ATTACHED FLOW
GUST PENETRATION LOADS

Lockheed
MISSILES & SPACE COMPANY
A GROUP DIVISION OF LOCKHEED AIRCRAFT CORPORATION
SUNNYVALE, CALIFORNIA

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Prepared by: I. M. Scholnick
I. M. Scholnick, Senior Engineer, Aerodynamics

Checked by: Lars Eric Ericsson
Lars Eric Ericsson, Staff Engineer, Sr. Flight Technology

Approved by: M. Tucker
M. Tucker, Manager, Flight Technology

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ABSTRACT

An attached flow theory for predicting the gust induced loads on slender bodies traveling at supersonic speeds is presented. It has been assumed that the vehicle remains undeflected as it traverses the gust. For arbitrary gust penetration depths, the method of characteristics has been employed. In the case of high frequencies, analytic solutions have been derived for both small and large gust penetration depths. In each instance, results for a sinusoidal gust are also presented.

CONTENTS

NOMENCLATURE

A	πR^2, Cross-sectional area, dimensionless
a	$\bar{a}/\bar{U}_\infty$, speed of sound, dimensionless
a_n	Constants defined in Appendix C
b_n	Constants defined in Appendix B
B	$\bar{B}/\bar{U}_\infty \bar{L}$, Axially symmetric component of the perturbation potential, dimensionless
C	$\bar{C}/\bar{U}_\infty \bar{L}$, Non-symmetric component of the perturbation potential, dimensionless
c_p	Pressure coefficient
d_n, D_n	Constants defined in Appendix B
e_n	Constants defined in Appendix B
F	$\bar{F}/\bar{q}$, Normal force, dimensionless
g, G	Wind gust functions, dimensionless
h_n	Constants defined in Appendix C
k	Fourier variable, dimensionless
K_1	Modified Bessel function of the first kind and of the first order
$\bar{L}$	Body length, m
L^{-1}	Inverse Laplace Transform operator
ℓ	$\bar{\ell}/\bar{L}$, Wavelength of sinusoidal gusts, dimensionless
M_∞	U_∞/a_∞, Mach number
p	$t - x$, Gust penetration parameter, dimensionless
p_o	$t - x - x_o$, Gust penetration parameter, dimensionless
$\bar{q}$	$\bar{\rho}_\infty \bar{U}_\infty^2/2$, Dynamic pressure, Kg/m^2
R	$\bar{R}/\bar{L}$, Local body radius, dimensionless
r	$\bar{r}/\bar{L}$, Radial coordinate, dimensionless, see Figure 1
s	Laplace variable, dimensionless
$S(y)$	Unit step function, $S(y) = 0$ for $y \le 0$ and $S(y) = 1$ for $y > 0$
t	$\bar{t}\,\bar{U}_\infty/\bar{L}$, Dimensionless time
U	$\bar{U}/\bar{U}_\infty$, Axial velocity ratio
u	$\partial C/\partial x'$, Velocity perturbation due to wind gust, dimensionless

NOMENCLATURE (Cont'd)

v	$\partial C/\partial r$, Velocity perturbation due to wind gust, dimensionless
W_o	$\overline{W}_o/\overline{U}_\infty$, Maximum amplitude of wind gust, dimensionless
x	$\bar{x}/\overline{L}$, Axial coordinate, dimensionless, see Figure 1
x_o	Dimensionless constant
α_n, β_n	Constants defined in Appendix B
β^2	$M_\infty^2 - 1$
γ	Ratio of specific heats
$\Gamma(y)$	Gamma function
∇^2	$\overline{L}^2 \overline{\nabla}^2$, Laplace Operator
ϵ	Dimensionless constant
η	Defines family of characteristics of slope -1
θ	Azimuthal coordinate, see Figure 1
κ	$K_1(M_\infty sR)/[K_1'(M_\infty sR)M_\infty sR]$
λ	$\bar{\lambda}/\overline{L}$, Slope of characteristic curves, dimensionless
ξ	Defines family of characteristics of slope +1
ρ	$\bar{\rho}/\bar{\rho}_\infty$, Density ratio
σ	Euler's constant, 0.5772156649
Φ	$\overline{\Phi}/\overline{U}_\infty \overline{L}$, Total velocity potential, dimensionless
φ	$\overline{\varphi}/\overline{U}_\infty \overline{L}$, Perturbation potential, dimensionless
ψ	Psi function
ω	Constant; frequency of sinusoidal gust, dimensionless
ω_r	$M_\infty R\omega$

SUBSCRIPTS

i, ℓ, m, n, N	Integer constants
k	k^{th} element of the Fourier spectrum
r	Partial derivative with respect to r
x, x'	Partial derivative with respect to x, x'
t, t'	Partial derivative with respect to t, t'
∞	Conditions upstream in undisturbed flow

SUPERSCRIPTS

$\overline{I}, \overline{R}$	Imaginary and real components respectively
$(\bar{\ })$	Dimensional quantity
$(\tilde{\ })$	Laplace transform with respect to t
*	Fourier transform with respect to t'
'	Derivative with respect to the total argument; transformed space as defined by Eq. A1
(n)	n^{th} approximation

ATTACHED FLOW GUST PENETRATION LOADS

I. INTRODUCTION

A linearized attached flow theory for the prediction of the aerodynamic forces (which would represent forcing functions in the equations of motion for an elastic launch vehicle) on bodies of revolution encountering arbitrary wind gusts is developed. Solutions to the full linearized potential equation for supersonic flow are found by employing the method of characteristics. For very slender bodies, a simplified form of the potential equation is solved through the application of Laplace Transform techniques. Solutions valid for small and large gust penetration depths are derived in analytic form.

