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ESTIMATION OF THE PARAMETERS OF

SAMPLED-DATA SYSTEMS BY STOCHASTIC APPROXIMATION

Technical Report

Caswell B. Neal

January 1969

Sponsored by

The National Aeronautics and Space Administration
under Grant No. NGR 05-018-022

ELECTRONIC SCIENCES LABORATORY

USC Engineering

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ABSTRACT

Various methods have been proposed to estimate the parameters of both open loop and closed loop sampled-data control systems. Generally speaking, these methods yielded approximate models of the system under study; the degree of approximation depending on the a priori knowledge of the system structure, state observation noise, system nonlinearities, and other factors. However, none of the methods has been applied to the problem of determining the sampling interval of either closed loop or open loop sampled-data control systems. This has been the task of the present study. Specifically, this dissertation is concerned with estimation of parameters in systems that have internal sampling, but have continuous input and output. The continuous portion of the sampled-data system is given by the differential equation

$$\frac{dz^*}{dt} = f(z, p, u(t)); \quad z(t=0) = \zeta$$

where z is an n dimensional vector of state, $f(\cdot)$ is the n dimensional vector of the dynamical system, p is a constant h dimensional vector of parameters, $u(t)$ is an r dimensional vector of piecewise continuous control functions, and ζ is the initial condition vector. For our results, $f(\cdot)$ was required to be of class C^1 in z and p. The differential equation is preceded by some form of data hold. The model-matching technique was used for parameter estimation. Methods were developed for determining not only the sampling

interval, but all the other parameters and initial conditions of the sampled-data system as well.

In this investigation, three methods were employed for the estimation of sampling intervals and other parameters of a sampled-data system. In all methods, the cost function was the integral of norm-squared error, where the error function was defined as the difference between the observed state vector of the system, and the state vector of the model.

The first method employed programmed search to vary the model parameters in order to minimize the cost function.

The second method employed iterative gradient search by means of discrete sensitivity difference equations for the various model parameters. The work of Bekey and Tomovic in connection with discrete sensitivity difference equations for the sampling interval was extended to all the other parameters of the system. Gradient search was then used for parameter estimates.

The third, and most important, method used was that of stochastic approximation. This permitted observation noise. The mean-square convergence of the model parameters to the true parameters of the system was proved under the following conditions: The system and model agreed in form and order, the data holds were identical, the observation noise had zero mean, finite variance, and was uncorrelated with both the system state vector and model state vector, $f(\cdot)$ was of class C^1 in z and p, and the partial derivative of the cost function with respect to the sampling interval existed and was bounded.

Stochastic approximation was then applied to the practically important problem of estimating the parameters of the human operator from records of scalar input and scalar output of the human operator operating in a closed loop configuration. Parameters were estimated successfully in both continuous and sampled-data models of human operators.

CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 General Statement Of The Problem

Various methods have been proposed to estimate the parameters of both open loop and closed loop sampled-data control systems. Generally, these methods yield approximate models of the system under study; the degree of approximation depending on a priori knowledge of the system structure, state observation noise, system nonlinearities, and other factors. However, at the present time not one of the current methods has been applied to the problem of determining the sampling interval of either closed loop or open loop sampled-data control systems. This is the task of the present study. Specifically, we will be concerned with systems that have internal sampling, but have continuous input and output. Refer to Figure 1.1 for a schematic diagram of such a system. The continuous portion of the sampled-data system is given by

$$\frac{dz}{dt} = f(z, p, u(t)), \quad z(t=0) = \zeta \tag{1.1}$$

where z is a n dimensional vector of state, f is the n dimensional vector of the dynamical system, p is a constant h dimensional vector of parameters, and $u(t)$ is an r dimensional vector of control. Note that $h \le n$. The solution to (1.1), written formally as $z(t; p, \zeta, u(t))$, will often be denoted by $z(t; p, \zeta)$, $z(t; p)$, or $z(t)$ as required by the particular treatment at the time. Thus, we will usually suppress notational dependence on initial conditions and/or parameters when they are not to be varied during the course of an estimation procedure, and will not always explicitly show the $u(t)$ dependence for reasons which will become clear later.

Proceeding informally for the present, we will assume that the f^i, $\partial f^i/\partial z^g$, and $\partial f^i/\partial p^j$ $(i,g=1, 2,\ldots,n;\ j=1, 2,\ldots,h)$ exist and are continuous functions of t, z, p, and u. Assuming, furthermore, that the data hold is of a given type, such as, for example, zero-order, we will treat the problem of estimating not only the sampling interval T of the sampled-data system of Figure 1.1, but also the components of the parameter vector p of the continuous system as well. The methods we develop can, in addition, be used to estimate the components of the initial condition vector ζ. However, modeling studies are limited to the estimation of p and T.

While it is clear that this is an important topic in estimation theory, it is of practical importance as well. For example, the application of analytical and computer techniques to process control is a challenging and important problem area in the modern control field. Generally, in order to control the plant in the desired manner, the parameters of the closed-loop system must be known. This study enlarges the set of plant parameters which may be estimated to include the sampling interval when the data hold is of constant characteristic and the differential equation of the plant satisfies equation (1.1).

In addition to process control, the study of manual control continues to be an important problem area in the synthesis of modern

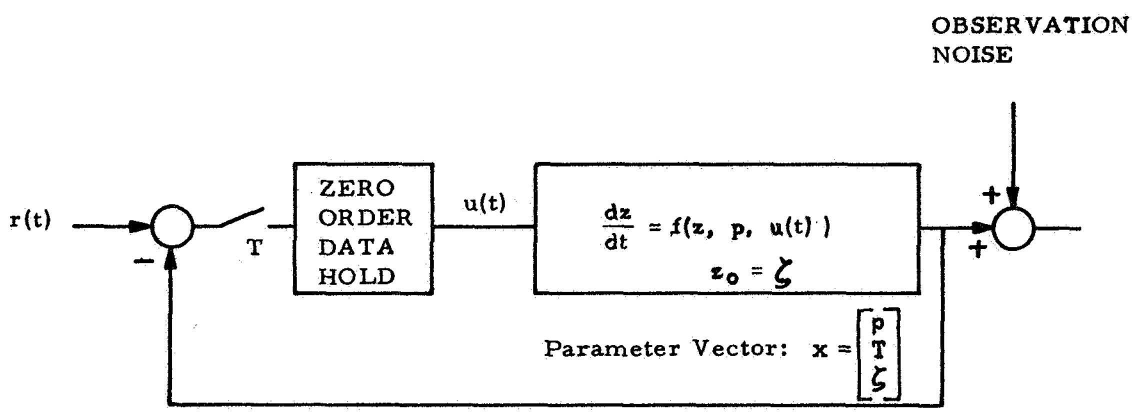


Figure 1.1 Sampled-Data System to be Estimated

aerospace vehicles. Early flights in the manned space vehicle program, including the Mercury and Gemini missions, have clearly demonstrated the importance of the human controller in the closed-loop control system configuration, and the consequent importance of precise knowledge of his dynamic characteristics to control system designers. As new space programs are formulated, it will become increasingly important to develop satisfactory techniques for determining accurately the dynamic characteristics of human performance in control tasks.

1.2 Some Definitions

At the outset we will adopt the following somewhat arbitrary definitions. In particular, they are concerned with the problem of determining the coefficients and/or states of a plant described by an ordinary differential equation.

Definition 1: An identification is here defined as the determination of the coefficients of the differential equation of the plant by means of some types of model adjustment technique when the exact form of the differential equation is known and when measurements of the observed quantities are noise-free. Under these restrictions, a plant is said to be identified when these coefficients are known exactly.

Definition 2: An optimal estimate is here defined as the determination, in some optimal sense, of the coefficients of a plant by means of model adjustment when the exact form of the differential equation of the plant is possibly unknown and when the observed quantities are noise-corrupted.

1.3 Background

The research activity reported in this dissertation is concerned with the problem of estimating sampling rates in sampled-data control systems whose output state variables are continuous functions of time. While the purpose is to develop a method which will ultimately be useful in estimating sampling intervals in a wide variety of sampled-data control systems, the approach taken here is rather general, being concerned with estimating a sampling-interval, as well as other parameters, of a sampled-data system in general.

However, a literature search discloses that all previously recorded attempts at such estimation have been concerned with the human operator. This is because of the relative importance of obtaining accurate models of the operator dynamics for use in control synthesis studies. Examples of these studies are found in connection with aircraft and spacecraft design.

Intermittency in human tracking behavior has been used as a basis for modeling of simple manual control systems in a number of early studies. This is discussed in papers by Ward [1], Bekey [3], Lemay and Naslin [4]. More recently, intermittent behavior has been reported by Pew, Duffendack, and Fensch [5]. In all these models, systematic techniques for the determination of the sampling interval have been lacking. The problem is further complicated by the fact that the studies of McRuer et al [6] as well as recent experiments by Jacobson, Biddle, and Bekey [7] have indicated that if sampling is present in human operator behavior, it does not consist of a simple periodic sampler, but rather a random sampler in which a mean sampling interval has superimposed upon it a random variability of magnitude sufficient to obscure the resulting periodicities in operator output spectra.

Recently, some progress has been made in the direction of obtaining methods for the estimation of the parameters of sampled models, including a quantitative measure of both a constant or a random sampling frequency. The work of Bekey and Tomovic [8] has shown that dynamic system sensitivity analysis can be applied to systems with variable sampling. They have furnished the mathematical formulation of the system sensitivity to sampling interval variations, and have shown how adaptive sampling may be implemented in adaptive control situations. More recently, Neal [9] has applied these results to the determination of constant sampling frequencies in linear noise-free closed-loop sampled-data control systems described by Figure 1.1. This work will be discussed in Chapter 2.

1.3.1 Brief Review Of Some Parameter Estimation Techniques

The purpose of this section is to provide a brief review of several parameter estimation techniques which are of current interest. For detailed accounts of a wide variety of parameter estimation techniques reference is made to the more detailed surveys of Nahi [66], Eveleigh [81], Cuenod and Sage [82], and Eykhoff [83]. Unless noted all vectors have the dimensions given in Section (1.1).

Kopp and Orford [13] describe a method for obtaining an iterative estimate of both the parameters and the state of linear models of possibly nonlinear time-varying systems described by ordinary differential equations. Such nonlinear systems are represented by

$$\frac{dz}{dt} = f(z, p(t), u(t), t), \qquad z(o) = \zeta \tag{1.2}$$

where the nomenclature is the same as that of equation (1.1). Basically, their technique is an extension of the Kalman iterative state estimation technique implemented, in this case, by adjoining a set of assumed linear differential equations for the parameters to the set of linear differential equations describing the linear model of the system. The differential equations for the unknown parameters are assumed to satisfy, for example, a model such as

$$\frac{dp^i}{dt} = \alpha^i(t)\left[p^i(t) - \hat{p}^i(t)\right] + w_n^i(t); \quad p^i(o) = \hat{p}_o^i \tag{1.3}$$

where $p^i(t)$ represents the i^{th} unknown parameter, $\alpha^i(t)$ is a given (assumed) time-varying coefficient, $\hat{p}^i(t)$ is the present estimate of the temporal history of $p^i(t)$, and $w_n^i(t)$ is the noise term of the i^{th} parameter with assumed properties:

$$E(w_n^i(t)) = 0, \quad (i = 1, 2, \ldots, h), \; h \le n,$$

$$E(w_n^i(t) w_n^i(\tau)) = \sigma_{w_n}^2 i \; \delta(t - \tau), \tag{1.4}$$

where $\delta(\cdot)$ is the Dirac delta function, and $\sigma_{w_n}^2 i$ is given. The initial conditions, ζ and p_o, for both the state differential equations (1.2) and the parameter differential equations (1.3) are drawn from a set of normally distributed random variables. Sequential linear regression is then used to obtain the estimates of the augmented state vector. Because linear regression is employed, the parameter estimates thus satisfy a minimum mean-square error criterion [66].

The quasilinearization method [14] has been applied to the estimation of the components of the constant parameter vector p and the initial condition vector ζ of equation (1.1)

$$\frac{dz}{dt} = f(z, p, u(t)), \qquad z(o) = \zeta \tag{1.1}$$

where the form of $f(\cdot)$ is assumed to be known, and it is assumed that noise-free observations of some of the states of (1.1) are available. By assuming

$$\frac{dp}{dt} = 0, \qquad p(o) = p_o \tag{1.5}$$

and adjoining (1.5) to (1.1) and regarding (1.1) as the forward loop control system, and u(t) as the sum-junction error signal, of a unity feedback closed loop control system, the new problem [14] becomes that of estimating the components of the augmented initial condition vector $\tilde{z}$ of the vector differential equation

$$\frac{d\tilde{z}}{dt} = f(\tilde{z}), \qquad \tilde{z}(o) = \tilde{\zeta} \tag{1.6}$$

where the $\tilde{z}(t)$ is an (n+h) dimensional vector. Observations $b^1(t)$ of only the first component of the original state vector $(z^1 = \tilde{z}^1)$ are required and the quasilinearization technique is then used to generate the $(k+1)^{st}$ sequential estimate time history of the augmented state vector, written as $\tilde{z}_{(k+1)}(t)$, so as to minimize

$$S = \sum_{i=1}^{N} (\tilde{z}^1_{(k+1)}(t_i) - b^1(t_i))^2. \qquad (1.7)$$

In order to start the procedure an initial estimate of $\tilde{z}(t)$ is assumed. The details of the quasilinearization technique are discussed in the work of Bellman, Kagiwada, and Kalaba [14]. The quasilinearization approach to parameter estimation has the weakness that convergence, in general, will occur only if the initial estimates of the components of $\tilde{z}(t)$ are sufficiently close to their respective true values.

Detchmendy and Sridhar [84] applied invariant imbedding to the estimation of noisy states and parameters in time-varying nonlinear dynamic systems. The form of the dynamic system is assumed to be known exactly and may be written, for example, as

$$\frac{dz}{dt} = f(z,t) + k(z,t)u(t), \qquad z(o) = \zeta, \qquad (1.8)$$

where $u(t)$ represents an r vector of unknown forcing functions. Also, equation (1.8) is here assumed to be already in augmented form, and hence contains the assumed differential equations for the parameter variations. Observations of the states z are expressed by the m vector

$$v(t) = h(z,t) + n(t) \qquad (1.9)$$

where $n(t)$ is the observation error m vector. No statistical assumptions are made concerning the unknown input vector $u(t)$ or the observation error vector n(t). The cost function

$$S = \int_0^{t_f} \left[\|v(t) - h(\hat{z},t)\|_Q^2 + \|\dot{\hat{z}} - f(\hat{z},t)\|_W^2 \right] dt \tag{1.10}$$

$$= \int_0^{t_f} \left[\|v(t) - h(\hat{z},t)\|_Q^2 + \|\hat{u}(t)\|^2_{k'Wk} \right] dt$$

is to be minimized with respect to z(t) and u(t) for $0 \leq t \leq t_f$ subject to the constraint differential equation

$$\frac{d\hat{z}}{dt} = f(\hat{z},t) + k(\hat{z},t)\hat{u}(t), \qquad \hat{z}(o) = \hat{\zeta}, \tag{1.11}$$

where $\hat{z}(t)$ and $\hat{u}(t)$ are the estimates of the state vector and unknown forcing function vector respectively, and Q and W are positive definite weighting matrices. The Hamiltonian for the system of (1.10) and (1.11) is then written and the maximum principle is used to obtain a two-point boundary value problem for which some of the boundary conditions are specified at t=0 and some are specified at $t = t_f$. Then, by using the invariant imbedding equations, a sequential estimator for the noisy states and noisy parameters is obtained. The derivation of the invariant imbedding equations is given in References [84] and [85]. The invariant imbedding approach to parameter estimation has the advantage that noisy parameters can be treated and, if the system is stable and observable, then convergence of the estimator equations to their minimizing (least-squares) values will occur over a wide range of initial conditions [85].

