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ABSTRACT

We discuss the numerical instability arising from the coupling between hydrostatic equilibrium and thermal processes in a star. Two alternative physical pictures are possible: the heat wave does or does not propagate through the adjacent shells in the star in a given time step (slow or rapid evolution). Correspondingly, we have two alternative approaches to the mathematical formulation. If the physical picture is wrong, we encounter a numerical instability.

In practice, different physical pictures are necessary for a model because of a great difference in the time scale of heat conduction between the core and the envelope of the star when we compute an advanced phase of evolution. After analyzing the nature of the instability, we show that a single mathematical scheme is possible which always meets the necessary physical picture required by rapid or slow evolution in the stellar core or envelope.

I. INTRODUCTION AND SUMMARY

An elegant, widely-used method for the automatic computation of stellar evolution was developed by Henyey, Forbes, and Gould (1964), herein referred to as the Henyey method. However, the numerical stability of stellar evolution computations has not yet been analyzed sufficiently. The problem of the computation of stellar evolution is characterized as a mixed initial-boundary value problem with four simultaneous differential equations, two describing hydrostatic equilibrium and the other two the thermal process. In the present paper we shall discuss the most serious numerical instability which arises from coupling between hydrostatic equilibrium and the thermal process.

a) Explicit and Implicit Methods

We shall consider a method of obtaining the stellar structure at time t, assuming that the preceding structure at time $t - \Delta t$ is known. The star is divided into a suitable number of shells. There are two alternative ways of solving the problem. One is called the explicit method, in which any quantity at t is expressed explicitly in terms of the quantities at $t - \Delta t$. The other is called the implicit method, in which the quantities at t are expressed implicitly with respect to the quantities at $t - \Delta t$, and some elimination method is required to obtain the quantities at t. The Henyey method is an implicit method in this sense.

There are mixed expressions; for example, when the thermal process is expressed explicitly but the hydrostatic equilibrium is expressed implicitly. Examples of such expressions were given by Schwarzschild and Selberg (1962), by Rakavy, Shaviv, and Zinamon (1967), and by Murai, Sugimoto, Hōshi, and Hayashi (1968). In these examples, the entropy density, $s\ (M_r,\ t)$, is calculated by using quantities determined only at $\underline{t} - \Delta \underline{t}$*, where M_r denotes the mass inside the radius $\underline{r}$. The notation in the present paper is the same as that given in the textbook by Schwarzschild (1958) unless otherwise indicated. By using $s\ (M_r,\ t)$, we can solve the two differential equations of hydrostatic equilibrium for time $\underline{t}$ separately from the equations for the thermal process. The heat flux, $L_r\ (t)$, is calculated from the temperature distribution after the structure at $\underline{t}$ has been obtained. In the following discussion, it is an essential point as to whether the thermal process is expressed explicitly or implicitly; the expression for the hydrostatic equilibrium does not matter. Thus we shall call a method "explicit" whenever the thermal process is expressed explicitly.

b) Stability Condition for a Purely Thermal Process

The stability of a heat-flow problem in a solid body has been well studied (see, e.g., Richtmyer and Morton 1967). The stability condition of the explicit method can be written as

$$2 \Delta t \leq \tau_h\ (\Delta r) \quad [\text{explicit}], \tag{1}$$

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*Rakavy et al. (1967) introduced a damping term in quasi-time, t_q. Thus the entropy density was expressed as $s\ (M_r,\ t;\ t_q)$, with $s\ (M_r,\ t) = \lim_{t_q \to \infty} s\ (M_r,\ t;\ t_q)$. They calculated $s\ (M_r,\ t;\ t_q)$ by using $s\ (M_r,\ t;\ t_q - \Delta t_q)$ and $s\ (M_r,\ t - \Delta t)$. This was an iteration of the explicit method for the thermal process, since the mechanical equilibrium was solved separately by assuming the entropy denisty.

where the time scale of heat conduction through a shell of radial width Δr is given by

$$\tau_h (\Delta r) \equiv \frac{1}{s_{rad}} \left(\frac{\partial s}{\partial \ln T} \right)_p \frac{\kappa \rho (\Delta r)^2}{c} . \tag{2}$$

In the above equation, s_{rad} denotes the entropy density of radiation $(4 a T^3/\rho)$. On the other hand, the explicit method is stable for <u>any</u> time step in a purely thermal problem.

