X-552-69-92

PREPRINT

NASA TM X- 63532

ERROR ANALYSES OF RESONANT ORBITS FOR GEODESY

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MARCH 1969

GSFC

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

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ABSTRACT

An error analysis of resonant orbits has indicated that attempts to recover high order resonant geopotential coefficients will be seriously hampered by errors in the non-resonant geopotential terms. This effect, plus the very high correlations (up to .999) of the resonant coefficients with each other and the orbital period in single satellite resonant solutions, makes individual resonant orbits nearly valuless for geodesy. Multiple-satellite, single-plane solutions are only a slight improvement over the single satellite case. Independent determinations of resonant terms from low altitude satellites require multiple orbital planes and small beat-periods to reduce correlations and effects of errors of non-resonant geopotential terms. Also, these unmodeled parameter effects on low-altitude resonant satellites make the use of tracking arcs exceeding two to three weeks of doubtful validity. Because high-altitude resonant orbits are less affected by non-resonant terms in the geopotential, longer tracking arcs can be used for them.

This project was sponsored by NASA under Contract No. NAS 5-9756-135.

ERROR ANALYSES OF RESONANT ORBITS FOR GEODESY

1. INTRODUCTION

This study concerns determining the realistic accuracy to which certain geopotential coefficients can be recovered by observing satellites on resonant orbits. The results will be useful to investigators contemplating either resonant satellite geodetic missions or analysis of existing resonant satellite orbits.

Reference 1 presented a "noise only" error analysis of resonant orbits. Designed to yield only relative information, that study did not produce absolute predictions of the quality of determinations of gravity coefficients from resonant orbits. Error analyses are almost always very optimistic unless we include the effects of errors in the unmodeled or unadjusted parameters of the problem. The resonant satellite geodesy problem is an extreme example of this phenomenon because the "noise only" standard deviations are as much as five orders of magnitude optimistic in some cases.

This study includes effects of errors in the following parameters:

- non-resonant geopotential coefficients;
- station location;
- unadjusted resonant geopotential coefficients; and
- GM, the product of the Earth's mass and the gravitational constant.

Errors in all of these quantities have large effects on the solution. Indeed, for low altitude (or high order) resonant satellites in a single plane, errors in the non-resonant geopotential coefficients entirely dominate the solution.

Finally, the causes of the large effects of the errors in the unadjusted parameters have been sought in terms of the physical characteristics of resonant orbits.

2. SHALLOW RESONANCE

For several reasons this study is confined to orbits in "shallow" resonance – approximate rather than exact or "deep" resonant orbits. First, very deep resonance is not required to achieve large perturbations such as several kilometers along track; second, deeply resonant orbits are achievable only after several orbit corrections. From a mission standpoint, such a requirement means greatly increased spacecraft complexity. Finally, this study has shown that for low and medium altitude satellites deeply resonant orbits are less desirable for satellite geodesy than orbits in shallow resonance because they are affected more by unmodeled parameters.

The terms "shallow" and "deep" resonance can be given a reasonably precise meaning in terms of satellite beat period. The condition for an exactly repeating ground track on the average, or deep resonance, is

$$\dot{D} = \dot{\omega}_c + \dot{M}_c + s(\dot{\Omega}_c - \dot{\theta}) = 0,$$

where the subscript c denotes secular rate, and s is the integer number of revs/day made by the satellite. ($\dot{D}$ is the drift rate of the orbit, $\dot{\omega}$, $\dot{\Omega}$ and $\dot{\theta}$ are the rates of rotation of perigee, the node and the earth. $\dot{M}$ is the mean motion of the satellite.) (See Reference 1.) For shallow resonance, $\dot{D}$ is not zero but is small compared to the satellite mean motion. For example, the GEOS-II satellite orbiting the Earth approximately thirteen times per day has

$$\dot{D} = -57.2^\circ/\text{day},$$

compared with

$$\dot{M}_s = 4619^\circ/\text{day}.$$

The beat period, $360^\circ/\dot{D}$, is a measure of the time for the ground track to repeat. For GEOS-II this is $6^{d}.3$.

The most important parameters to consider when selecting orbits for resonant satellite geodesy are beat period, inclination, and eccentricity. The influence of these parameters will be demonstrated.

The solutions for perturbations of orbital elements for deeply resonant orbits are different from those for non-resonant orbits. The non-resonant solutions assume small perturbations, and we can treat them as linear forced oscillations about an intermediary, or reference, trajectory. Of course, deep resonance violates this condition because the satellite may be perturbed up to 90° along track, not a small change. However, Palmiter and Gedeon in Reference 2 have shown that the non-resonant perturbation solutions are quite accurate for small beat periods and are in error only a few percent for beat periods near 100 days. Thus, we define shallow resonance as corresponding to repeating ground track beat periods of about 100 days or less and reserve the term deep resonance for longer beat periods. However, using non-resonant formulas on orbits with beat periods near 100 days requires considerable caution because the observable resonant perturbations on an orbit may vary in magnitude by a factor of 100 or more. Obviously, even a one percent error in calculating the larger effects may seriously degrade the quality of determining the sources of the smaller effects. Numerical integration is probably the only reasonable procedure for beat periods greater than approximately fifty days.

The graphs in Reference 1 show how the acceleration of mean anomaly caused by dominant resonant geopotential coefficients varies with inclination. These graphs indicate that high inclinations, $i \gtrsim 40°$, are required to achieve large perturbations for medium and low altitude resonant orbits, $s \geq 8$ revs/day. Our present report exhibits similar information for several selected representative cases, yet shows variation with eccentricity and beat period as well. To accomplish this, we lumped the along-track effects of many or all of the resonant coefficients into a Root-Sum-of-Squares (RSS). The along-track effect was estimated from

$$\Delta(\text{along-track}) \simeq a(\Delta\omega + \Delta M + \Delta\Omega \cos i)$$

where $\Delta\omega$, ΔM, and $\Delta\Omega$ appear in Reference 3 as

$$\Delta\omega_{\ell mpq} = \mu a_e^{\ell} \frac{\left[(1-e^2)^{\frac{1}{2}} e^{-1} F_{\ell mp} (\partial G_{\ell pq}/\partial e) - \cot i (1-e^2)^{-\frac{1}{2}} (\partial F_{\ell mp}/\partial i) G_{\ell pq}\right] S_{\ell mpq}}{n a^{\ell+3} [(\ell - 2p)\dot{\omega}_e + (\ell - 2p + q)\dot{M}_e + m(\dot{\Omega}_e - \dot{\theta})]},$$

$$\Delta\Omega_{\ell mpq} = \mu a_e^{\ell} \frac{(\partial F_{\ell mp}/\partial i) G_{\ell pq} \bar{S}_{\ell mpq}}{n a^{\ell+3} (1-e^2)^{\frac{1}{2}} \sin i [(\ell - 2p)\dot{\omega}_e + (\ell - 2p + q)\dot{M}_e + m(\dot{\Omega}_e - \dot{\theta})]},$$

$$\Delta M_{\ell mpq} = \mu a_e^{\ell} \frac{[-(1-e^2) e^{-1} (\partial G_{\ell pq}/\partial e) + 2(\ell + 1) G_{\ell pq}] F_{\ell mp} \bar{S}_{\ell mpq}}{n a^{\ell+3} [(\ell - 2p)\dot{\omega}_e + (\ell - 2p + q)\dot{M}_e + m(\dot{\Omega}_e + \dot{\theta})]} + \qquad (2\text{-}1)$$

$$- \frac{3 \mu a_e^{\ell} F_{\ell mp} G_{\ell pq} (\ell - 2p + q) \bar{S}_{\ell mpq}}{a^{\ell+3} [(\ell - 2p)\dot{\omega}_e + (\ell - 2p + q)\dot{M}_e + m(\dot{\Omega}_e - \dot{\theta})]^2},$$

with

$$S_{\ell mpq} = J_{\ell m} \begin{bmatrix} \cos \\ \sin \end{bmatrix}_{(\ell - m)\ \text{odd}}^{(\ell - m)\ \text{even}} \{(\ell - 2p)\omega + (\ell - 2p + q)M + m(\Omega - \theta - \lambda_{\ell m})\}$$

or

$$S_{\ell mpq} = \begin{bmatrix} C_{\ell m} \\ -S_{\ell m} \end{bmatrix}_{(\ell - m)\ \text{odd}}^{(\ell - m)\ \text{even}} \cos\{(\ell - 2p)\omega + (\ell - 2p + q)M + m(\Omega - \theta)\}$$

$$+ \begin{bmatrix} S_{\ell m} \\ C_{\ell m} \end{bmatrix}_{(\ell - m)\ \text{odd}}^{(\ell - m)\ \text{even}} \sin\{(\ell - 2p)\omega + (\ell - 2p + q)M + m(\Omega - \theta)\} .$$

The quantity $\bar{S}_{\ell mpq}$ is the integral of $S_{\ell mpq}$ with respect to its argument. The p, q indices identify particular harmonic components of a spherical harmonic (ℓ, m). The functions $F_{\ell mp}(i)$ and $G_{\ell pq}(e)$ are defined in Reference 3. $G_{\ell pq}(e)$ is $O(e^{|q|})$ for low and moderate e.

In Reference 1 and elsewhere, the resonant harmonic components (ℓ, m, p, q) are given by the condition that

$$\ell - 2p + q = \frac{m}{s}$$

For example, the resonant (ℓ, m, p, q) sets for a satellite with mean motion at or near fifteen teims per day are

15, 15, 6, -2

15, 15, 7, 0

15, 15, 8, 2

. . . .

. . . .

. . . .

16, 15, 7, -1

16, 15, 8, 1

. . . .

. . . .

. . . .

In the present work, all resonant (ℓ, m, p, q) sets to (ℓ, m) = (30, 30) were computed and included in the perturbation calcuations.

As a consequence of the treatment of tesseral harmonic perturbations as linear forced oscillations we also speak of a beat period for each resonant (ℓ, m, p, q) component calculated from $360°/\dot{D}_{\ell mpq}$, where

$$\dot{D}_{\ell mpq} = (\ell - 2p)\,\dot{\omega}_c + (\ell - 2p + q)\dot{M}_c + m(\dot{\Omega}_c - \dot{\theta}).$$

For a circular orbit, or one at the critical inclination, all resonant terms have the same beat period, the repeating groundtrack value. This does not apply to eccentric orbits at arbitrary inclination because of the secular advance of perigee caused by J_2. The resonant (ℓ, m, p, q) sets for the example above would yield the corresponding drift rates $\dot{D}_{\ell mpq}$:

$$\dot{D}_{15,15,6,-2} = 3\dot{\omega}_c + \dot{M}_c + 15(\dot{\Omega}_c - \dot{\theta}),$$

$$\dot{D}_{15,15,7,0} = \dot{\omega}_c + \dot{M}_c + 15(\dot{\Omega}_c - \dot{\theta}),$$

$$\dot{D}_{16,15,7,-1} = 2\dot{\omega}_c + \dot{M}_c + 15(\dot{\Omega}_c - \dot{\theta}), \text{ etc.}$$

Only in the q = 0 circular case is the beat period of a resonant (ℓ, m, p, q) harmonic component the same as the repeating groundtrack beat period. The drift rates and the resulting beat periods for the various components vary according to the contribution of (ℓ - 2p)$\dot{\omega}$. Of course, the larger the beat period, the greater the effect of any particular (ℓ, m, p, q) component. For a beat period > 5 days the term with the quadratic divisor in $\Delta M_{\ell mpq}$ will dominate; thus, the effect will increase quadratically with beat period as long as Equations 2-1 are valid.

$\dot{\omega}_c$ is always small compared to $\dot{M}_c$ or $\dot{\theta}$; thus, whether (ℓ - 2p) is 0, 2, or 3 does not have much effect for very shallow resonance ($\leq$ 10 days). However, for somewhat deeper resonance, especially for beat periods > 30 days, $\dot{\omega}_c$ is comparable to $(\ell - 2p + q)\dot{M}_c + m(\dot{\Omega}_c - \dot{\theta})$. The presence or absence of $\dot{\omega}$ may drastically alter the beat period of a particular (ℓ, m, p, q) component compared to the repeating groundtrack beat period. Figure 2-1 show how the RMS of the beat periods of the resonant terms varies with repeating ground track beat period, and inclination for an orbit resonant with 10th order of geopotential terms. The identifying numbers on each curve indicate the repeating groundtrack beat period for the case.

