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THE USE OF MODERN TURBULENCE THEORY FOR CALCULATING EDDY DIFFUSIVITIES*

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ABSTRACT

The prospects are examined for obtaining analytical (as opposed to empirical) solutions to turbulent transport problems. The direct interaction approximation, a modern turbulence theory, is applied to the heat transfer boundary value problem and is seen to provide a reasonable generalization of the concept of eddy diffusivity. The calculation requirements are very great for present day computers.

SUMMARY

The formal turbulent heat transfer problem is posed: given only the boundary data and certain statistical information about the turbulent velocity field, calculate the total rate of heat transport across the boundaries. The difficulty in calculating the joint velocity field-temperature field statistical moments stems from the closure problem of the dynamical moment equations. Of all the modern closure methods of turbulent theory, that one which seems most likely to have success in realistic heat transfer problems is the direct interaction approximation (DIA) of Kraichnan. The DIA happens to be the exact solution for a model dynamical system and, consequently, cannot give the nonphysical behavior of other approximations not having positive definite probability density functionals. Furthermore, it properly allows the existence of an eddy diffusivity tensor in the case of a uniform mean temperature gradient. The latter can exist if the integral length scales of the turbulence are much smaller than any of the scales of statistical nonhomogeneity.

The DIA equations involve the Green's function $\langle G \rangle$ for the mean temperature field. In physical coordinates $\langle G \rangle$ is initially a delta-function and, consequently, the source of severe numerical difficulties. In wave-number coordinates these difficulties are avoided, as $\langle G \rangle$ is initially unity or at least diagonal with respect to wave-numbers in directions of nonhomogeneity. Unfortunately, the nonlinear convolution term requires six additional integrations for each direction of nonhomogeneity. This number can be reduced to two for a suitable expansion of the velocity covariance.

In either coordinate representation the DIA places severe demands on the largest and fastest computers available. Because the DIA can be made energetically consistent for a finite mode system, it is recommended that the Fourier representation be used for computation in spite of the greater number of integrations compared to the physical coordinates representation.

CONTENTS

THE USE OF MODERN TURBULENCE THEORY

FOR CALCULATING EDDY DIFFUSIVITIES

I. INTRODUCTION

It is not possible at the present time to perform analytical calculations of turbulent transport problems with the same degree of confidence and rigor one would have for problems in ordinary molecular diffusion, heat conduction, or laminar convection. Unless the turbulent velocity fluctuations are known with absolute certainty, the mathematical problem may not even be solvable in principle.

Consider the familiar heat conduction equation

$$\frac{\partial T}{\partial t} - \kappa \nabla^2 T = 0$$

where T is the temperature and κ is the constant molecular thermal diffusivity (or thermometric conductivity). Many analytic solutions are known for this equation. The situation is less hopeful with the addition of a laminar convection velocity $\mathbf{U}(\mathbf{x}, t)$

$$\frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla T - \kappa \nabla^2 T = 0 .$$

Only a few solutions for special $\mathbf{U}$ are known, although approximate solutions may be obtained numerically for a wide class of problems. When $\mathbf{U}$ takes on random or turbulent values $\mathbf{u}$, or $\mathbf{U} = \langle\mathbf{U}\rangle + \mathbf{u}$, where the angular brackets denote the average or known part, the situation is almost hopeless, as the random coefficient $\mathbf{u}$ in the equation

$$\frac{\partial T}{\partial t} + \langle\mathbf{U}\rangle \cdot \nabla T - \kappa \nabla^2 T = - \mathbf{u} \cdot \nabla T \tag{1}$$

induces a random part θ of T, or $T = \langle T\rangle + \theta$.

The reason that $\mathbf{u}$ is considered unknown and not merely complicated lies in our inability to solve the Navier-Stokes equations with sufficient accuracy.