II. GENERAL FORMULATION

By limiting our attention to inviscid, irrotational, isentropic flows, the potential equation is[1]

$$\bar{a}^2 \nabla^2 \bar{\Phi} = \bar{\Phi}_{\bar{t}\bar{t}} + \left[\frac{\partial}{\partial \bar{t}} + \frac{\nabla\bar{\Phi} \cdot \nabla}{2}\right](\nabla\bar{\Phi})^2 \tag{1}$$

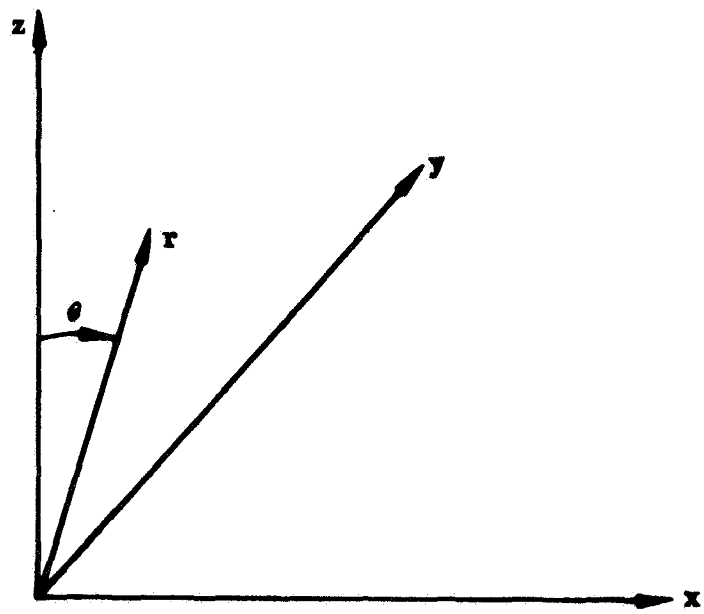
For pointed bodies of sufficiently small slope, we introduce a body fixed perturbation potential.

$$\bar{\Phi} = \bar{U}_\infty \bar{x} + \bar{\varphi} \tag{2}$$

Relative to the cylindrical coordinate system shown in Fig. 1, the linearized form of Eq. (1) takes the following form

FIGURE 1

CYLINDRICAL COORDINATE SYSTEM



$$\bar{a}_\infty^{-2}\left(\bar{\varphi}_{\bar{x}\bar{x}} + \bar{\varphi}_{\bar{r}\bar{r}} + \frac{1}{\bar{r}}\bar{\varphi}_{\bar{r}} + \frac{1}{\bar{r}^2}\bar{\varphi}_{\theta\theta}\right) = \bar{\varphi}_{\bar{t}\bar{t}} + 2\bar{U}_\infty \bar{\varphi}_{\bar{x}\bar{t}} + \bar{U}_\infty^2 \bar{\varphi}_{\bar{x}\bar{x}} \tag{3}$$

The boundary conditions associated with Eq. (3) for an axisymmetric body in axial supersonic flight encountering a lateral wind gust $\bar{W}_o G(\bar{x},\bar{t})$ parallel to the $\bar{z}$-axis are*

$$\bar{\varphi}_r = \bar{U}_\infty \frac{d\bar{R}}{d\bar{x}}(\bar{x}) - \bar{W}_o\, G(\bar{x},\bar{t}) \cos\theta \tag{4}$$

at the body surface $\bar{r} = \bar{R}(\bar{x})$ and

$$\bar{\varphi}_{\bar{r}} = 0 \text{ at } \bar{r} = \infty \tag{5}$$

In terms of dimensionless quantities, the mathematical statement of the problem is of the form,

$$\beta^2 \varphi_{xx} - \varphi_{rr} - \frac{1}{r}\varphi_r - \frac{1}{r^2}\varphi_{\theta\theta} = -M_\infty^2(\varphi_{tt} + 2\varphi_{xt}) \tag{6}$$

$$\varphi_r(r = R) = \frac{dR}{dx} - W_o G(x,t) \cos\theta \tag{7a}$$

$$\varphi_r = 0 \text{ at } r = \infty \tag{7b}$$

The pressure coefficient as derived from Bernoulli's equation for nonsteady flow is

$$c_p = \frac{2}{\gamma M_\infty^2}\left\{\left[1 - \frac{(\gamma - 1)}{2} M_\infty^2\left(2\Phi_t + (\nabla\Phi)^2 - 1\right)\right]^{\gamma/\gamma-1} - 1\right\} \tag{8}$$

*The potential $\bar{\varphi}$ describes a missile in an axial flow suddenly being subjected to a lateral $\bar{z}$ displacement per unit time equal to $\bar{W}_o G(\bar{x},\bar{t})$.

The linearized form for c_p, correct to the first order is

$$c_p = -2(\varphi_x + \varphi_t) \tag{9}$$

and the force per unit length in the z-direction is of the form

$$\frac{\partial F}{\partial x} = \int_0^{2\pi} c_p(r = R) R \cos\theta \, d\theta \tag{10}$$

III. METHOD OF CHARACTERISTICS SOLUTION

To effect a solution to Eq. (6), subject to the constraints of Eqs. (7a, b), the perturbation potential φ may be written as

$$\varphi(x, r, \theta, t) = B(x, r) + C(x, r, t) \cos\theta \tag{11}$$

Inserting Eq. (8) into Eq. (6) yields the two equations

$$\beta^2 B_{xx} - B_{rr} - \frac{1}{r} B_r = 0 \tag{12a}$$

$$\beta^2 C_{xx} - C_{rr} - \frac{1}{r} C_r + \frac{C}{r^2} = -M_\infty^2 (C_{tt} + 2C_{xt}) \tag{12b}$$

dividing the problem into two separate problems; the flow past a body of revolution, (a) at zero angle-of-attack and (b), at an unsteady angle-of-attack. The boundary conditions associated with Eqs. (9a) and (9b) are respectively

$$B_r(r = R) = \frac{dR}{dx} \tag{13a}$$

$$C_r(r = R) = -W_o G(x,t) \tag{13b}$$

A detailed description of the coordinate transformation and numerical method utilized in obtaining a method of characteristics solution to Eq. (9) is found in Appendix A.