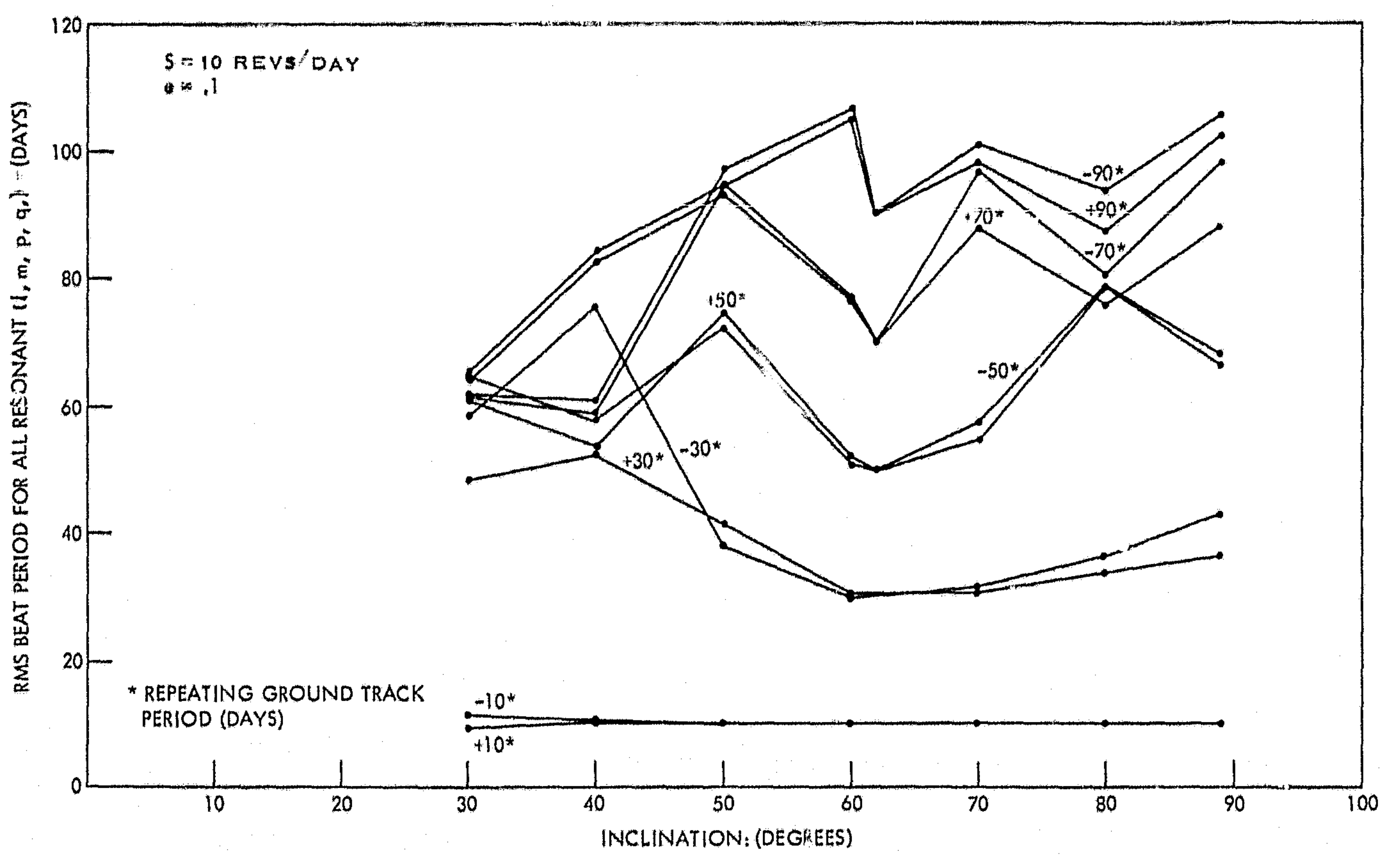

Figure 2-1–Effect of Inclination on RMS Beat Period of Resonant (ℓ, m, p, q) Components for S = REVS/DAY

In Figure 2-1, first note that the ten day repeating groundtrack beat period case shows almost no variation of resonant beat periods with inclination. Longer repeating groundtrack beat periods show the RMS beat period varying considerably both with inclination and beat period except at $i = 63°45$, the critical inclination. At $i = 63°45$, $\dot{\omega}_c = 0$, and all beat periods are the same. Of course, all resonant terms beating with the same frequency is not conducive to separating the terms from one another. For high altitude satellites, this situation is aggravated at all inclinations because $\dot{\omega}_c$ declines rapidly with an increasing semi-major axis.

Finally, an orbit with a large negative beat period is close in semi-major axis to an orbit with a large positive beat period. Thus, the values of perturbations vary irregularly at long beat periods because a small change in repeating groundtrack beat period may cause a large number of resonant (ℓ, m, p, q) components to radically alter their beat period, even changing sign. Several following graphs will depict this irregularity.

The effect of eccentricity on the resonance phenomenon is considerable. Reference 1 conveys that maximizing eccentricity maximizes resonant perturbations; however, an excessive eccentricity might be poor because too many terms might have perceptible effects. Table 2-1 from Reference 4 shows an extreme example of this phenomenon. This is an analysis of some of the resonant perturbations on 1966 92a, a Russian twelve-hour communication satellite. Second order terms from (2,2) to (19,2) produce effects in the transverse (along-track) direction larger than 10 km!

Figure 2-2 shows how the RSS of the dominant resonant perturbations along-track vary as a function of beat period for several eccentricities and inclinations for an orbit resonant with 10th order terms. The y axis shows the Root-Sum-of-Squares of the along-track perturbations of (10,10) through (15,10).

A 10th order resonance was the basis of Figure 2-2; however, the along-track displacements are typical of orbits in shallow resonance. For high inclination orbits of moderate to high eccentricity, a beat period of twenty days virtually guarantees effects of several km along-track. The effects for negative beat periods – mean motion <u>less</u> than an integer times the rotation rate of the Earth – are comparable. Thus, deep resonance is <u>not</u> required to produce easily observable effects and is even detrimental to determining geopotential coefficients, as will be seen later. Figures 2-3, 2-4 and 2-5 show similar information for satellites resonant with fourth order terms. Note the very large effects for e = .55.

Table 2-1

SATELLITE 66 92A

S = 2 REV./DAY PERIGEE HEIGHT = 269. KM.

A = 4.1605 E.R. E = 0.7495 I = 64.90 DEG.

L, M, P, Q	BEAT PERIOD (DAYS)	CENTRAL ANGLE (DEGREES)	ALONG-TRACK (METERS)	DELTA A (METERS)
2, 2, 0, -1	490.8	9.12D 00	4.23D 06	2.86D 03
2, 2, 1, 1	468.2	5.92D 01	2.74D 07	1.95D 04
2, 2, 2, 3	447.6	5.10D-02	2.36D 04	1.76D 01
3, 2, 0, -2	503.0	2.30D-01	1.07D 05	7.06D 01
3, 2, 1, 0	479.3	3.41D 00	1.58D 06	1.10D 03
3, 2, 2, 2	457.7	7.72D 00	3.58D 06	2.60D 03
4, 2, 1, -1	490.8	1.70D 00	7.88D 05	5.35D 02
4, 2, 2, 1	468.2	1.31D 00	6.05D 05	4.30D 02
4, 2, 3, 3	447.6	1.31D 00	6.07D 05	4.51D 02
5, 2, 1, -2	503.0	5.35D-01	2.48D 05	1.64D 02
5, 2, 2, 0	479.3	3.45D 00	1.60D 06	1.11D 03
5, 2, 3, 2	457.7	7.34D-01	3.40D 05	2.47D 02
5, 2, 4, 4	438.0	3.20D-01	1.48D 05	1.13D 02
6, 2, 1, -3	515.7	6.14D-02	2.84D 04	1.84D 01
6, 2, 2, -1	490.8	8.79D-01	4.07D 05	2.77D 02
6, 2, 3, 1	468.2	1.58D 00	7.32D 05	5.21D 02
6, 2, 4, 3	447.6	1.84D-01	8.53D 04	6.36D 01
6, 2, 5, 5	428.7	3.59D-02	1.66D 04	1.29D 01
7, 2, 2, -2	503.0	3.74D-01	1.73D 05	1.15D 02
7, 2, 3, 0	479.3	1.64D 00	7.58D 05	5.28D 02
7, 2, 4, 2	457.7	1.39D 00	6.46D 05	4.71D 02
7, 2, 5, 4	438.0	1.12D-01	5.20D 04	3.97D 01
8, 2, 3, -1	490.8	1.03D-01	4.75D 04	3.24D 01
8, 2, 4, 1	468.2	2.31D-01	1.07D 05	7.64D 01
8, 2, 5, 3	447.6	1.21D-01	5.58D 04	4.17D 01
9, 2, 4, 0	479.3	8.70D-02	4.03D 04	2.81D 01
9, 2, 5, 2	457.7	1.40D-01	6.49D 04	4.74D 01
9, 2, 6, 4	438.0	5.21D-02	2.41D 04	1.84D 01
10, 2, 4, -1	490.8	3.02D-02	1.40D 04	9.52D 00
10, 2, 5, 1	468.2	6.08D-02	2.82D 04	2.01D 01
10, 2, 6, 3	447.6	1.08D-01	5.01D 04	3.74D 01
10, 2, 7, 5	428.7	3.18D-02	1.47D 04	1.15D 01
11, 2, 4, -2	503.0	5.02D-02	2.32D 04	1.55D 01
11, 2, 5, 0	479.3	7.36D-02	3.41D 04	2.38D 01
11, 2, 7, 4	438.0	2.70D-02	1.25D 04	9.58D 00
12, 2, 4, -3	515.7	2.97D-02	1.37D 04	8.92D 00
12, 2, 5, -1	490.8	8.58D-02	3.97D 04	2.71D 01
12, 2, 6, 1	468.2	8.68D-02	4.02D 04	2.88D 01
13, 2, 5, -2	503.0	2.91D-02	1.35D 04	8.99D 00
13, 2, 6, 0	479.3	5.48D-02	2.54D 04	1.78D 01
13, 2, 7, 2	457.7	3.88D-02	1.80D 04	1.32D 01
14, 2, 6, -1	490.8	4.98D-02	2.31D 04	1.58D 01
14, 2, 7, 1	468.2	6.37D-02	2.95D 04	2.12D 01
14, 2, 8, 3	447.6	3.35D-02	1.55D 04	1.16D 01
15, 2, 7, 0	479.3	4.52D-02	2.10D 04	1.47D 01
15, 2, 8, 2	457.7	4.07D-02	1.89D 04	1.38D 01
16, 2, 7, -1	490.8	2.78D-02	1.29D 04	8.80D 00
16, 2, 8, 1	468.2	4.61D-02	2.14D 04	1.53D 01
16, 2, 9, 3	447.6	3.39D-02	1.57D 04	1.18D 01
19, 2, 9, 0	479.3	2.53D-02	1.17D 04	8.24D 00
4, 4, 1, 0	239.6	7.44D-01	3.45D 05	4.79D 02
4, 4, 2, 2	234.1	1.36D 00	6.28D 05	8.93D 02
4, 4, 3, 4	228.8	6.32D-02	2.93D 04	4.26D 01

