300. The largest values for both y^+_{max} and y^-_{max} also were listed. Finally, the sample probability density functions $\hat{p}\,(y^+_{max})$ and $\hat{p}\,(y^-_{max})$ were determined and percentile estimates made by

$$y^{+}_{P\max} = \beta^{+} \left| \int_{-\infty}^{\beta^{+}} \hat{p}(y^{+}_{\max})\, d(y^{+}_{\max}) = \hat{P} \right.$$

$$(-\infty < y^{+}_{\max} < \beta^{+}) \qquad (55)$$

$$y^{-}_{P\max} = \beta^{-} \left| \int_{-\infty}^{\beta^{-}} \hat{p}(y^{-}_{\max})\, d(y^{-}_{\max}) = 1 - \hat{P} \right.$$

where the probability P varies over the range $0 \leq P \leq 1$.

6. NOISE BURST RESULTS

The results of the digital simulation study are shown as Figures 13 through 18. All are plots of $\hat{P}_{t_0}(|\beta| \leq \beta_0)$ for either Q = 5 or Q = 50 where the exceedance level β_0 is plotted versus the parameter $f_n t_0/Q$. Data are shown for the five modulation functions and the stationary response of the system to white noise is included for purposes of comparison.

Figures 13 and 14 are plots of the average response maxima for Q = 5 and Q = 50, respectively. The dashed curves define the variance extremes of the output data. That is, the upper curve is the variance of y(t) for the rectangular step modulation while the lower curve is the variance of y(t) for the terminal sawtooth modulation. The response to the stationary input is the response maxima corresponding to the fiftieth percentile, i.e., $\hat{P}_T(|\beta| \leq \beta_0) = 0.50$.

Figures 15 and 16 are plots of the fiftieth percentile response maxima for Q = 5 and Q = 50. Figures 17 and 18 are plots of the ninety-fifth percentile response maxima for Q = 5 and Q = 50. As expected, due to the white noise normalization σ_y, the nonstationary response values do not exceed the stationary response values. Moreover, they generally follow the trend of the stationary results. This suggests the stationary response maxima may be used as conservative bounds for response maxima due to the modulated noise inputs. The degree of conservatism varies with the shape of the modulation function; the greatest deviations are those for the sawtooth functions. The least deviation is for the rectangular step modulation.