Stochastic approximation, which will be discussed at length in Chapter 3, has been suggested or used by Bertram [17], Sakrison [18,19], Ho and Whelan [20], Kushner [21,22], Ho and Lee [23], Kirvaitis [24], Holmes [25], and others for parameter estimates in both open loop and closed loop linear and nonlinear continuous control systems. However, up to the present time, no application of this technique has been made to determining sampling intervals. In this dissertation, we will apply stochastic approximation to the problem of estimating sampling intervals and other parameters of closed loop sampled-data systems.

1.4 Objectives Of The Study

From the foregoing discussion it is clear that many techniques have been successfully applied to the task of estimating the parameters of controlled systems. Some of these can also be used to estimate the parameters of closed loop control systems. Until the present study it has not been shown that any of the previous methods could be used to identify either deterministic or random sampling intervals in closed loop sampled-data control systems. Therefore the objectives of this study are as follows:

Given the sampled-data control system of Figure 1.1, with the properties given in Section 1.1, it is desired to develop an estimation technique which will ultimately lend itself to the estimation of all the parameters of the sampled-data system, including the sampling interval. In order to accomplish this objective, consider the model-matching least-square parameter estimation configuration of either

Figure 1.2, or Figure 1.3, consisting of a closed loop sampled-data system, which, in practice, might have unknown parameters, and a model of that system which will be designated as the sampled-data model. Both sampled-data system and sampled-data model are driven by the scalar function r(t). The sampled-data system and sampled-data model consist of a closed loop configuration of sampler, data-hold, and continuous system. In the sampled-data system, the sampling is assumed to be periodic with period T, and the data-hold is assumed to be of zero order. Similarly, the sampled-data model has periodic sampling, of period $\hat{T}$, and has a zero-order data hold. The continuous system is, in general, not perfectly known, and our broad objective is to develop ways for estimating its parameters as well as the sampling interval T. For purposes of later analysis, we will require that the continuous model satisfy the continuity and differentiability requirements listed in Section 1.1. The continuous model is given by

$$\frac{d\hat{z}}{dt} = \hat{f}(\hat{z};\, \hat{p},\, \hat{u}(t)), \qquad \hat{z}\,(t=0) = \hat{\zeta}\,, \tag{1.12}$$

where $\hat{z}$ and $\hat{f}$ are n dimensional vectors, $\hat{p}$ is an h dimensional vector of parameters, and $\hat{u}(t)$ is a piecewise continuous scalar control variable. Note that $h \leq n$. In general, superscripts will refer to components of vectors, e.g., $\hat{z}^1$ is defined as the output component of the vector $\hat{z}$. The purpose of the modeling procedure is to construct a continuous model which is of the same form as the continuous system. Therefore, because of the above analytical requirements imposed on the continuous model, we will also impose the same continuity and differentiability requirements on the continuous system. The continuous system is hence assumed to be of the form

$$\frac{dz}{dt} = f(z, p, u(t)) ; \qquad z(t=0) = \zeta \tag{1.13}$$

where z and f are n dimensional vectors, and the vector of constant parameters p is h dimensional. Define the sampled-data system $(h + 1 + n)$ dimensional parameter vector by

$$x = (p, T, \zeta)', \tag{1.14}$$

where $'$ indicates transpose, and define the sampled-data model $(h+1+n)$ dimensional parameter vector by

$$\hat{x} = (\hat{p}, \hat{T}, \hat{\zeta})' . \tag{1.15}$$

Note that $(h+1+n) \leq 2n + 1$. <u>Henceforth, we will describe (1.14) and (1.15) as m dimensional vectors, where $m \leq 2n + 1$.</u> The model-matching configuration of either Figure 1.2, or Figure 1.3 will be driven by $r(t)$, a scalar function, which is required to be non-zero over the constant iteration interval τ. At the end of a particular iteration, the components of the parameter vector $\hat{x}$ will be adjusted to new values according to the particular algorithm used in the study, then the integration will begin over again. Define the vector error function by

$$\epsilon(t; x, \hat{x}, r(t)) = v(t; x, r(t)) - \hat{z}(t; \hat{x}, r(t)), \tag{1.16}$$

where

$$v(t; x, r(t)) = z(t; x, r(t)) + n_1(t), \tag{1.17}$$

and where $z(t;\ x,\ r(t))$ and $\hat{z}(t;\ \hat{x},\ r(t))$ are the state vectors of the sampled-data system and the sampled-data model respectively, and $n_1(t)$ is the state observation noise vector. Define the cost function

$$J(\tau;\ x,\ \hat{x},\ r(t)) = \int_0^{\tau} \epsilon'(t;\ x,\ \hat{x},\ r(t))\ W\ \epsilon(t;\ x,\ \hat{x},\ r(t))\ dt \qquad (1.18)$$

where W is a positive definite weighting matrix and τ is the constant iteration interval. (In the sequel, we will often indicate (1.18) by either $J(\tau;\ \hat{x},\ r(t))$, or $J(\tau;\ \hat{x})$, since x is a constant parameter vector, whereas $\hat{x}$ may be adjusted after each iteration. Likewise, equation (1.16) will be indicated by $\epsilon(t;\ \hat{x})$.

(I) Using the estimation configuration of Figure 1.2:

(a) Determine conditions under which equation (1.18) has a unique minimum over $\hat{T}$ when the continuous system and the continuous model have the same form and when

$$(p,\ \zeta)' = (\hat{p},\ \hat{\zeta})'. \qquad (1.19)$$

(b) Suppose the continuous system is not modeled corrently so that either the continuous model and continuous system do not agree in form, or if they do agree in form, then the parameter vectors $(p,\ \zeta)' \neq (\hat{p},\ \hat{\zeta})'$. Determine whether the cost equation (1.18) then has a minimum over $\hat{T}$.

(c) Investigate conditions for convergence of the estimate $\hat{T}$ to the true value of T when a steep descent approach using the sensitivity difference equations is employed in conjunction with an iterative adjustment scheme. As in (a)

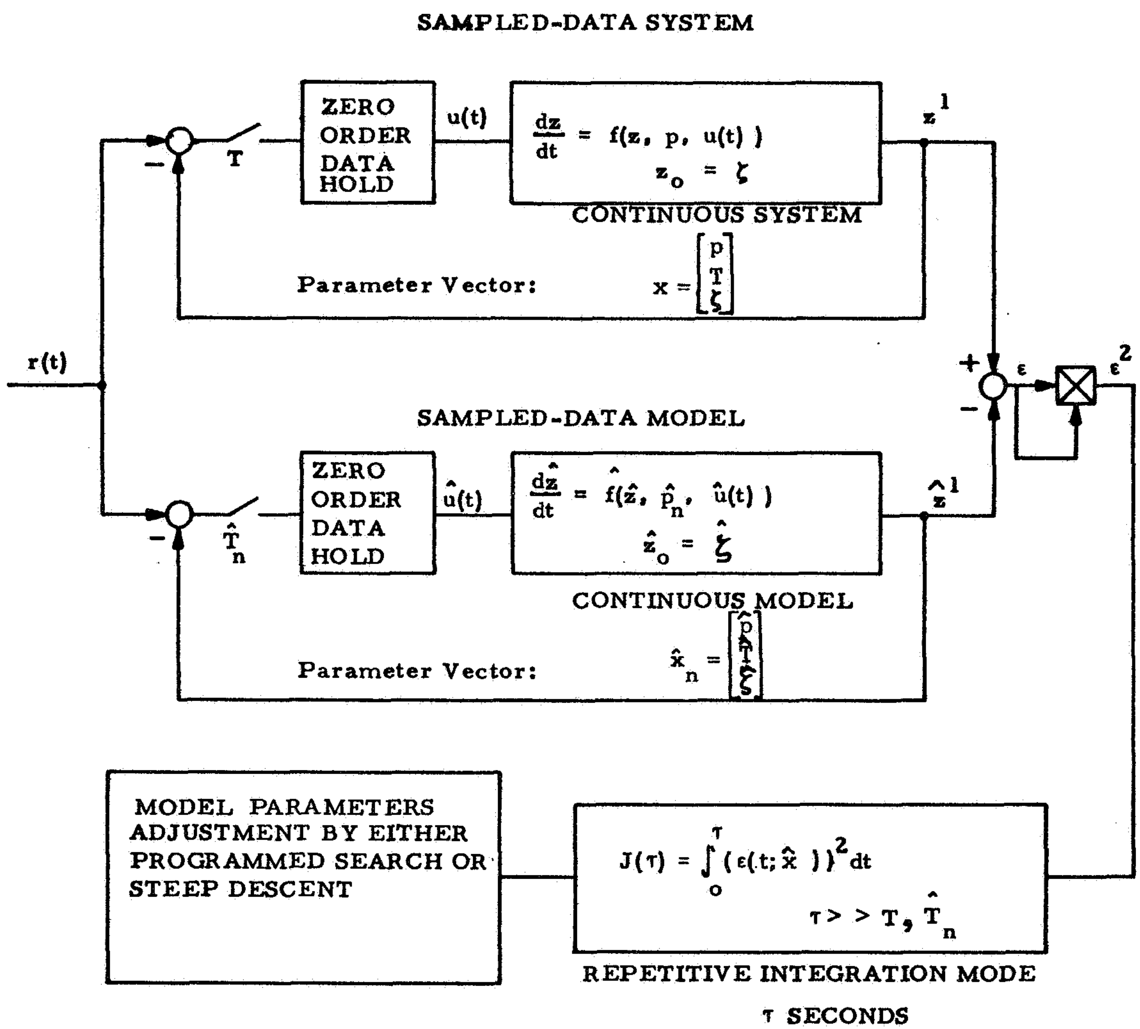


Figure 1.2 Parameter Estimation of a Sampled-Data System by Model Adjustment

assume that sampled-data system and sampled-data model are identical except for the sampling intervals.

(II) Using the estimation configuration of Figure 1.3:

(d) Study the application of stochastic approximation to the problem of estimating the sampling interval T as well as other parameters of the sampled-data system; i.e., obtain estimates $\hat{x}$ of the complete sampled-data system parameter vector x. Assume that the noise $n_1(t)$ corrupts the observations of the system state vector z(t).

(e) Study the effect on parameter estimation caused by introducing a random noise component into each of the parameters of the sampled-data system.

(III) Using data obtained from human operator experiments (Figure 1.4):

(f) Determine whether the human operator has a sampled-data property by employing stochastic approximation to obtain parameter estimates after constructing models to be used in the configuration of Figure 1.3.

(g) By using stochastic approximation, attempt to improve the best estimates of human operator models currently available in the literature.

1.5 Organization Of The Dissertation

This dissertation is organized into five chapters and several appendices. Chapter 1 gives the general problem statement, background material relevant to the study, objectives of the study, and

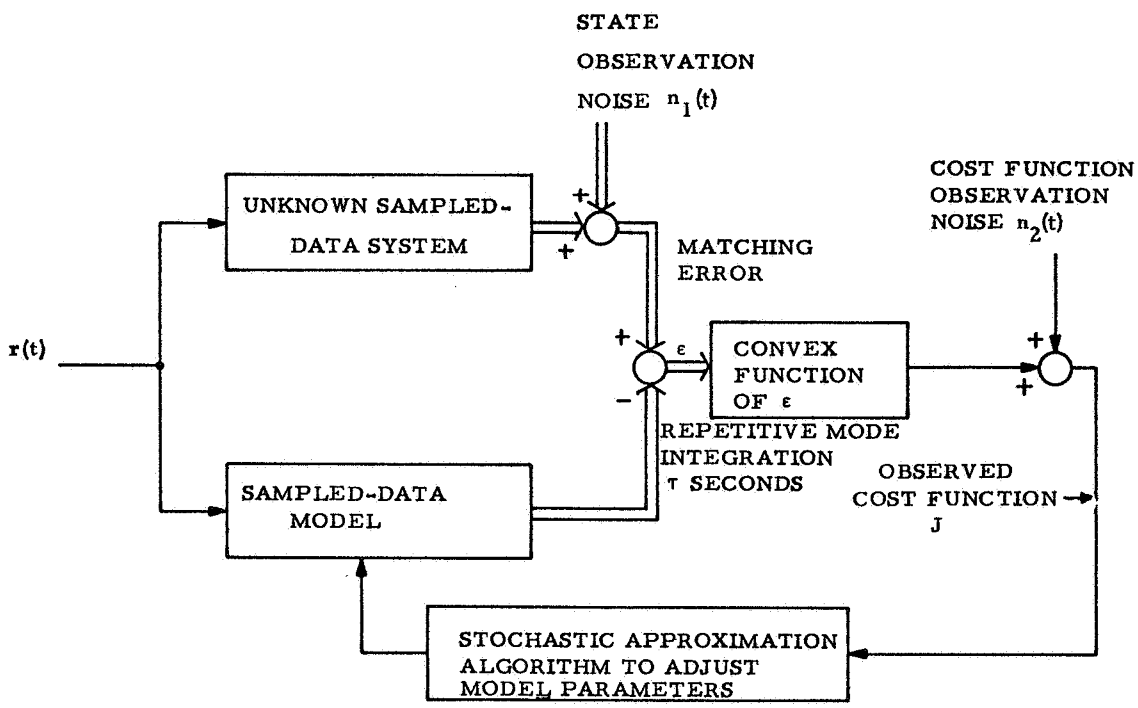
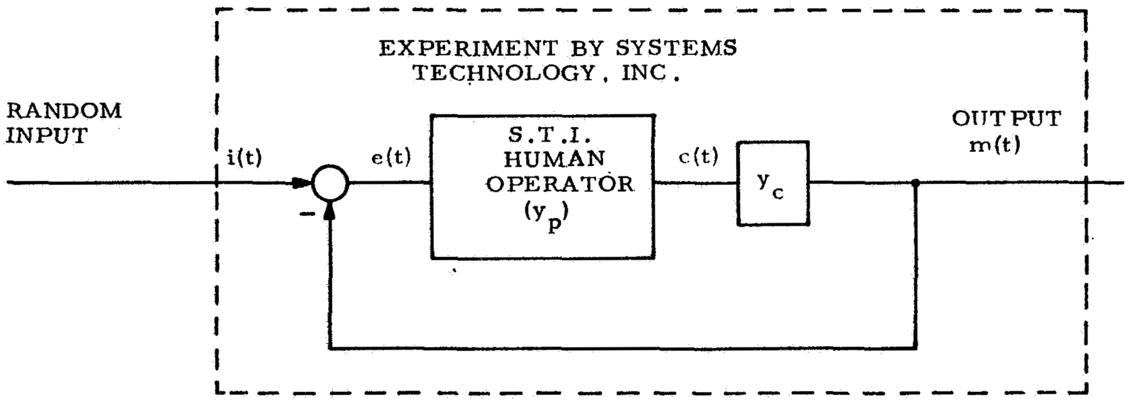


Figure 1.3 Parameter Estimation of an Unknown Sampled-Data System by Stochastic Approximation



AVAILABLE DATA: $i(kT_q)$, $e(kT_q)$, $c(kT_q)$, $m(kT_q)$;
T_q = 0.05 sec.

Figure 1.4 Human Operator Experiment Showing Quantized Data Points.

restrictions placed on the study. This chapter concludes with comments on the importance and applicability of the research and its influence on the current state of parameter estimation and human operator modeling.