Because of their nonlinearity and nonlocalness the equations of motion are extremely sensitive to initial conditions. An infinitessimal error will generate at some time a completely different flow. This is related to the observation that fluid particle trajectories tend statistically to wander apart. Alternatively, we can look at this from the point of view of Heisenberg;[1] a fluid is a mechanical system with an infinite number of degrees of freedom, all of which must be specified initially to determine the motion uniquely. Only when the viscosity is high enough (or the Reynolds number low enough) will the number of degrees of freedom reduce to one (laminar flow), for which U can be calculated unambiguously.

Although an exact solution to the turbulent heat transfer problem is hopeless, a statistical one may not be. Here we will be concerned with the statistical theory of turbulent convection, in which statistical averages, denoted by angular brackets, are taken over an infinite collection or statistical ensemble of flows. Each member flow [n] is called a realization. We have, then,

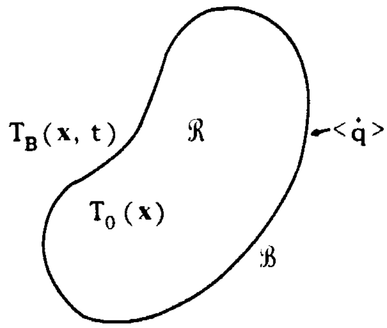
$$\langle U(x, t)\rangle \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} U^{[n]}(x, t)$$

for the mean of U and similar definitions for averages of other quantities.

Typically, eddy diffusivities and eddy viscosities do not arise in modern theories of turbulence and turbulent convection. They were introduced by Boussinesq[2] in 1877 and given physical basis by Prandtl[3] with the crude concept of mixing length. These approaches have been more successful so far than any modern analytical theories because of the presence of adjustable empirical constants, rather than their being good representations of the physics (Reference 4, p. 275).

II. THE PROBLEM

To define the problem let us consider as an example the case of heat convection by an incompressible single-phase fluid in a region $\mathcal{R}$ having boundaries $\mathcal{B}$. We assume constant physical properties with no viscous dissipation. The problem is to determine the heat transfer across $\mathcal{B}$, $\langle\dot{q}\rangle$, given only the initial and boundary temperature distributions $T_0(x)$ and $T_B(x, t)$, the Prandtl number, and the Reynolds number. We assume that the Reynolds number determines all the statistics of U and that these are known. Because $\langle\dot{q}\rangle = -\kappa \frac{\partial}{\partial n}\langle T\rangle_B$, the problem reduces to finding the average temperature distribution $\langle T(x, t)\rangle$ throughout $\mathcal{R}$.



The fundamental difficulty of solving the statistical problem is illustrated by the moment formulation of the problem. Rewrite Equation (1) as

$$\mathfrak{L}\, T = -\mathbf{u} \cdot \boldsymbol{\nabla}\, T$$

where $\mathfrak{L}$ is the nonrandom operator

$$\mathfrak{L} = \left\{ \frac{\partial}{\partial t} + \langle \mathbf{U} \rangle \cdot \boldsymbol{\nabla} - \kappa \nabla^2 \right\}.$$

Averaging Equation (1), we get

$$\mathfrak{L} \langle T \rangle = -\boldsymbol{\nabla} \cdot \langle \mathbf{u}\, \theta \rangle \qquad (2)$$

which involves an undetermined quantity, the velocity-temperature correlation. This equation was given by Kampé de Fériet[5] in 1937. A closed equation for $\langle T \rangle$ may be formed from Equation (2) with the introduction of an "eddy diffusivity" ε defined by*

$$\langle \mathbf{u}\, \theta \rangle \equiv -\varepsilon\, \boldsymbol{\nabla} \langle T \rangle$$

and which must be evaluated empirically. This approach is very crude, inasmuch as Equation (2) must be solved—empirically, at least—in order to determine ε. Alternatively, we may use Equation (1) to write equations for the higher order undetermined moments that appear in Equation (2) and similar equations. An unclosed hierarchy of moment equations results, as in writing an equation for

*In general ε must be considered a second order tensor (Cf. Equation (6) and Reference 4, p. 279).