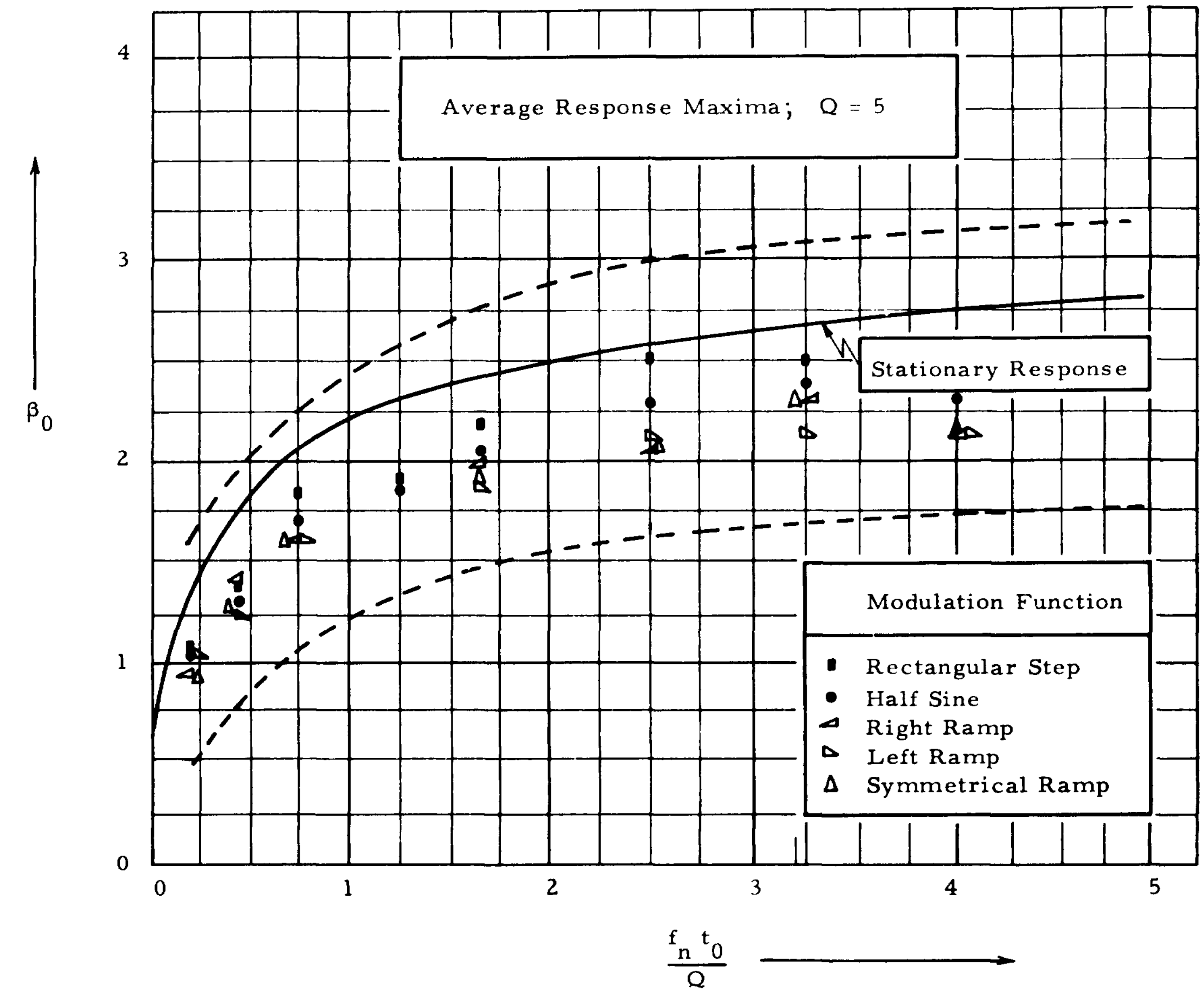


Figure 13. Average Response Maxima of a Single Degree-of-Freedom System to a Burst of Amplitude Modulated White Noise