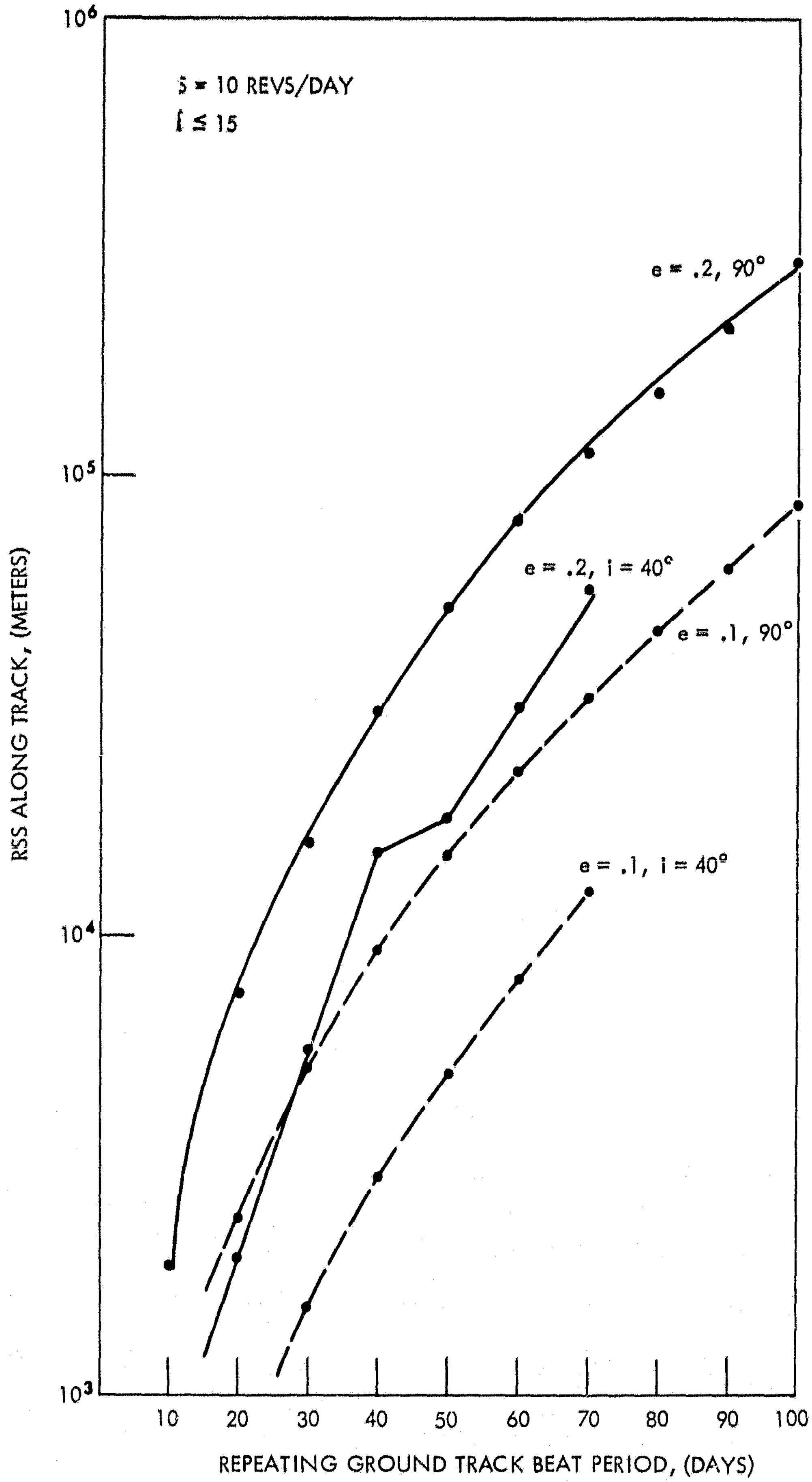

Figure 2-2—RSS Along-Track Perturbations for Resonant Terms with S = 10 REVS/DAY

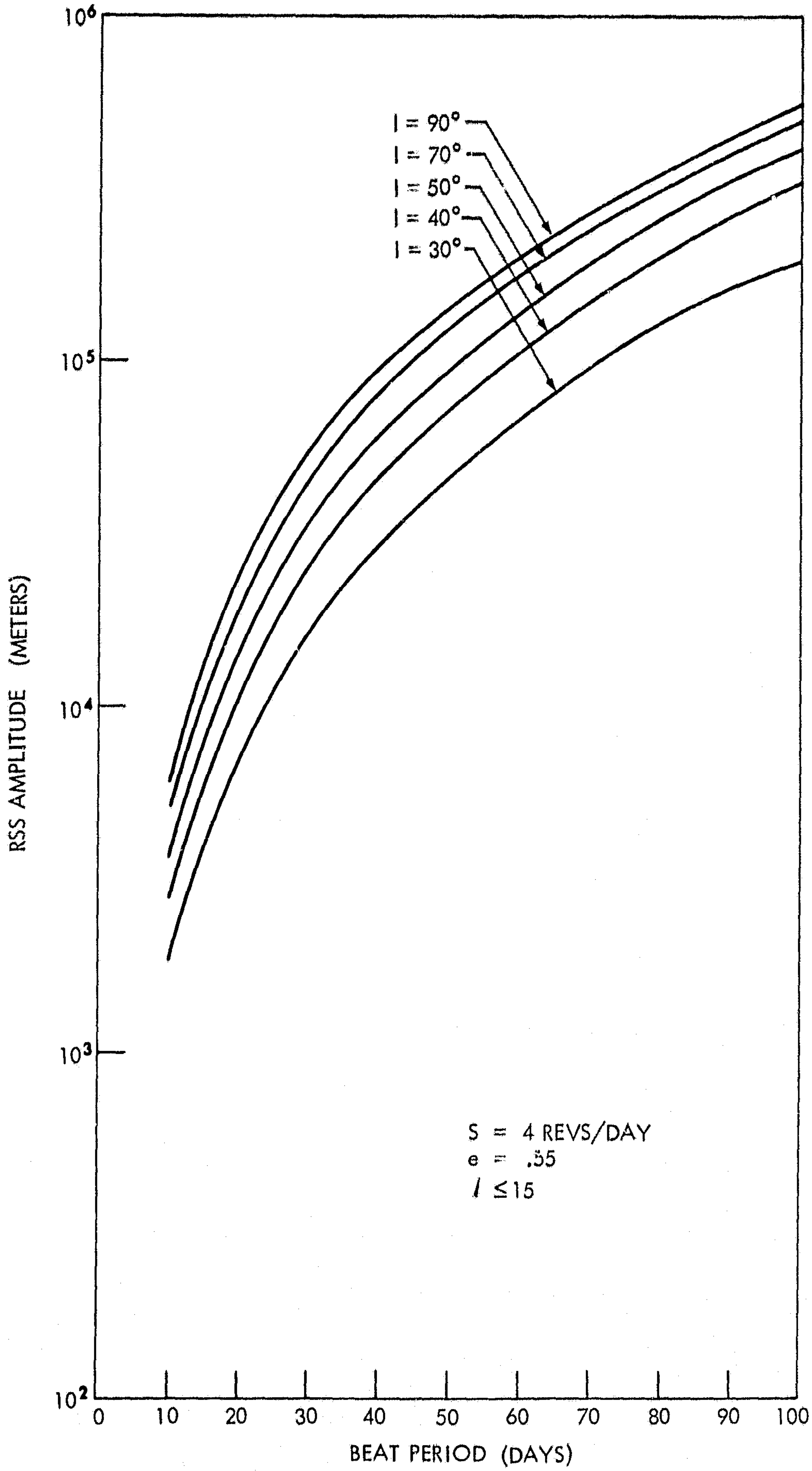

Figure 2-3—Effect of Inclination on Along-Track RSS Perturbations for S = 4 REVS/DAY, e = .55

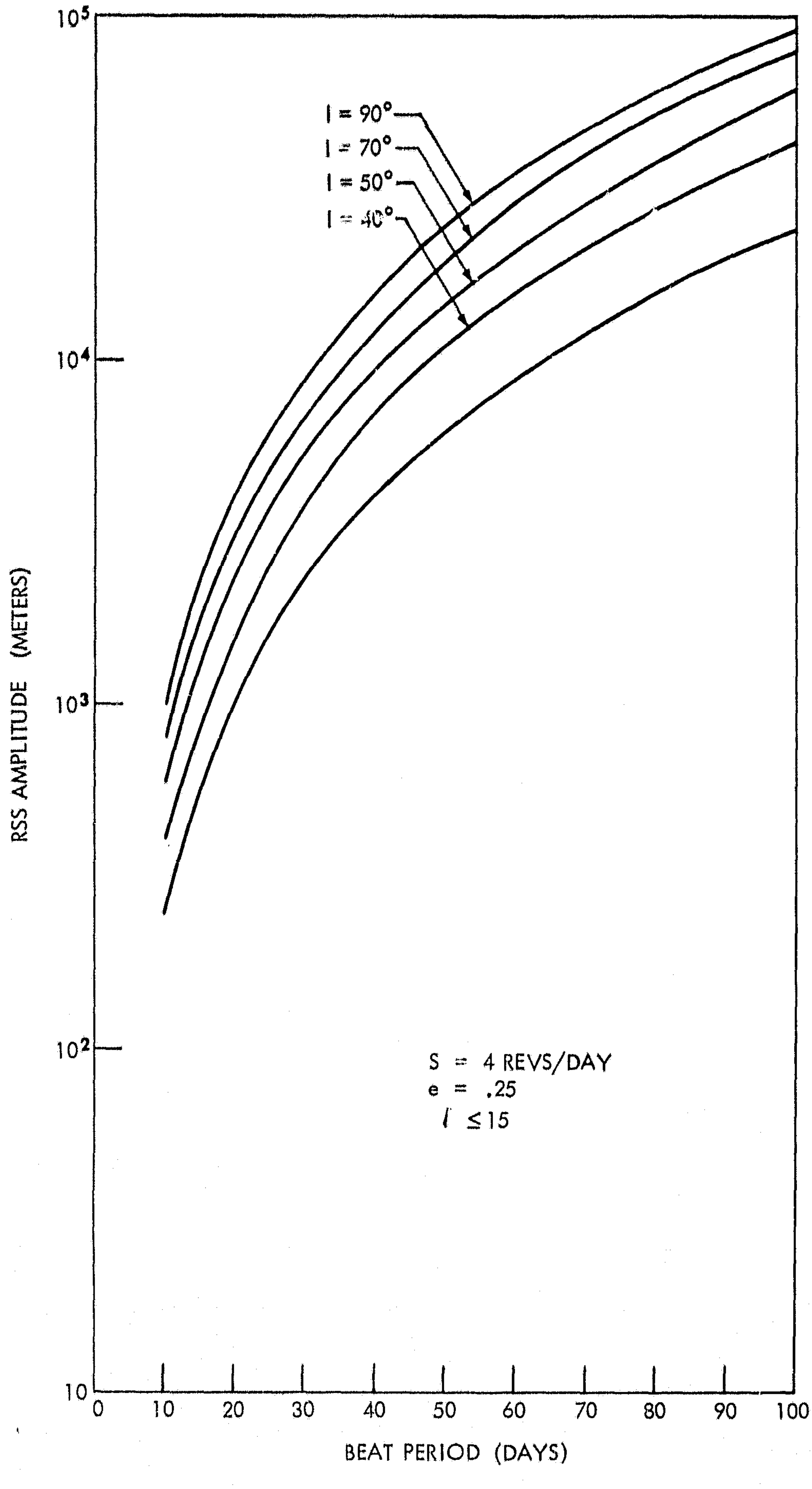

Figure 2-4—Effect of Inclination on Along-Track RSS Perturbations for S = 4 REVS/DAY, e = .25

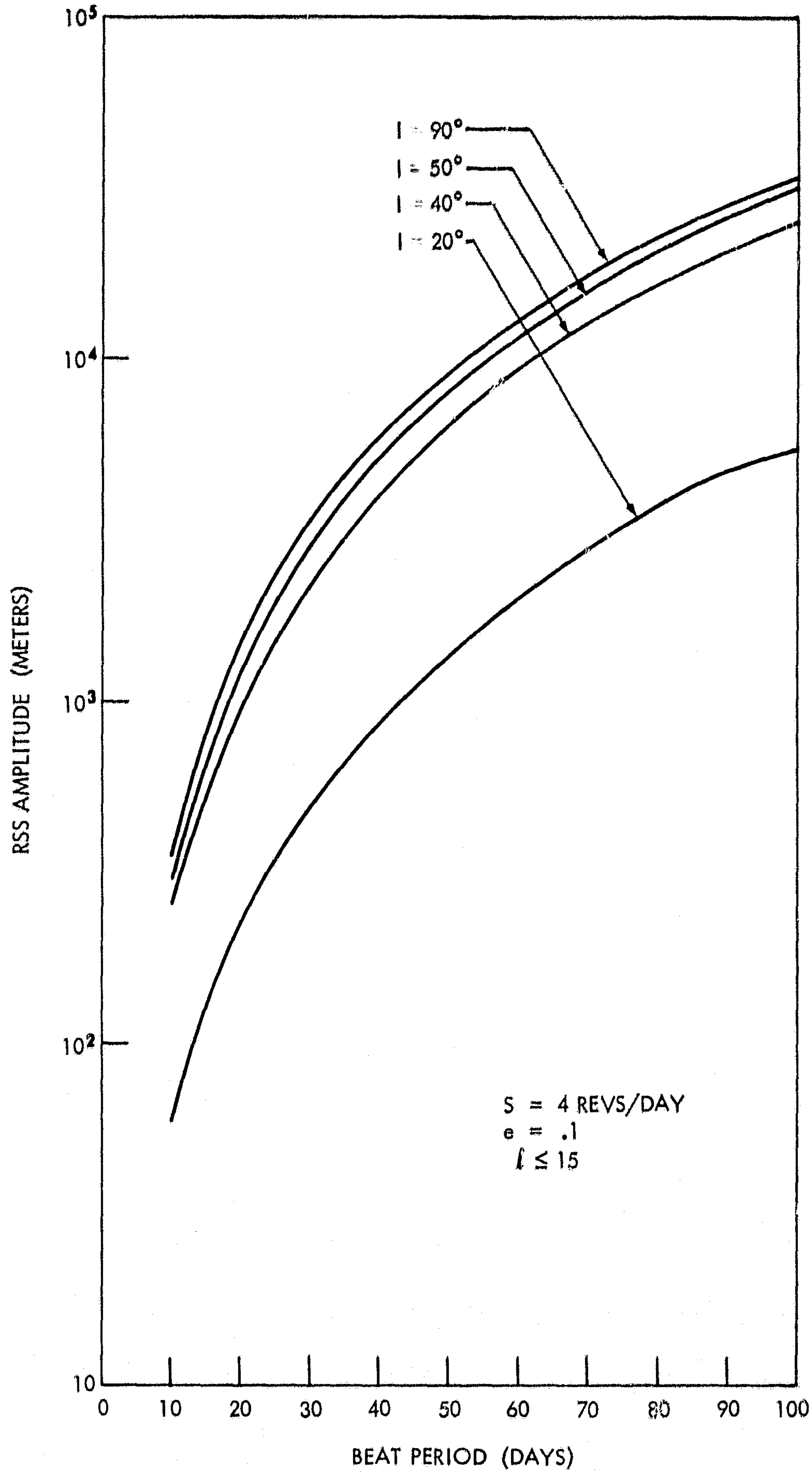

Figure 2-5—Effect of Inclination on Along-Track RSS Perturbations for S = 4 REVS/DAY, e = .1

Choosing orbital eccentricity therefore involves two important factors. First, the eccentricity must be large enough so that the eccentric resonant terms, ℓ even for s odd and vice versa, have observable effects. However, if eccentricity is too large, a solution for any subset of the resulting numerous resonant terms will be corrupted by the unmodeled ones. Figures 2-6 and 2-7 show the ratios of the RSS along-track effects due to circular and eccentric resonant terms for the tenth order resonant terms with $\ell \leq 15$ for e = .05 and .1. Note that for e = .1, the circular and eccentric resonant terms are comparable in importance. Large eccentricity is not required to make the eccentric resonant terms visible. Figure 2-8 for the fourth order resonance shows a similar result. Note in Figure 2-8 the lack of irregularity in the curves. For a satellite resonant with fourth order terms, λ_e is very small and has little effect on beat periods.