the higher order moment that occurs in the preceding equation, the nonlinearity of dynamical quantities always produces another unknown. We can obtain in the present problem the hierarchy

$$\mathfrak{L} \langle T \rangle = - \nabla \cdot \langle \mathbf{u}\, \theta \rangle$$

$$\mathfrak{L} \langle \mathbf{u}'\, \theta \rangle = - \nabla \cdot \left[\langle \mathbf{u}\mathbf{u}'\, \theta \rangle + \langle \mathbf{u}\mathbf{u}' \rangle \langle T \rangle \right]$$

$$\cdots$$

$$\mathfrak{L} \langle \underbrace{\mathbf{u}' \ldots \mathbf{u}''}_{n}\, \theta \rangle = - \nabla \cdot \left[\langle \underbrace{\mathbf{u}\mathbf{u}' \ldots \mathbf{u}''}_{n+1}\, \theta \rangle - \langle \underbrace{\mathbf{u}' \ldots \mathbf{u}''}_{n} \rangle \langle \mathbf{u}\, \theta \rangle + \langle \underbrace{\mathbf{u}\mathbf{u}' \ldots \mathbf{u}''}_{n+1} \rangle \langle T \rangle \right]$$

$$\cdots$$

where we have used nonsimultaneous and noncoincident $\mathbf{u}$'s (denoted with primes), that are transparent to the $\mathfrak{L}$ and ∇ operators, in order to avoid the Navier-Stokes equations. In this formulation certain space-time arguments are made to coincide before substitution into the lower order moment equation. The task of suitably truncating this set of equations to a deterministic set is called the closure problem.

The closure problem of the moment equations illustrates the fundamental difficulty of turbulence and turbulent convection theory. In each realization T is a functional of the infinitely complicated function $\mathbf{U}(\mathbf{x}, t)$. The infinite complexity of the turbulence $\mathbf{U}(\mathbf{x}, t)$ and its effect on $T(\mathbf{x}, t)$ cannot be removed simply by averaging.[6] This difficulty persists regardless of the formulation of the problem. In the moment formulation, besides the closure problem of the hierarchy of moment equations, iteration series (expansion in terms of a turbulent Péclét number) do not converge. In the probability functional or characteristic functional approaches, multimode interactions are inseparable in wavenumber coordinates and nonlocal interactions are nondiagonalizable in physical coordinates.

III. APPLICATION OF A MODERN TURBULENCE THEORY

A. Status of the Theory

How will modern turbulence theory help us? Usually the same approaches are used for turbulent convection as for describing turbulence itself, although one must be careful, because the nature of the nonlinearities are different. Most of these approaches are not satisfactory. This is not surprising, since almost all of them are based in some sense on perturbation expansions about the case of zero turbulence. Any success with these in real problems must be considered luck.*

One lucky theory is the direct interaction approximation (DIA). It is probably the most satisfactory of modern closure procedures as far as its ability to give sensible results is concerned and is the one we will discuss here. It was invented by Kraichnan[8,9] in 1958, given for the initial-value scalar convection problem in 1961,[10,11] and can be easily generalized to the arbitrary boundary-value heat transfer problem.† It has been derived in several different ways.

Although there is some disagreement over the extent to which the DIA is a "rational approximation",[14-16] we believe it to be a good approximation, or at least one of the best available, because of its ability to satisfy a number of consistency requirements and properties of the true solution. These desiderata for closure theories should be satisfied by any turbulent convection theory that is expected to give good results. The following list contains some of the more important requirements.‡ Unfortunately, no existing closure scheme satisfies them all.