c) Stability Condition for a Coupled Hydrostatic-Thermal System (Real Star)

For the explicit method the stability condition is the same as in relation (1), since the hydrostatic equilibrium is solved separately from the thermal process, as discussed in § I, a above. For the implicit method, on the other hand, the condition that the system of four differential equations be stable is expressed roughly by

$$\Delta t \gtrsim \tau_h (r) \quad [\text{implicit}] , \tag{3}$$

where $\tau_h (r)$ denotes another time scale of heat conduction expressed by equation (2) with Δr replaced by r. We call a system of differential equations unstable when it has a branch of a solution growing so strongly that it is practically impossible to compute all of the independent solutions numerically (Wendroff 1966). This point will be discussed in more detail in § II.

d) Problem Encountered in Practice

We shall now consider the computation of stellar evolution through a phase with a reasonable number of time steps. As may be seen from the stability conditions (1) and (3), the explicit method is suitable for rapid evolution (Δt small compared with $\tau_h(\Delta r)$), e.g., for evolutionary phases with extensive neutrino loss or thermal instability, but, on the other hand, the implicit method is suitable for relatively slow evolution, e.g., in phases of nuclear burning with negligible neutrino loss.

The time scale of heat conduction varies greatly through the star, especially in a red giant star with a helium zone and hydrogen envelope.* For a reasonable division into shells, $\tau_h(\Delta r)$ has its smallest value at the outermost shell in radiative equilibrium, even where condition (1) must be satisfied in the explicit method. (In a convective region, τ_h can be considered to vanish.) For a reasonable value of Δt, in some cases, we encounter a contradictory requirement that the implicit method is necessary in the outer region but the explicit method is required in the central region. Typical examples are the phases of and near carbon burning.

e) Stability of Difference Equations and Conclusion

Fortunately, such a contradictory requirement can be satisfied, as will be shown in § III. We shall look more carefully into the behavior of the system of

*For example $\tau_h(\Delta r)$ for a unit scale height of pressure, h, is 2×10^{13} sec at the central region ($r = h$), 8×10^{11} sec at the helium-burning shell, and 3×10^{10} sec at the hydrogen-burning shell of a 15 $M_\odot$ star just before the carbon-burning phase (stage 5, Table 7-6 of Hayashi et al. (1962)).

difference equations of the Henyey method. In the limit of an infinitesimal time step but keeping spatial step finite, the difference equations no longer approximate the original system of four differential equations.* It is shown that, with a properly formulated Henyey method, we can obtain physically significant branches of solutions by avoiding the unstable branches of solution which the original system of four differential equations has in the above limit. Solution by such a method can be expressed by an expansion in which the leading term is a solution of the explicit method and the first-order term represents the coupling between a change in the heat flux and a change in the hydrostatic equilibrium. Since this method is essentially a form of the Henyey method, it is stable for slow evolution. Thus, we can solve stably both the core and envelope, as well as both rapid and slow evolution, in a single scheme of computation. Application of this method to a physical problem will be given in a separate paper (Sugimoto 1969).

II. NATURE OF THE DIFFERENTIAL EQUATIONS FOR STELLAR EVOLUTION

a) Differential Equations for Stellar Structure

The structure of a star in quasi-static equilibrium is determined by the following four differential equations:

$$\frac{\partial \ln p}{\partial \ln M_r} = - \frac{G M_r^2}{4\pi r^4 p} , \tag{1a}$$

*In order to approximate the original differential equations, both the time step and the spatial step must become infinitesimal simultaneously, keeping $\Delta t/\Delta r$ finite.

$$\frac{\partial \ln T}{\partial \ln M_r} = \frac{1}{n+1} \frac{\partial \ln p}{\partial \ln M_r} , \tag{4b}$$

$$\frac{\partial \ln r}{\partial \ln M_r} = \frac{M_r}{4\pi r^3 \rho} , \tag{4c}$$

$$\frac{\partial L_r}{\partial \ln M_r} = \left(- T \frac{\partial s}{\partial t} + \epsilon_n - \epsilon_\nu \right) M_r , \tag{4d}$$

where

$$\frac{1}{n+1} = \min \left\{ \frac{1}{(n+1)_{ad}} , \frac{1}{(n+1)_{rad}} \right\} , \tag{5a}$$