IV. STRIP THEORY SOLUTION

By assuming the body to be very slender and using a high frequency approximation, the gust loading problem can be reduced to a strip theory formulation. Mathematically this is effected by ignoring x-derivatives relative to t- and r-derivatives in the equation for the perturbation potential. Under this approximation Eq. (12b) reduces to

$$C_{rr} + \frac{1}{r} C_r - \frac{C}{r^2} - M_\infty^2 C_{tt} = 0 \tag{14}$$

subject to the boundary condition given by Eq. (13b).

A solution to Eq. (14) can be obtained through application of the theory of Laplace transforms. If we define

$$\tilde{C}(r,x,s) = \int_0^\infty e^{-st} C(r,x,t)\,dt$$

and apply this transform to Eqs. (14) and (13b) we have,

$$r^2 \tilde{C}_{rr} + r\tilde{C}_r - \left(1 + M_\infty^2 s^2 r^2\right)\tilde{C} = 0 \tag{15a}$$

$$\tilde{C}_r(r = R) = -W_o \tilde{G}(x,s) \tag{15b}$$

The solution to the above pair of equations, which remains bounded at infinity, can be written as[4]

$$\widetilde{C} = \frac{-W_0 \widetilde{G}(x, s) K_1 (M_\infty s r)}{M_\infty s K_1' (M_\infty s R)} \tag{16}$$

where K_1 is the first order modified Bessel Function of the first kind and primes indicate differentiation with respect to the total argument.

Upon substituting Eqs. (9) and (11) into Eq. (10) and applying the Laplace transform, we have

$$\frac{\partial \widetilde{F}}{\partial x}(x, s) = \left\{ \left[\left(1 + M_\infty^2 s^2 R^2 \right) \kappa^2 - \kappa \right] W_0 \widetilde{G} A' - 2 A W_0 \kappa \left[\widetilde{G}_x + s \widetilde{G} \right] \right\} \tag{17}$$

The desired solution is obtained by taking the inverse Laplace transform of Eq. (17) for specific wind profiles $G(x, t)$. The inversion can be carried out through the application of the theory of residues which involves the numerical evaluation of several integrals[6]. In view of the more general solution discussed in Appendix A this will not be done; instead, expansions of the right hand side of Eq. (17) will be made which enable us to obtain analytic results valid for small and large gust penetrations.

IV-A. Series Solution for Gust Induced Loads Near the Gust Front

The general form for the wind gust relative to a coordinate system attached to a body moving with a velocity U_∞ into the gust is

$$G(x,t) = g\left[\omega(t - x - x_0)\right] S(t - x) \tag{18}$$

and the associated Laplace transform of G is[3]

$$\widetilde{G} = \frac{e^{-s(x+x_o)}}{\omega} \tilde{g}\left(\frac{s}{\omega}\right) + \frac{e^{-s(x+x_o)}}{\omega} \int_{-x_o\omega}^{0} g(z)\, e^{-z(s/\omega)}\, dz \tag{19}$$

For the form given by Eq. (18), the transform of the loading reduces to

$$\frac{\partial \widetilde{F}}{\partial x} = W_o \widetilde{G} A' \left[\left(1 + M_\infty^2 s^2 R^2\right)\kappa^2 - \kappa\right] \tag{20}$$

Asymptotic expansions of κ, κ^2 and $\widetilde{G}$ in terms of s yields series approximations which are easily inverted through application of the Convolution Theorem[3]. The details of the expansion and subsequent inversion for small values of the gust penetration parameter appear in Appendix B; the result is of the form

$$\frac{\partial F}{\partial x} \frac{\omega_R}{W_o A'}$$

$$\begin{aligned}
&\simeq S(p_o)\Bigg\{\omega_R d_1 + d_0 D_1 + \left(\frac{p_0}{M_\infty R}\right)\left(\omega_R^2 d_2 + \omega_R d_1 D_1 + d_0 D_2\right) + \frac{1}{2}\left(\frac{p_0}{M_\infty R}\right)^2 \left(\omega_R^3 d_3\right. \\
&\quad \left. + \omega_R^2 d_2 D_1 + \omega_R d_1 D_2 + d_0 D_3\right) + \frac{1}{6}\left(\frac{p_0}{M_\infty R}\right)^3 \left(\omega_R^4 d_4 + \omega_R^3 d_3 D_1 + \omega_R^2 d_2 D_2\right. \\
&\quad \left. + \omega_R d_1 D_3 + d_0 D_4\right)\Bigg\} + O\left[\left(\frac{p_0}{M_\infty R}\right)^4\right] + S(p)\Bigg\{\omega_R e_1 + e_0 D_1 + \left(\frac{p}{M_\infty R}\right)\left(\omega_R^2 e_2 + \omega_R e_1 D_1\right. \\
&\quad \left. + e_0 D_2\right) + \frac{1}{2}\left(\frac{p}{M_\infty R}\right)^2 \left(\omega_R^3 e_3 + \omega_R^2 e_2 D_1 + \omega_R e_1 D_2 + e_0 D_3\right) + \frac{1}{6}\left(\frac{p}{M_\infty R}\right)^3 \left(\omega_R^4 e_4\right. \\
&\quad \left. + \omega_R^3 e_3 D_1 + \omega_R^2 e_2 D_2 + \omega_R e_1 D_3 + e_0 D_4\right)\Bigg\} + O\left[\left(\frac{p}{M_\infty R}\right)^4\right]
\end{aligned} \tag{B6}$$