Chapter 2 is concerned with estimating the sampling interval in noise-free systems. Starting with some additional definitions, a mathematical basis is developed for conditions under which identification of sampling intervals is possible in noise-free sampled-data systems. Simulation results are presented for both identification and estimation of sampling intervals. Two methods are used: Programmed search over a variable set of parameters, and iterative steep descent using the sensitivity difference equations of the sampling interval and other parameters.

Chapter 3 introduces the method of stochastic approximation for estimating parameters and presents a convergence theorem for the estimation problem indicated in Figure 1.3 together with the stochastic approximation algorithm to be used in subsequent studies.

Chapter 4 is concerned with the results of a variety of simulations involving parameter estimation by means of stochastic approximation. The system complexity ranges from noise-free linear systems to both noisy linear and nonlinear systems. In the noisy systems, all of the parameters, including the sampling interval, have random components. In addition, a discussion is given of the influence of the character of the input signal and observation noise on biasing the parameter estimates when either of these signals has nonzero mean value.

Chapter 5 presents the results of applying the stochastic approximation algorithm to the special problem of estimating human operator model parameters from actual human operator experimental data. The data were taken from compensatory tracking studies and were generated according to the arrangement of Figure 1.4.

1.6 Limitations Of The Study

A number of limitations apply to the broad objectives stated above. These restrictions fall into three categories: (1) Restrictions imposed by the estimation algorithm. (2) Restrictions imposed by the form of the model. (3) Restrictions imposed by the type of experiment performed to furnish the operator data.

1.6.1 Restrictions Imposed By The Estimation Algorithm

In this study three algorithms are employed for parameter estimation in sampled-data systems:

(a) Programmed search for the set of parameters which minimize the cost function equation (1.18). Reference Figure 1.2.

(b) Parameter sensitivity difference equations together with steep descent to minimize equation (1.18). Reference Figure 1.2.

(c) Stochastic approximation using equation (1.18) as the basis of the algorithm. Reference Figure 1.3.

While the first technique could conceivably be used in the actual case of noisy observations of the sampled-data system output; i.e., according to Figure 1.3, no convergence theorem for the parameter estimates has been developed for this application.

The second technique has been used for systems with noisy observations, however, no convergence theorem is available for this application either. Furthermore, the mathematical complexity associated with obtaining the difference equations for high order models is time-consuming and error-fraught.

The third technique, stochastic approximation is a method for estimating the parameters of systems under theoretical restrictions which, in practice, are often realizable. In general, the cost function must be convex, and must have a unique minimum. Also, the observation noise must have zero mean value and must be uncorrelated with either the outputs of the sampled-data system or the sampled-data model. If the cost function has local minima, then a preliminary search can be employed to identify them [30]. After that step, stochastic approximation can be used to improve the parameter estimates by using a suitable initial parameter estimate vector. Stochastic approximation has the advantage over the previously mentioned techniques that a convergence theorem for the parameter estimates is available. This theorem, to be proved in Chapter 3, shows that under the above restrictions on noise, assuming the unique minimum, and with the restrictions on system and model given in Section 1.4, that $\lim_{n\to\infty} E(\hat{x}_n - x)^2 = 0$, where $E(\cdot)$ is the expectation operator. In addition, for sampled-data systems, simulation results indicate that the driving signal, r(t) of Figure 1.3 should also have zero mean value. Simulation results corroborate analysis and indicate that if the mean value of the observation noise is not zero, then a bias in the parameter estimates will occur.

Other parameter estimation schemes were not tried because of the success enjoyed with stochastic approximation, and because of its suitability to the real-world modeling and parameter estimation problem.

1.6.2 Restrictions Imposed By The Form Of The Model

In connection with programmed search, it will be shown in Chapter 2 that the set of model parameters which minimize the cost function is not unique, but depends on the model chosen. Hence biased parameter estimates, may occur if the continuous system and continuous model do not agree in form and initial conditions and unless the properties of the data hold of the model agree with those of the sampled-data system. However, sensitivity of parameter estimates to model structure was not analyzed in general, although some numerical examples are given.

Likewise, in connection with the application of stochastic approximation (S.A.) it is clear that biased parameter estimates may occur if the form of the sampled-data model and initial conditions do not agree with the form of the sampled-data system and initial conditions. Furthermore, in the practical case where one is trying to estimate the parameters of an unknown sampled-data system from input - output data, neither the form, nor the initial conditions, of the differential equation of the continuous system, nor the properties of the data hold may be known. Under these circumstances, one concludes that biased estimates of parameters of the sampled-data system will be the rule. However, this is not a weakness of the stochastic approximation method; rather, it is due to uncertainty in the modeling. In an effort to overcome this restriction, the technique employed when using stochastic approximation to estimate the parameters of an unknown sampled-data system, was to first choose a closed-loop model, adjust the model parameters by S.A. and record the minimum cost function along with the minimizing parameter vector of the model. Other models were then tried and S.A. was used to adjust the parameters of each model. This procedure of modeling and subsequent parameter estimation was continued until the point of diminishing returns was reached. Examples of this procedure, used in connection with modeling input-output data from human operator experiments, are given in Chapter 5.

1.6.3 Restrictions Imposed By The Human Operator Tracking Experiment

For an actual application of the stochastic approximation method it was decided to use data from an experiment where a human operator controlled dynamic elements in a closed loop tracking situation as shown in Figure 1.4. The modeling technique outlined above was employed with considerable success. This is evidenced by the fact that by using stochastic approximation to adjust the parameters of a simple model of the human operator that a decrease in the cost function was obtained as compared to the best previous estimate published in the literature. Further decreases were realized when more complicated models were used. Despite this success, we must point out the limitations in estimates of the parameters of the human operator induced by the human operator tracking experiment. These are as follows:

(a) The operator performed a specific tracking task. The test results, and the parameter estimates derived from them, might have been different had the operator been performing a number of tracking tasks in some repetitive sequence.

(b) Because of the limited amount of test data used in the modeling and parameter estimation, no account is given of the operator's possibly time-varying behavior.

1.7 Applications Of This Dissertation

Stochastic approximation is a very general technique for estimating the parameters of sampled-data, as well as continuous control systems. While it is applied in this dissertation to the problem of estimating the sampling interval and other parameters of the human operator, it can just as well be applied to problems of parameter estimation in all sorts of continuous and sampled-data processes.

Also, the relatively large improvement (decrease in cost function) accomplished in this study by using stochastic approximation to adjust the parameters of one of the best current models of the human operator suggest the possible improvement to be realized in subsequent applications of this technique to the whole gamut of human operator modeling problems including multi-axis control.

CHAPTER 2

ESTIMATION OF SAMPLING INTERVALS AND OTHER PARAMETERS IN NOISE-FREE SAMPLED-DATA SYSTEMS

2.1 Introduction

This chapter presents the results of the initial phase of the investigation into ways for estimating the parameters of a closed-loop sampled-data system.

The configuration of Figure 2.1 is used and represents the estimation problem discussed in Chapter 1. In this chapter, the parameter estimates are obtained by either programmed search over the variable parameters of the model, or by iterative steep descent based on using the sensitivity difference equations of the variable parameters of the model. With either method, the purpose is to obtain the parameter vector $\hat{x}$ of the sampled-data model which minimizes the cost function

$$J\,(\tau;\, x,\, \hat{x},\, r(t)) = \int_0^T (z^1(t;\, x,\, r(t)) - \hat{z}^1(t;\, \hat{x},\, r(t)))^2\, dt \qquad (2.1)$$

where the notation is that given in Chapter, and where z^1 and $\hat{z}^1$ are the (scalar) outputs of system and model respectively. We will here define the minimizing vector $\hat{x}$ as the optimal estimate of the parameter vector x of the sampled-data system.

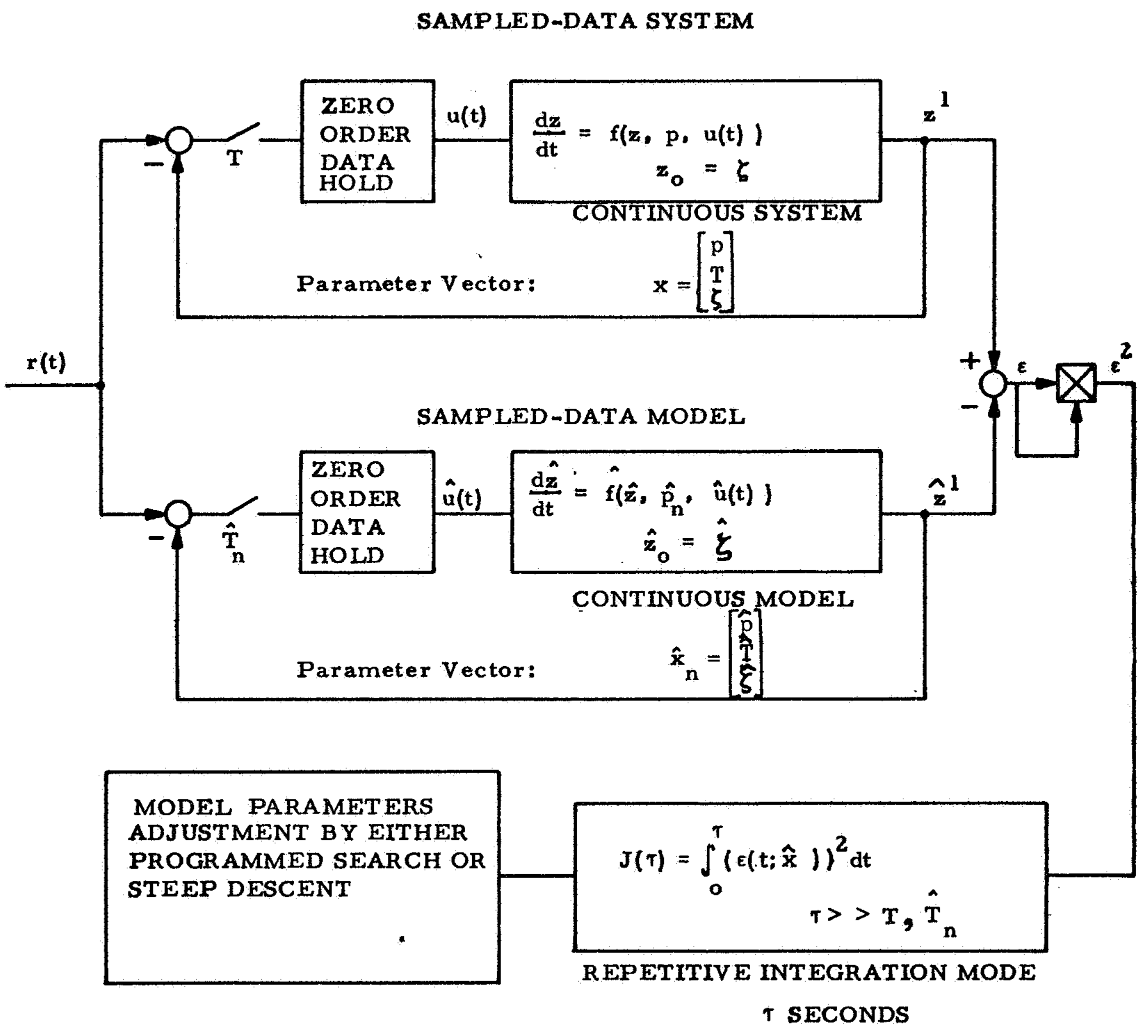


Figure 2.1 Parameter Estimation of a Sampled-Data System by Model Adjustment

2.2 Problems For Investigation

The problems attacked in this chapter are those which have been outlined in Section 1.4a, b, c. We assume that the estimator configuration of Figure 2.1 is used and that the continuous system

$$\frac{dz}{dt} = f(z, p, u(t)), \qquad z(t=0) = \zeta \tag{2.2}$$

and continuous model

$$\frac{d\hat{z}}{dt} = f(\hat{z}, \hat{p}, \hat{u}(t)), \qquad \hat{z}(t=0) = \hat{\zeta} \tag{2.3}$$

both have the continuity and differentiability properties described in Section 1.1. Further, we assume that the data holds of sampled-data system and sampled-data model (see Figure 2.1) are of zero order, and assume that all parameters p of the continuous system (1.1) are constant and that the sampling interval T of the sampled-data system is also constant. We assume that $r(t)$ is a suitable nonzero function and that the phase of the sampling of the model is adjustable so that the sampling of model and system can be made synchronous when $\hat{T} = T$.

In this section we seek to analyze the following problems:

(1) Assuming that the continuous system and continuous model have identical differential equations and that $(\hat{p}, \hat{\zeta})' = (p, \zeta)'$, then determine conditions under which the cost function (2.1) will have a unique minimum over the estimate $\hat{T}$ of the sampling interval T as $\hat{T}$ ranges over $(0 \le \hat{T} < \infty)$.

(2) Assuming that the system is either not modeled correctly, so that for all choices of $\hat{p}$ and $\hat{\zeta}$ and for all nonzero r(t) the functions $\hat{f}(\cdot)$ and $f(\cdot)$ are not the same, or, if it is modeled correctly, then $(\hat{p}, \hat{\zeta})' \neq (p, \zeta)'$, then determine whether the cost function (2.1) will have a minimum over $\hat{T}$ for $(0 \leq \hat{T} < \infty)$.

(3) Assuming that the form of the continuous model agrees with that of the continuous system, so that if $\hat{u}(t) = u(t)$ and $\hat{p} = p$ then

$$\hat{f}(\hat{z}, \hat{p}, \hat{u}(t)) = f(z, p, u(t)) \tag{2.4}$$

and further assuming that $(\hat{p}, \zeta)' = (p, \zeta)'$, then represent the resulting minimum value of (2.1) over $\hat{T}$ by

$$J_1 = \min_{\hat{T}} J(\tau; x, \hat{x}, r(t)) \tag{2.5}$$

Assuming next that $(\hat{p}, \hat{\zeta})' \neq (p, \zeta)'$, then represent the resulting minimum value of (2.1) over $\hat{T}$ by

$$J_2 = \min_{\hat{T}} J(\tau; x, \hat{x}, r(t)) \tag{2.6}$$

Develop an analytical relationship between J_1 and J_2.

2.3 Reference Mathematical Basis

The solutions to the above problems will be obtained after we first establish a reference mathematical basis for the identification of the unknown sampling interval T by means of programmed search and the estimation configuration of Figure 2.1. We will first need some additional definitions to those already given in Chapter 1. We assume the estimation configuration of Figure 2.1.

2.3.1 Additional Definitions

Definition 3: We say that we have an optimal estimate $\hat{T}$ of an unknown sampling interval T when the minimization of the cost function (2.1) has been carried out over some restricted set of candidate models and parameter vectors denoted by $\{\hat{f}(\hat{z}, \hat{p}, \hat{u}(t)); \hat{x}\}_{r_v}$ An example of a restricted set of models is the set of second order systems with variable coefficients, variable initial conditions, and variable transport lag together with specified sets of these parameters.

Definition 4: We say that we have an optimum estimate $\hat{T}$ of an unknown sampling interval T when the minimization of the cost function (2.1) has been carried out for all possible choices of candidate models, parameter vectors and initial conditions. (Note: From definition 1, Chapter 1, it is clear that the above optimum estimate for the noise-free case considered in this chapter is the same as the identification of Chapter 1.)

2.3.2 The Differential Equation Of The Continuous System

For our results in sampling interval identification, we will require a unique solution of (2.1) for specified parameter vector p, initial condition vector ζ, and control vector function u(t). In addition, for the treatment of the deterministic gradient method in this chapter, as well as the treatment by stochastic approximation in Chapter 3, we will require that the partial derivatives of the solution of (2.1), with respect to parameters and initial conditions, exist and be continuous. The following theorem is essentially stated in [32]. The extension to include controls is stated in [33].