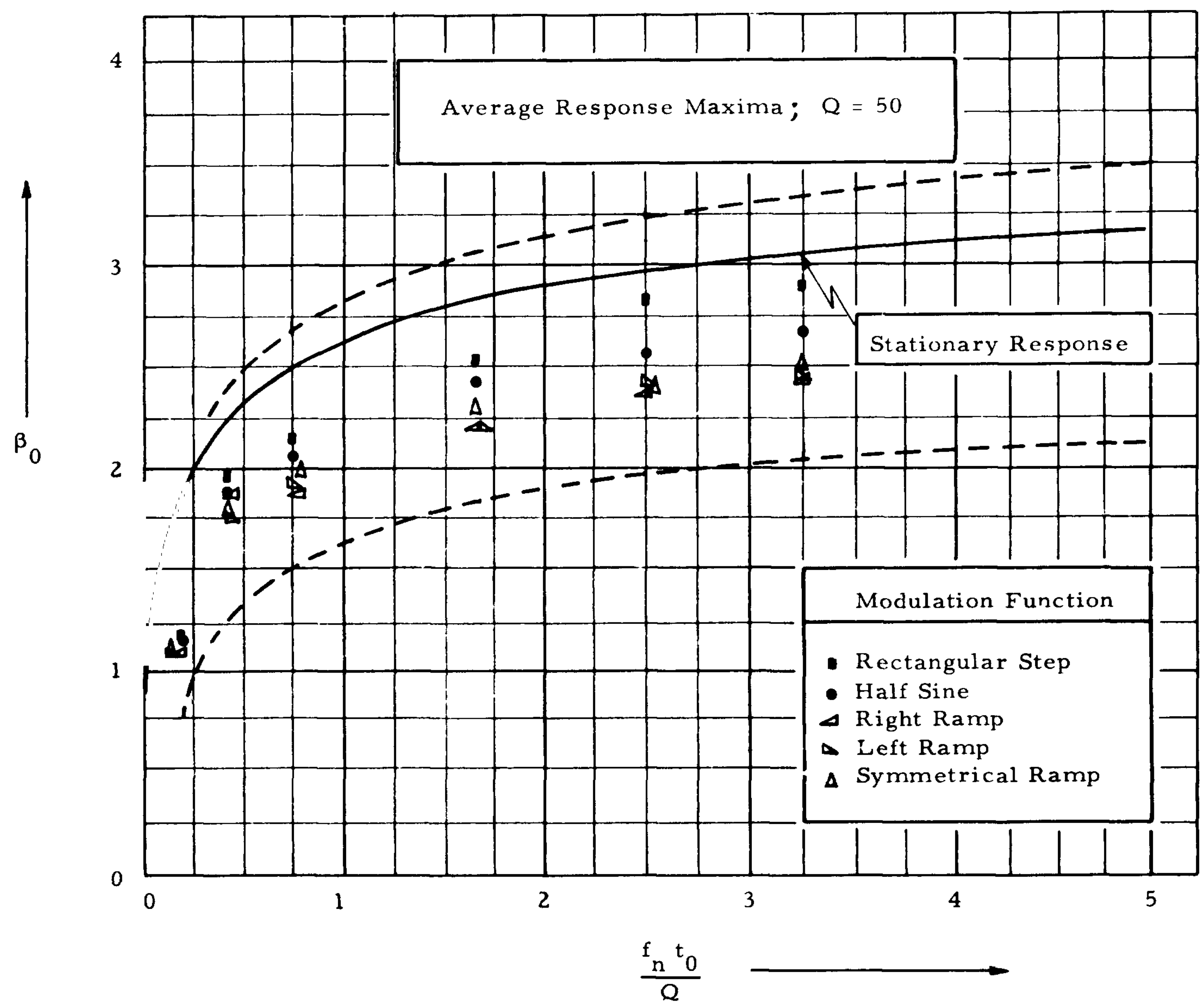


Figure 14. Average Response Maxima of a Single Degree-of-Freedom System to a Burst of Amplitude Modulated White Noise

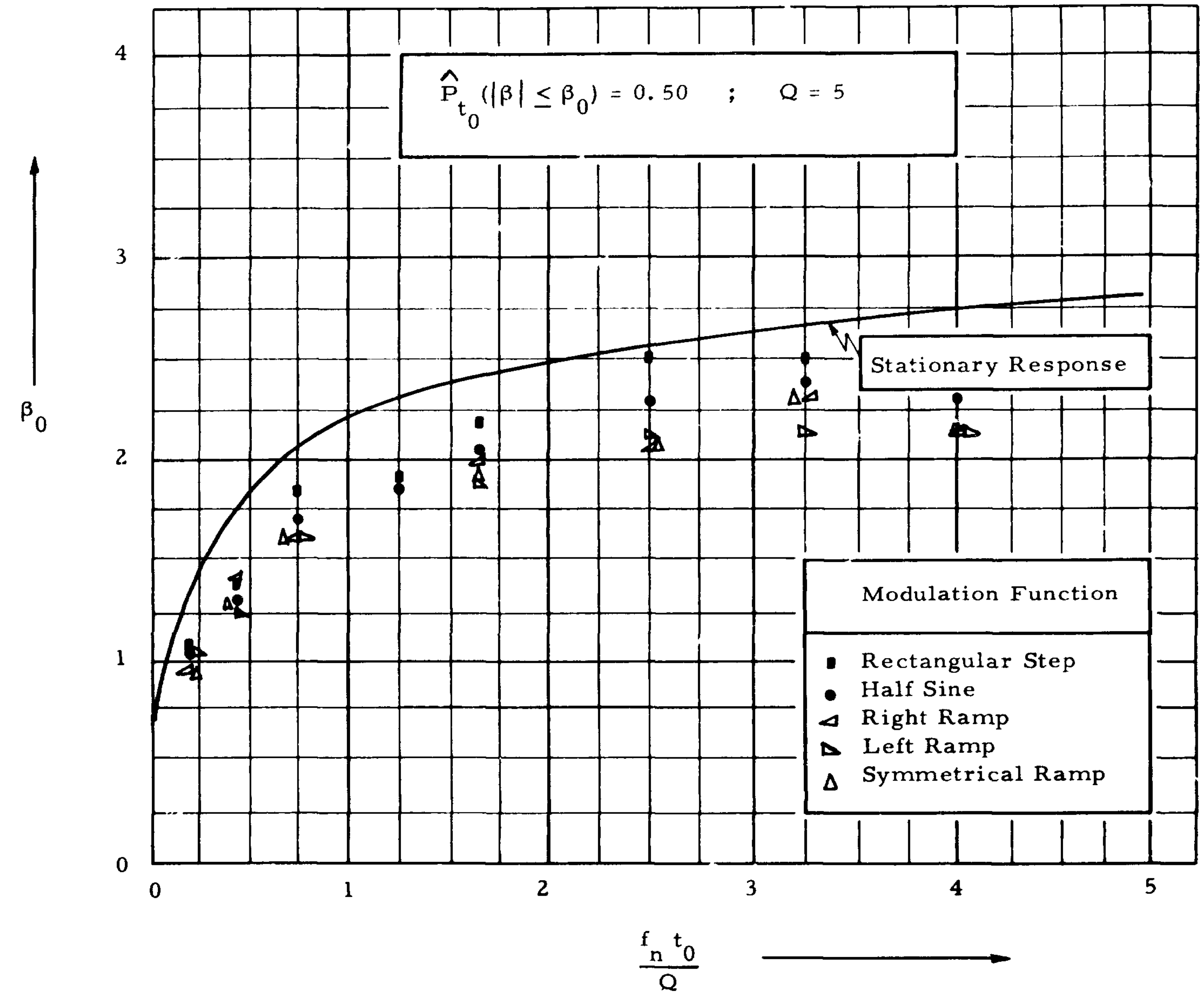


Figure 15. Response Maxima of a Single Degree-of-Freedom System to a Burst of Amplitude Modulated White Noise