$$\frac{1}{(n+1)_{rad}} = \frac{3}{16 \pi acG} \frac{p}{T^4} \frac{\kappa L_r}{M_r} \tag{5b}$$

and $1/(n+1)_{ad}$ is the adiabatic temperature gradient. For the sake of definiteness in the following discussion, we take the dependent variables as $\ln p$ (pressure), $\ln T$ (temperature), $\ln r$, and L_r; these will be denoted as y_i, with $i = 1, 2, 3,$ and 4, respectively. Hereafter we shall denote a matrix by a capital letter, a vector by a lower case letter, and their elements with subscripts. The independent variable, $\ln M_r$, will be denoted by $\underline{x}$. The nuclear energy generation rate, ϵ_n, the energy loss rate by neutrinos, ϵ_ν, and the opacity, κ, are usually expressed in terms of ρ and $\underline{T}$. The density, ρ, the entropy, and the adiabatic temperature gradient are in principle expressible as functions of $\underline{p}$ and $\underline{T}$. Equations (4a)-(4d) are rewritten as

$$\frac{\partial y}{\partial x} = \varphi (y, x) . \tag{6}$$

Of course, transformations of the dependent variables can be incorporated, but an essential point in the following discussion is that the equations for hydrostatic equilibrium involving φ_1 and φ_3 do not contain y_4. There are two boundary conditions at the center: $r = L_r = 0$ at $M_r = 0$. There are two other boundary conditions at the surface: $p = T = 0$ at $M_r = M$ (total mass of the star), when the surface region is in radiative equilibrium.

b) Relaxation Method

The relaxation method was first introduced by Henyey et al. (1959), for solving problems in stellar evolution. The time derivative in equation (4) $\partial s/\partial t$, is replaced by the forward difference, i.e., by $\{s(t) - s(t - \Delta t)\}/\Delta t$. Except for this term, equation (4) does not contain any time derivative. We shall henceforth write d/dx instead of $\partial/\partial x$. We assume a trial solution $y^{(0)}$ and assume that

$$y = y^{(0)} + \delta y \tag{7}$$

satisfies equation (6). Assuming that δy is a small correction to $y^{(0)}$, then substituting equation (7) into equation (6), and linearizing it, we obtain

$$\frac{d\delta y}{dx} = A\, \delta y + b, \tag{8a}$$

$$A_{1,4} = A_{3,4} = A_{4,4} = 0, \tag{8b}$$

where $A_{i,j} = (\partial \varphi_i/\partial y_j)^{(0)}$ and $b_i = \varphi_i\,(y^{(0)}, x) - dy_i^{(0)}/dx$ are numerically known functions of $\underline{x}$. After having solved equation (8a) and obtained $\underline{y}$ by (7),

$y^{(0)}$ is replaced by $\underline{y}$. The whole procedure is repeated until δy becomes suitably small. Thus the problem is how to solve equation (8).

c) Nature of the Differential Equations for the Correction Term

We consider only a region in radiative equilibrium, since in a convective region all of the $A_{i,4}$ vanish and y_4 is separable from the others. If a finite efficiency of the convective heat-transport is taken into account by means of mixing-length theory, for example, it can be treated in the same way as the radiative equilibrium in the sense that $1/(n+1)$ depends upon L_r. Important quantities in the following discussion are written as

$$A_{2,4} = -\frac{\kappa M_r}{16\,\pi^2\, c\,\rho\, T\, s_{rad}\, r^4},$$

$$A_{4,1} = \left\{-\frac{T}{\Delta t}\left(\frac{\partial s}{\partial \ln p}\right)_T + \left(\frac{\partial \epsilon_n}{\partial \ln p}\right)_T - \left(\frac{\partial \epsilon_\nu}{\partial \ln p}\right)_T\right\} M_r,$$

$$A_{4,2} = \left[-\frac{T}{\Delta t}\left\{s(t) - s(t-\Delta t) + \left(\frac{\partial s}{\partial \ln T}\right)_p\right\} + \left(\frac{\partial \epsilon_n}{\partial \ln T}\right)_p - \left(\frac{\partial \epsilon_\nu}{\partial \ln T}\right)_p\right] M_r. \tag{9}$$