(1). Global Conservation Laws. The DIA and the true system give the same result when averaged over $\mathfrak{R}$. The DIA preserves global conservation of thermal energy[12]

$$\frac{d}{dt}\int_{\mathfrak{R}} \langle T\rangle \, d\mathfrak{R} + \int_{\mathfrak{B}} \langle T\rangle\langle U\rangle \cdot d\mathfrak{B} = \kappa \int_{\mathfrak{B}} \nabla \langle T\rangle \cdot d\mathfrak{B}$$

and of an "entropy"

$$\frac{d}{dt}\int_{\mathfrak{R}} \langle T^2\rangle \, d\mathfrak{R} + \int_{\mathfrak{B}} \langle T^2\rangle\langle U\rangle \cdot d\mathfrak{B} = \kappa \int_{\mathfrak{B}} \nabla \langle T^2\rangle \cdot d\mathfrak{B} - 2\kappa \int_{\mathfrak{R}} \langle |\nabla T|^2\rangle \, d\mathfrak{R}.$$

*This pessimistic view is due to Kraichnan.[7] Some of the approaches are referenced in Appendix A.

†Kraichnan[12] had considered the boundary-value problem but did not properly take into account Equation (3). See also Reference 13.

‡This list is taken from Kraichnan[7] and from Orszag.[17]

In these equations we have assumed that fluctuations in $\mathbf{u}$ disappear on $\mathcal{B}$.

(2). Realizability. All probabilities associated with the random variables T and U must be positive. This requires positive-definiteness of power spectra, bounds on the skewness, and inequalities for all the other moments.[18,19] The DIA satisfies this requirement because it is the exact solution to a model system in which the infinite ensemble of realizations is broken up into an ensemble of collections of realizations where $\mathbf{u}$ and T randomly interact. This situation is analogous to dime-flipping experiments in which the outcome of one toss depends on all the others. The behavior of T in realization $[n]$ is given by the equation[12]

$$\mathcal{L}\, T^{[n]} = - \sum_{k,m} \Phi^{n}_{k,m}\, \mathbf{u}^{[k]} \cdot \nabla T^{[m]}$$

where

$$\Phi^{n}_{k,m} = N^{-2} \sum_{\beta,\gamma} \exp\left\{\frac{2\pi i}{N}\left[\beta(k-n) + \gamma(m-n)\right]\right\} \varphi_{\beta+\gamma,\beta,\gamma}\,.$$

N is the number of realizations in a collection, and the φ's are restricted by

$$|\varphi| = 1$$

$$\varphi_{\alpha,\beta,\gamma} = 1 \ (\alpha, \beta, \text{ or } \gamma = 0)$$

$$\varphi_{\alpha,\beta,\alpha-\beta} = \varphi_{\alpha,\alpha-\beta,\beta} = \varphi^{*}_{-\alpha,-\beta,-\alpha+\beta} = \varphi^{*}_{\alpha-\beta,-\beta,\alpha}$$

but otherwise take on random phases throughout the collections.

(3). Galilean Covariance. The DI equations do not transform properly under Galilean coordinate transformations (convection by a uniform velocity) in homogeneous turbulence.[20] This is perhaps the main failing of the DIA. The invariance of the dynamical equations to uniform convection should be retained to properly describe the convection of small scales by large scales, as necessary for the derivation of Kolmogorov's law in ordinary turbulence theory.

(4). Stochastic Irreversibility. It is important that relaxation effects be described correctly, at least qualitatively. In particular, we believe that the statistical equations should exhibit the property of a mechanical system with a huge number of coupled degrees of freedom to approach an asymptotic statistical state that is independent of the initial conditions.[17] The relaxation is controlled by two factors: one is the explicit irreversibility of the conduction term which provides a sink for the dissipation of temperature fluctuations and which can be diagonalized or localized by using wavenumber coordinates. The other is the nonlocalness in both physical and wavenumber coordinates of the convection term, which, we believe, tends to drive the system irreversibly towards equipartition of the temperature fluctuations (destruction of spatial correlations).* The DIA retains both mechanisms.

(5). Quantitative Contact with Dynamical Equations. Besides the qualitative features we have mentioned, it is important that the DIA be in some sense an approximation to the dynamical equations (Equation (1) plus the Navier-Stokes equations). This condition is met, as the DIA becomes exact as a perturbation solution in the limit that the turbulence becomes weak — that is, in the limit that the right hand side of Equation (1) may be considered a perturbation.[10] Requirement (1) also serves to provide some contact.