Theorem 2.1 [32, 33, 80]: Let Z^n, and P^h be open sets in the Euclidian spaces E^n and E^h respectively. Let (T_1, T_2) be an open t interval. Let $u(t)$ be a piecewise continuous function from (T_1, T_2) into E^r. For any t in (T_1,T_2), define the vector of values of $u(t)$ by u ; $u \epsilon E^r$. Consider the

$$\frac{dz}{dt} = f(t, z, p, u(t)); \qquad z(t=0) = \zeta , \tag{2.7}$$

where z and f are n vectors, p is a constant parameter vector belonging to P^h, and ζ is a constant initial condition vector belonging to Z^n. Suppose the functions f^i, $\partial f^i/\partial z^g$, and $\partial f^i/\partial p^j$ are continuous from $(T_1,T_2) \times Z^n \times P^h \times E^r$ into E^1 $(i, g = 1, 2, \ldots, n)$, $(j = p, 2,\ldots,h)$. Let p_0 belong to P^h and t_0 belong to (T_1, T_2). Let $u_0(t)$ be a chosen piecewise continuous function taking its vector of values u_0 in E^r. Choose a fixed $p = p_0$. Let ψ be the solution of (2.7) on a t interval $(t_1 \le t \le t_2)$ belonging to (T_1, T_2). Then there exists a $\delta > 0$ such that for any (τ, ζ, p, u) belonging to a domain Q_1, where

$$Q_1: \; t_1 \leq \tau < t_2, \; \|\psi(\tau) - \zeta\| + \|p - p_0\| + \|u(\tau) - u_0(\tau)\| < \delta , \tag{2.8}$$

there exists a unique solution ϕ of (2.7) on $t_1 \le t \le t_2$, where (t_1, t_2) is a subset of (T_1, T_2), satisfying $\phi(0; p, u(t), \zeta) = \zeta$.

Moreover, ϕ is of class C^1 on Q_2; i.e., the partial derivatives $\partial\phi^i/\partial z^g$, $\partial\phi^i/\partial t$, and $\partial\phi^i/\partial p^j$ are continuous functions on the $(n + h + r + 2)$ dimensional domain Q_2, where

$$Q_2: \ (t_1 < t < t_2) \quad \text{and } (\tau, \zeta, p, u) \text{ belong to } Q_1.$$

<u>Remark 1:</u> The theorem simply states that if a solution exists, then it is unique and has the properties described.

<u>Remark 2:</u> The continuous model $\hat{f}(\cdot)$ is assumed to be identical in form to the continuous system $f(\cdot)$, hence the same theorem applied to it also.

<u>Remark 3:</u> The existence and continuity of the partial derivatives $\partial\phi^i/\partial t$ (of the solution) will be required later in this chapter when we treat dynamic sensitivity difference equations and employ the gradient search technique to obtain parameter estimates.

<u>Remark 4:</u> The existence and continuity of the partial $\partial\phi^i/\partial z^g$ implies the existence and continuity of the partials $\partial\phi^i/\partial\zeta^g$ with respect to initial conditions [80]. The existence and continuity of the latter as well as the existence and continuity of the partials $\partial\phi^i/\partial p^j$ will be required when treating the estimation of the sampling interval and other parameters of the sampled-data system by means of the sensitivity difference equations and gradient technique later in this chapter. The same comments apply to the treatment of the estimation problem by stochastic approximation; this will be considered in Chapter 3.

<u>Remark 5:</u> When (2.7) is a linear system, the above results are global; i.e., they hold for all p, ζ, and choice of piecewise continuous

control function u(t) [33].

Remark 6: The proof (Reference [33]) requires that the components of z, p, and u(t) lie in closed balls in Z^n, P^h and E^r respectively. Closed balls are compact and convex [67], hence p must belong to a compact convex set.

The above theorem will now be applied to the problem of identifying an unknown sampling period.

2.3.3 Theorems For The Identification Of A Sampling Period When Using Noise-Free Model-Matching

Consider the sampled-data system and sampled-data model in the model-matching configuration of Figure 2.1 where each consists of a periodic sampler, data-hold, and continuous dynamic system in a closed loop configuration with negative feedback from the scalar output variable. When the sampling interval T is the only unknown, we have the following theorems:

Theorem 2.2 Assume the model-matching configuration of sampled-data system and sampled-data model described by Figure 2.1. Assume that the continuous system and continuous model are of identical form, with equal parameter vectors, exclusive of the sampling intervals T and $\hat{T}$, and with equal initial condition vectors. Assume that the sampling pulse train of the sampled-data model is given by

$$p(t;\, T) = \sum_{k_2=0}^{\infty} \delta(t - k_2 T - \gamma T) \tag{2.9}$$

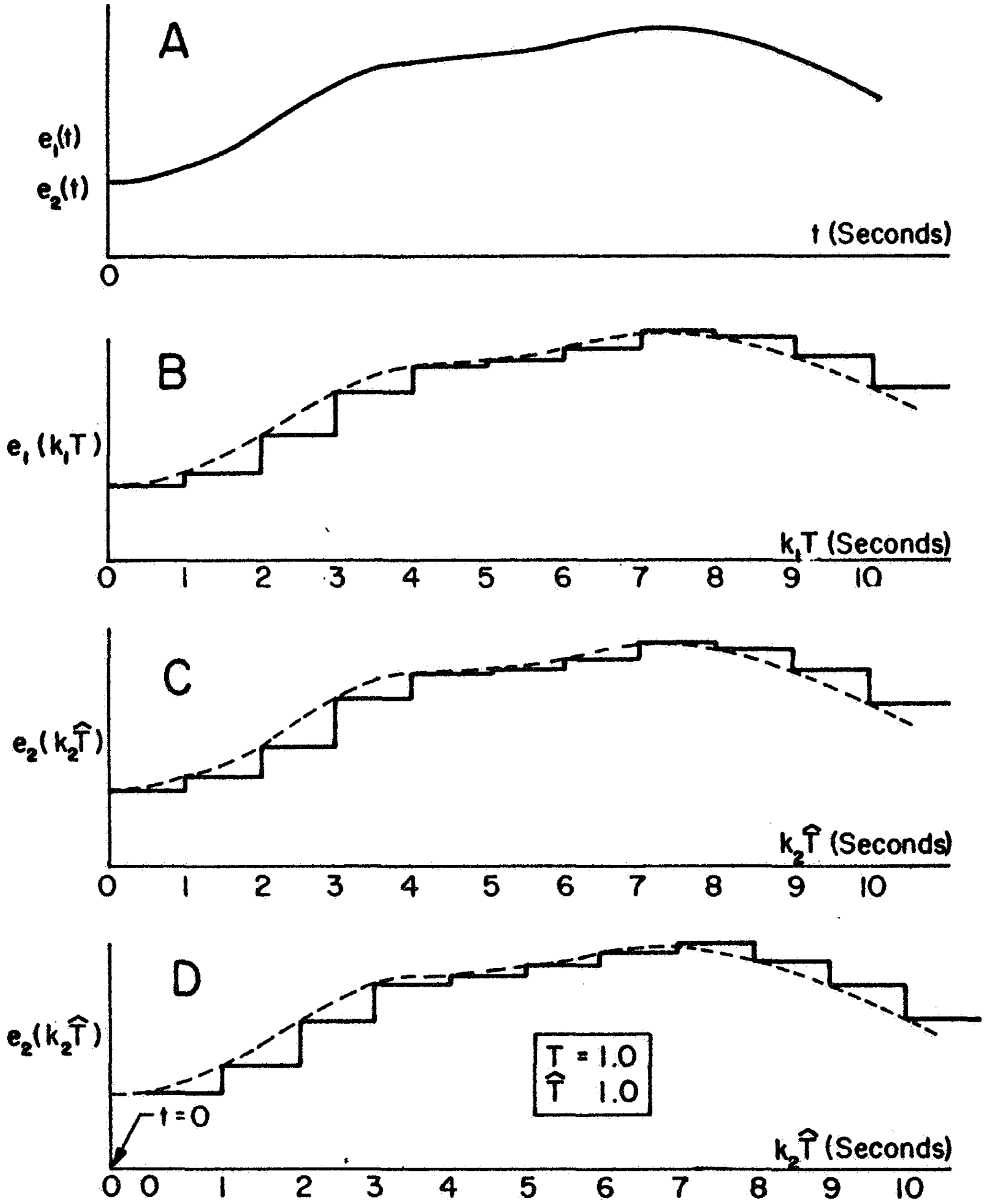
where $[-1/2 \le \delta \le 1/2]$ and $\delta(\cdot)$ is the Dirac delta function, and k_2 is an integer 0, 1, 2, ..., so that it is possible to make the sampling instants of sampled-data system and sampled-data model synchronous when $\hat{T} = T$ by adjusting the phase by $\pm|\gamma|\hat{T}$. Assume that r(t) is a non-zero piecewise continuous function, and assume that $f(\cdot)$ and $\hat{f}(\cdot)$ are as described in Theorem 2.1. Let $\tau \gg T$, and $\tau \gg \hat{T}$. Then necessary and sufficient to identify the unknown sampling interval T is that (2.1) is zero for $\tau > 0$; i.e.,

$$J(\tau;\, x,\, \hat{x},\, r(t)) = \int_0^\tau (z^1(t;\, x,\, r(t)) - \hat{z}^1(t;\, \hat{x},\, r(t)))^2 dt = 0 \quad (2.10)$$

and T is given by the $\hat{T}$ for which (2.10) is true.[1]

Proof (Sufficiency): Suppose $\hat{T} = T$ and for $[-1/2 \le \gamma \le 1/2]$ that the impulse trains are synchronous, yet $J(\tau;\, x,\, x,\, r(t)) \ne 0$. From the hypothesis, the solutions of the differential equations of the continuous system and continuous model, (2.2) and (2.3) respectively, are deterministic and identical when started from identical initial conditions and when the system and model are noise free. Consider the sampling intervals following the initial output from the hold devices. These may be visualized by reference to Figure 2.2. (The data holds have been taken as zero-order.) The initial sampling impulses are coincident. From the

[1] Since T and $\hat{T}$ are the only parameters of interest here, we will here designate $z(t;\, x,\, r(t))$ by $z(t;\, T)$. Similarly, for $\hat{z}(t;\, \hat{x},\, r(t))$ we will use $\hat{z}(t;\, \hat{T})$.



Notes: 1) Sampling occurs synchronously between B and C.

2) Sampling does not occur synchronously between B and D.

Figure 2.2 Examples of Synchronous and Non-Synchronous Sampling uniqueness theorem, the outputs $z(t; T)$ and $\hat{z}(t; \hat{T})$ must be identical for identical initial conditions and parameters since they are the solutions of identical differential equations; i.e., the solutions of (2.2) and (2.3) are

$$z(t; T) = \int_0^t f(z, p, u(k_1T))d\sigma + \zeta \tag{2.11}$$

and

$$\hat{z}(t; \hat{T}) = \int_0^t \hat{f}(\hat{z}, \hat{p}, \hat{u}(k_2\hat{T}))d\sigma + \zeta \tag{2.12}$$

respectively, where k_1 and k_2 are integers belonging to the sequence (0, 1, 2, ...). Now recall the feedback relationships

$$u(k_1T) = r(k_1T) - z^1(k_1T; T) \tag{2.13}$$

and

$$\hat{u}(k_2\hat{T}) = r(k_2\hat{T}) - \hat{z}^1(k_2\hat{T}; \hat{T}) \tag{2.14}$$

Since $k_1 = k_2$, and $\hat{T} = T$, then $z^1(k_1T; T) = \hat{z}^1(k_2\hat{T}; \hat{T})$ again from the uniqueness of the solutions. (This is clear if we consider that both systems are started together at $k_1T = 0 = k_2\hat{T}$.) Also, $r(k_1T) = r(k_2T)$; hence

$$u(k_1T) = \hat{u}(k_2\hat{T}), \tag{2.15}$$

Thus,

$$\|z(t; T) - \hat{z}(t; \hat{T})\| \leq \int_0^t \|f(z,p,u(k_1T)) - \hat{f}(\hat{z},\hat{p},\hat{u}(k_2\hat{T}))\| d\sigma = 0. \tag{2.16}$$

as a consequence of the uniqueness theorem. But $\|\cdot\|$ cannot be < 0, hence

$$\| z(t;\, T) - \hat{z}(t;\, \hat{T}) \|^2 = 0 \tag{2.17}$$

which implies that each component of the vector $z(k_1T;\, T) - \hat{z}(k_2\hat{T};\, \hat{T})$ is zero. Therefore, from (2.1)

$$J(\tau;\, x,\, \hat{x},\, r(t)) = \int_0^{\tau} (z^1(t;\, T) - \hat{z}^1(t;\, \hat{T}))^2 dt = 0 \tag{2.18}$$

and this contradicts the assumption that $J(\tau;\, x,\, \hat{x},\, r(t)) \neq 0$ for all $t \in [0, \tau]$, and for $\tau > 0$ and both T and $\hat{T} \ll \tau$.

<u>Necessity:</u> Suppose, from (2.1), that (2.13) holds, but $T \neq \hat{T}$. Then, from (2.13),

$$z^1(t;\, T) - \hat{z}^1(t;\, \hat{T}) = 0 \qquad \text{(a.e.)} \tag{2.19}$$

for $t \in [0, \tau]$ where $\tau \gg T,\, \hat{T}$. But $z^1(t;\, T)$ and $\hat{z}^1(t;\, \hat{T})$ are respectively the first components of the solution vectors of (2.2) and (2.3) for identical initial conditions and parameters, but with possibly different control signals $u(k_1\, T)$ and $u(k_2\hat{T})$. Note: $\hat{T} \neq T$ implies $\hat{u}(k_2\hat{T})$ does not always equal $u(k_1T)$. From the hypothesis on $r(t)$ we know that $z^1(t;\, T)$ and $z^1(t;\, T)$ cannot be zero on the entire interval $[0, \tau]$. From the uniqueness theorem and the hypothesis on the adjustability of the phase of the impulse train of the model with reference to the impulse train of the system sampler, (2.9), this means a contradiction: That is, assuming identical initial conditions, then the hypothesis of (2.18) can be rewritten as

$$J(\tau;x,\hat{x},r(t)) = \int_0^T\left[\left(\int_0^T\left(f(z,p,u(k_1T)) - \hat{f}(\hat{z},\hat{p},\hat{u}(k_2\hat{T}))\right)dt\right)^1\right]^2 dt=0 \tag{2.20}$$

where $(\cdot)^1$ here indicates the first component (output) of the difference of the solution vectors. Then (2.20) implies that the integrand is zero

$$\int^T(f(z,\ p,\ u(k_1T)) - \hat{f}(\hat{z},\ \hat{p},\ \hat{u}(k_2\hat{T})))dt = 0 \tag{2.21}$$

But since each differential equation (2.2) and (2.3), has a unique solution for a particular $u(t)$, (2.21) implies that $u(k_1T) = \hat{u}(k_2\hat{T})$ and therefore that $k_1T = k_2\hat{T}$, and hence that $\hat{T} = T$, since we start with $k_1 = k_2$ and the same initial data and parameters.

Theorem 2.3: Assuming the hypothesis of Theorem 2.2, then (2.1)

$$J(\tau;\ x,\ \hat{x},\ r(t)) \neq 0 \tag{2.22}$$

on a $\hat{T}$ interval; i.e., $J(\tau;\ x,\ \hat{x},\ r(t))$ is zero for one value of $\hat{T}$ only.

Proof: This follows directly from the uniqueness of the solutions of (2.2) and (2.3). First, the initial conditions and the parameters of the sampled-data system and the sampled-data model are the same except possibly $\hat{T} \neq T$. Start at $t = 0 = k_1T = k_2\hat{T}$. The solutions can be identical only if $\hat{T} = T$, and for no other value of $\hat{T}$. Hence, there is no neighborhood of $\hat{T}$ for which $J(\tau;x,\ \hat{x},\ r(t))$ can be zero for the above construction.