Figures 2-9 and 2-10 show how eccentricity affects the number of resonant terms that have large effects. These figures are plots of the ratio of the RSS of the resonant terms with $\ell - m \leq 5$ to resonant terms with $5 < \ell - m \leq 20$ for an orbit resonant with tenth order terms. Note that for e = .2 and medium inclination, $i \simeq 40°$, this ratio is about 1:1. This is a poor situation. The high degree terms would certainly corrupt the determination of the terms with $\ell \leq 5$. However, at high inclinations, the terms with small $(\ell - m)$ dominate, and the ratio increases to about ten for e = .2 and 20-30 for e = .1. Thus, an eccentricity of about .1 appears to be a good choice for low altitude resonant orbits. It is high

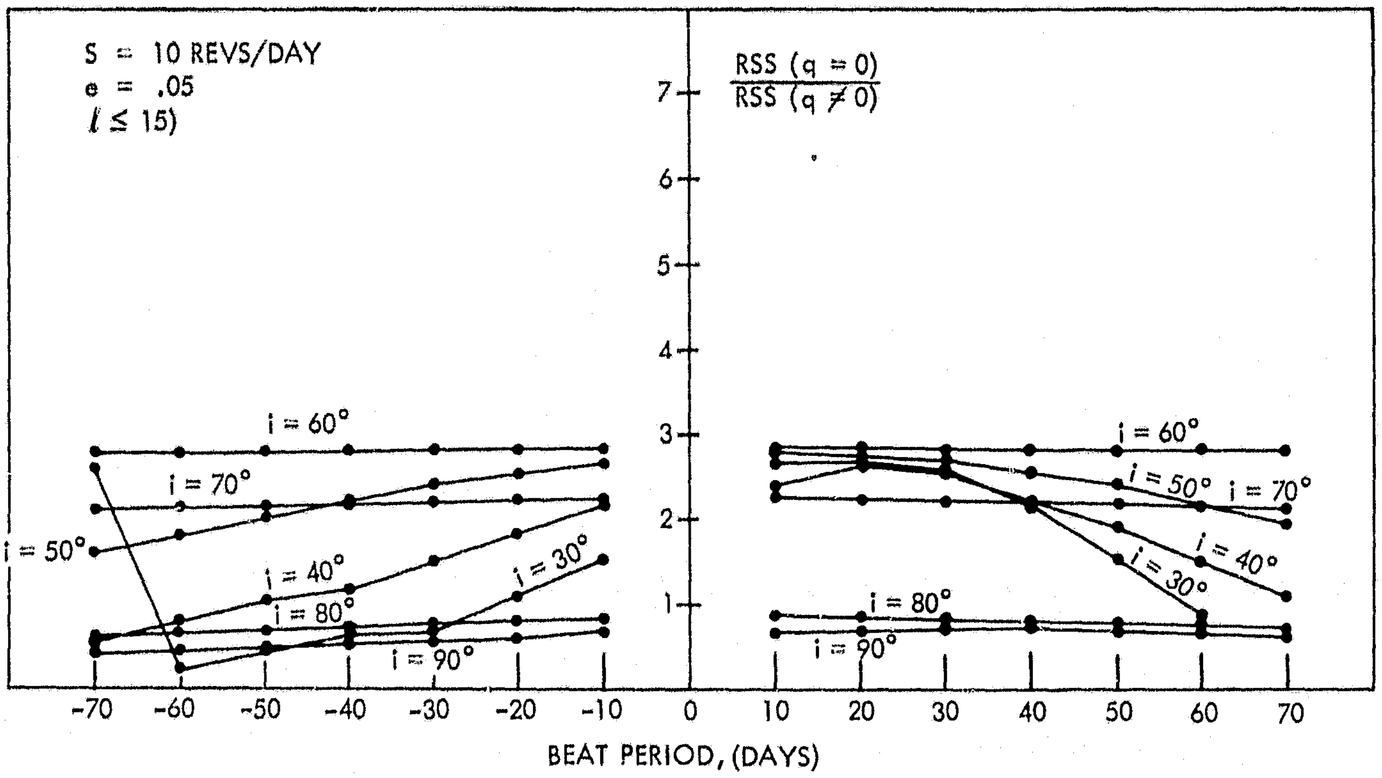

Figure 2-6—Effect of Inclination on the Relative Strength of Eccentric Resonant Terms; S = 10 REVS/DAY, e = .05

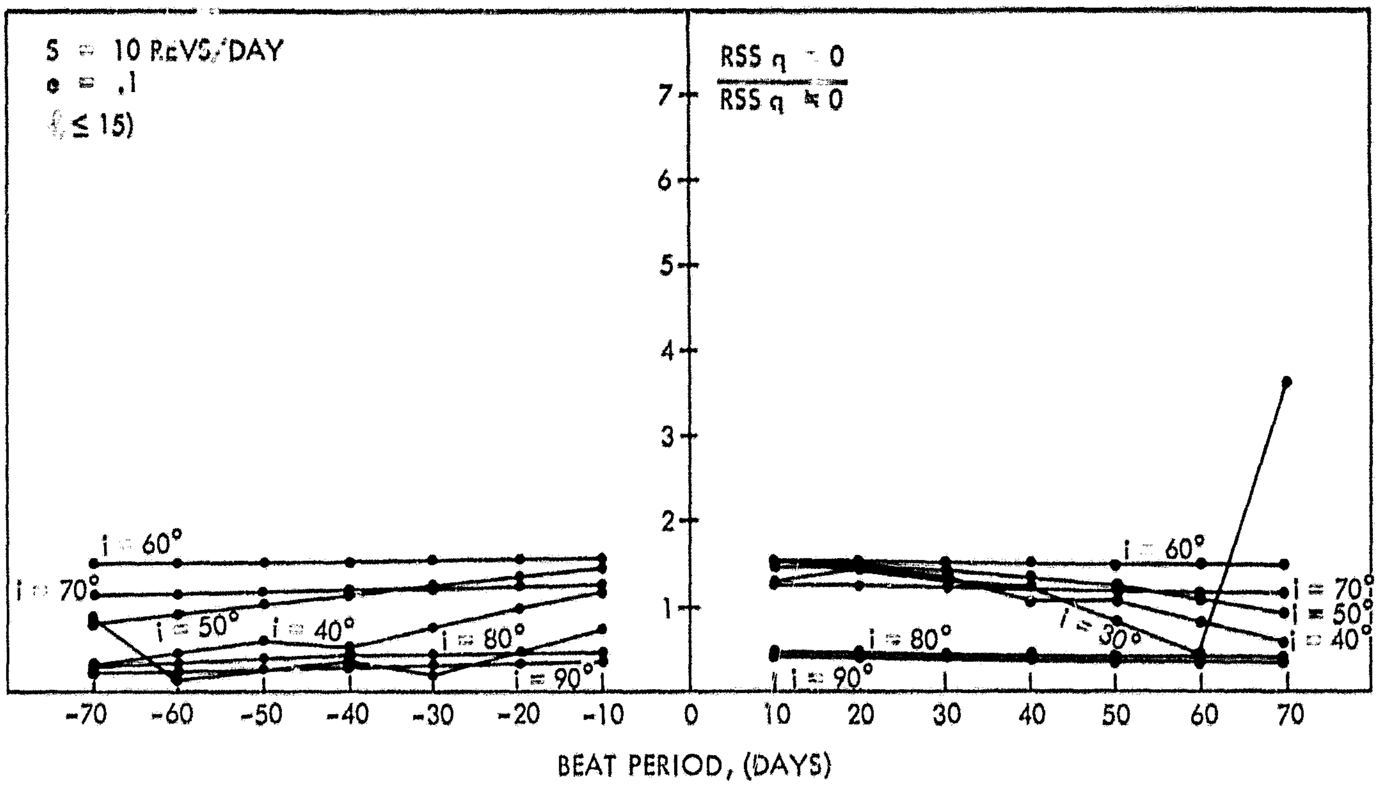

Figure 2-7—Effect of Inclination on the Relative Strength of Eccentric Resonant Terms; S = 10 REVS/DAY, e = .1

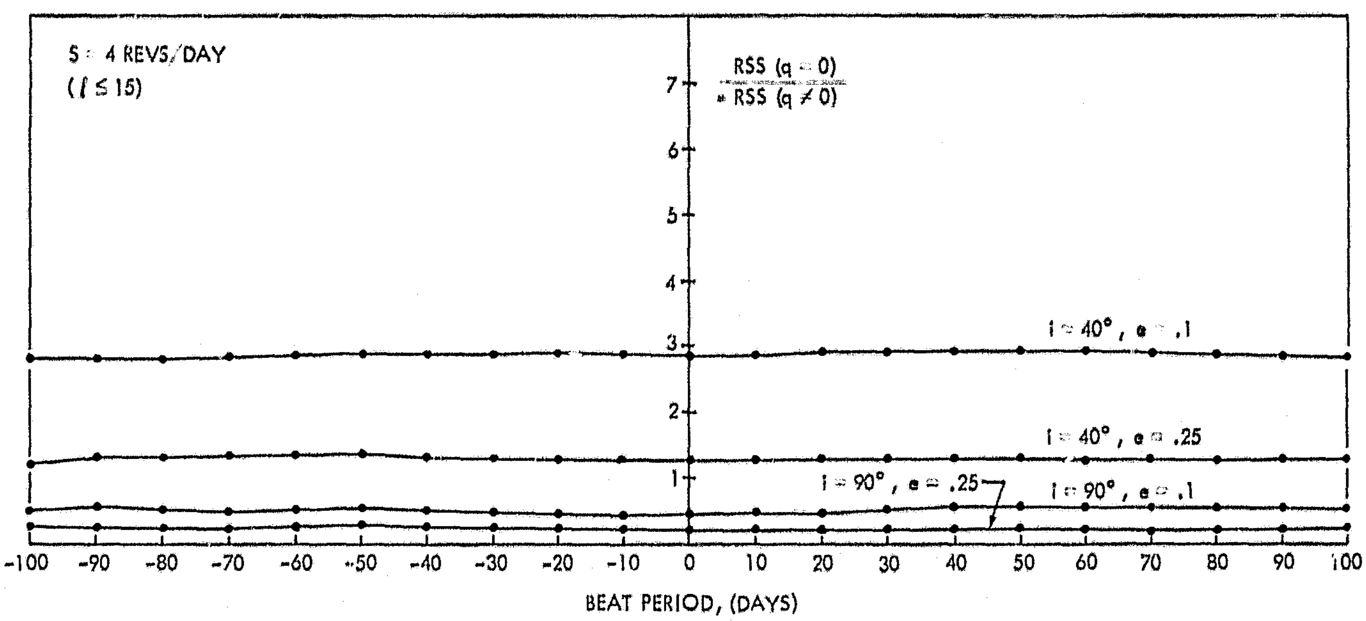

Figure 2-8—Effect of Eccentricity and Inclination on the Relative Strength of Eccentric Resonant Terms for S = 4 REVS/DAY

enough so that eccentric orbit resonant terms have effects as large as the circular orbit resonant terms, yet not so large that an unmanageable number of terms exist. A similar situation also generally applies to all other eccentric resonant orbits.