B. The Direct Interaction Equations

Before writing down the DI equations, we need to define the Green's function for the temperature field. The Green's function G is defined to be the solution of Equation (1) for the special case of delta function initial conditions and zero boundary conditions — that is, G is the response to a unit pulse of temperature when the initial temperature is otherwise zero and the boundaries are infinitely conducting. Hence we have two problems, which we write symbolically as

$$
\begin{array}{lccc}
 & \underline{\text{T - problem}} & & \underline{\text{G - problem}} \\
\mathfrak{R}: & \mathfrak{N}T = 0 & & \mathfrak{N}G = 0 \\
\mathfrak{B}: & T = T_B & & G_B = 0 \\
t = 0: & T = T_0 & & G_{initial} = \delta(\ )
\end{array}
$$

*These ideas have recently been incorporated in a phenomenological approach by Leith.[21]

where

$$\mathfrak{N} = \left\{\frac{\partial}{\partial t} + \mathbf{U} \cdot \boldsymbol{\nabla} - \kappa \nabla^2\right\}.$$

The temperature field T can then be recovered by integrating over the initial and boundary data weighted by the Green's function,[22] which result we write symbolically*

$$T = \int_{\mathcal{R}} T_0 \, G \, d\mathcal{R} - \kappa \int_0^t \int_{\mathcal{B}} T_B \, \boldsymbol{\nabla} G \cdot d\mathcal{B} \, dt \,. \tag{3}$$

We actually need $\langle G \rangle$, however, for which there is an almost identical closure problem as for $\langle T \rangle$. To simplify matters we use a maximal randomness condition that T_0 and T_B be statistically independent of the $\mathbf{u}$ field to all orders of $\mathbf{u}$.[13] This guarantees that in Equation (3) G will be independent of T_0 and T_B, and hence $\langle G \rangle$ can be determined independently of $\langle T \rangle$.

The DIA yields the following closed integrodifferential equation for $\langle G \rangle$ involving velocity field statistics to only second order[10]

$$\mathfrak{L} \langle G \rangle = \left[\boldsymbol{\nabla} \cdot \langle \mathbf{u}\mathbf{u}' \rangle \langle G \rangle\right] * \left[\cdot \, \boldsymbol{\nabla} \langle G \rangle\right]. \tag{4}$$

Here the asterisk denotes a space-time convolution integral. Equation (4) may be integrated over T_0 and T_B (Equation 3) to obtain an equation for $\langle T \rangle$, which, by comparison with Equation (2), yields[12]

$$\langle \mathbf{u}\theta \rangle = - \left[\langle \mathbf{u}\mathbf{u}' \rangle \langle G \rangle\right] * \left[\cdot \, \boldsymbol{\nabla} \langle T \rangle\right]. \tag{5}$$

This goes to the form

$$\langle \mathbf{u}\theta \rangle = - \boldsymbol{\varepsilon} \cdot \boldsymbol{\nabla} \langle T \rangle \tag{6}$$

*In Equation (3) the del operates on the source point. Equations (3)–(7) are written out more explicitly in Appendix B. See also Reference 13.

in the special case that $\nabla\langle T\rangle$ is approximately constant, where $\boldsymbol{\epsilon}$ is effectively an eddy diffusivity tensor[11, 12]

$$\boldsymbol{\epsilon} = \iint \langle \mathbf{u}\mathbf{u}' \rangle \langle G \rangle \, d\mathcal{R} dt \, . \tag{7}$$

We see that the DIA provides a generalization of the classical ideas of eddy diffusivity that is consistent with the physics of the problem, that turbulence is a somewhat nonlocal phenomenon. In general the turbulent temperature flux $\langle \mathbf{u}\,\theta \rangle$ is not directly proportional to $\nabla\langle T\rangle$ but depends on the entire temperature field and on the turbulence throughout $\mathcal{R}$. An eddy diffusivity does exist only if $\nabla\langle T\rangle$ is uniform over the region in space-time in which the velocity is correlated. This can occur at steady state in situations where the integral length scale of the turbulence is much smaller than any scales of statistical nonhomogeneity or, crudely speaking, if the turbulent eddies are much smaller than the dimensions of the flow system. The eddy diffusivity is tensorial and goes to the scalar form only if the turbulence is isotropic. These restrictions and generalizations on the existence of a classical eddy diffusivity seem to be qualitatively correct, and are analogous to the conditions for existence of a molecular thermal conductivity in the statistical theory of transport in gases.