Conjecture: When (2.2) and (2.3) are each linear systems, and the parameter vectors and initial condition vectors are respectively equal, then $J(\tau; x, \hat{x}, r(t))$ is convex in T. A number of demonstrations of this conjecture are given in the sequel.

2.4 Simulation Results for Programmed Search

Experimental digital studies were made to record the cost function $J(\tau; x, \hat{x}, r(t))$ as a function of the various parameters of the continuous model for the case of close model matching and also for the case of poor model matching. Transfer functions used are given in Table 2.1.

Table 2.1: Transfer Functions Of Continuous System And Continuous Model.

Programmed Search For Optimal Estimate Of T.		
System	Model	Figure Number
$\frac{1.0}{s}$	$\frac{\hat{K}}{s}$	2.3 2.4
$\frac{(s+2)}{s(s+1)}$	$\hat{K}\frac{(s+2)}{s(s+1)}$	2.5 2.6
$e^{-0.1s}\frac{(s+2)}{s(s+1)}$	$\hat{K}\frac{(s+2)}{(s+1)}$	2.7
$e^{-0.1s}\frac{(s+2)}{s(s+1)}$	$\hat{K}e^{-0.1s}\frac{(s+2)}{s(s+1)}$	2.8
$\frac{(s+2)}{s(s+1)}$	$\frac{\hat{K}}{s}$	2.9
Note: All initial conditions are zero.		

Model parameters which were varied included sampling interval T, gain, and transport lag. The simulations used impulse sampling and zero-order data holds. The sampling interval was held constant over each iteration interval (τ). The sampling instants were synchronous when $\hat{T} = T$ in all cases. It was found that non-synchronous sampling, when $\hat{T} = T$, had very little effect on the graphical results, and therefore these results are not reported here.

Figures 2.3 and 2.5 verify the Identification Theorems. These figures also show that when the system and model agree in form but differ by gain, then the cost curve is minimized at some $\hat{T}$ other than $\hat{T} = T$. This is also the case when the form of the model does not match the system, as in the case for Figures 2.7 and 2.9. Note, in Figure 2.7, that the presence of a transport lag in the system (but not in the model) causes a bias in the estimate of T.

Figure 2.9 shows the effect of a large mismatch between continuous system and continuous model. While the cost curves are convex, the relatively shallow minimum indicates the mismatch.

2.5 Iterative Gradient Search

Again, consider the noise-free modeling scheme of Figure 2.1. Assume a zero-order data hold and periodic impulse sampling with unknown period T and that the form and order of the continuous system is known; however, the coefficients of the differential equation of that system must be estimated. The sampling interval T is unknown and it is desired to develop a method for determining an

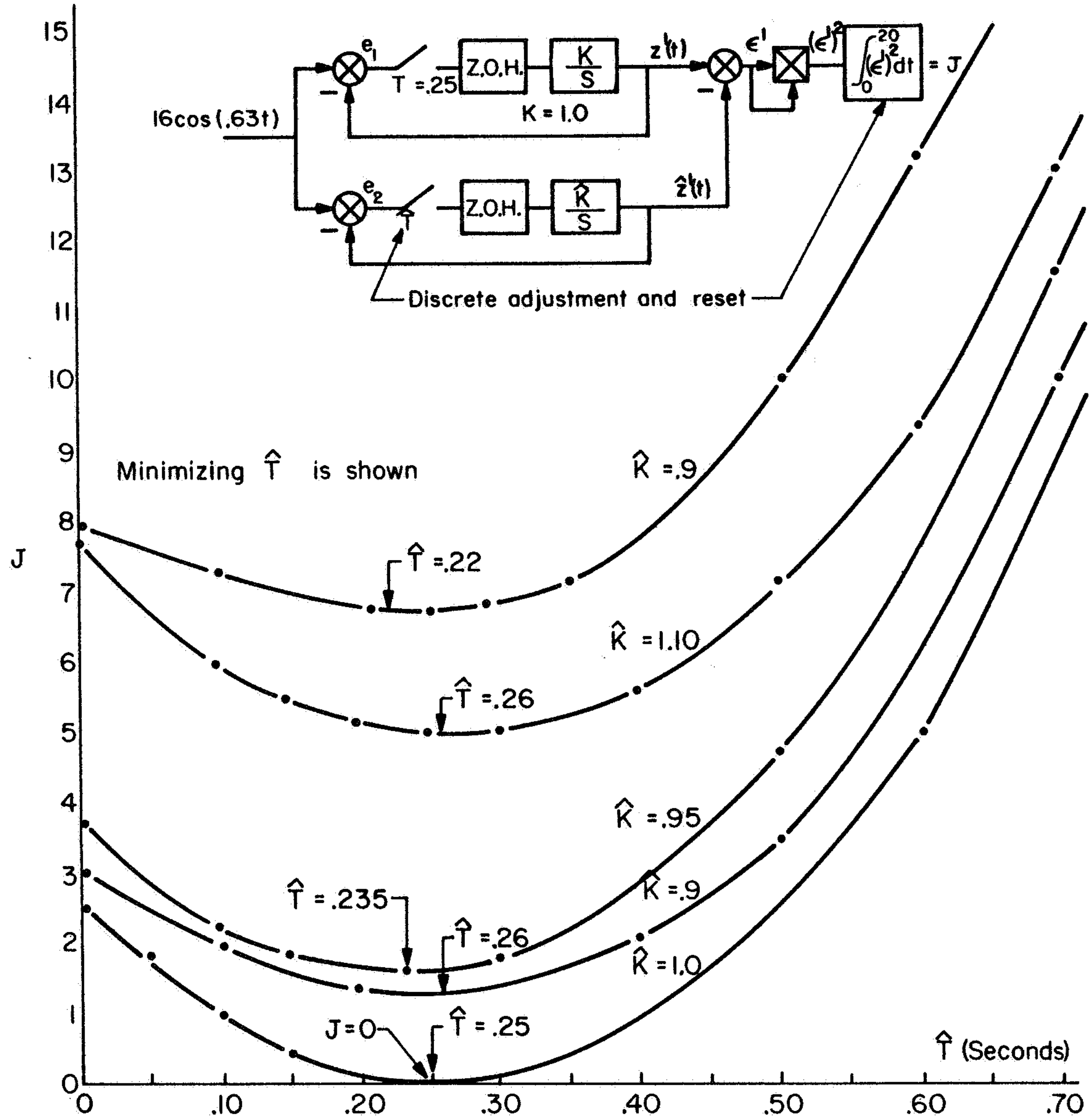


Figure 2.3 Programmed Search For T - First Order System - Model Match

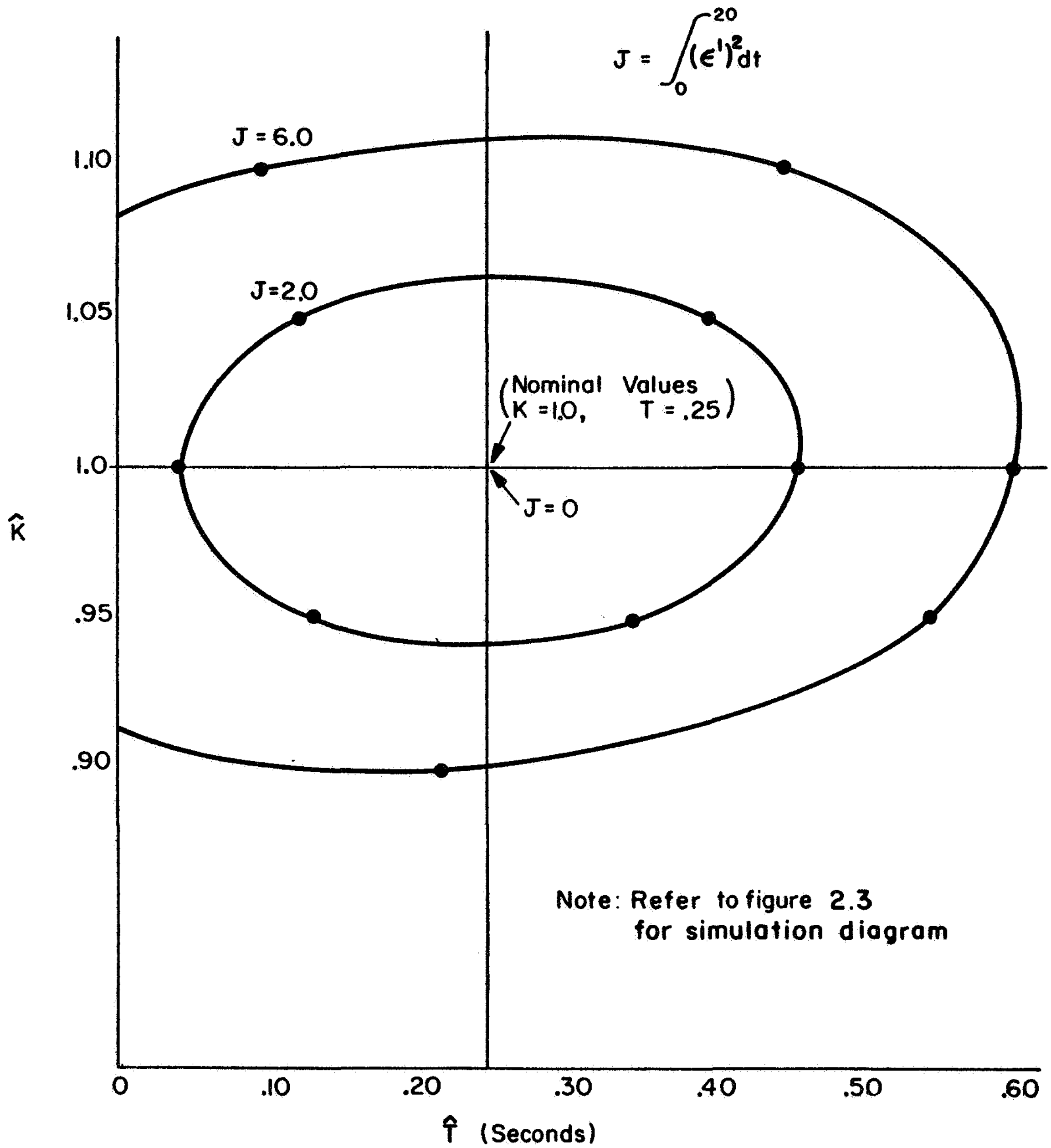


Figure 2.4 Constant Cost Contours - First Order System Matched By First Order Model

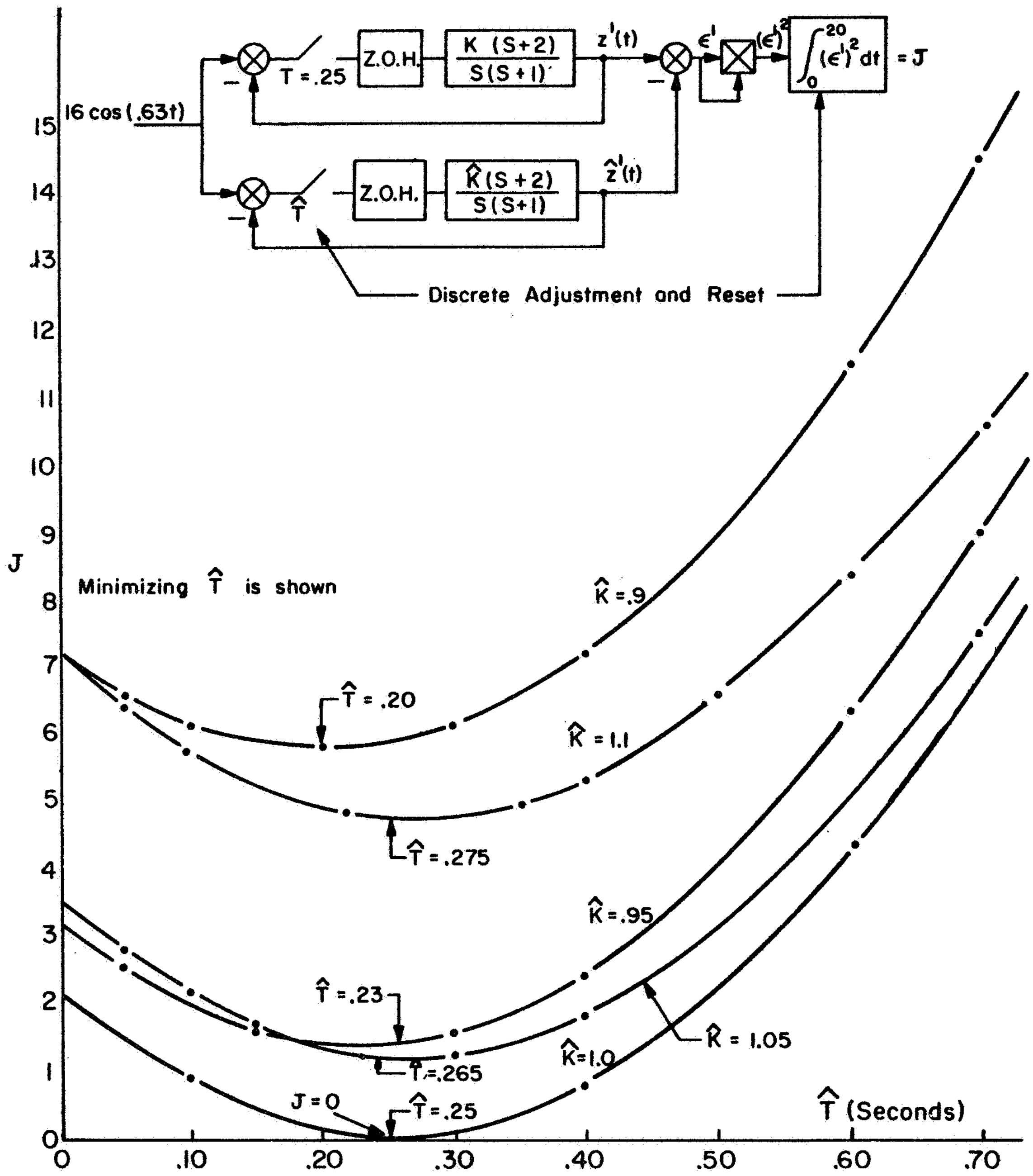


Figure 2.5 Programmed Search For T. Second Order System And Model.