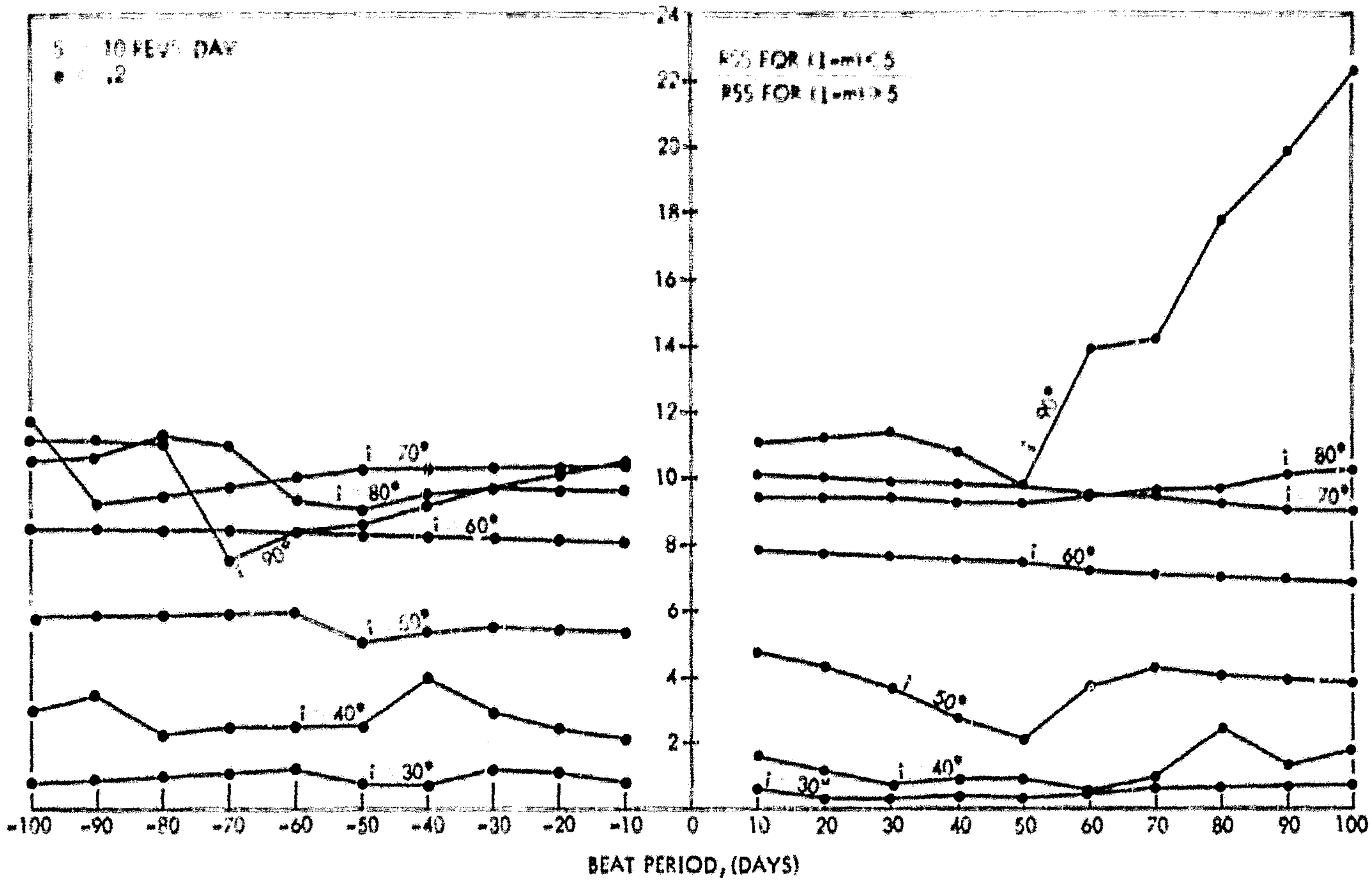

Figure 2-9—Effect of Inclination on the Number of Resonant Terms with Large Perturbations for S = 10 REVS/DAY, e = .2

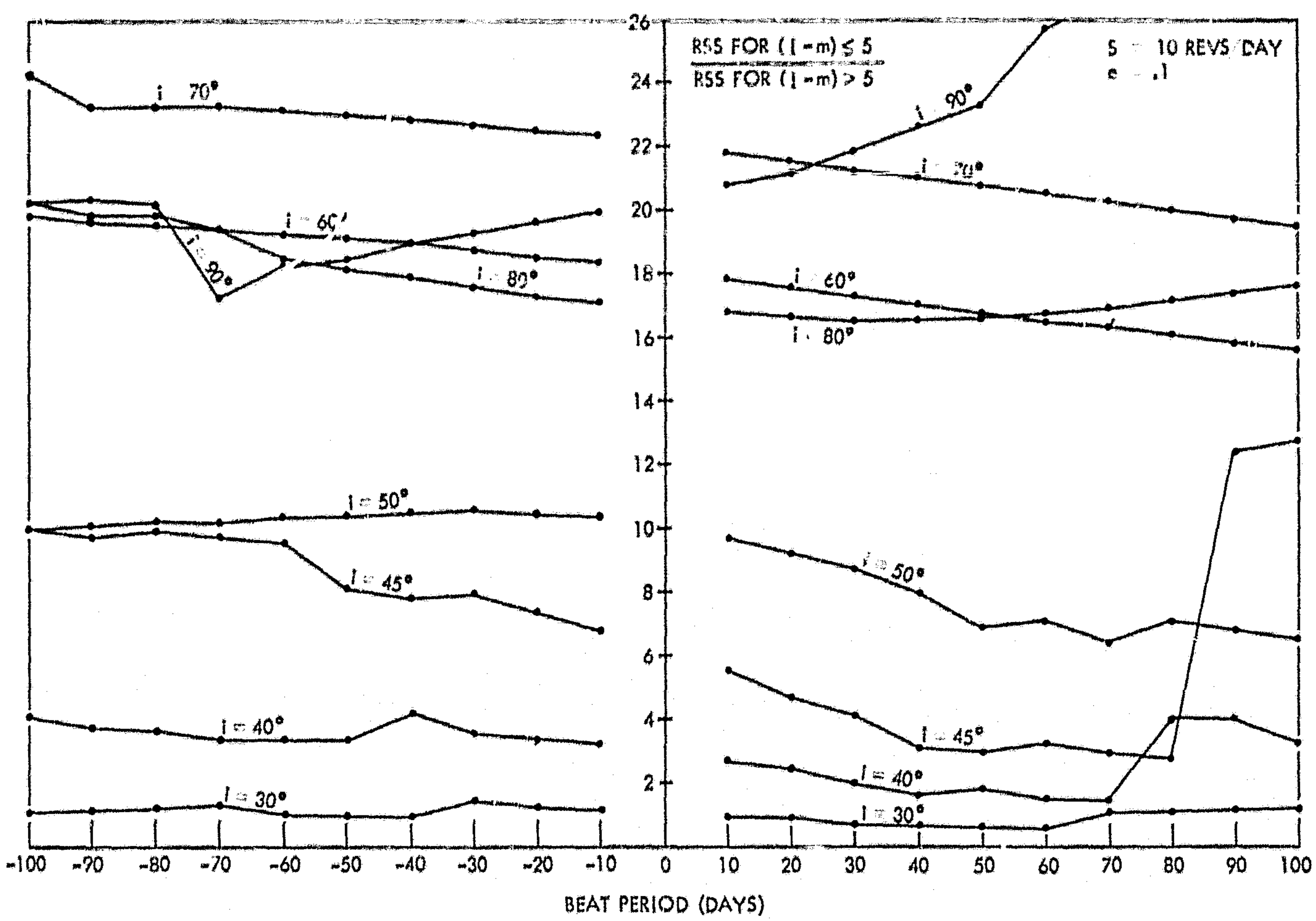

Figure 2-10—Effect of Inclination on the number of Resonant Terms with Large Perturbations for S = 10 REVS/DAY, e = .1

We are not yet tempted to draw conclusions about the ideal orbit elements for resonant satellite geodesy. The conclusions would in many cases be quite erroneous. We can only draw conclusions about the "best" orbits after considering the effect of the other important factor in the problem, namely the inevitable errors in the parameters not part of the solution vector.

3. ERROR ANALYSIS OF RESONANT ORBITS

Resonant orbits are extremely attractive for geodesy. Resonance enhances perturbations caused by tesseral harmonics and is especially important for high order terms. In the non-resonant case, terms beyond the eighth order may have an effect on an orbit of only a few meters. Furthermore, it is very difficult at present to extract spherical harmonic coefficients of this order from gravimetry. Thus, accurate coefficients from resonant orbits will provide important standards for evaluation of gravimetric results in addition to improving our capability to determine orbits.

We have attempted to discover the degree of accuracy that we may determine geopotential coefficients by using high quality observations of satellites in resonant orbits. As in many other "real world" analyses, the data quality is not the limiting factor; the errors in the unmodeled (unadjusted) parameters cause severe degradation of the solution over the "noise only" situation. A detailed description of how the effects of errors in unmodeled parameters are propagated into the estimates of the variances of the adjusted parameters appears in the Appendix.

In this study, the adjusted parameters were generally a subset of the infinite number of resonant geopotential coefficients for the orbit in question, and the elements of the orbit or orbits used in the solution. The unmodeled parameters were

- unadjusted resonant terms,
- unadjusted non-resonant terms,
- station location errors, and
- uncertainty in GM.

Table 3-1 lists the a priori uncertainties in these quantities and the tracking station complement and schedule.

A priori estimates of the errors of the unmodeled parameters are somewhat controversial because we tend to question the raw variances that may result from

Table 3-1
Unadjusted Parameters

S = 10 revs/day		S = 4 revs/day		All Cases	
Parameter	A Priori	Parameter	A Priori	Parameter	A Priori
$C_{14,10}$	2×10^{-18}	$C_{8,4}$	4×10^{-11}	Δ(GM)	10^{-6}
$S_{14,10}$	2×10^{-18}	$S_{8,4}$	2.2×10^{-10}	ΔCOM *X	-20m
$C_{18,10}$	8×10^{-19}	$C_{9,4}$	4×10^{-11}	ΔCOM Y	-20m
$S_{15,10}$	8×10^{-19}	$S_{9,4}$	1.6×10^{-11}	ΔCOM Z	-20m
$C_{16,10}$	6×10^{-19}	$C_{10,4}$	2×10^{-11}	Δgeopotential (non-resonant)	.25 (APL-SAO)
$S_{16,10}$	6×10^{-19}	$S_{10,4}$	6.5×10^{-11}		

Tracking Stations	Latitude	Longitude (E.)	Height above Spheroid (m)
Wallops Station	37°859	284.490	- 36.40
Winkfield (UK)	51.445	359.302	76.00
Hawaii	21.521	202.003	368.31
Tananerive	-19.009	47.300	1355.87
Western Test Range	37.500	237.502	195.03
Carnarvon	-24.903	113.716	10.54

Data quality; σ range = 1.4m, σ angles = 20" @ 1 obs/sec.

Tracking Schedule**

Start	Stop
0^d	4^d
6^d	$9^d.5$
12^d	16^d
18^d	22^d
24^d	28^d
30^d	

*Center of Mass
**2 week cases cut off at $14^d.0$

a least squares fit. Our a priori estimates of uncertainty represent what we believe is reasonable or will be attained in the near future.

The a priori values of the unadjusted tenth order resonant terms have been taken as the expected order of magnitude of these terms. The earth's gravity coefficients generally follow the rule

$$\bar{C}_{\ell m}, \bar{S}_{\ell m} \approx 0 \left(\frac{10^{-5}}{\ell^2}\right),$$

where the overbars indicate normalized coefficients. For the unadjusted fourth order terms, essentially the same rule was used. This may be pessimistic because there are published values of many fourth order coefficients. In both cases the list of unadjusted resonant terms was truncated at a degree for which $\ell - m > 7$. For low and moderate eccentricities, terms for which $\ell - m > 7$ have enormously reduced effects compared to terms for which $\ell = m$ or $\ell = m + 1$.

Determining the error in the non-resonant geopotential terms is difficult. We chose to assume that the term by term differences between gravity models are some measure of their uncertainty. We have used ϵ (APL 3.5 - SAO M1), where ϵ is a scale factor. Because the SAO M1 model is more recent and yields better results for orbit determination than the APL 3.5, ϵ was taken as .25. The uncertainty in GM was taken as 1:10^6.

We chose six tracking stations having good distribution in longitude and highly accurate tracking equipment. We have assumed that GSFC Laser or FPQ-6 radar quality ranges were taken with $\sigma_{range} \simeq 1.4$ m at a data rate of 1 obs/sec. The angular data were assumed good to 20"; however, in the two to four week arcs considered, angular data of this quality have an entirely negligible effect on the solution. The a priori estimates of uncertainty of the station positions were taken as ~20 m in each center-of-mass coordinate. Biases in the range data were not considered because biases for these instruments are known to be much smaller than the station location error figure of ~20 m.

A few simulations of the recovery of resonant geopotential coefficients showed that data quality is not a limiting factor for resonant satellite geodery, at least within reasonable limits. The contribution of the pure noise to the estimates after two weeks of tracking is always two to three orders of magnitude less than the order of magnitude of the coefficients themselves. Thus we did not evaluate factors such as the effect of adding more tracking stations and different data types. Resonance effects are so large that with modern tracking systems, the random noise does not significantly affect the estimates of the coefficients compared to the effects of the unmodeled parameters.

Determining geopotential coefficients is plagued by the problem of separability. The principal non-resonant effect of a geopotential term of order m is an m times daily oscillation. Thus, we require many different orbital planes to provide the information for separating the effects of the various terms of a given order.

A similar situation exists for circular resonant orbits or for resonant orbits at the critical inclination, wherein all resonant terms produce perturbations of the same frequency. Using one of these orbits, we cannot determine more than a single "lumped" coefficient. Contrarily, the frequency content of an eccentric resonant orbit, not at $i = i_c$, with a beat period greater than ten to fifteen days, is considerable. The present authors and others have suggested the possibility of recovering a large number of geopotential coefficients from a single eccentric resonant orbit. However, simulated satellite tracking on a .1 eccentricity orbit having a -30 day beat period resonant with tenth order terms for four weeks has yielded very poor results due to errors in the unmodeled parameters, especially errors in the non-resonant geopotential coefficients. No significant information concerning individual resonant terms was obtainable from single resonant orbits.