At this point it is natural to ask what sort of physical picture is represented by the DIA? The answer is difficult, as from what we have said the DIA seems to be an arbitrary mathematical model involving unrealistically coupled realizations. However, consider the case of homogeneous turbulence for which we write the Fourier transform of Equation (1) as*

$$\dot{\hat{T}}(\mathbf{k}) + \kappa k^2 \hat{T}(\mathbf{k}) = -i \, \mathbf{k} \cdot \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \hat{\mathbf{u}}(\mathbf{k}') \, \hat{T}(\mathbf{k}''). \tag{8}$$

The model equation for which the DIA is the exact statistical solution is

$$\dot{\hat{T}}(\mathbf{k}) + \kappa k^2 \hat{T}(\mathbf{k}) = i \, \mathbf{k} \cdot \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \phi(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \, \hat{\mathbf{u}}(\mathbf{k}') \, \hat{T}(\mathbf{k}'')$$

*This form is suitable if we let the T and $\mathbf{u}$ fields be periodic over arbitrarily large box-volumes.

where $|\phi| = 1$, ϕ(any argument = 0) = 1, $\phi(\mathbf{k}, \mathbf{k}', \mathbf{k}'') = \phi^*(\mathbf{k}'', -\mathbf{k}', \mathbf{k}) = \phi(\mathbf{k}, \mathbf{k}'', \mathbf{k}') = \phi^*(-\mathbf{k}, -\mathbf{k}', -\mathbf{k}'')$ and otherwise the phases of ϕ are random. The model, then, replaces the true wavenumber interactions by fictitious interactions whose strengths remain identical to those of the true system (because of the constraints on ϕ) but whose phases are randomized. The DIA gets its name by treating each elementary triad wavenumber interaction (a direct interaction) in the sum term of Equation (8) as a perturbation against the background of all the indirect interactions, and assuming that only the sum of the DI's survives the summation in Equation (8). With this mechanism excitations are thus relaxed by dynamical interaction with the full turbulent field, and not just by conductive decay.

IV. COMPUTATIONAL CONSIDERATIONS

We must be able to solve Equation (4) for $\langle G \rangle$ to obtain any practical information from the DIA. In order to test the approximation we will consider a brute force calculation before attempting such ad hoc simplifications as lower dimensional models or special assumptions on the form of $\langle G \rangle$.

The usual procedure for numerically approximating an integrodifferential equation is to set up finite difference equations on a space-time or wavenumber-time mesh and to march the solution forward from the initial time layer. These difference equations should be consistent with Equation (4) as the mesh density increases and should preserve the desiderata that we listed previously. We usually have to resort to a posteriori evidence of numerical stability and assume convergence to the true solution as a matter of faith.

The physical demands presented by Equation (4) on existing high-speed electronic computers are nearly excessive. Huge amounts of storage are needed because $\langle G \rangle$ has up to 8 independent variables. Great speed is necessary for evaluation of the convolution integral term, which requires a 4-fold integration (or summation) in $\mathbf{x}$-space or up to a 22-fold integration in $\mathbf{k}$-space.

We have found severe numerical difficulties in attempting to calculate $\langle G \rangle$ in physical coordinates ($\mathbf{x}$-space).[13] Most of these are associated with the fact that $\langle G \rangle$ is initially a delta function and is difficult to approximate numerically. The symptoms are large oscillations that decay no faster than $\langle G \rangle$ but which can be reduced by decreasing the time increments. The oscillations are induced at small times by truncation of the differential operators in Equation (4) and can be important at larger times because of the time convolution. An equally serious difficulty is failure to preserve the consistency requirements listed earlier.