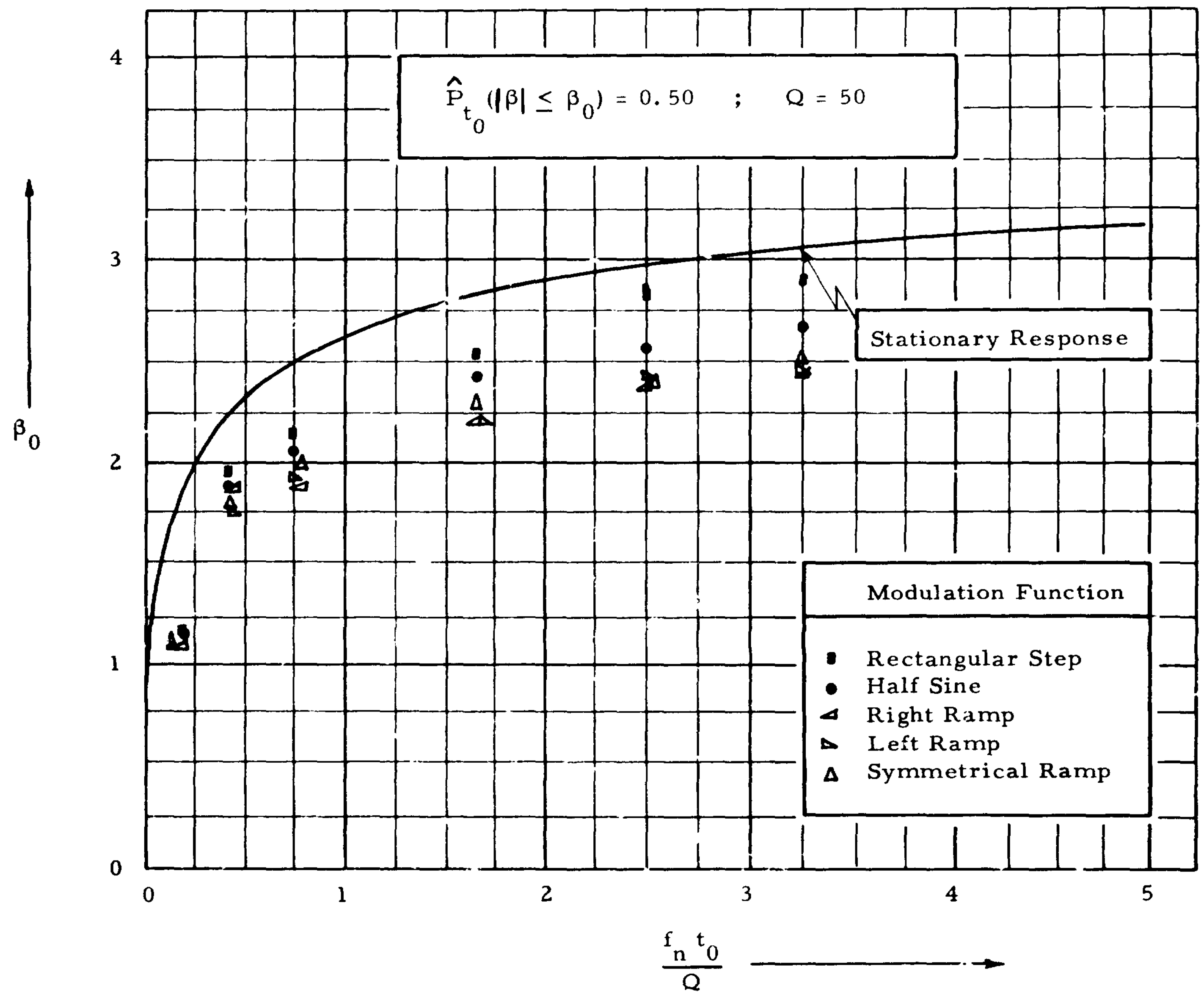


Figure 16. Response Maxima of a Single Degree-of-Freedom System to a Burst of Amplitude Modulated White Noise