IV-B. Series Solution for Gust Induced Loads Far From the Gust Front

If the functions κ, κ^2 and $\tilde{G}$ which appear on the right hand side of Eq. (20) are replaced by their Taylor Series' approximation, a Laplace inversion valid for large values of the gust penetration parameter can be effected. The solution is discussed in Appendix C and the final form for the loading is

$$\frac{\partial F}{\partial x} \frac{\omega_R}{W_0 A'}$$

$$\simeq S(p_0) \left\{ -6 E_0 \left(\frac{p_0}{M_\infty R} \right)^{-3} + \frac{18}{\omega_R} E_1 \left(\frac{p_0}{M_\infty R} \right)^{-4} - 1034\, E_0 \left(\frac{p_0}{M_\infty R} \right)^{-7} \ln^2 \left(\frac{2p_0}{M_\infty R} \right) \right.$$

$$\left. + 62.5\, E_0 \left(\frac{p_0}{M_\infty R} \right)^{-5} \ln \left(\frac{2p_0}{M_\infty R} \right) - 24 \left(\frac{p_0}{M_\infty R} \right)^{-5} \left[2(4\sigma + 1) E_0 + \frac{3E_2}{\omega_R^2} + 5E_0 \psi(5) \right] \right\}$$

$$+ O\left(\frac{p_0}{M_\infty R} \right)^{-6} \qquad \text{(C6)}$$

IV-C. Series Solution For A Sinusoidal Wind Gust

The solution for the gust loading due to a sinusoidal wind is obtained from Eqs. (B6) and (C6) when using the high frequency approximation for the potential equation. The gust is described by Eq. (18) with g defined as,

$$g = \sin \left[\omega (t - x - x_0) \right] \qquad (21)$$

and $\tilde{g}(s/\omega)$ as[3],

$$\tilde{g}(s/\omega) = \left[1 + (s/\omega)^2 \right]^{-1} \qquad (22)$$

For small values of the gust penetration parameter, corresponding to stations near the gust front, Eq. (B6) reduces to the form

$$\frac{\partial F}{\partial x} \frac{\omega_R}{W_0 A'}$$

$$\simeq S(p)\left\{-\omega_R \sin \omega x_0 + \omega_R\left(\frac{p}{M_\infty R}\right)\left[\omega_R \cos \omega x_0 - D_1 \sin \omega x_0\right]\right.$$

$$+ \frac{\omega_R}{2}\left(\frac{p}{M_\infty R}\right)^2 \left[\omega_R^2 \sin \omega x_0 + \omega_R D_1 \cos \omega x_0 - D_2 \sin \omega x_0\right]$$

$$\left. + \frac{\omega_R}{6}\left(\frac{p}{M_\infty R}\right)^3 \left[-\omega_R^3 \cos \omega x_0 + D_1 \omega_R^2 \sin \omega x_0 + \omega_R D_2 \cos \omega x_0 - D_3 \sin \omega x_0\right]\right\}$$

$$+ O\left[\left(\frac{p}{M_\infty R}\right)^{-4}\right] \tag{23}$$

where D_n is defined in Appendix B.

For large values of the gust penetration parameter, corresponding to stations far from the gust front, the asymptotic form for the gust loading is given by Eq. (C6)

$$
\frac{\partial F}{\partial x} \frac{\omega_R}{W_0 A'}
$$

$$
\simeq S(p_0) \Bigg\{ -6 \left(\frac{p_0}{M_\infty R}\right)^{-3} \cos \omega x_0 + \frac{18}{\omega_R} \left(\frac{p_0}{M_\infty R}\right)^{-4} (\omega x_0 \cos \omega x_0 - \sin \omega x_0)
$$

$$
- 1034 \cos \omega x_0 \left(\frac{p_0}{M_\infty R}\right)^{-7} \ln^2 \left(\frac{2p_0}{M_\infty R}\right) + 62.5 \cos \omega x_0 \left(\frac{p_0}{M_\infty R}\right)^{-5} \ln \left(\frac{2p_0}{M_\infty R}\right)
$$

$$
- 24 \left(\frac{p_0}{M_\infty R}\right)^{-5} \Bigg[\cos \omega x_0 \Big(5\psi(5) + 8\sigma + 2 \Big) + \frac{3}{\omega_R^2} \Big((\omega x_0)^2 \cos \omega x_0 - \cos \omega x_0
$$

$$
- \omega x_0 \sin \omega x_0 \Big) \Bigg] \Bigg\} + O \left[\left(\frac{p_0}{M_\infty R}\right)^{-6} \right] \qquad (24)
$$

V. CONCLUSION

This report is concerned with the first phase of a study of the effects of wind gusts on missile dynamics. We have restricted our attention to gust induced aerodynamic forcing functions on very slender bodies traveling at supersonic speeds. In this first segment the aerodynamic loading, assuming attached flow, has been derived employing the method of characteristics and transform techniques. Solutions for arbitrary wind gusts have been found for all penetration depths; in the case of high frequencies, analytic solutions for the gust induced local loads have been derived for stations both near and far from the gust front.