Solution of Equation (4) in wavenumber coordinates (k-space) is much more attractive. First, the same set of consistency requirements can be satisfied — even with a finite mode system. Second, the initial singularity of $\langle G \rangle$ is avoided, because its initial value is unity for all wavenumbers or at least diagonal with respect to wavenumbers in directions of nonhomogeneity. Unfortunately, more integrations are required for the convolution term in nonhomogeneous turbulence. In going to k-space, for each dimension of nonhomogeneity each gradient operator produces one additional integration and the $\langle \mathbf{uu}' \rangle \langle G \rangle$ product produces 4. The total number (seven) can be reduced to 3 for a suitable Fourier expansion of $\langle \mathbf{uu}' \rangle$.

At the present time we are completing the calculation of an eddy diffusivity for the case of a steady isotropic turbulence and a uniform mean temperature gradient.* In this calculation $\langle G \rangle$ is a function of 2 independent variables, and the convolution requires 3 integrations (5 nested FORTRAN DO-loops). For comparison there is one data point for the measurement of velocity-temperature correlation in a grid-generated turbulence heated to a self-maintained uniform temperature gradient.[24] The space-time velocity correlation coefficient may be estimated from the data of Favre[25] or Frenkiel and Klebanoff.[26]

The next problem to consider is steady heat transfer to fully-developed turbulent channel flow, for which $\langle G \rangle$ is a function of 5 independent variables and the convolution requires 6 integrations (a reduction from 10) — 11 nested DO-loops. Once $\langle G \rangle$ is obtained, the Nusselt number for a variety of boundary conditions may be determined. One case, the flow between parallel plates at different temperatures, may be compared to the data of Page, et al.[27-29] Another case is the turbulent Graetz problem. We expect the channel flow problem to be very difficult.

V. CLOSING

Although progress is being made towards the goal of solving turbulent transport problems analytically, the picture looks somewhat grim at the present time. Perhaps the situation will improve with the next generation of high-speed computers or when simplified approximations to the DIA are found.

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*This problem was invented by Corrsin[23] in 1952.

APPENDIX A

Recently, there have appeared several reviews and comparisons of some of the analytical turbulence and turbulent convection theories (References 6, 7, 16, 17, 18 Ch. 3, 30, 31). Since no really comprehensive and critical survey has appeared, closure techniques are still in the hands of specialists (except for the mixing length theories). Here we do not attempt a complete list of techniques but only indicate the diversity of approaches — many of which have not been carried out to completion, and some of which are merely alternate ways of setting up the same problem.

We regard the more popular approaches originating before 1958 as the classical ones. These include — besides the phenomenological methods[4, 32, 33] — the moment discards,[34, 35, 36] * various types of quasinormal or fourth cumulant discard,[32, 37-44] and the maximum dissipation hypothesis of Malkus.[45, 46]

The level of sophistication suddenly increased with the inception of Kraichnan's direct interaction approximation.[8, 9] Kraichnan also developed a random coupling model[10] for the DIA, an ad hoc modification of a Lagrangian form of the DIA,[47] and some variations. A great number of papers have been generated from the DIA (the present one included) — too many to list here.

Work has continued on the use of characteristic functionals and functional probability formulations (Hopf,[48] Lewis and Kraichnan,[49] Rosen,[50, 51] Edwards' random phase approximation,[52] Herring's self-consistent-field approximation[53, 54]) and the setting down of BBGKY-type hierarchies.[55-57]

Recently, Wiener-Hermite expansions of the random variable have been examined.[58-64] In these investigations the expansions were made about "white noise", although expansion about "red noise" would seem preferable for describing relaxation effects.