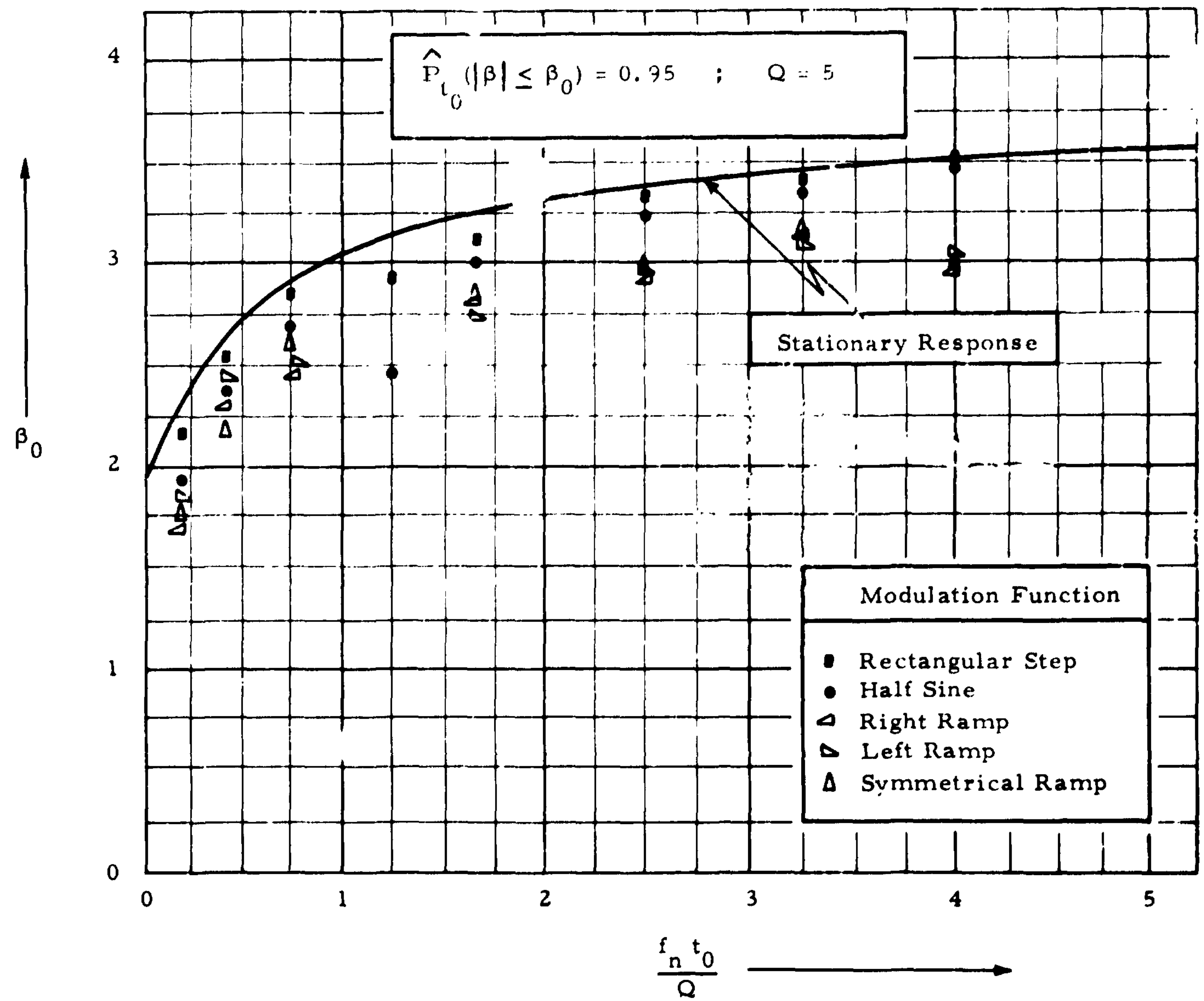


Figure 17. Response Maxima of a Single Degree-of-Freedom System to a Burst of Amplitude Modulated White Noise