Failure to obtain useful information from a single eccentric resonant orbit led to attempted multiple-satellite solutions. Figure 3-1 compares the single-satellite solution with a three-satellite solution. The solution vector contains the orbital elements and the resonant terms from (10, 10) - (13, 10). The solutions shown for (10, 10) and (13, 10) are typical. The contribution to the solution uncertainty of the noise is off the bottom of the scale.

The units on the y-axis of Figure 3-1 and of subsequent similar figures are those of standard deviations (σ). On the left of each bar indicated by arrows are the a priori σ, and the σ of the solution including the effects of the unmodeled parameters. On the right side of the bars are contribution levels of the various unmodeled parameters. The total effect of all of these is their Root-Sum-of-Squares. Rather than give the effect of the error of each center-of-mass (COM) component, the contributions in each direction are combined as COM X, COM Y, COM Z.

In Figure 3-1 we see the startling result that the σ's of the fits are in some cases larger than the a priori values. But the σ's shown include the effects of the unadjusted parameters. The noise-only σ's are always two to three orders of magnitude smaller than the a priori information. The appearance of σ's that include unadjusted parameter effects larger than the a priori values implies that the corrections from an attempted fit would be so bad that one would have been better off not making the fit. For an example, if one knew a station position

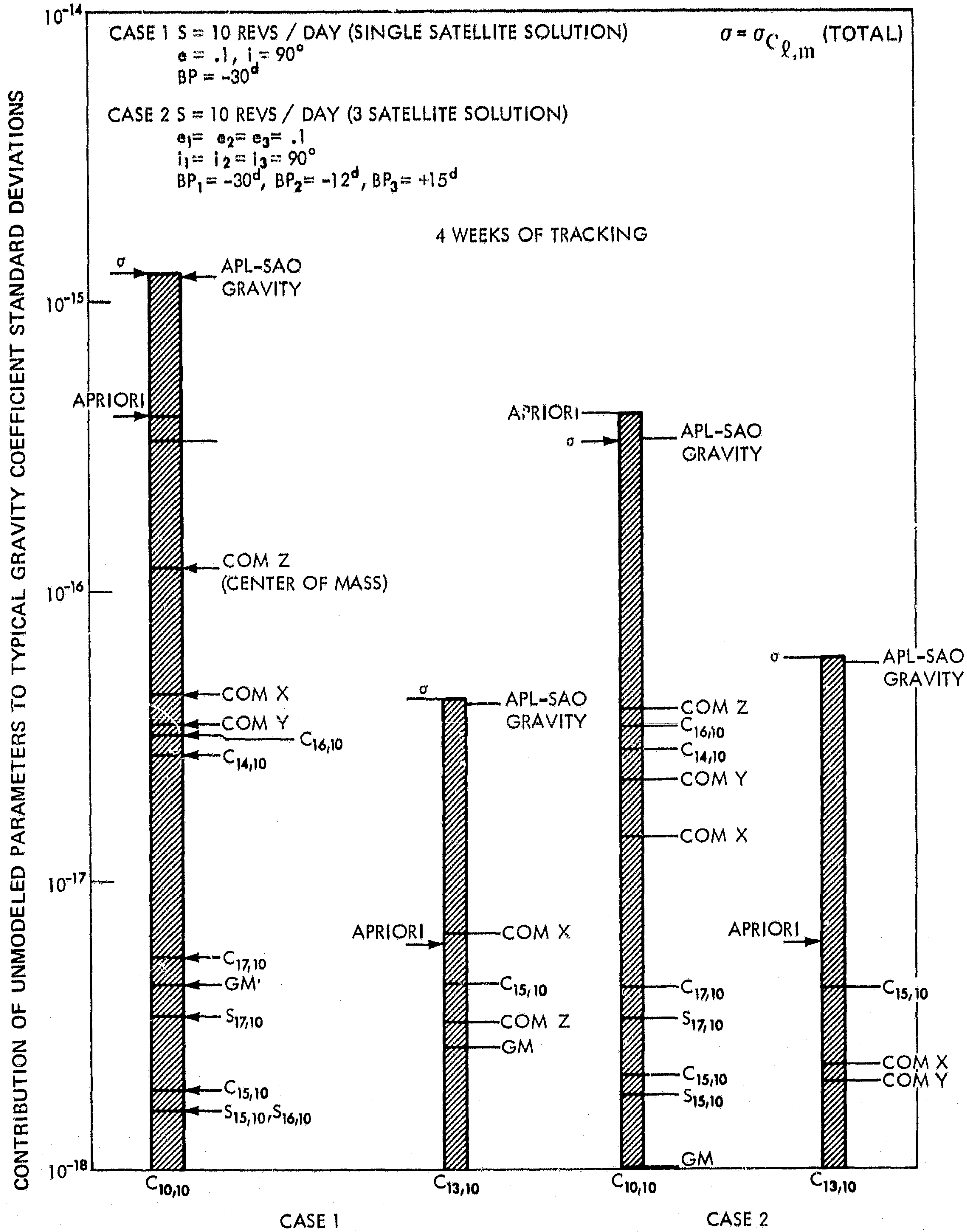

Figure 3-1—Single and Multiple Satellite Solutions for 10th Order Terms

within 20 meters and tried to improve the location by using near-Earth satellite data yet ignoring the effect of J_2, he would obviously degrade his knowledge with such a fit.

Figure 3-1 shows that although the noise-only solutions may be a great improvement over the a priori variances of the constants, including the effects of unadjusted parameters would not permit much more than order-of-magnitude information for more than a few coefficients even with three satellites. Furthermore, the correlations of the constants with each other and/or the orbital semi-major axes are in many cases .95 or higher.

The unmodeled parameters causing the largest degradation of the solution are the unadjusted non-resonant terms. Their contribution to the total is denoted "APL-SAO gravity."

The degree of the effect of errors in the non-resonant terms was unexpected; however, further investigations revealed the reason for this large effect.

We know that errors in the gravity field prevent accurately determining orbital energy, thus along-track position. The resonance effect manifests itself almost entirely along-track and is extraordinarily sensitive to semi-major axis for beat periods ≥10-15 days (as discussed in Section 2). This fact is also revealed by very high correlations ≥.9 between many of the resonant constants and semi-major axes. Thus, we expect the errors of the non-resonant geopotential terms to have a large and increasing effect with increasing beat period. Figure 3-2 illustrates this. Here we are comparing identical two-satellite solutions for tenth order resonant terms. In both cases, orbital eccentricities were 1/10, and inclinations were 90° during four weeks of simulated tracking. The long beat period multiple satellite solution with one beat period >100 days has superior noise-only results to the short beat period case of ±10 days. However, the effect of the unadjusted non-resonant gravity terms is vastly decreased in the dominant (10, 10) resonant term for the short beat period, seen by comparing the values denoted by "APL-SAO gravity." Furthermore, as expected, the correlation coefficients for the resonant constants and the semi-major axes of the orbits is reduced for the small periods, some of them approaching .9 for the ±10 day beat period in contrast to about .97 for the long beat period case.

Figures 3-1 and 3-2 show the real problem confronting the resonant satellite geodicist is finding orbital situations for combined solutions that minimize the effect of the unmodeled parameters. Figures 3-3, 3-4, and 3-5 show the results of attempts at reduction of unmodeled parameter effects by variation of various orbit and other parameters.