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Appendix A
METHOD OF CHARACTERISTICS SOLUTION

To first order, the pressure perturbations associated with the axial flow [Eq. (12a)] do not contribute to the aerodynamic loading. Therefore, attention will be focused on the flow due to the wind gust which is characterized by the following equations

$$\beta^2 C_{xx} - C_{rr} - \frac{1}{r} C_r + \frac{C}{r^2} = -M_\infty^2 (C_{tt} + 2C_{xt}) \tag{12b}$$

$$C_r(r = R) = -W_0 G(x,t) \tag{13b}$$

Equation (9a) can be reduced to a more convenient form by making a transformation from the (x,t) plane to the (x',t') plane

$$x' = \beta^{-1} x \tag{A1a}$$

$$t' = (\beta/M_\infty)\left(t - M_\infty^2 \beta^{-2} x\right) \tag{A1b}$$

With this transformation, Eqs. (9a) and (9b) become

$$C_{x'x'} - C_{rr} - \frac{1}{r} C_r + \frac{C}{r^2} = C_{t't'} \tag{A2a}$$

$$C_r(r = R) = -W_0 G(x',t') \tag{A2B}$$

Upon taking the Fourier Transform of the above set of equations with respect to t', we have

$$C^*_{x'x'} - C^*_{rr} - \frac{1}{r} C^*_r + \frac{C^*}{r^2} + k^2 C^* = 0 \tag{A3a}$$

$$C^*_r(r = R) = -W_0 G^*(x', k) \tag{A3b}$$

where

$$C^*(x', r, k) = \int_0^\infty e^{-ikt'} C(x', r, t') dt'$$

and similarly for G^*.

Equation (A3a) is classified as a quasi-linear hyperbolic second-order equation and can be solved numerically through application of Masseau's method.[2] In this method, appropriate finite difference equations are substituted for the characteristic differential equations, which are then solved by an iterative scheme.

The two families of characteristic lines are defined by the equations

$$\xi = r - x' \tag{A4a}$$

$$\eta = r + x' \tag{A4b}$$

where ξ and η are constants. The slope's associated with the families ξ = constant and η = constant are +1 and -1 respectively. The characteristic differential expressions valid along the direction $\lambda = +1$ are

$$du^* - dv^* - f^* dx' = 0 \tag{A5a}$$

$$dC^* = u^* dx' + v^* dr \tag{A5B}$$

where

$$f^* = \left(\frac{C^*}{r}\right)_r - k^2 C^*$$

Along the direction $\lambda = -1$, the characteristic differential expressions are

$$du^* + dv^* - f\,dx' = 0 \tag{A6a}$$

$$dC^* = u^* dx' + v^* dr \tag{A6b}$$

Interior Point Solution

At an interior point, an iterative solution can be obtained from Eqs. (A5) and (A6). Single subscripted variables imply evaluation at the field point denoted by the subscript (Fig. A1); a doubly subscripted variable represents the average value between the two points. A quantity without a superscript is exact, while one with a superscript (in brackets) implies that it is an element of a sequence of approximate values obtained through iteration.

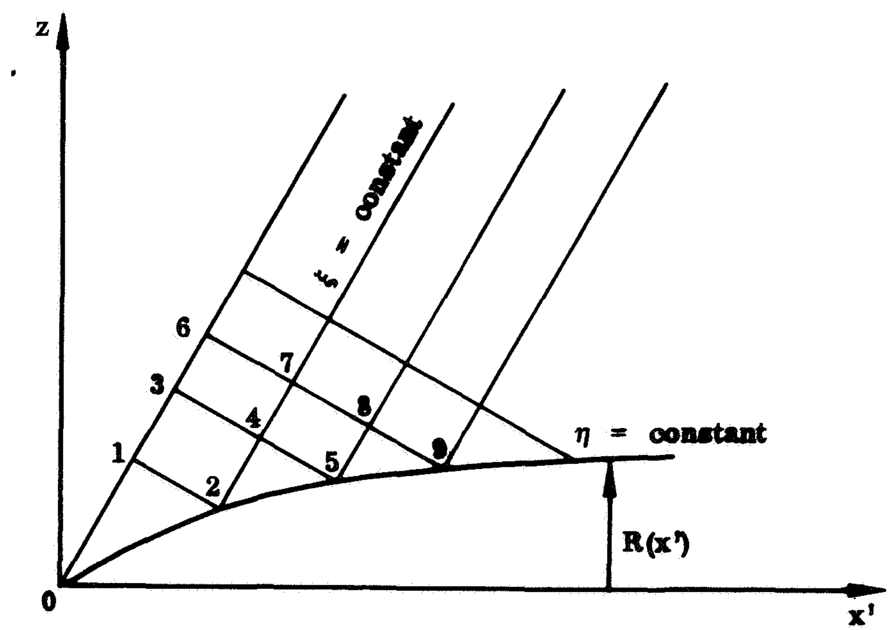

Fig. A1 CHARACTERISTIC GRID

The method of solution for point 4 in Fig. A1, assuming conditions at points 2 and 3 are known, will be described. The characteristic lines, ξ = constant and η = constant, are straight lines; thus the coordinates of point 4 (x'_4, r_4) is obtained from the equations

$$(x'_4 - x'_2) = (r_4 - r_2) \tag{A7a}$$

$$(x'_4 - x'_3) = -(r_4 - r_3) \tag{A7b}$$

The finite difference forms of Eqs. (A5a) and (A6a) together with the average of the difference forms of Eqs. (A5b) and (A6b) are used to generate the sequences $u_4^{*(n)}$, $v_4^{*(n)}$, $C_4^{*(n)}$.