We consider the computation of a phase of relatively rapid evolution with $\underline{N}$ time-steps. Where the heat conduction is negligible, the left hand side of equation (4d) should be small compared with each term in its right hand side. There is convection wherever the nuclear energy generation is large. Such a convective region does not contribute to the numerical instability. Therefore the neutrino loss, ϵ_ν, is approximately equal to $\{s(t) - s(t - \Delta t)\} T/\Delta t$. The change of entropy density, $|s(t) - s(t - \Delta t)|$, is at most about $(\partial s/\partial \ln T)_p/N$, and less than this for a shell in which the neutrino loss is relatively small as compared with other shells in the star. Thus the terms proportional to $(\partial s/\partial \ln p)_T$ and $(\partial s/\partial \ln T)_p$ are most dominant in $A_{4,1}$ and $A_{4,2}$. Taking into account only these terms, we have

$$\Omega \equiv (A_{2,4}\, A_{4,2})^{\frac{1}{2}} \simeq \frac{1}{U} \left\{ \frac{\tau_h(r)}{\Delta t} \right\}^{\frac{1}{2}}, \tag{10}$$

where $U \equiv d \ln M_r/d \ln r$ is three times the ratio of the density to the mean interior density as seen in equation (4c). In the limit of infinitely rapid evolution or an infinitesimal time step, Ω is infinitely large, while $A_{4,1}/A_{4,2}$ remains finite.

We consider a range of $\underline{\underline{x}}$, where $\underline{A}$ can be considered to be practically constant. The nature of the solution for the homogeneous part of equation (8a) is understood by the secular equation,

$$|A - \lambda| = 0. \tag{11}$$

Taking into account equation (8b), it is easily shown for large Ω that two eigenvalues are given by

$$\lambda_4 = - \lambda_3 = \Omega + 0\,(\Omega^\circ). \tag{12}$$

The other two are given by

$$\lambda_k = \lambda_k^{(e)} + 0\,(\Omega^{-2}),\ \ k = 1, 2, \tag{13}$$

where $\lambda_k^{(e)}$ denotes eigenvalues of the equations

$$\frac{d\,\delta y_i}{dx} = \sum_{j=1}^{3} A_{i,j}\ \delta y_j,\ \ i = 1, 3, \tag{14a}$$

$$\sum_{j=1}^{3} A_{4,j}\ \delta y_j = 0. \tag{14b}$$

These are the eigenvalues of the two differential equations for hydrostatic equilibrium, i.e., equation (14a), together with a given distribution of entropy as a subsidiary condition (14b); they are the eigenvalues of the explicit method. Only these two eigenvalues are physically significant in the limit of large Ω.

The general solution of equation (8a) contains strongly growing branches, exp $(\pm\Omega x)$, and the differential equations themselves are unstable. Thus, it is impossible to obtain independent solutions by numerical integration. Even if one has started from different sets of initial values, the unstable branch overcomes the other branches after some steps of integration, and these solutions are no longer practically independent of each other (Wendroff 1966).*

A measure of the criterion for equation (8) to be practically stable is given by using equation (10) in the form

$$I = \int \Omega\, dx \simeq \int_{r=0}^{\text{surface}} \{\tau_h(r)/\Delta t\}^{1/2}\, d\ln r \sim 0\,(1). \tag{15}$$

*Even when we start with properly selected initial values so as not to contain an unstable branch, approximation with a finite step of integration introduces the unstable branch.

The stability condition (3) given in § I, c is another expression of the above condition. This means that a heat wave must propagate throughout the star in a time Δt, as required from the physical picture of equation (4). It is to be noticed that the integral in the above equation applies only in the radiative region.

III. DIFFERENCE EQUATIONS

We consider only the homogeneous part of equation (8) with a difference equation of the following type:

$$- \delta y_i^{k+1} + \delta y_i^k + \sum_{j=1}^{4} \beta_i A_{i,j}^k \Delta x^k \delta y_j^k$$

$$+ \sum_{j=1}^{4} (1-\beta_i) A_{i,j}^{k+1} \Delta x^k \delta y_j^{k+1} \simeq 0, \tag{16}$$

where the superscript denotes the spatial mesh points, $k = 1, 2, \ldots, K-1$ and

$$\Delta x^k = x^{k+1} - x^k . \tag{17}$$

We are concerned only with the coupling between hydrostatic equilibrium and thermal process so that we put $\beta_1 = \beta_3 = \frac{1}{2}$.