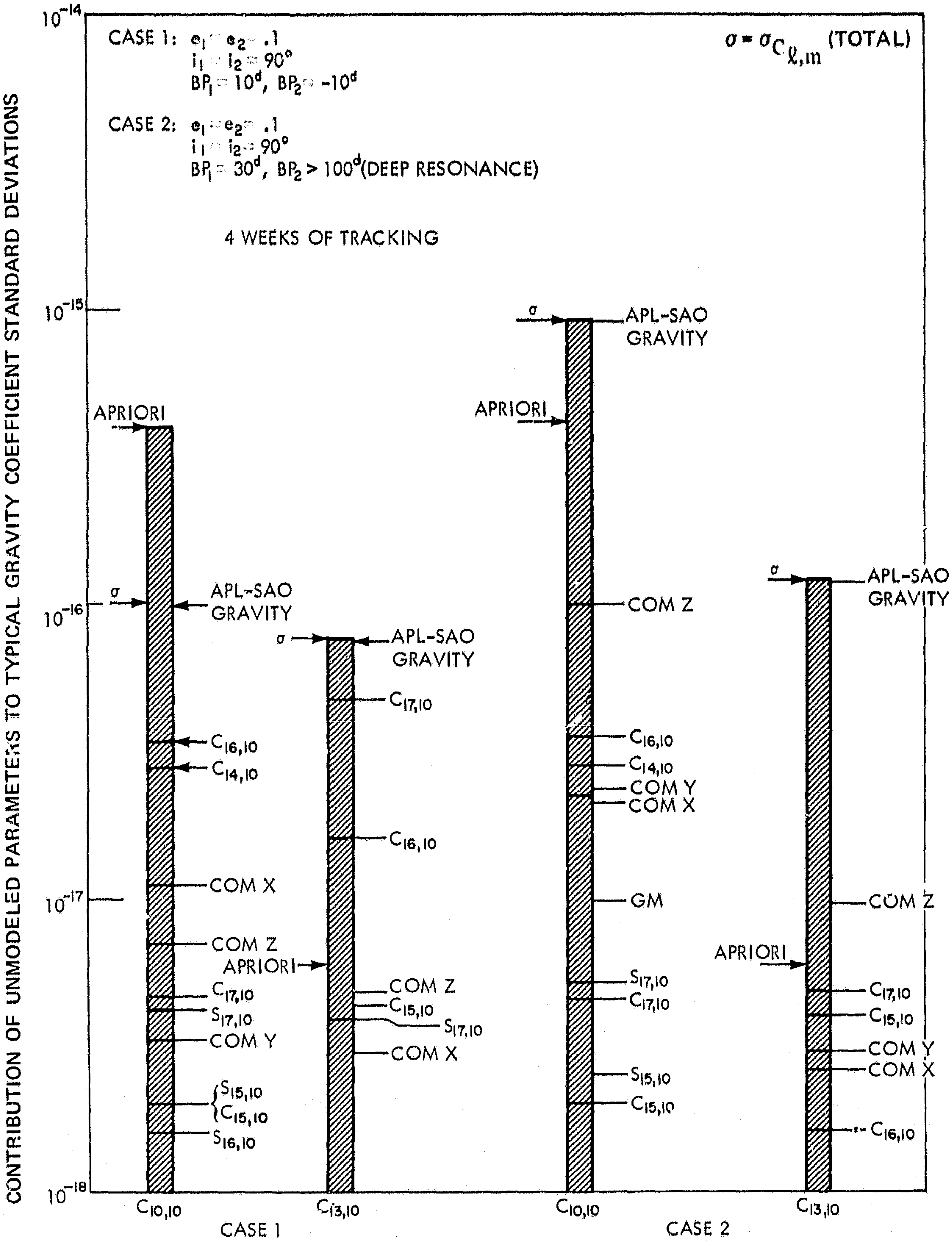

Figure 3-2–Effect of Beat Period on the Solution for 10th Order Terms

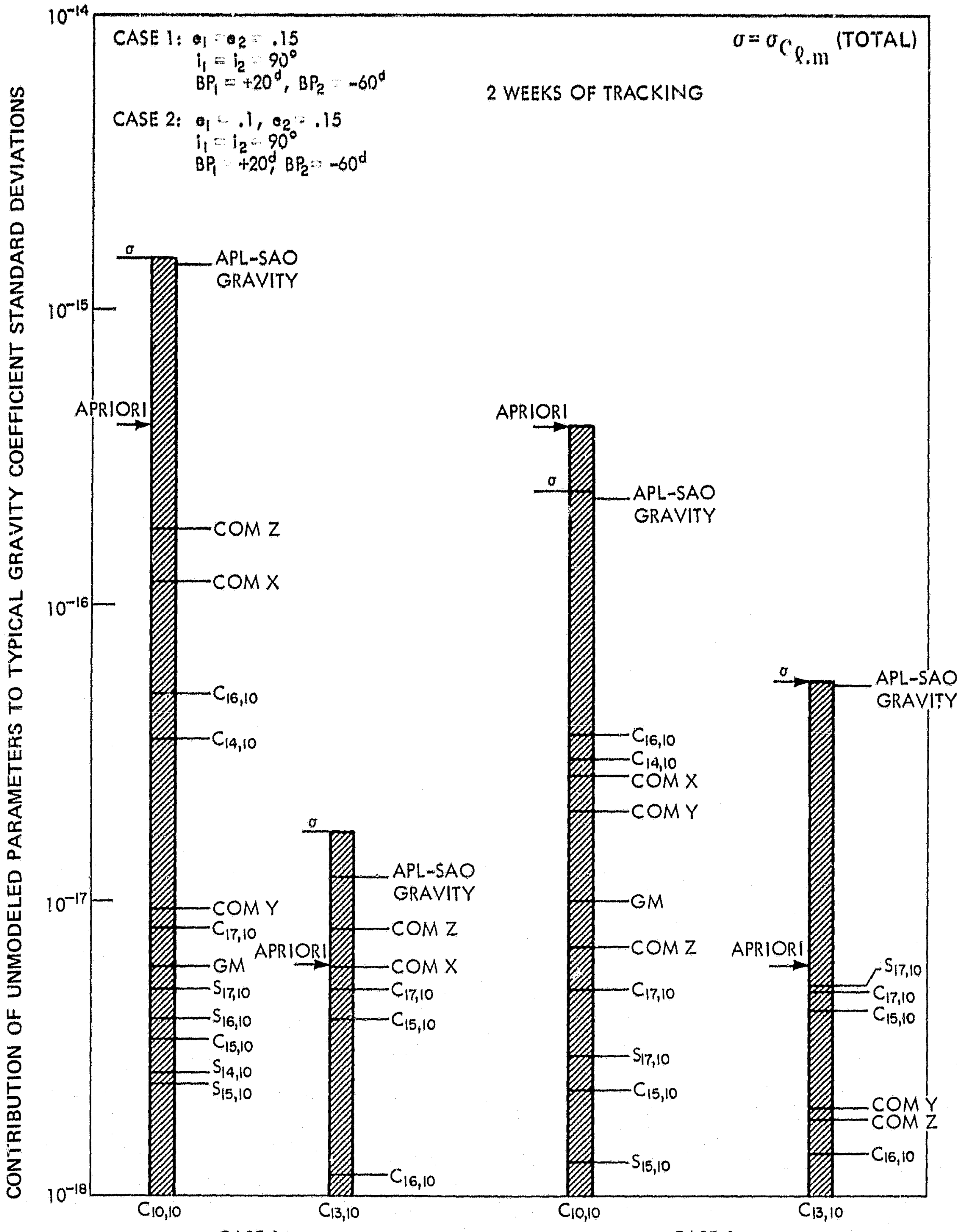

Figure 3-3–Effect of Variation in Eccentricity on Multiple Satellite Solutions for 10th Order Terms

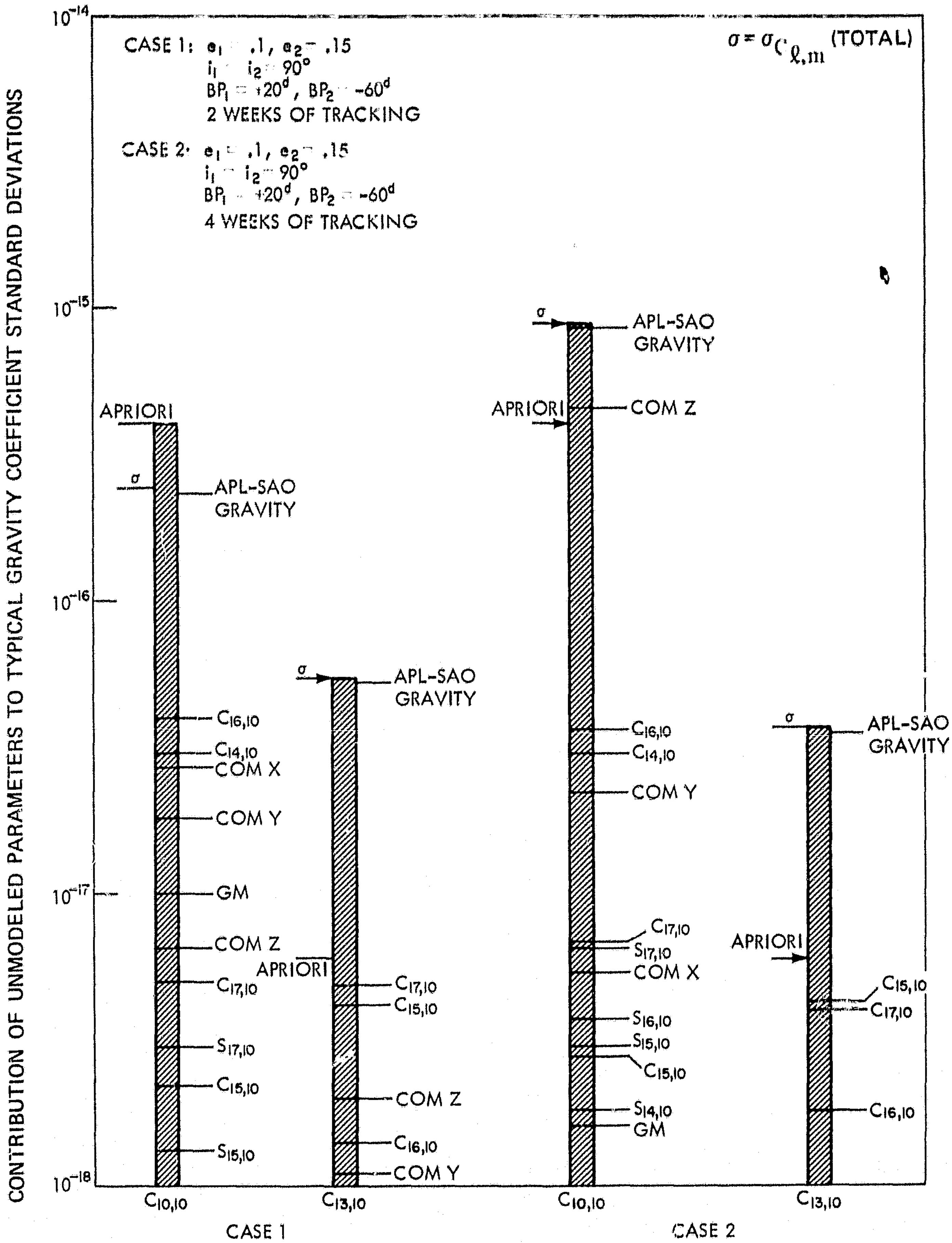

Figure 3-4—Effect of Variation in Arc Length on Multiple Satellite Solution for 10th Order Terms

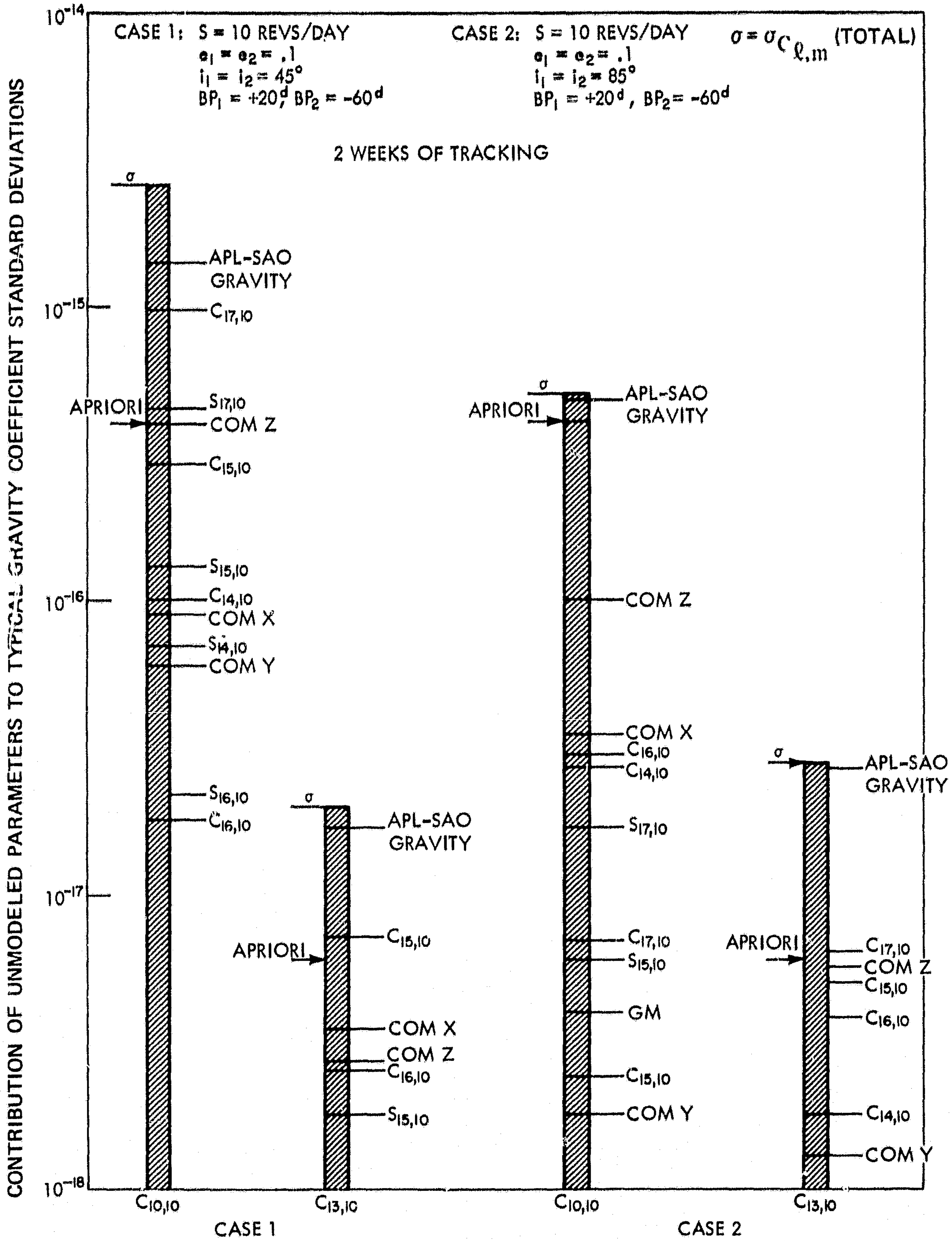

Figure 3-5–Effect of Inclination on Multiple Satellite Solution for 10th Order Terms

Figure 3-3 compares the results of two, two-satellite solutions. Eccentricities for case 1 were the same at .15 and for case 2 were set at .1 and .15. Two weeks of tracking were assumed for these polar orbits. A combined solution with different eccentricities sometimes produces improved results although there is no meaningful information with this change for terms with $(\ell - m) > 1$.

Another attempt with varied orbital geometry used identical eccentricity of .15 and inclination of 90°; however, arguments of perigee and node were 90° and 135° apart, respectively. Again, varying the orbital elements produced a slight improvement only for small $(\ell - m)$.

Figure 3-4 shows the results of observing multiple satellites in the same plane for two and four week periods. Note that the value of $\sigma(C_{10,10})$ is worse for the longer arc. This phenomenon is often observed in orbit determination problems. Two long a tracking arc may degrade a solution because of the effects of the unmodeled parameters.

Inclination significantly affects the resonance phenomenon. Figure 3-5 shows the results of two-satellite solutions at 45° and 85°. The solution at the high inclination is superior for the dominant $C_{10,10}$ term, as would be expected. At the lower inclination, the effects of the various resonant terms are more nearly equal in magnitude than at the high inclination where terms with $(\ell - m)$ equal to 0 or 1 dominate. At the low inclination the effects of the unmodeled resonant terms are very large for this reason.

The single plane cases in Figures 3-1 through 3-5 have a very common factor: The errors in the non-resonant gravity terms dominate the estimates. We can control the effect of the unmodeled resonant terms by selecting eccentricity. Station location errors and GM error are seldom serious. The large effect of the unmodeled non-resonant terms is caused by the sensitivity of the resonance phenomenon to the orbital energy or period.

Having failed to achieve meaningful results with a single plane, multiple satellite solution, we next attempted multiple orbital planes. Figure 3-6 compares the previous three-satellite, single-plane solution with a two-satellite, two-plane solution. The effects of the unadjusted non-resonant gravity terms have decreased greatly over the single plane solution, particularly for large $(\ell - m)$. This multiple plane solution has been the first to yield substantial information about a number of resonant coefficients. Even more important, the correlations of the geopotential coefficients with the orbital semi-major axes were reduced to .8 at the most.