For the n^{th} approximation we have,

$$\left(u_4^{*(n)} - u_2^*\right) - \left(v_4^{*(n)} - v_2^*\right) - \left(\frac{C_4^{*(n)}}{r_4} - \frac{C_2^*}{r_2}\right) + k^2 C_{24}^{*(n)} (x'_4 - x'_2) = 0 \tag{A8a}$$

$$\left(u_4^{*(n)} - u_3^*\right) + \left(v_4^{*(n)} - v_3^*\right) + \left(\frac{C_4^{*(n)}}{r_4} - \frac{C_3^*}{r_3}\right) + k^2 C_{34}^{*(n)} (x'_4 - x'_3) = 0 \tag{A8b}$$

$$\begin{aligned} C_4^{*(n)} - \frac{(C_2^* + C_3^*)}{2} = {} & \frac{1}{2} u_{24}^{*(n)} (x'_4 - x'_2) + \frac{1}{2} v_{24}^{*(n)} (r_4 - r_2) \\ & + \frac{1}{2} u_{34}^{*(n)} (x'_4 - x'_3) + \frac{1}{2} v_{34}^{*(n)} (r_4 - r_3) \end{aligned} \tag{A8c}$$

where for $n = 1$,

$$C_{24}^{*(1)} = C_2^* \;,\quad u_{24}^{*(1)} = u_2^* \;,\quad v_{24}^{*(1)} = v_2^*$$

$$C_{34}^{*(1)} = C_3^* \;,\quad u_{34}^{*(1)} = u_3^* \;,\quad v_{34}^{*(1)} = v_3^*$$

and for $n > 1$,

$$\begin{aligned}
C_{24}^{*(n)} &= \frac{1}{2}\left(C_2^* + C_4^{*(n-1)}\right) \\
u_{24}^{*(n)} &= \frac{1}{2}\left(u_2^* + u_4^{*(n-1)}\right) \\
v_{24}^{*(n)} &= \frac{1}{2}\left(v_2^* + v_4^{*(n-1)}\right) \\
C_{34}^{*(n)} &= \frac{1}{2}\left(C_3^* + C_4^{*(n-1)}\right) \\
u_{34}^{*(n)} &= \frac{1}{2}\left(u_3^* + u_4^{*(n-1)}\right) \\
v_{34}^{*(n)} &= \frac{1}{2}\left(v_3^* + v_4^{*(n-1)}\right)
\end{aligned} \tag{A6}$$

The iteration is continued until successive values for $u_4^{*(n)}$, $v_4^{*(n)}$, and $C_4^{*(n)}$ respectively are sufficiently close to each other.

Boundary Point Solution

To obtain solutions along the body surface, the above procedure must be modified in the following manner.[5] Values for the flow properties at point 5 will be found assuming their values are known at points 2 and 4. Along the η = constant line we have

$$x'_5 - x'_4 = -(r_5 - r_4) \tag{A9a}$$

$$\left(u_5^{*(n)} - u_4^*\right) + (v_5^* - v_4^*) + \left(\frac{C_5^{*(n)}}{r_5} - \frac{C_4^{*(n)}}{r_4}\right) + k^2 C_{45}^{*(n)} (x'_5 - x'_4) = 0 \quad \text{(A9b)}$$

where $r_5 = R(x'_5)$ and $v_5^* = -W_0 G^*(x'_5)$

The total differential of the function $C^*(x', r)$ is

$$\begin{aligned} dC^* &= \frac{\partial C^*}{\partial x'} dx' + \frac{\partial C^*}{\partial r} dr \\ &= u^* dx' + v^* dr \end{aligned} \quad \text{(A10)}$$

Using the boundary condition, Eq. (A3b), and the equation for the body radius $R(x')$, we have along the surface

$$\frac{dC^*}{dx'} = u^* - W_0 \frac{dR}{dx'} G^*(x') \quad \text{(A11)}$$

The finite difference form associated with the above equation for the n^{th} approximation to the flow properties at point 5 is

$$C_5^{*(n)} - C_2^* = u_{25}^{*(n)} (x'_5 - x'_2) - W_0 \int_{x'_2}^{x'_5} G^*(x') \frac{dR}{dx'} dx' \quad \text{(A12)}$$

Hence Eqs. (A9) and (A12) define sequences of solutions for the flow quantities at the body surface.

Upstream of the $\xi = 0$ characteristic line the perturbations are zero.

Normal Force Distribution

The pressure coefficient in the (x', t') plane is obtained from Eqs. (7d), (A1a), and (A1b)

$$c_p = -2\beta^{-1}\left(\varphi_{x'} - M_\infty^{-1}\varphi_{t'}\right) \tag{A13}$$

In terms of the axial and transverse perturbation potentials B and C, c_p is of the form

$$c_p = -2\beta^{-1} B_{x'} - 2\beta^{-1}\cos\theta\left(C_{x'} - M_\infty^{-1} C_{t'}\right) \tag{A14}$$

Upon substituting Eq. (A14) into Eq. (7e) the Fourier transformed expression for the normal force distribution is*

$$\frac{\partial F^*}{\partial x'} = -2\pi R\left(u^* - ikM_\infty^{-1} C^*\right) \tag{A15}$$

where u^* and C^* are to be evaluated at the body surface.