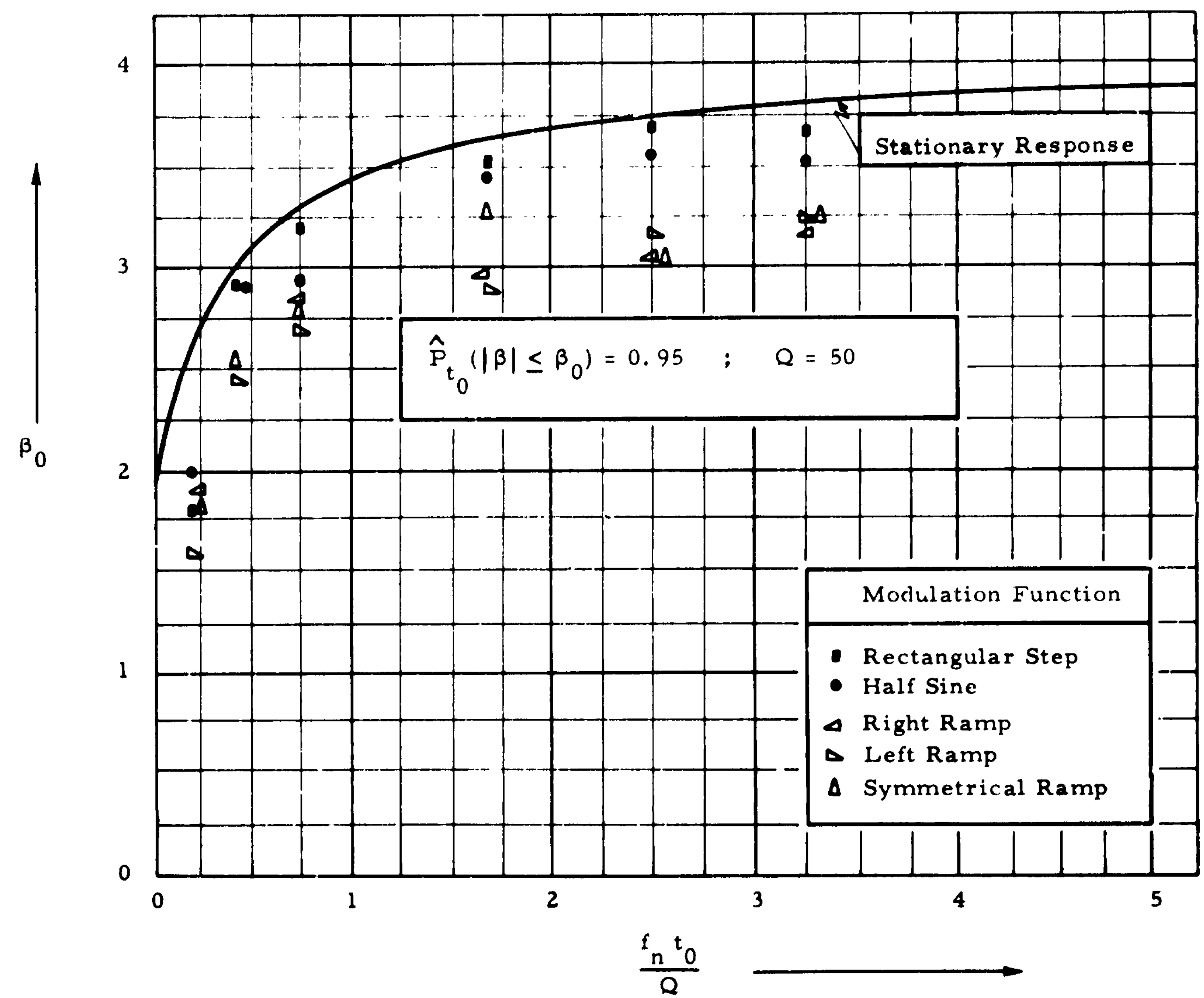


Figure 18. Response Maxima of a Single Degree-of-Freedom System to a Burst of Amplitude Modulated White Noise

Some error is introduced into our results due to discretizing both the system equation of motion and the input forcing function. This usually can be made negligibly small; it is estimated to be less than two-percent in this study. One discretization error that needs further discussion is the bias error in locating the true peak value of y(t). Another error that needs comment is the statistical error associated with random data fluctuations.

For narrowband noise, let us assume the error in detecting a peak value of y(t) is of the same order of magnitude as that of locating the peak value of a simple harmonic signal which has been digitally sampled. By assuming a uniformly distributed phase angle, the bias error $Er\,[Y_{peak}]$ when $Y = A \cos \omega_0(t + t')$ is given by*

$$Er[Y_{peak}] = A \frac{\sin(\omega_0 \Delta t/2)}{(\omega_0 \Delta t/2)} \tag{56}$$

where Δt is the time increment between two adjacent points of the discretized signal. Figure 19 shows the effect of Eq. (56). Now with $\Delta t = 0.0005$ sec and $f_0 = 100$ Hz. $Er\,[Y_{peak}]$ is less than one percent.