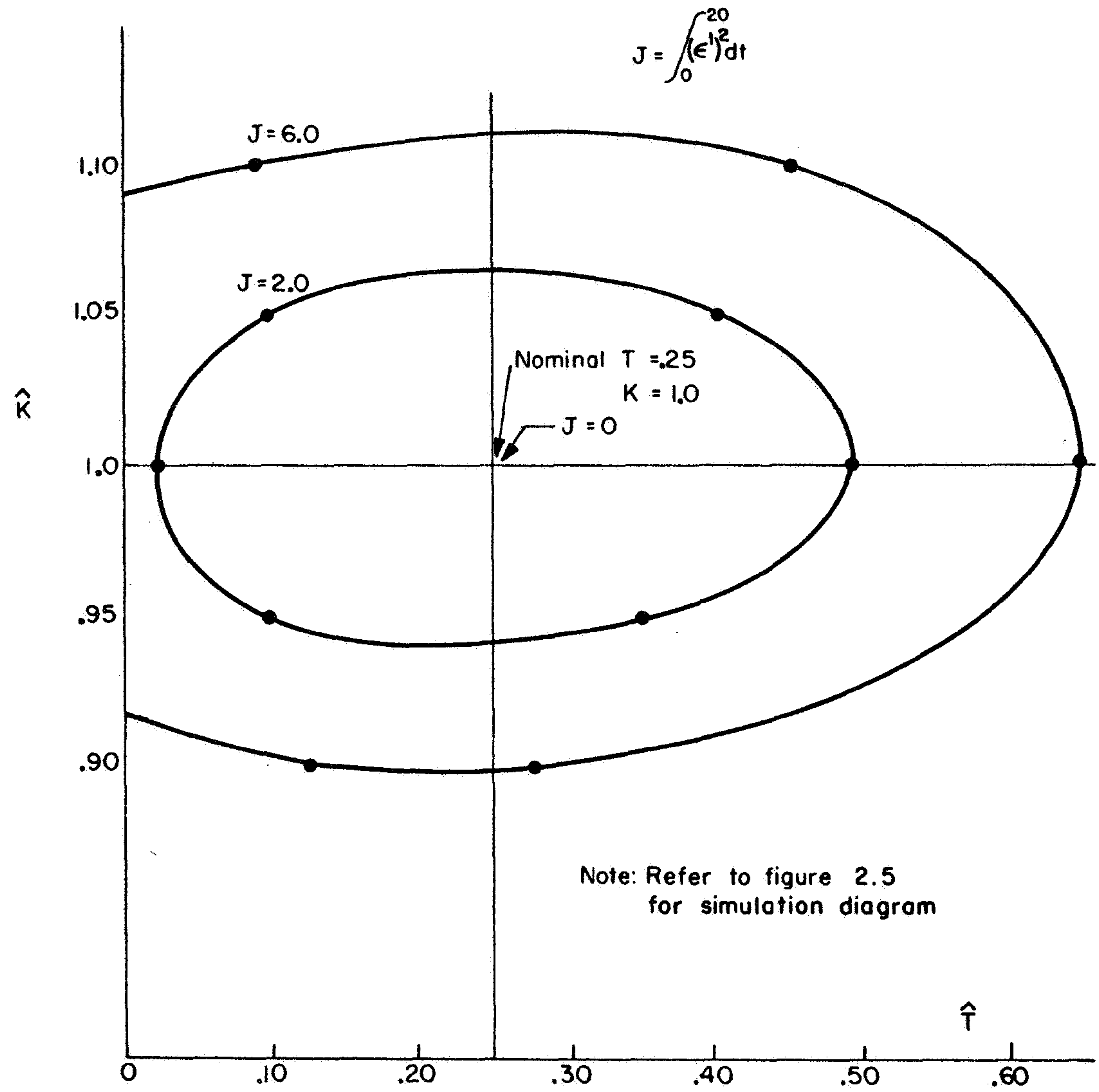


Figure 2.6 Constant Cost Contours-Second Order System Matched By Second Order Model

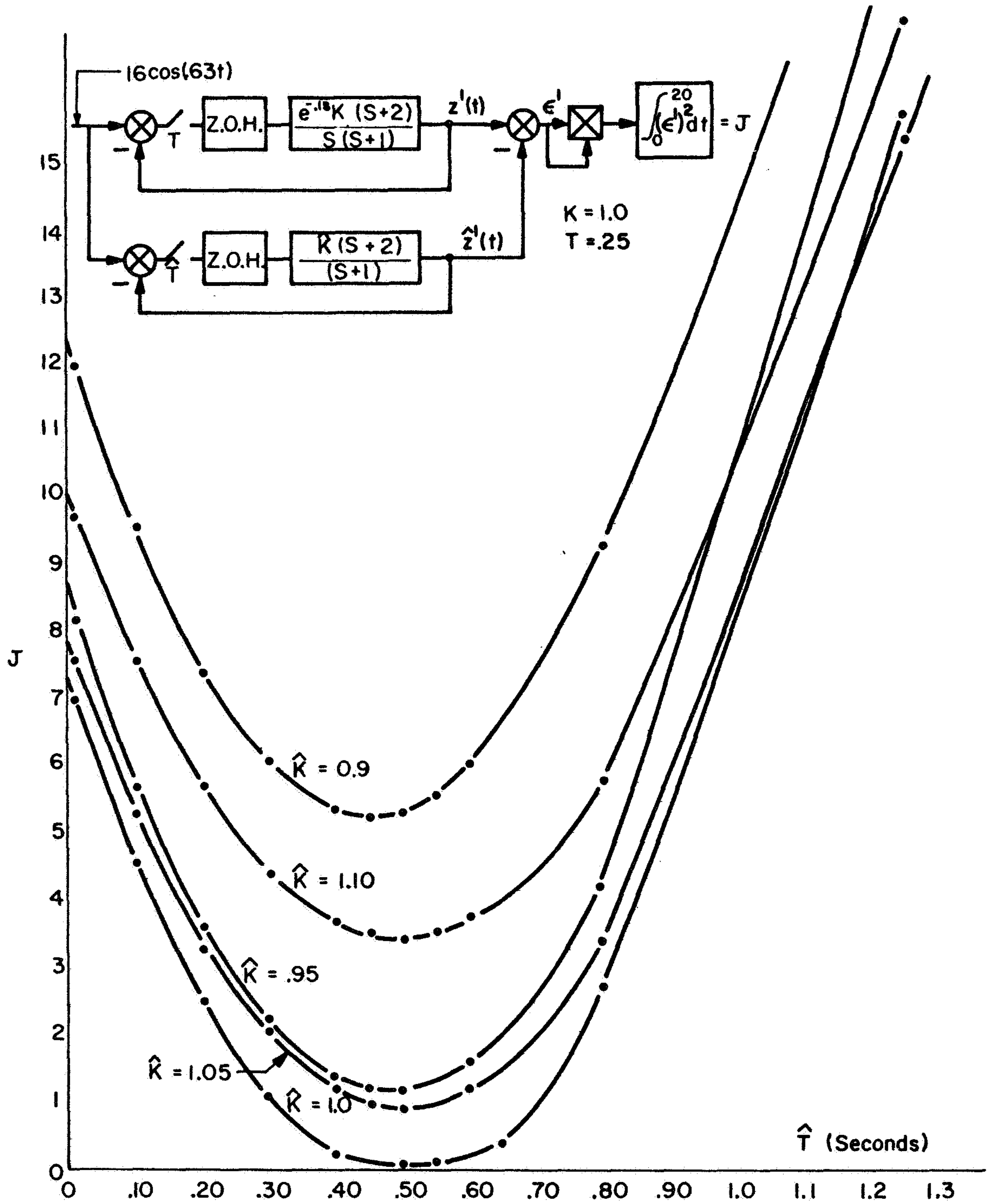


Figure 2.7 Programmed Search For T. System With Transport Lag-Model Without Transport Lag.

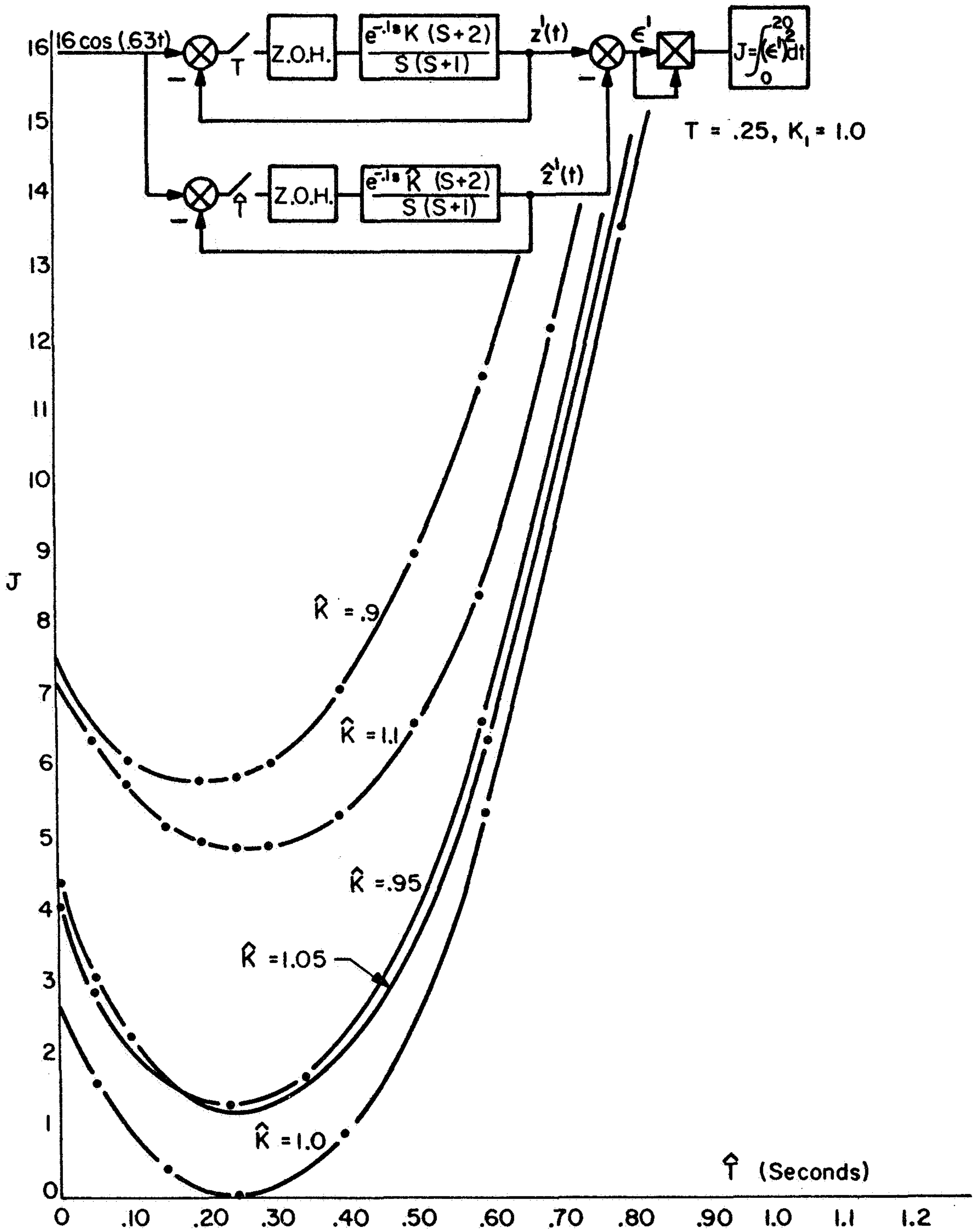


Figure 2.8 Programmed Search For T. Both System And Model Have Transport Lag.

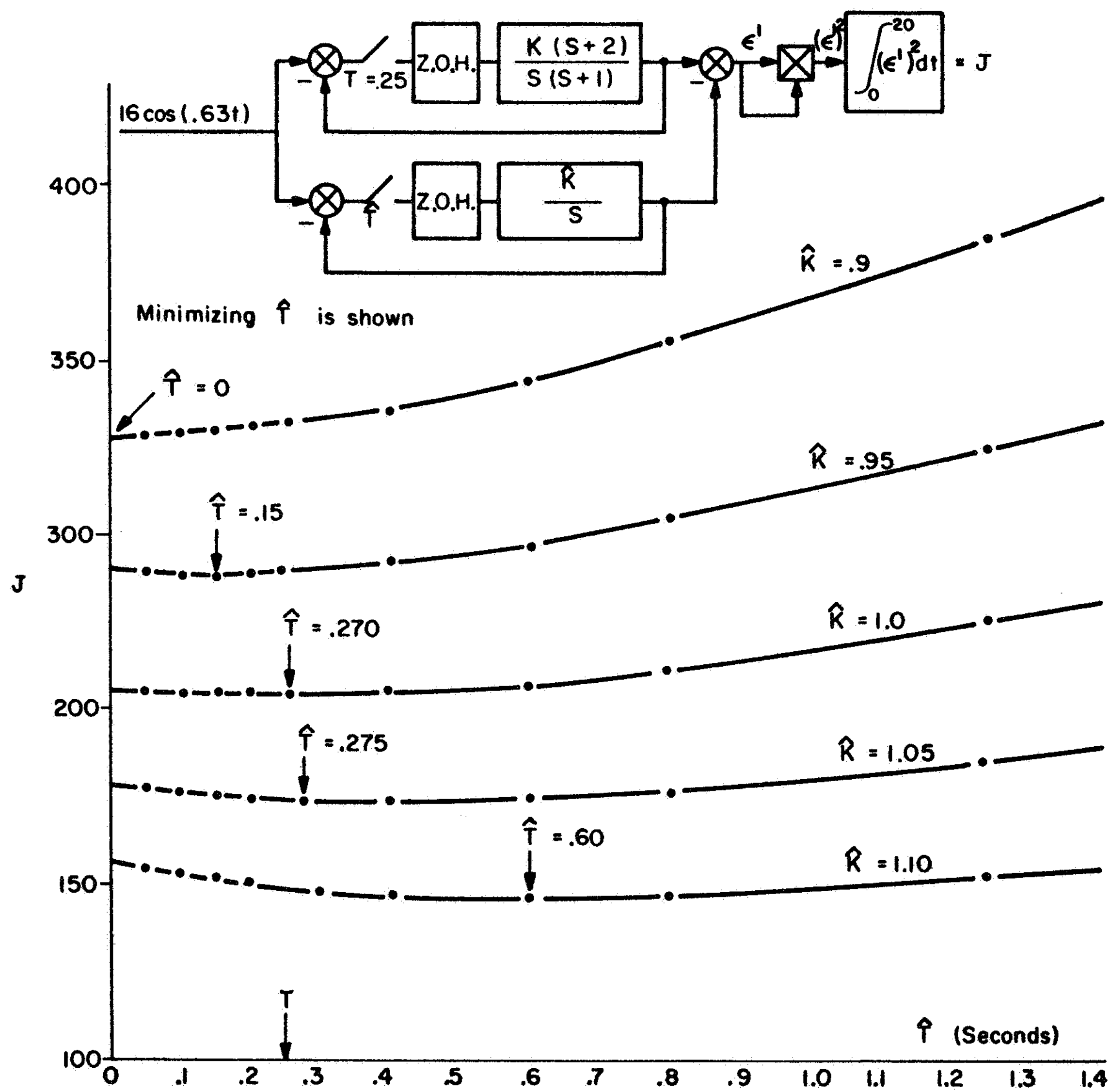


Figure 2.9 Programmed Search For T. Mismatch Of Second Order System By First Order Model

estimate of T as well as other system parameters by employing a gradient search procedure. As before, it is assumed that the system is noise-free and that only the output variable of the system is available. A discrete gradient method will be used in order to avoid the mathematical difficulty encountered when a gradient operation is attempted on either a time-varying scalar or vector [34]. The procedure will be to find the gradient of the cost function with respect to the variable model parameters and then increment each of the model parameters by an amount proportional to the gradient in order to eventually minimize the cost function. The model parameters and the sampling interval $\hat{T}$ are varied, as necessary, only at the end of each iteration cycle and are then held fixed during the next iteration cycle. While discrete gradient adjustment techniques have been used previously for model coefficient adjustment and subsequent system parameter identification [34], the extension to the problem of determining a unknown sampling frequency has not been previously reported. The sampling interval global sensitivity function which is employed was defined and discussed by Bekey and Tomovic [8].