Sinusoidal Wind Gust

The solution for the aerodynamic loading due to a sinusoidal wing gust is readily obtained from the above equations. A gust traveling at a relative speed $\bar{U}_\infty$ in the axial direction is of the form

$$G(\bar{x}, \bar{t}) = \sin\left[\frac{2\pi}{\ell}(\bar{U}_\infty \bar{t} - \bar{x})\right] S(\bar{U}_\infty \bar{t} - \bar{x}) \tag{A16}$$

*Fourier Inversion Formula

$$C(x', r, t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikt'} C^*(x', r, k)\, dk$$

where the unit step function S has been introduced to account for penetration transients. In terms of dimensionless variables and complex notation we have

$$G(x,t) = \operatorname{Im} e^{i\frac{2\pi}{\ell}(t-x)} \tag{A17}$$

for $t - x > 0$. Upon making the transformation from the (x,t) plane to the (x', t') plane the equation for the gust is

$$G(x', t') = \operatorname{Im} e^{ikt'} e^{ikx'/M_\infty} \tag{A18}$$

where $k = 2\pi M_\infty/\ell\beta$. In terms of the Fourier transform formulation we can write

$$G(x', t') = G_k^*(x', k)e^{ikt'} \tag{A19}$$

where $G_k^*(x', k)$ is the amplitude of a single element of the entire spectrum. If we define

$$C(x', t') = C_k^*(x', k)e^{ikt'} \tag{A20}$$

and substitute into Equations (A2a) and (A2b) we have

$$C^*_{k_{x'x'}} - C^*_{k_{rr}} - \frac{1}{r}C^*_{k_r} + \frac{C^*_k}{r^2} + k^2 C^*_k = 0 \tag{A21a}$$

$$C^*_{k_r}(r = R) = -W_0 G^*_k(x', k) \tag{A21b}$$

The solution to the above pair of equations is transformed to the (x', t') plane upon multiplying by $e^{ikt'}$. In general, this product will have real and imaginary components; in view of the form for $G(x', t')$, Eq. (A18), the imaginary component of Eq. (A20) is the physically significant portion of the result.

In order to solve the pair of Eqs. (A21a) and (A21b), we introduce the real and imaginary components for C^* and G^*

$$C_k^* = C_k^{\overline{R}} + i C_k^{\overline{I}}$$

$$G_k^* = G_k^{\overline{R}} + i G_k^{\overline{I}}$$

Equations (A21a) and (A21b) are then replaced by the sets,

$$C_{k_{x'x'}}^{\overline{R}} - C_{k_{rr}}^{\overline{R}} - \frac{1}{r} C_{k_r}^{\overline{R}} + \frac{C_k^{\overline{R}}}{r^2} + k^2 C_k^{\overline{R}} = 0 \qquad \text{(A22a)}$$

$$C_{k_r}^{\overline{R}} (r = R) = -W_0 \cos (kx'/M_\infty) \qquad \text{(A22b)}$$

and

$$C_{k_{x'x'}}^{\overline{I}} - C_{k_{rr}}^{\overline{I}} - \frac{1}{r} C_{k_r}^{\overline{I}} + \frac{C_k^{\overline{I}}}{r^2} + k^2 C_k^{\overline{I}} = 0 \qquad \text{(A23A)}$$

$$C_{k_r}^{\overline{I}} (r = R) = - W_0 \sin (kx'/M_\infty) \qquad \text{(A23b)}$$

Each of the above sets of equations can be solved through application of the numerical technique described in the previous sections.

Appendix B

SERIES EXPANSION FOR SMALL VALUES OF THE GUST PENETRATION PARAMETER

Solutions for the gust loading, relative to the wind profile given by Eq. (18), are obtained from Eq. (20) through a Laplace Inversion. For large values of S , the functions κ and G can be replaced by their asymptotic forms[4]

$$\left[1 + (M_\infty s R)^2\right]\kappa^2 - \kappa = \sum_{n=0}^{\infty} D_n (M_\infty s R)^{-n} \tag{B1}$$

$$\tilde{G}(x,s) = \frac{e^{-s(x+x_0)}}{\omega} \sum_{n=0}^{N} d_n\left(\frac{s}{\omega}\right)^{-n} + \frac{e^{-sx}}{\omega}\sum_{n=1}^{N}\left[g^{(n-1)}(-x_0\,\omega)\right]\left(\frac{s}{\omega}\right)^{-n}$$

$$+ \frac{e^{-s(x+x_0)}}{\omega}\left(\frac{s}{\omega}\right)^{-N}\int_{-x_0\omega}^{0} g^{(N)}(y)\, e^{-\frac{sy}{\omega}}\, dy \tag{B2}$$

where b_i is defined by the expansion for $\tilde{g}$

$$\tilde{g}\left(\frac{s}{\omega}\right) = \sum_{n=0}^{\infty} b_n\left(\frac{s}{\omega}\right)^{-n} \tag{B3}$$

and

$$d_0 = b_0 \quad , \quad d_n = b_n - g^{(n-1)}(0)$$

$$c_n = \alpha_n + \sum_{i=1}^{n-1} \alpha_n \beta_{n-i} + \beta_n \quad , \quad \alpha_n = \frac{1}{n!\, 8^n} \prod_{i=1}^{n} \left[4 - (2i - 1)^2\right]$$

$$\beta_1 = -7/8 \ , \ \beta_2 = 41/128$$

$$\beta_3 = 895/1024 \ , \ \beta_4 = 44973/32768$$

$$D_0 = 1 \ , \ D_1 = 2c_1 - 1 \qquad D_2 = 1 + 2c_2 + c_1^2 - c_1$$

$$D_3 = 2c_3 + 2c_1 + 2c_1 c_2 - c_2$$

$$D_4 = 2c_4 + 2c_2 + 2c_1 c_3 + c_2^2 - c_3$$

An expression for $\partial\tilde{F}/\partial x$, which is valid for large values of s , is obtain upon substituting Eqs. (B1) and (B2) into Eq. (20). The transformation from s to t is easily accomplished through application of the inverse transforms[3]

$$L^{-1}\{s^{-n}\} = t^{n-1}/(n - 1)! \tag{B4}$$

$$L^{-1}\{e^{-\epsilon s}\tilde{f}(s)\} = f(t - \epsilon)S(t - \epsilon) \tag{B5}$$