Tolerance and/or confidence limits may be used to assess the statistical variability of the data. Such limits, however, are extremely difficult to formulate since the true distributions of y^+_{max} and y^-_{max} are not available. Although nonparametric as well as parametric statistical methods [5] can be used, the resultant limits generally are much too wide for practical application. Perhaps the

*Personal communication with R. K. Otnes (see Reference 11).

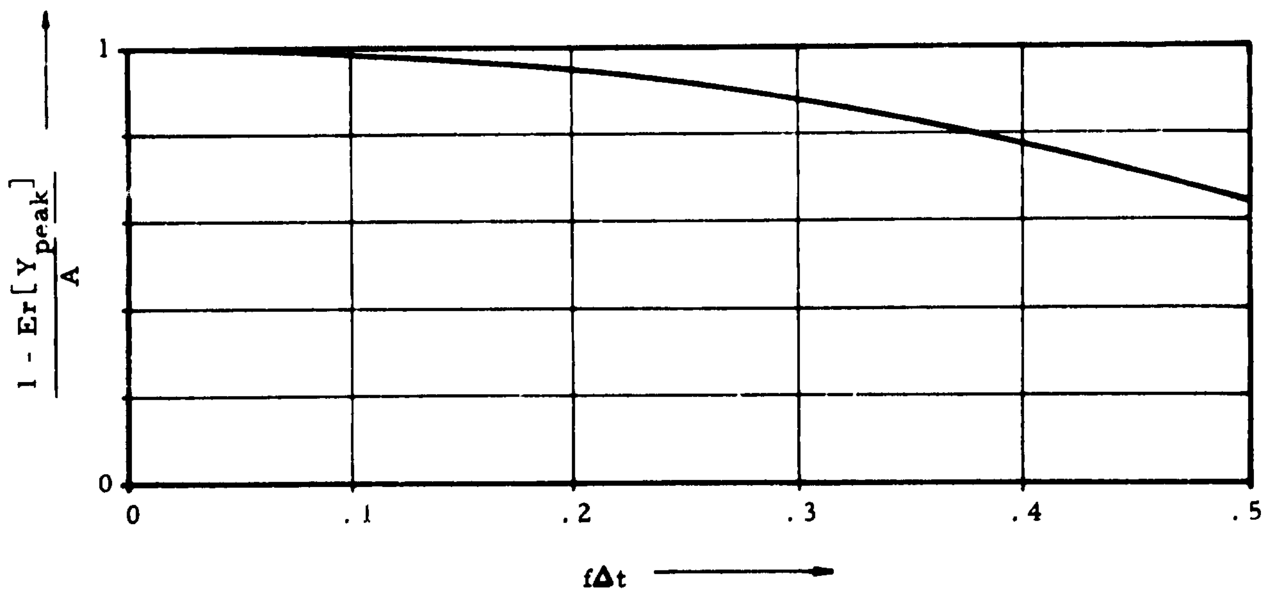


Figure 19. Bias Error in Estimating the Peak Value of a Simple Harmonic

simplest approach is to increase substantially the number of data points, then use χ^2 goodness-of-fit tests. Since we are interested primarily in the scatter of extremal data where $P_{t_0}(|\beta| \leq \beta_0) > 90\%$, an all-digital approach (as used here) generally will prove expensive as a large amount of computer time is required to produce such extremals. A seemingly attractive procedure for acquiring and processing extremal data is by means of a hybrid computer. The analog computer can be used to simulate the physical system and produce the extremal response values (since running time comparatively is inexpensive); the digital computer then used to perform the data analysis.

7. CONCLUDING REMARKS

Probabilistic shock spectra are presented for five noise modulation functions. Such spectra are normalized plots which characterize the single highest response of a single degree-of-freedom mechanical system to a burst of amplitude modulated random noise. They are functions of exceedance probability, the system natural frequency and damping, the time duration of the burst, and the degree of correlation of the input noise.

Probabilistic shock spectra have application in the design of unimodal mechanical systems in nonstationary environments where a single exceedance level (say, a peak stress or maximum deflection) may be used as a design criterion. Their application to distributed structural systems appears limited to conditions whereby the system response may be assumed to consist of a summation of independent contributions from each of the normal modes.

Several investigations seem warranted at this point. One is to investigate more fully the condition for which the nonstationary mean square response exceeds its stationary or steady state value. A second is to examine the effect of independent single degree-of-freedom systems with closely spaced natural frequencies on the response maxima statistics. A third is to study the effect of typical spatial correlation functions on the time varying mean square response of a distributed system.