a) Henyey-Type Elimination

We introduce new independent variables,

$$\delta \eta_i^k = \delta y_i^k, \ i = 1, 2; \ \delta \eta_i^k = \delta y_i^{k+1}, \ i = 3, 4. \tag{18}$$

Equation (16) then becomes

$$P^k \, \delta \eta^{k-1} + Q^k \, \delta \eta^k + R^k \, \delta \eta^{k+1} = 0. \tag{19}$$

Of course the first two columns of the matrix P^k and the last two columns of R^k are zeros. Introducing another matrix, Γ^k, of which the last two columns are zeros, and which satisfies

$$\delta \eta^{k-1} = \Gamma^k \, \delta \eta^k, \tag{20}$$

the solution of equation (19) is given by

$$\Gamma^{k+1} = (-P^k \Gamma^k - Q^k)^{-1} R^k, \tag{21a}$$

$$\delta \eta^k = \Gamma^{k+1} \Gamma^{k+2} \cdots \Gamma^K \, \delta \eta^K. \tag{21b}$$

We have two independent choices of $\delta \eta^K$, since the choice of $\delta \eta_3^K$ and $\delta \eta_4^K$ is meaningless. On the other hand, $\Gamma^1_{i,j}$ (i = 1,2, j = 1,2) does not enter into equation (21a). Thus we have two independent choices of Γ^1, for which the determinant of the submatrix consisting of $\Gamma^1_{i,j}$ (i = 3,4, j = 1,2) does not vanish. Consequently, we have two independent sets of Γ^k. Thus we have four particular solutions.*

b) Limit to a Small Time Step with Fixed Mesh-Points

We now consider the case with a large value of Ω. If we solve equation (16) with mesh-points satisfying $\omega \equiv \Omega \Delta x \ll 1$, the difference equation (16) approximates

*Usually, the two boundary conditions at the center are incorporated in equation (19) for k = 1, so that P^1 vanishes and the choice of Γ^1 is meaningless. Then, we obtain only one set of Γ^k, i.e., two particular solutions which satisfy the two boundary conditions at the center of the star.

the differential equation (8) and we pick up the unstable branch. Thus, we examine equation (21) for the case of $\omega \gg 1$, i.e., for rapid evolution with a finite number of mesh-points. The result depends upon the detailed formulation of the difference equations as well as upon the method of elimination.* It is difficult to show generally what the solution (21) approaches in the limit of a large value of ω. However, numerical experiments show that it approaches the solution of the explicit method when we choose

$$\beta_2 = 0 \text{ and } \beta_4 = 1, \tag{22a}$$

or

$$\beta_2 = 1 \text{ and } \beta_4 = 0. \tag{22b}$$

We shall examine mathematically the first case, only because a mathematical proof is easiest in this case. We drop the superscript when no confusion is anticipated. Taking account of equations (8b), and (22a), the matrix elements are written as

$$W = -P^k \Gamma^k - Q^k$$

*For example, if we assume two sets of values for δy^1, which satisfy the boundary condition at the center of the star, we can solve equation (16) numerically for the two particular solutions. Numerical experiments show that these two solutions are not practically independent at the stellar surface, reflecting the nature of the differential equation (8).

$$= \begin{pmatrix} -P_{1,3}\Gamma_{3,1}-Q_{1,1}, & -P_{1,3}\Gamma_{3,2}-Q_{1,2}, & -Q_{1,3}, & 0 \\ -Q_{2,1}, & -Q_{2,2}, & -Q_{2,3}, & -Q_{2,4} \\ -P_{3,3}\Gamma_{3,1}-Q_{3,1}, & -P_{3,3}\Gamma_{3,2}-Q_{3,2} & -Q_{3,3}, & 0 \\ -P_{4,3}\Gamma_{3,1}-\Gamma_{4,1}-Q_{4,1}, & -P_{4,3}\Gamma_{3,2}-\Gamma_{4,2}-Q_{4,2}, & 0, & 1 \end{pmatrix} \tag{23}$$