From (2.1) the scalar cost function is

$$J(\tau;x,\hat{x},r(t)) = \int_0^{\tau} (z^1(t;x,r(t)) - \hat{z}^1(t;\hat{x},r(t)))^2 dt \qquad (2.23)$$

Fixing τ and x, (2.23) will be designated by $J(\hat{x}, r(t))$. Note that z^1 and $\hat{z}^1$ are the scalar output variables of the nth order system and the nth order model respectively. We apply the gradient operator (with respect to the sampled-data model adjustable parameter vector $\hat{x}$) to J in order to obtain the m dimensional gradient vector

$$\nabla_{\hat{x}}\left[J(\hat{x}, \quad r(t))\right] = \int_0^T \nabla_{\hat{x}}\left[z^1(t;x,r(t)) - \hat{z}^1(t;\hat{x},r(t))\right]^2 dt \quad (2.24)$$

corresponding to the components of the m dimensional model parameter vector

$$\hat{x} = [\hat{p}, \hat{T}, \hat{\zeta}]' \quad (2.25)$$

Recall from Chapter 1, that $m = (h+1+n) \leq 2n+1$. The components of $\hat{x}$ will then be adjusted in accordance with the sign and magnitude of the components of (2.24) and the iteration over $[0,\tau]$ will begin again.

Two distinct methods of calculating the components of (2.24) will be described. The first is an approximate method [34] yielding the discrete approximation to the i^{th} component of the parameter vector $\hat{x}$ for the j^{th} iterative computation of the parameter vector. With appropriate notational simplification, this is given by

$$\frac{\partial J}{\partial \hat{x}^i}(\hat{x}_j, r(t)) \simeq \frac{J(\hat{x}_j^{\,1}, \ldots, \hat{x}_j^{\,i-1}, \hat{x}_j^{\,i} + \Delta\hat{x}_j^{\,i}, \ldots, \hat{x}_j^{\,m}) - J(\hat{x}_j)}{\Delta\hat{x}_j^{\,i}} \quad (2.26)$$

$$(i = 1, 2, \ldots, m)$$

Note that if the parameter vector is m dimensional, then m+1 computations of (2.26) are required. This method is well-suited to hybrid computation.

The second method, better suited to strictly iterative mode analog computation, will extend use of the discrete sensitivity difference equations as defined by Bekey and Tomovic [8]. The development leading to them is as follows: Perform the differentiation indicated by (2.24) to obtain

$$\nabla_{\hat{x}}[J(\hat{x},r(t))] = -2\int_0^T (z^1(t;x,r(t)) - \hat{z}^1(t;\hat{x},r(t)))\nabla_{\hat{x}}\left[\hat{z}^1(t;\hat{x},r(t))\right] dt \quad (2.27)$$

Details of calculating the vector $[\nabla_{\hat{x}}\, \hat{z}^1(\,\cdot\,)]$ will subsequently be discussed. We first point out that the iterative adjustment procedure is carried out in the steps

(a) Start with an initial parameter vector

$$\hat{x}_1 = (\hat{p}_1, \hat{T}_1, \hat{\zeta}_1)' \quad (2.28)$$

where the $(\,\cdot\,)_1$ refers to the first iteration.

(b) Obtain the components of the graident vector from either (2.26) or (2.27). Call this $\nabla_{\hat{x}}\,[J(\hat{x}_1, r(t))]$

(c) Compute the new parameter vector from the iterative steep descent equation [34]

$$\hat{x}_2 = \hat{x}_1 - K_1\nabla_{\hat{x}}\left[J(\hat{x}_1, r(t))\right] \quad (2.29)$$

where K_1 is a matrix, in general. When K_1 is optimally selected, (2.29) is called the steepest descent equation [37].

(d) The general parameter correction formula is

$$\hat{x}_{n+1} = \hat{x}_n - K_n\nabla_{\hat{x}}[\,J(\hat{x}_n, r(t))] \quad (2.30)$$

There are a variety of ways of selecting K_n and Table 2.2 is a collection of some of the expressions for this matrix [34,35,36,37]. See the Appendix for details. Mention should also be made of the optimum gradient method of McGhee [38] although the scope of the present study and space limitations make it unsuited for discussion here.

Table 2.2: Gain Matrix Expressions

Newton-Raphson:	$K_n = 2H_n^{-1}$
Gauss-Newton:	$K_n = \left[2\int_0^T \sigma(k_2\hat{T})(\sigma(k_2\hat{T}))' dt\right]^{-1}$
Gauss-Newton (modified):	$K_n = k\left[2\int_0^T \sigma(k_2\hat{T})(\sigma(k_2\hat{T}))' dt\right]^{-1}$
Steep Descent:	$K_n = kI; \quad k \geq 0$
Notes	1) $H_n = \nabla_{\hat{x}}\left[(\nabla_{\hat{x}}(J(\hat{x}_n, r(t)))'\right]$
	2) $\sigma(k_2\hat{T})$ is solution of the dynamic sensitivity difference equation of the model.
	3) I is the unit matrix.
	4) n is the iteration number.

The components of the gradient vector $\nabla_{\hat{x}}[z^1(\cdot)]$ in (2.27) can be evaluated at the end of every iteration interval $[0,\tau]$ by using the approach suggested by Bekey and Tomovic [8] which employs sensitivity analysis and difference equations. As pointed out in [34] the parameters must be held constant over an interation interval $\left[0, \tau\right]$, otherwise the gradient operation is not defined. The difference equation approach is well-suited to this requirement, and is formulated as follows: [34,86,87]: To the solution of the vector differential equation to the model of Figure 2.1 and for the initial conditions vector $\hat{\zeta}$, there is a vector difference equation representing the solution at the particular sampling instant $t = k_2\hat{T}$; $(k_2 = 0, 1, 2, \ldots)$. From (2.3) the differential equation of the continuous model for $t \geq k_2\hat{T}$ is

$$\frac{d\hat{z}}{dt} = \hat{f}(\hat{z}, \hat{p}, \hat{u}(t)), \qquad \zeta(k_2\hat{T}) = z(k_2\hat{T}) \tag{2.31}$$

The difference equation representation of (2.31) is chosen in such a way that it describes the solution of (2.31) at the sampling instants. One way to obtain the difference equation is to use the continuous solution of (2.31) for $t \epsilon (k_2\hat{T}, (k_2+1)\hat{T})$. This is

$$\hat{z}(t; k_2\hat{T}, \hat{p}, \hat{\zeta}(k_2\hat{T}), \hat{u}(t)) = \int_{k_2\hat{T}}^{t} \hat{f}(\hat{z}, \hat{p}, \hat{u}(\sigma))\, d\sigma + \hat{z}(k_2\hat{T}) \tag{2.32}$$

For the feedback configuration of Figure 2.1, and for $t = ((k_2+1)\hat{T} - \epsilon)$, where ϵ is small and positive, we will represent (2.32) by the difference equation

$$\hat{z}((k_2+1)\hat{T}; k_2\hat{T}, \hat{x}, \hat{z}(k_2\hat{T}), r(k_2\hat{T})) \triangleq \hat{F}\left[\hat{z}(k_2\hat{T}), \hat{p}, \hat{T}, r(k_2\hat{T})\right] \tag{2.33}$$

The correspondence between the terms of (2.33) and (2.32) is clear. Note that (2.33) is an n vector. Following Bekey and Tomovic [8], the vector sensitivity difference equations required for (2.27) are next obtained. We first translate (2.33) back in time [87] to obtain

$$\hat{z}(k_2\hat{T};(k_2-1)\hat{T},\hat{x},\hat{z}((k_2-1)\hat{T}),r((k_2-1)\hat{T}))=\hat{F}\left[\hat{z}((k_2-1)\hat{T}),\hat{p},\hat{T},r((k_2-1)\hat{T})\right] \tag{2.34}$$

and then apply the parameter gradient operator, defined by

$$\nabla_{\hat{x}}=\left[\frac{\partial}{\partial\hat{p}^1},\frac{\partial}{\partial\hat{p}^2},\ldots\ldots,\frac{\partial}{\partial\hat{p}^h},\frac{\partial}{\partial\hat{T}},\frac{\partial}{\partial\hat{\zeta}_0^1},\frac{\partial}{\partial\hat{\zeta}_0^2},\ldots\ldots,\frac{\partial}{\partial\hat{\zeta}_0^n}\right]' \tag{2.35}$$

to (2.34). (In (2.35), $\hat{\zeta}_0 \triangleq \hat{\zeta}(k_2\hat{T}=0)$). Adopting a more concise notation, the three sets of differential equations resulting from applying (2.35) to (2.34) are written

$$\frac{\partial\hat{z}^i(k_2\hat{T})}{\partial\hat{p}^j}=\sum_{k=1}^{n}\frac{\partial\hat{F}^i(\cdot)}{\partial\hat{z}^k((k_2-1)\hat{T})}\cdot\frac{\partial\hat{z}^k((k_2-1)\hat{T})}{\partial\hat{p}^j}+\frac{\partial\hat{F}^i(\cdot)}{\partial\hat{p}^j} \tag{2.36}$$

$$\frac{\partial\hat{z}^i(k_2\hat{T})}{\partial\hat{T}}=\sum_{k=1}^{n}\frac{\partial\hat{F}^i(\cdot)}{\partial\hat{z}^k((k_2-1)\hat{T})}\cdot\frac{\partial\hat{z}^k((k_2-1)\hat{T})}{\partial\hat{T}}+\frac{\partial\hat{F}^i(\cdot)}{\partial\hat{T}}$$
$$+\frac{\partial\hat{F}^i(\cdot)}{\partial r((k_2-1)\hat{T})}\cdot\frac{\partial r((k_2-1)\hat{T})}{\partial\hat{T}} \tag{2.37}$$

$$\frac{\partial\hat{z}^i(k_2\hat{T})}{\partial\hat{\zeta}_0^g}=\sum_{k=1}^{n}\frac{\partial\hat{F}^i(\cdot)}{\partial z^k((k_2-1)\hat{T})}\cdot\frac{\partial\hat{z}^k((k_2-1)\hat{T})}{\partial\hat{\zeta}_0^g}+\frac{\partial\hat{F}^i(\cdot)}{\partial\hat{\zeta}_0^g} \tag{2.38}$$

where $(i, g = 1, 2, \ldots, n)$ and $(j = 1, 2, \ldots, h)$, and where the partial derivatives (influence functions) of $\hat{z}$ with respect to the parameters, initial conditions, and sampling interval are to be regarded as the perturbations of the solutions $\hat{z}$ when evaluated at $k_2\hat{T} \in [0, \tau]$ due to perturbation of the particular parameter at $k_2\hat{T} = 0$. Thus, we define

$$u^i_{\hat{p}^j}(k_2) \triangleq \frac{\partial \hat{z}^i (k_2\hat{T})}{\partial \hat{p}^j} \tag{2.39}$$

$$u^i_{\hat{\zeta}^g_0}(k_2) \triangleq \frac{\partial \hat{z}^i (k_2\hat{T})}{\partial \hat{\zeta}^g_0} \tag{2.40}$$

$$(i, g = 1, 2, \ldots, n); \ (j = 1, 2, \ldots, h)$$

as the discrete sensitivity functions due to parameter and initial condition variations. The existence and continuity of the above derivatives is guaranteed by the requirements on $\hat{f}(\cdot)$ stated in Theorem 2.1. Again, simplifying notational dependence, we define

$$u^i_{\hat{T}}(k_2) = \lim_{\Delta\hat{T}\to 0} \frac{(\hat{z}^i(k_2 (\hat{T}+\Delta\hat{T})) - \hat{z}^i(k_2 \hat{T}))}{\Delta\hat{T}}, \tag{2.41}$$

$$= \lim_{\Delta\hat{T}\to 0} \frac{(\hat{F}^i(k_2,(\hat{T}+\Delta\hat{T})) - \hat{F}^i(k_2, \hat{T}))}{\Delta\hat{T}}, \tag{2.42}$$

$$(i = 1, 2, \ldots, n)$$

as the discrete sensitivity function due to sampling interval variation. The existence and continuity of this derivative is assured if we require that $\hat{F}^i(\cdot)$ be differentiable with respect to $\hat{T}$.

Bekey and Tomovic [8] have termed (2.42) the global sensitivity function for the sampling interval.

The initial conditions for the discrete sensitivity functions (2.39), (2.40), and (2.41) are obtained by determining the effect of changing either a parameter, the sampling interval, or an initial condition at the beginning of the iteration interval, i.e., when $k_2\hat{T} = 0$. Thus

$$u^i_{\hat{p}^j}(0) = 0 \qquad (i = 1, 2, \ldots, n), \; (j = 1, 2, \ldots, h) \qquad (2.43)$$

$$u^i_{\hat{T}_i}(0) = 0 \qquad (i = 1, 2, \ldots, n) \qquad (2.44)$$

$$\begin{aligned} u^i_{\hat{\zeta}_0^g}(0) &= 1.0 \quad (i = g) \quad (i, g = 1, 2, \ldots, n) \\ &= 0 \quad (i \neq g) \end{aligned} \qquad (2.45)$$

We can now write the difference equations (2.36), (2.37), and (2.38) in discrete sensitivity function notation as

$$u^i_{\hat{p}^j}(k_2) = \sum_{k=1}^{n} \frac{\partial \hat{F}^i\left[\hat{z}((k_2-1)\hat{T}), \hat{p}, \hat{T}, r((k_2-1)\hat{T})\right]}{\partial \hat{z}^k((k_2-1)\hat{T})} u^k_{\hat{p}^j}(k_2-1) + \frac{\partial \hat{F}^i\left[\hat{z}(k_2-1)\hat{T}), \hat{p}, \hat{T}, r((k_2-1)\hat{T})\right]}{\partial \hat{p}^k};$$

$$u^i_{\hat{p}^j}(0) = 0. \qquad (2.46)$$

$$u^i_{\hat{T}}(k_2) = \sum_{k=1}^{n} \frac{\partial \hat{F}^i\left[\hat{z}((k_2-1)\hat{T}), \hat{p}, \hat{T}, r((k_2-1)\hat{T})\right]}{\partial \hat{z}^k((k_2-1)\hat{T})} u^k_{\hat{T}}(k_2-1)$$

$$+ \frac{\partial \hat{F}^i\left[\hat{z}((k_2-1)\hat{T}), \hat{p}, \hat{T}, r((k_2-1)\hat{T})\right]}{\partial \hat{T}}$$

$$+ \frac{\partial \hat{F}^i\left[\hat{z}((k_2-1)\hat{T}), \hat{p}, \hat{T}, r((k_2-1)\hat{T})\right]}{\partial r((k_2-1)\hat{T})} \cdot \frac{\partial r((k_2-1)\hat{T})}{\partial \hat{T}};$$

$$u^i_T(0) = 0 \qquad (2.47)$$

$$u^i_{\hat{\zeta}^g_0}(k_2) = \sum_{k=1}^{n} \frac{\partial \hat{F}^i\left[\hat{z}((k_2-1)\hat{T}), \hat{p}, \hat{T}, r((k_2-1)\hat{T})\right]}{\partial \hat{z}^k((k_2-1)\hat{T})} u^k_{\hat{\zeta}^g_0}(k_2-1)$$

$$+ \frac{\partial \hat{F}^i\left[\hat{z}((k_2-1)\hat{T}, \hat{p}, \hat{T}, r((k_2-1)\hat{T})\right]}{\partial \hat{\zeta}^g_0};$$

$$u^i_{\hat{\zeta}^g_0}(0) = 1.0 \quad (i = g)$$
$$= 0 \quad (i \neq g) \qquad (2.48)$$

$(i, g = 1, 2, \ldots, n)$, $(j = 1, 2, \ldots, h)$.

These are the discrete sensitivity difference equations for the model-matching configuration of Figure 2.1.