The Taylor Series form for the aerodynamic loading associated with a wind gust is

$$
\begin{aligned}
&\frac{\partial F}{\partial x}\,\frac{\omega_R}{W_o A'} \\
&\simeq S(p_0)\left\{\omega_R d_1 + d_0 D_1 + \left(\frac{p_0}{M_\infty R}\right)\left(\omega_R^2 d_2 + \omega_R d_1 D_1 + d_0 D_2\right) + \frac{1}{2}\left(\frac{p_0}{M_\infty R}\right)^2\left(\omega_R^3 d_3\right.\right. \\
&\quad + \omega_R^2 d_2 D_1 + \omega_R d_1 D_2 + d_0 D_3\Big) + \frac{1}{6}\left(\frac{p_0}{M_\infty R}\right)^3\Big(\omega_R^4 d_4 + \omega_R^3 d_3 D_1 + \omega_R^2 d_2 D_2 \\
&\quad \left.\left. + \omega_R d_1 D_3 + d_0 D_4\right)\right\} + O\left[\left(\frac{p_0}{M_\infty R}\right)^4\right] + S(p)\left\{\omega_R e_1 + e_0 D_1 + \left(\frac{p}{M_\infty R}\right)\left(\omega_R^2 e_2\right.\right. \\
&\quad + \omega_R e_1 D_1 + e_0 D_2\Big) + \frac{1}{2}\left(\frac{p}{M_\infty R}\right)^2\left(\omega_R^3 e_3 + \omega_R^2 e_2 D_1 + \omega_R e_1 D_2 + e_0 D_3\right) \\
&\quad \left. + \frac{1}{6}\left(\frac{p}{M_\infty R}\right)^3\left(\omega_R^4 e_4 + \omega_R^3 e_3 D_1 + \omega_R^2 e_2 D_2 + \omega_R e_1 D_3 + e_0 D_4\right)\right\} + O\left[\left(\frac{p}{M_\infty R}\right)^4\right]
\end{aligned}
\tag{B6}
$$

where p and p_0 are gust penetration parameters and $e_0 = 0$, $e_n = g^{(n)}(-x_0\omega)$.

Appendix C

SERIES EXPANSION FOR LARGE VALUES OF THE GUST PENETRATION PARAMETER

In order to obtain solutions for large values of p_0, the functions κ, κ^2, and G which appear in Eq. (20) are replaced by their Taylor Series' approximations[4]

$$\left[1 + (M_\infty s R)^2\right]\kappa^2 - \kappa$$

$$= 2 + 3(M_\infty s R)^2 \ln\left(\frac{M_\infty s R}{2}\right) + (3\sigma + 1)(M_\infty s R)^2 + 2.5(M_\infty s R)^4 \ln^2\left(\frac{M_\infty s R}{2}\right)$$

$$+ 2(4\sigma + 1)(M_\infty s R)^4 \ln\left(\frac{M_\infty s R}{2}\right) + 1.375(M_\infty s R)^6 \ln^3\left(\frac{M_\infty s R}{2}\right) + \ldots\ldots \tag{C1}$$

$$\tilde{G}(x, s) = \frac{e^{-s(x+x_0)}}{\omega} \sum_{n=0}^{\infty} E_n \left(\frac{s}{\omega}\right)^n \tag{C2}$$

where a_i is defined by the expansion for $\tilde{g}$

$$\tilde{g}\left(\frac{s}{\omega}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{s}{\omega}\right)^n \tag{C3}$$

and

$$h_n = \int_{-x_0\omega}^{0} y^n g(y)\, dy$$

$$E_n = a_n + (-1)^n \frac{h_n}{n!}$$

An expression for $\partial F/\partial x$ valid for small values of s is obtained upon substituting the above expansions into Eq. (20). The resultant form involves functions of the form $s^n \ln^m s$. These can be inverted through application of the following inverse transform and its derivatives with respect to m.

$$L^{-1}\{s^{n-m+1}\} = (-1)^{n+1} \Gamma(n+2-m) \frac{\sin(\pi m)}{\pi} t^{m-n-2} \tag{C4}$$

The first derivative of Eq. (C4) with respect to m yields the following inverse Laplace Transform[4]

$$L^{-1}\{-s^{n-m+1} \ln s\} = (-1)^{n+1} t^{m-n-2} \left\{ \Gamma(n+2-m) \cos \pi m + \frac{\sin(\pi m)}{\pi} \left[\ln t - \Gamma(n+2-m)\psi(n+2-m) \right] \right\} \tag{C5}$$

Additional inversion formulas can be derived by differentiation of each new form. These equations together with Eq. (B5) are sufficient for evaluating the asymptotic form for the aerodynamic loading associated with a wind gust,

$$\frac{\partial F}{\partial x} \frac{\omega_R}{W_0 A'}$$

$$\simeq S(p_0) \left\{ -6 E_0 \left(\frac{p_0}{M_\infty R} \right)^{-3} + \frac{18}{\omega_R} E_1 \left(\frac{p_0}{M_\infty R} \right)^{-4} - 1034\, E_0 \left(\frac{p_0}{M_\infty R} \right)^{-7} \ln^2 \left(\frac{2p_0}{M_\infty R} \right) \right.$$

$$\left. + 62.5\, E_0 \left(\frac{p_0}{M_\infty R} \right)^{-5} \ln \left(\frac{2p_0}{M_\infty R} \right) - 24 \left(\frac{p_0}{M_\infty R} \right)^{-5} \left[2(4\sigma + 1) E_0 + \frac{3E_2}{\omega_R^2} + 5 E_0 \psi(5) \right] \right\}$$

$$+ O\left[\left(\frac{p_0}{M_\infty R} \right)^{-6} \right] \tag{C6}$$