We split the matrix $\underline{W}$ into two parts so that $|W| = |W^{(e)}| + |W^{(\ell)}|$: The matrices $W^{(e)}$ and $W^{(\ell)}$ denote the same matrix as $\underline{W}$ but the fourth rows are replaced by $(-Q_{4,1}, -Q_{4,2}, 0, 0)$ and $(-P_{4,3}\Gamma_{3,1} - \Gamma_{4,1}, -P_{4,3}\Gamma_{3,2} - \Gamma_{4,2}, 0, 1)$, respectively. Hereafter, we shall not write Δx explicitly, since it is fixed. The element, $-Q_{2,1}$, appears if we do not substitute equation (4a) into (4b). It should be noticed that $Q_{4,2}\,Q_{2,4}$ is equal to ω^2 and is invariant for a scale change of $\delta\eta_j$.

We assume that $Q_{2,4}\Gamma^k_{4,1}, Q_{2,4}\Gamma^k_{4,2}$ and the other $\Gamma^k_{i,j}$ are small compared with ω^2 as should be the case for $k = 1$. Denoting the co-factor of $W_{i,j}$ by $|\hat{W}_{i,j}|$, we have

$$|W| = -Q_{2,4}\,|\hat{W}^{(e)}_{2,4}| \left[1 + \left\{ \frac{|\hat{W}^{(\ell)}_{2,4}|}{|\hat{W}^{(e)}_{2,4}|} - \frac{|\hat{W}_{4,4}|}{Q_{2,4}\,|\hat{W}^{(e)}_{2,4}|} \right\} \right], \tag{24}$$

Since $|\hat{W}^{(e)}_{2,4}|$ is a determinant which appears in the explicit method, it does not vanish insofar as the explicit method is stable. Thus, $Q_{2,4}\,|\hat{W}^{(e)}_{2,4}|$ is of the order of ω^2 and the quantity in the curly brackets in equation (24) is of the order of ω^{-2}.

We denote a matrix as $W(R; i, j)$ which is the same matrix as $\underline{W}$ but with the i-th column of W replaced by the j-th column of $\underline{R}$. Using equations (21a) and (24), and noting $R_{4,1} = R_{4,2} = 0$ because of equation (22a), the other matrix elements of Γ can be expanded in ω^{-2} as

$$\Gamma^{k+1}_{i,j} = \frac{|\hat{W}^{(e)}(R;i,j)_{2,4}|}{|\hat{W}^{(e)}_{2,4}|}\left[1+\left\{\frac{|\hat{W}^{(\ell)}(R;i,j)_{2,4}|}{|\hat{W}^{(e)}(R;i,j)_{2,4}|}\right.\right.$$

$$\left.-\frac{|\hat{W}^{(\ell)}_{2,4}|}{|W^{(e)}_{2,4}|}-\frac{|\hat{W}(R;i,j)_{4,4}|}{Q_{2,4}\,|\hat{W}^{(e)}(R;i,j)_{2,4}|}+\frac{|\hat{W}_{4,4}|}{Q_{2,4}\,|\hat{W}^{(e)}_{2,4}|}\right\}$$

$$\left.+\,0(\omega^{-4})\right],\quad i = 1,2,3;\; j = 1,2, \tag{25}$$

$$\Gamma^{k+1}_{4,j} = -\frac{\sum_{i=1}^{3} R_{i,j}\,|\hat{W}^{(e)}_{i,4}|}{Q_{2,4}\,|\hat{W}^{(e)}_{2,4}|}\left[1+\left\{\frac{\sum_{i=1}^{3} R_{i,j}\,|\hat{W}^{(\ell)}_{i,4}|}{\sum_{i=1}^{3} R_{i,j}\,|\hat{W}^{(e)}_{i,4}|}\right.\right.$$

$$\left.\left.-\frac{|\hat{W}^{(\ell)}_{2,4}|}{|W^{(e)}_{2,4}|}+\frac{|\hat{W}_{4,4}|}{Q_{2,4}\,|\hat{W}^{(e)}_{2,4}|}\right\}+0(\omega^{-4})\right],\quad j = 1,2, \tag{26}$$

where the quantities in the curly brackets are of the order of ω^{-2}. Thus, all of $\Gamma^{k+1}_{i,j}$ and $Q^{k+1}_{2,4}\Gamma^{k+1}_{4,j}$ remain finite since $Q^{k+1}_{2,4}/Q^{k}_{2,4}$ is finite.