There are still other approaches. Soon after the DIA appeared, Wyld[65] and L. Lee[66] derived consolidated perturbation expansions for the triple correlations in diagram notation. (See also Kraichnan[10] and Edwards.[52]) J. Lee[31] has modified the quasinormal approximation in the scalar mixing problem by comparison with the DIA. Orszag[17, 18] has used "inequality preserving" closures, based on consistency requirement #2, for the cumulant equations in homogeneous turbulence. Kampé de Fériet[67] has set up the Gram-Charlier approximation for bivariate distributions, intended for application to homogeneous turbulence.

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*There have been several applications to shear turbulence, heat transfer, etc.

This short list does not exhaust the number of possible approaches. For example, variational and information-theoretic methods that have found use in other fields of statistical physics (e.g., Jaynes[68]) have only been discussed sparingly.[8, 17, 69] The possibility of singular perturbation expansions about infinite Reynolds number should also be considered — requiring first a theory for infinite Reynolds number turbulence involving, say, distinct separation of energy and dissipation spectra. Finally, because of dynamic passivity of T and the absence of a feedback effect on errors in **u**, the use of model random fields **u** should be investigated for "exact" solution of Equation (1).

APPENDIX B

The equations in the text that involved G were only written symbolically. More accurately, let $G(\mathbf{x}, t \mid \mathbf{x}', t')$ be the temperature at the point $\mathbf{x} \in \mathfrak{R}$ at time t for the response to a unit pulse of temperature at the point $\mathbf{x}' \in \mathfrak{R}$ at the time $t' < t$. The equation for G is

$$\mathfrak{N}(\mathbf{x},t)\, G(\mathbf{x},t \mid \mathbf{x}',t') = 0 \qquad \text{for } \mathbf{x}, \mathbf{x}' \in \mathfrak{R} \text{ and } t > t'$$

with the initial condition

$$G(\mathbf{x},t' \mid \mathbf{x}', t') = \delta(\mathbf{x} - \mathbf{x}') \qquad \text{for } \mathbf{x}, \mathbf{x}' \in \mathfrak{R}$$

and boundary condition

$$G(\mathbf{x}, t \mid \mathbf{x}', t') = 0 \qquad \text{for } \mathbf{x} \text{ or } \mathbf{x}' \in \mathfrak{B}.$$

The last condition is sufficient for reciprocity. Equation (3) is then

$$T(\mathbf{x},t) = \int_{\mathfrak{R}} d^3x'\, T_0(\mathbf{x}')\, G(\mathbf{x}, t \mid \mathbf{x}', 0) - \kappa \int_0^t dt' \int_{\mathfrak{B}} d^2x'\, T_B(\mathbf{x}', t') \frac{\partial}{\partial n'} G(\mathbf{x}, t \mid \mathbf{x}', t')$$

where $\mathbf{n}$ is the outward unit normal on $\mathfrak{B}$. Equation (4) becomes

$$\mathfrak{L}(\mathbf{x},t)\langle G(\mathbf{x},t \mid \mathbf{x}',t')\rangle = \int_{t'}^t dt'' \int_{\mathfrak{R}} d^3x''\, \nabla_{\mathbf{x}} \cdot \langle \mathbf{u}\mathbf{u}''\rangle \langle G(\mathbf{x},t \mid \mathbf{x}'',t'')\rangle \cdot \nabla_{\mathbf{x}''} \langle G(\mathbf{x}'', t'' \mid \mathbf{x}', t')\rangle,$$

Equation (5) is

$$\langle \mathbf{u}(\mathbf{x},t)\, \theta(\mathbf{x},t)\rangle = -\int_0^t dt' \int_{\mathfrak{R}} d^3x' \langle \mathbf{u}\mathbf{u}'\rangle \langle G(\mathbf{x},t \mid \mathbf{x}', t')\rangle \cdot \nabla_{\mathbf{x}'} \langle T(\mathbf{x}', t')\rangle ,$$

and Equation (7) is

$$\boldsymbol{\epsilon}(\mathbf{x},t) = \int_0^t dt' \int_{\mathfrak{R}} d^3x' \langle \mathbf{u}\mathbf{u}'\rangle \langle G(\mathbf{x},t \mid \mathbf{x}', t')\rangle .$$

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