In the multiple plane solution we did not expect but did obtain an increased effect of the unadjusted resonant terms. However, one of the planes was inclinded

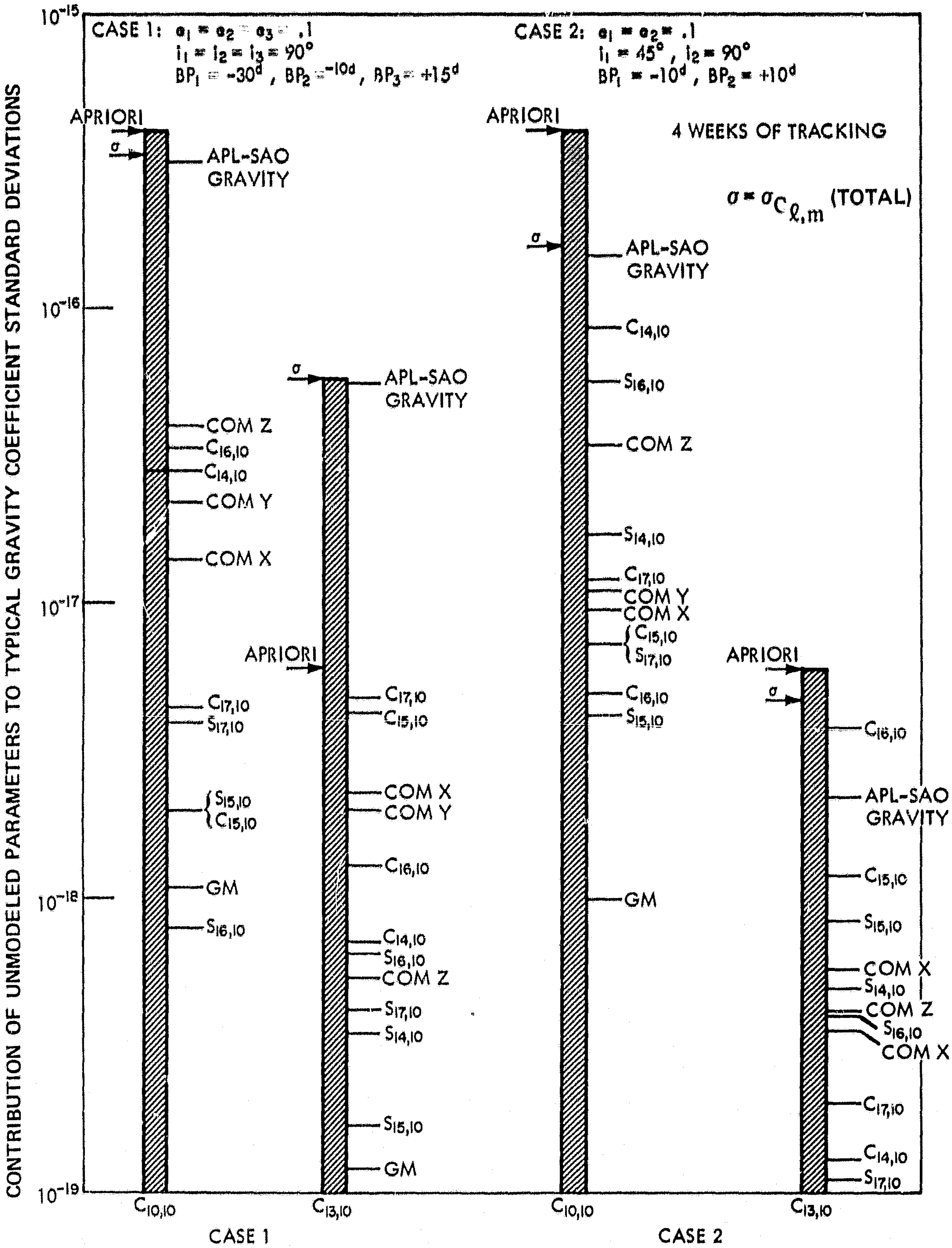

Figure 3-6–Single and Multiple Plane Solutions for 10th Order Terms

45°. Recall that low and middle inclinations are less favorable in this respect because the resonant perturbations in these orbits are more nearly equal for both small and large values of $(\ell - m)$.

The problem of correlations appears to be worse for high altitude orbits. A correlation coefficient of .999 was observed between $C_{4,4}$ and $C_{7,4}$ for case one of Figure 3-7. Furthermore, the correlation coefficients with the semi-major axis, were .9 in some cases, in spite of the multiple orbit planes. Therefore, constants determined from this situation would have no meaningful applicability other than to these orbits. Many orbit planes are necessary for high altitude resonant orbits to yield independent estimates of the gravity field.

4. CONCLUSIONS

We have ascertained that the random noise level required for resonant satellite geodesy is not stringent; moreover, resonant satellite geodesy does not require continuous tracking by numerous stations. However, data biases, like station locations may be important if excessive. Further, tracking arcs longer than a few weeks for low altitude resonant orbits are undesirable because of the effects of errors in the non-resonant geopotential coefficients.

We have also demonstrated that single, eccentric resonant orbits alone are not valuable to geodesy if more than order-of-magnitude estimates of geopotential coefficients are required. The estimates of the errors of the geopotential coefficients obtained from a single resonant satellite are almost perfectly correlated with each other as well as with the estimates of the error of the semi-major axis. Thus, the coefficients from a single satellite will be applicable only to the orbit from which they were obtained. This fact, while unfortunate for geodesy, suggests that for operational purposes, when resonances exist, we can obtain good results for prediction purposes by solving for one or two "lumped" resonant coefficients. We have in addition shown that the single plane multiple satellite solutions are only a slight improvement over single satellite solutions. As in the single satellite case, the controlling factors are the errors in the unmodeled non-resonant terms and the very high correlations of the coefficients with each other and with orbital semi-major axes.

Finally, small ($\simeq$10 day) beat periods are desirable to uncouple the geopotential coefficients from the orbital semi-major axes. The low beat periods also mean that period errors of at least several minutes can be tolerated in a resonant satellite geodesy mission.

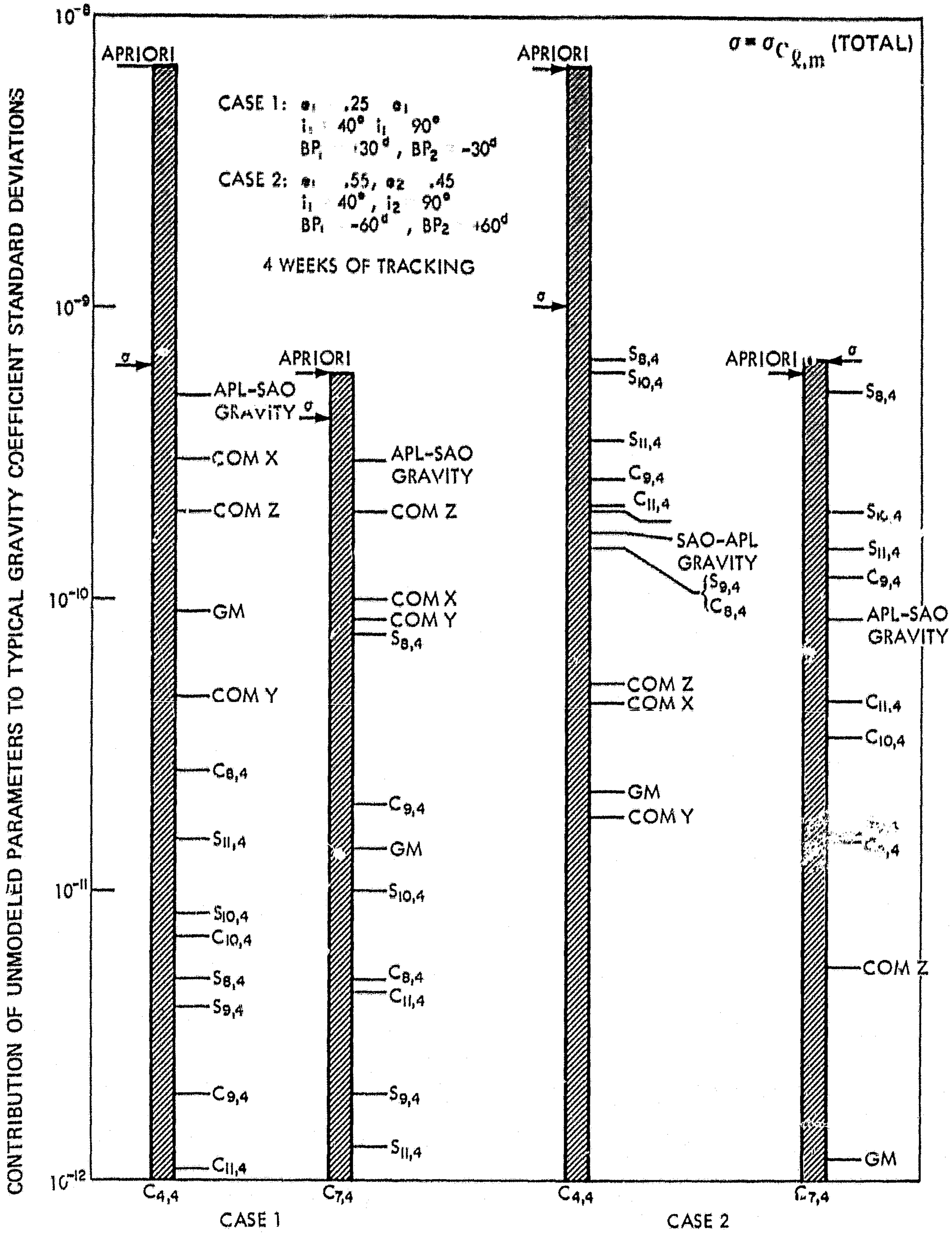

Figure 3-7–Two Plane Solutions for 4th Order Terms

Thus, resonant satellites in a single plane must be used in a combined solution with non-resonant satellites or gravimetry or both to make a meaningful contribution to geodesy Satellites of low to moderate eccentricity in several orbital planes (with some of high inclination), all resonant with terms of the same order, can produce good independent estimates of geopotential coefficients.

APPENDIX

ERROR ANALYSIS

The ORAN (Orbital Analysis) computer program used for this study was designed for computing the effects of random and systematic errors on minimum variance orbit determinations. Systematic errors can be in the form of either adjusted or unadjusted parameters, with the effects of the latter broken down into effects of the individual error sources. The program computes the effects of the unadjusted parameters on both the recovered parameters and the orbit, with the orbital effects propagated from epoch to any desired prediction time.

The program is configured for multiple arcs, with some error model parameters such as station positions constrained to be common to all arcs, and other parameters, such as measurement biases, which differ from arc to arc.

Force model errors can arise from uncertainties in geopotential coefficients through degree and order 20. Uncertainties in up to 44 individual coefficients can be carried, and any of these may be either adjusted or their unadjusted effects propagated. Alternately, or in addition, the force model error can be carried as the differences between complete gravity models in which case the restriction to 44 parameters does not apply. The SAO, APL, and NWL models are built into the program and the differences between any two of these three, or any complete model supplied as input, are available as force model errors. Note that the gravity model difference is treated as a single parameter, and 43 geopotential parameters may also be considered as adjustable. Of course, adjusting a geopotential coefficient removes it from the model difference set.

Mathematically, the unmodeled error propagation is based on the following observations. The minimum variance orbit determination uses the basic equation

$$\delta O = A \delta a + e \qquad (1)$$

to relate discrepancies (δO) between measured and calculated observations to discrepancies (δa) between true and a priori estimates of the set of parameters to be recovered. The set δa includes the six orbital elements but may also include other parameters. The matrix A is the set of partial derivatives of the measurements with respect to the adjustable parameters, and e is a vector of measurement "noise." When the least squares criterion is used to solve (1) for the best estimate of a, the result is

$$\delta \hat{a} = (A^T W A)^{-1} A^T W \delta 0, \tag{2}$$

where W is the matrix of measurement weights. For the solution to be minimum variance, the weight matrix must be chosen such that

$$W^{-1} = E(ee^t). \tag{3}$$

That is, W must be the inverse of the variance covaraince matrix of measurement noise. In the normal data reduction programs, W is generally so chosen because it actually is measurement random error, in which case W is rather accurately expressed as a diagonal matrix.

For various reasons, the set of parameters adjusted in data reduction programs is only a subset of those parameters having some error. For example, our knowledge of geopotential coefficients is by no means complete. Yet a truncated model is always (of necessity) used, and the error in all coefficients used is ignored in all variance computations. Because the net effect is that e is not random yet contains definite systematic components, we can obtain a more accurate representation of the measurement discrepancy vector by expressing e as

$$e = K\gamma + \epsilon, \tag{4}$$

where γ is a set of errors in parameters previously ignored, K is the matrix of partial derivatives of the measurements with respect to these parameters, and ϵ is the vector of measurement random noise upon which W is still based. Substitution of (4) into (1) gives

$$\delta 0 = A\delta a + K\gamma + \epsilon. \tag{5}$$

If the weight matrix for the measurements is based on ϵ and is the same as that used in the data reduction program, it follows that the solution for $\delta\hat{a}$ actually being obtained is not that given by (2), but actually is a "biased" solution given by

$$\delta \hat{a} = (A^T W A)^{-1} A^T W (\delta 0 - K\gamma). \qquad (6)$$

From this relation, we may obtain by differentiation the effects of "unit" values of the set of γ parameters,

$$\frac{\partial \delta \hat{a}}{\partial \gamma} = -(A^T W A)^{-1} A^T W K. \qquad (7)$$

It follows that if the matrix K can be obtained, the effects of unit values of the γ parameters are obtained by substituting K for the $\delta 0$ vector used in the data reduction program. A priori estimates of errors in the γ parameters lead to an estimate of the magnitudes of the effects on recovered parameters, and the trajectory, of each γ parameter.

Uncertainties in the γ's are generally uncorrelated. If their correlations are known or can otherwise be accounted for, an estimate of the total or overall accuracy of the orbital solution is readily obtainable. For this study, errors in station locations, GM, and the geopotential were considered. (See Table 3-1.)

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