c) Physical Interpretation

The leading terms of equation (25) and of $Q_{2,4}\Gamma^{k+1}_{4,j}$ in equation (26) contain neither the (2,4)-element, $\Gamma_{4,j}$, nor any co-factor of the (4,4)-element. Moreover, the leading term of equation (25) does not contain the (2,j) elements. Thus, the leading terms are solutions to be obtained by the explicit method, i.e., by letting $P_{4,4}$ and $Q_{4,4}$ vanish. In spite of a large value of ω, these are correct solutions of equation (14a) with the subsidiary condition (14b), since

$A_{i,j}\Delta x$ (i,j = 1,3) can be small compared with unity. The δy_4 is calculated from $d\delta y_2/dx$. Thus, the leading terms describe solutions for given distribution of entropy density as discussed in § I, a. The first-order terms in ω^{-2} describe effect of heat flow, and thus these are proportional to the co-factors of the (2,4) or the (4,4)-elements.

Thus, the mathematical structure of the above scheme corresponds to a good physical picture that the heat wave does not propagate in a given time interval through a shell in the limit of large ω. On the other hand, when ω is small, this scheme reduces to the usual implicit method and thus corresponds to a good physical picture that the heat wave propagates well throughout the star. The case of an intermediate value of ω will be discussed in § III, d.

We must now discuss the number of independent solutions and the boundary condition for a large value of ω throughout the star. We easily find that

$$\begin{vmatrix} \Gamma^k_{1,1} & \Gamma^k_{1,2} \\ \Gamma^k_{2,1} & \Gamma^k_{2,2} \end{vmatrix} = 0\ (\omega^{-2}), \tag{27}$$

which means that the first two columns of $\Gamma^k\,\Gamma^{k+1}\ldots\Gamma^K$ are not independent in the limit of infinite ω. Thus we have only two degrees of freedom in the choice of a solution, i.e., only in the choice of Γ^1. This corresponds to the fact that we have only two differential equations of the explicit method in this limit. One of the boundary conditions in this limit at the center, $L_r = 0$, has nothing to do with the hydrostatic equilibrium. One of the boundary conditions at the surface becomes a relation for the entropy at the outermost point. Thus we have only two boundary conditions — one at the center and another at the outermost point — which are satisfied by using the two independent solutions. If ω may be regarded as infinite in the shells for $k = 1,2, \ldots, k_0$ but small in the shells for $k = k_0 + 1, \ldots, K$, we have four independent solutions in the outer shells but these degenerate into two independent solutions in the inner shells. Thus the two boundary conditions at the outermost shell and one boundary condition at the center can be satisfied correctly. Although we have not discussed the inhomogeneous term in equation (8), it does not alter the above discussion.

d) Discussion and Comments

It is difficult to treat the case when ω takes an intermediate value. However, the following discussion will give some idea regarding a practical case. We must consider a region where $1 \lesssim \omega^2 \lesssim 10$. In this region the strongly-growing branches (equation [12]) of the differential equation cannot be solved exactly, since $\omega = \Omega \Delta x$ is too large. Thus the integral (15) must be replaced by ω^n, where n is the number of such shells. If ω^n is smaller than 10^{14}, we can extract the physically significant branches (equation [13]) by using the double precision computation. Although branch (12) is not solved exactly, there will be no problem, since dy_4/dx will be small enough compared with $|A_{4,j}\, y_j|$ in such a region.

We have not discussed other choices for equations (18) and (22). For example, the choice of $(\delta\eta_i^k)$ as $(\delta y_3^k, \delta y_4^k, \delta y_1^{k+1}, \delta y_2^{k+1})$ prevents subtraction in calculating δy_4^k from other variables by using equation (20). However, it is to be noticed that even in the choice of equation (18), the rounding-off error does not accumulate. For a phase or a region in which ω is small enough, $\beta_2 = \beta_4 = \frac{1}{2}$ or other weighting gives a better approximation and a more rapid convergence of the relaxation computation than equation (22). However, it is only a matter of programming technique to take this into account.

In summary, the Henyey method is a good mathematical scheme based on good physical pictures of both slow and rapid evolution if it is properly formulated. We can solve stably the radiation flux, L_r , as well as other variables, even for rapid evolution. The stability of L_r is a good measure of whether the method has been formulated in a given case so as to represent a good physical picture.

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