REFERENCES

1. Barnoski, R. L., "On the Single Highest Peak Response of a Dual Oscillator to Random Excitation," Transactions of the ASME, Journal of Applied Mechanics, June 1968, p 414.

2. Barnoski, R. L., "The Maximum Response of Distributed Structures with Rectangular Geometry to Random Excitation," Journal of Sound and Vibration, Vol. 7, No. 3, 1968, p 333.

3. Barnoski, R. L., "The Maximum Response of a Linear Mechanical System to Stationary and Nonstationary Random Excitation," NASA CR-340, December 1965.

4. Barnoski, R. L. and J. R. Maurer, "Mean Square Response of Unimodal Mechanical Systems to Nonstationary Random Excitation," MAC Report No. 809-02, August 1968, Measurement Analysis Corporation (A Digitek Corporation), 4818 Lincoln Blvd., Marina del Rey, California 90291

5. J. S. Bendat and A. G. Piersol, Measurement and Analysis of Random Data, New York, John Wiley and Sons, Inc., 1966, p 141.

6. Caughey, T. K. and A. H. Gray, [Discussion of Rosenblueth and Bustamente, "Distribution of Structural Response to Earthquakes"] Journal of Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 89, No. EM2, 1963, p 159.

7. Caughey, T. K. and H. J. Stumpf, "Transient Response of a Dynamic System under Random Excitation," Transactions of the ASME, Journal of Applied Mechanics, December 1961, p 563.

8. Crandall, S. H., "On the Distribution of Maxima in the Response of an Oscillator to Random Excitation," Rept # DSR 78867-3, Acoustics and Vibration Laboratory, Massachusetts Institute of Technology, Cambridge, Mass., August 1968.

9. Crandall, S. H., Chandirami, K. L., and R. G. Cook, "Some First-Passage Problems in Random Vibration," Transactions of the ASME, Journal of Applied Mechanics, September 1966, p 532.

10. Crandall, S. H., and W. D. Mark, Random Vibration in Mechanical Systems, New York, Academic Press, Inc., 1963.

11. Enochson, L. D., and R. K. Otnes, Programming and Analysis for Digital Time Series Data, Shock and Vibration Monograph No. 3, Defense Documentation Center, Cameron Station, Alexandria, Virginia 22314.

12. Gray, A. H., Jr., "First-Passage Time in a Random Vibrational System," Transactions of the ADME, Journal of Applied Mechanics, Paper No. 65-WA/APM-18.

13. Janssen, R. A., and R. F. Lambert, "Numerical Calculation of Some Response Statistics for a Linear Oscillator Under Impulsive-Noise Excitation," The Journal of the Acoustical Society of America, Vol. 41, No. 4, 1967, p 827.

14. Lin, Y. K., Probabilistic Theory of Structural Dynamics, McGraw-Hill Book Co., Inc., New York, 1967.

15. Longuet-Higgins, M. S., "The Distribution of the Intervals Between Zeros of a Stationary Random Function," Philosophical Transactions of the Royal Society, Series A, Vol. 354, 1962, p 557.

16. Lyon, R. H., "On the Vibration Statistics of a Randomly Excited Hand-Spring Oscillator," Journal of the Acoustical Society of America, Vol. 33, 1961, pp 1395-1403.

17. Mark, W. D., "On False-Alarm Probabilities of Filtered Noise," Proceedings of the IEEE, Vol. 54, No. 2, 1966, p 316.

18. Rice, J. R., and F. P. Beer, "First Occurrence Time of High-Level Crossings in a Continuous Random Process," Journal of the Acoustical Society of America, Vol. 39, 1966, p 323.

19. Rice, S. O., "Mathematical Analysis of Random Noise," in Selected Papers on Noise and Stochastic Processes, ed. N. Wax, New York, Dover Press, 1954.

20. Roberts, J. B., "An Approach to the First-Passage Problem in Random Vibration," Journal of Sound and Vibration, Vol. 8, No. 2, 1968, p 301.

21. Rosenblueth, E., and J. L. Bustamente, "Distribution of Structural Response to Earthquakes," Journal of the Engineering Mechanical Division, Proceedings of the American Society of Civil Engineers, Vol. 88, No. EM3, June 1962, p 75.

22. Shinozuka, M., and J. T. P. Yao, "On the Two-Sided Time-Dependent Barrier Problem," Journal of Sound and Vibration, Vol. 6, No. 1, 1967, p 98.

23. Srinavasan, S. K., Subramanian, R., and S. Kumawaswamy, "Response of Linear Vibratory Systems to Nonstationary Stochastic Impulses," Journal of Sound and Vibration, Vol. 6, No. 2, 1967, p 169.