It is shown in the Appendix that for a simple sinusoidal driving function

$$r(t) = A \sin \omega t \qquad (2.49)$$

that the corresponding derivative term of (2.47) is

$$\frac{\partial r\left[(k_2-1)\hat{T}\right]}{\partial \hat{T}} = \frac{1}{\hat{T}}\left[t\,\frac{\partial r(t)}{\partial t}\right]_{t=(k_2-1)\hat{T}}$$

$$= \frac{1}{\hat{T}}\left[(k_2-1)\hat{T}\,\dot{r}\,((k_2-1)\hat{T})\right] \tag{2.50}$$

Since this holds for a simple sinusoid, then for any $r(t)$ having a Fourier series expansion (in terms of sines and cosines) it is clear that (2.50) would also apply.

Recalling that we desire the vector $\nabla_{\hat{x}}\left[\hat{z}^1(t;\, \hat{x},\, r(t))\right]$ for use in (2.27), we can set up the discrete sensitivity equations (2.46), (2.47), and (2.48), along with (2.50) and solve for $u^i_{\hat{p}^j}(k_2)$, $u^i_{\hat{T}}(k_2)$, and $u^i_{\hat{\zeta}^g_0}(k_2)$. Then the components $u^1_{\hat{p}^j}(k_2)$, $u^1_{\hat{T}}(k_2)$, and $u^1_{\hat{\zeta}^g_0}(k_2)$ would be used in (2.27). It is helpful to observe [8] that the structures of the models necessary to generate $u^i_{\hat{p}^j}(k_2)$ and $u^i_{\hat{\zeta}^g_0}(k_2)$ are the same as the model of Figure 2.1. The model required to generate $u^i_{\hat{T}}(k_2)$ is complicated, however, by the second and third terms of (2.47). This will be made clearer when dealing with the example to follow.

We can now write the representation for (2.27) in terms of the discrete sensitivity functions so that

$$\nabla_{\hat{x}}[J(\hat{x}, r(t)] = -2\int_0^T (z^1(t; x, r(t)) - \hat{z}^1(t;\hat{x},r(t)))\nabla_{\hat{x}}[\hat{z}^1(t;\hat{x},r(t))]\, dt \qquad (2.27)$$

can be represented by

$$\nabla_{\hat{x}}[J(\hat{x},r(t))] \simeq -2\int_0^T (z^1(t;x,r(t)) - \hat{z}^1(t;\hat{x},r(t))) \begin{bmatrix} u^1_{\hat{p}^1}(k_2) \\ \cdot \\ \cdot \\ \cdot \\ u^1_{\hat{p}^h}(k_2) \\ \hline u^1_{\hat{T}}(k_2) \\ \hline u^1_{\hat{\zeta}_0^1}(k_2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u^1_{\hat{\zeta}_0^n}(k_2) \end{bmatrix} dt, \qquad (2.51)$$

where k_2 is such that $k_2\hat{T} \epsilon [0, \tau]$. When $k_2\hat{T} = \tau$, the parameter vector is updated via (2.30) and the next iteration is begun. The mechanization of (2.30) and (2.51) will be illustrated by an example. Before presenting that example however, it is pertinent to remark that the difference equation representation for linear and nonlinear systems leading to the general equations (2.33) and (2.34) has been discussed by Kalman and Bertram [86]. Bekey [87] has shown how to obtain the difference equation (2.33) by using either the z-transform of the continuous linear elements, or by working directly from the control system diagram by first assigning state variables. The latter method is particularly well-suited to setting up the difference equations for nonlinear systems where the z-transform does not, in general, exist for every element. Note that once (2.51) has been calculated, then the updated parameter estimate can be obtained from (2.30):

$$\hat{x}_{n+1} = \hat{x}_n - K_n \nabla_{\hat{x}} \left[J(\hat{x}, r(t)) \right] \tag{2.30}$$

2.5.1 Example of Gradient Search

Results are available in the study of the use of the gradient technique to identify the unknown parameters of a closed loop sampled data system when these parameters include the unknown sampling interval T.

Example 1: Let the continuous system and continuous model of Figure 2.1 be linear with differential equations as follows:

$$\text{System:} \quad \frac{dz}{dt} = K\,u(t); \qquad z(t=0) = 0 \tag{2.52}$$

$$\text{Model:} \quad \frac{d\hat{z}}{dt} = \hat{K}\,\hat{u}(t); \qquad \hat{z}(t=0) = 0 \tag{2.53}$$

It is desired to estimate the sampling interval T of the sampled-data system of Figure 2.1 and the gain K. A steep descent mechanization will be used to vary the estimates $\hat{T}$ and $\hat{K}$ of the model.

Using the zero-order data hold, the output at the sampling instants $k_2\hat{T}$ of the model loop is obtained, in this case, by z-transforming the combination of the Laplace transform of (2.53) and the zero-order data hold with the result:

$$Z\left[\left(\frac{1-z^{-1}}{s}\right)\cdot\frac{\hat{K}}{s}\right]=\frac{\hat{T}\hat{K}}{(z-1)}, \tag{2.54}$$

where $Z(\cdot)$ indicates the z-transform operation. Using (2.54), the forward loop transfer function of the model is

$$\hat{z}^1(z) = \frac{\hat{T}\hat{K}\hat{u}(z)}{(z-1)}, \tag{2.55}$$

The resulting difference equation for the forward loop is

$$\hat{z}^1((k_2+1)\hat{T}) = \hat{z}^1(k_2\hat{T}) + \hat{T}\hat{K}\hat{u}(k_2\hat{T}). \tag{2.56}$$

$$(k_2 = 0, 1, 2, \ldots)$$

Now

$$\hat{u}(k_2\hat{T}) = r(k_2\hat{T}) - \hat{z}^1(k_2\hat{T}). \tag{2.57}$$

$$(k_2 = 0, 1, 2, \ldots)$$

Substituting (2.57) into (2.56), the output is

$$\hat{z}^1((k_2+1)\hat{T}) = \hat{z}^1(k_2\hat{T}) + \hat{T}\hat{K}\left[r(k_2\hat{T}) - \hat{z}^1(k_2\hat{T})\right], \tag{2.58}$$

$$= \hat{z}^1(k_2\hat{T})\left[1 - \hat{T}\hat{K}\right] + \hat{T}\hat{K}r(k_2\hat{T}), \tag{2.59}$$

$$(k_2 = 0, 1, 2, \ldots)$$

with the initial condition $\hat{z}^1(t=0) = 0$.

The associated sensitivity difference equations are obtained by using (2.46), (2.47), and (2.48) along with (2.39), (2.40), and (2.41), and (2.50). The sensitivity difference equation for the model sampling interval $\hat{T}$ is

$$u^1_{\hat{T}}((k_2+1)\hat{T}) = \left[1 - \hat{T}\,\hat{K}\right] u^1_{\hat{T}}(k_2\hat{T}) + \hat{T}\,\hat{K}\left[\frac{1}{\hat{T}}\; t\dot{r}\;(t)\right]_{t=k_2\hat{T}}$$

$$+ \hat{T}\,\hat{K}\left[\frac{r(k_2\hat{T}) - \hat{z}^1(k_2\hat{T})}{\hat{T}}\right] \;;\; u^1_{\hat{T}}(0) = 0, \qquad (2.60)$$

where $(k_2 = 0, 1, 2, \ldots)$.

The sensitivity difference equation for the model gain is

$$u^1_{\hat{K}}\;((k_2+1)\hat{T}) = (1-\hat{T}\,\hat{K})\,u^1_{\hat{K}}\;(k_2\hat{T}) + \hat{T}\,\hat{K}\left[\frac{r(k_2\hat{T})-\hat{z}^1(k_2\hat{T})}{\hat{K}}\right];\; u^1_{\hat{K}}(0) = 0.$$

$$(k_2 = 0, 1, 2, \ldots) \qquad (2.61)$$

As remarked previously, the structure of the sensitivity model for this parameter is identical to the structure of the original model (2.59). Shifting (2.60) and (2.61) backward, as was done with (2.34) when developing the theoretical sensitivity difference equations, we have

$$u^{1}_{\hat{T}}(k_2\hat{T}) = \left[1 - \hat{T}\,\hat{K}\right] u^{1}_{\hat{T}}((k_2-1)\hat{T}) + \hat{T}\,\hat{K}\left[\frac{1}{\hat{T}}\ t\ \dot{r}\ (t)\right]_{t=(k_2-1)\hat{T}}$$

$$+ \hat{T}\,\hat{K}\left[\frac{r((k_2-1)\hat{T}) - \hat{z}^{1}((k_2-1)\hat{T})}{\hat{T}}\right] ,$$

$$u^{1}_{\hat{T}}(0) = 0, \qquad (2.62)$$

$$(k_2 = 1,\ 2,\ \ldots\ldots),$$

and

$$u^{1}_{\hat{K}}(k_2\hat{T}) = \left[1 - \hat{T}\,\hat{K}\right] u^{1}_{\hat{K}}((k_2-1)\hat{T})$$

$$+ \hat{T}\,\hat{K}\left[\frac{r((k_2-1)\hat{T}) - \hat{z}^{1}((k_2-1)\hat{T})}{\hat{K}}\right] ;$$

$$u^{1}_{\hat{K}}(0) = 0, \qquad (2.63)$$

$$(k_2 = 1,\ 2,\ 3,\ \ldots).$$

These are the discrete sensitivity equations which are actually solved, and furnish a concrete example of the abstract equations given by (2.46) and (2.47). The equations are solved by simulation and the solutions are substituted into (2.51). The parameter vector $\hat{x}_{n+1}$ for the estimate of x is then obtained from the algorithm (2.30).

The difference equations were programmed for solution in this case by noting the similarity of (2.62) and (2.63) to (2.59). The latter is a difference equation representation of a continuous system at sampling instants; therefore, the sensitivity difference equations were also programmed as continuous systems. The schematic of the iterative adjustment scheme for $\hat{T}$ alone is shown in Figure 2.10, and the schematic for the iterative adjustment scheme for both $\hat{T}$ and $\hat{K}$ is given by Figure 2.11.

Example 2: Let the continuous sytem and continuous model of Figure 2.1 be nonlinear with differential equations as follows:

$$\text{System:} \quad \frac{dz}{dt} = K[u(t)]^3 ; \qquad z(t=0) = 0 \tag{2.64}$$

$$\text{Model:} \quad \frac{d\hat{z}}{dt} = \hat{K}[\hat{u}(t)]^3 ; \qquad \hat{z}(t=0) = 0 \tag{2.65}$$

The parameters to be estimated are T and K. The estimates are $\hat{T}$ and $\hat{K}$.

This example will be limited to showing the formulation of the discrete sensitivity difference equations for a nonlinear system. No simulation results will be presented. Following Bekey [8], the difference equation describing the output of the model at the sampling instants $t=(k_2+1)\hat{T}$ can be obtained directly from Figure 2.1 after substituting (2.65) into the loop:

$$\hat{z}^1((k_2+1)\hat{T})=\hat{z}^1(k_2\hat{T}) + \hat{K}\hat{T}\left[r(k_2\hat{T}) - \hat{z}^1(k_2\hat{T})\right]^3; \; \hat{z}(0) = 0 . \tag{2.66}$$

Shifting backward to obtain the difference equation as a function of the last available samples of r(t) and $z^I(t)$

$$\hat{z}^1(k_2\hat{T}) = \hat{z}^1((k_2-1)\hat{T} + \hat{K}\,\hat{T}\left[r((k_2-1)\hat{T}) - \hat{z}^1((k_2-1)\hat{T})\right]^3 ;$$

$$\hat{z}^1(0) = 0 \qquad (2.67)$$

$$(k_2 = 1, 2, 3, \ldots).$$

Hence, from (2.33), we can identify

$$\hat{F}\left[\hat{z}((k_2-1)\hat{T}), \hat{p}, \hat{T}, r((k_2-1)\hat{T})\right]$$

$$= \hat{z}^1((k_2-1)\hat{T}) + \hat{K}\,\hat{T}\left[r((k_2-1)\hat{T}) - \hat{z}^1((k_2-1)\hat{T})\right]^3$$

$$\hat{z}^1(t=0) = 0 \qquad (2.68)$$

where $\hat{p}$ is the scalar parameter $\hat{K}$, and where $(k_2 = 1, 2, 3, \ldots.)$. Using (2.67) and (2.68), and employing (2.46) - (2.48) along with (2.39) - (2.41) and (2.43) - (2.45) and (2.50), the sensitivity difference equations for the parameters $\hat{T}$ and $\hat{K}$ are

$$u^1_{\hat{T}}(k_2\hat{T}) = u^1_{\hat{T}}((k_2-1)\hat{T})\left\{1 - 3\,\hat{K}\,\hat{T}\left[r((k_2-1)\hat{T}) - \hat{z}^1((k_2-1)\hat{T})\right]^2\right\}$$

$$+ \hat{K}\left[r((k_2-1)\hat{T}) - \hat{z}^1((k_2-1)\hat{T})\right]^3 \qquad (2.69)$$

$$+ 3\,\hat{K}\,\hat{T}\left[r((k_2-1)\hat{T}) - z^1((k_2-1)\hat{T})\right]^2\left[\frac{t}{\hat{T}}\;\dot{r}(t)\right]_{t=(k_2-1)\hat{T}} ;$$

$$u^1_{\hat{T}}(0) = 0$$

$$(k_2 = 1, 2, 3, \ldots..)$$

and

$$u^1_{\hat{K}}(k_2\hat{T}) = \left[1 - 3\hat{K}\hat{T}\left[r((k_2-1)\hat{T}) - \hat{z}^1((k_2-1)\hat{T}\right]^2\right]u^1_{\hat{K}}((k_2-1)\hat{T}) \qquad (2.70)$$
$$+ \hat{T}\left[r((k_2-1)\hat{T}) - \hat{z}^1((k_2-1)\hat{T})\right]^3 \ ; \ u^1_{\hat{K}}(0)=0,$$
$$(k_2 = 1, 2, 3, \ldots\ldots).$$

The same procedure would be employed to solve these sensitivity difference equations and use their solution to obtain components of the parameter correction gradient vector (for the new parameter estimate $\hat{x}_{n+1}$) as was done with Example 1.

2.5.2 Results Of Gradient Search Studies (Example 1 Only)

The gradient search studies were divided into two phases; the first was a gradient search over $\hat{T}$ alone with $\hat{K}$ held fixed and equal to $K = 1.0$. The second was a simultaneous gradient search over both $\hat{T}$ and $\hat{K}$. In both phases the results were obtained via the DSL/90 simulation program. The results of the gradient search over $\hat{T}$ alone are shown in Figure 2.10. The gain factor K_1 of (2.30) was selected as a fixed constant which means that a steep descent parameter adjustment scheme was followed.

Figure 2.11 shows the schematic for the two parameter gradient search; i.e., over both $\hat{T}$ and $\hat{K}$.

Figure 2.12 shows the results of the two parameter gradient search for Example 1.

It is felt that these results are more of academic interest than practical interest at the present time because of the following reasons:

(1) It is generally easier and more economical of programming effort and computer time to utilize programmed search to both obtain the optimal set of model parameters for a given model and than it is to construct separate gradient tracker programs for each model under consideration.

(2) There is considerable coupling between the parameters in even the simple case of the gradient search over two parameters. For example, it was found that convergence would not occur for every set of initial values $(\hat{T}_1, \hat{K}_1,)$ without the incorporation of considerable logic to automatically adjust the gain factor $\hat{K}_1$ as well as prevent $\hat{T}$ from going negative. (The latter event caused the search to terminate by the nature of the DSL/90 program.)

(3) Gradient optimization techniques are really best suited to situations where a model or system of fixed form but variable coefficients must be adjusted to satisfy some optimization criterion. The present task initiated in this report is somewhat broader in scope: It is to find the combination of model form and parameter values together with the value of sampling interval which yields the absolute minimum of $J(\cdot)$.

(4) The sensitivity difference equation approach is not suited to modeling situations where system observations are noisy. No convergence proof is available. A more

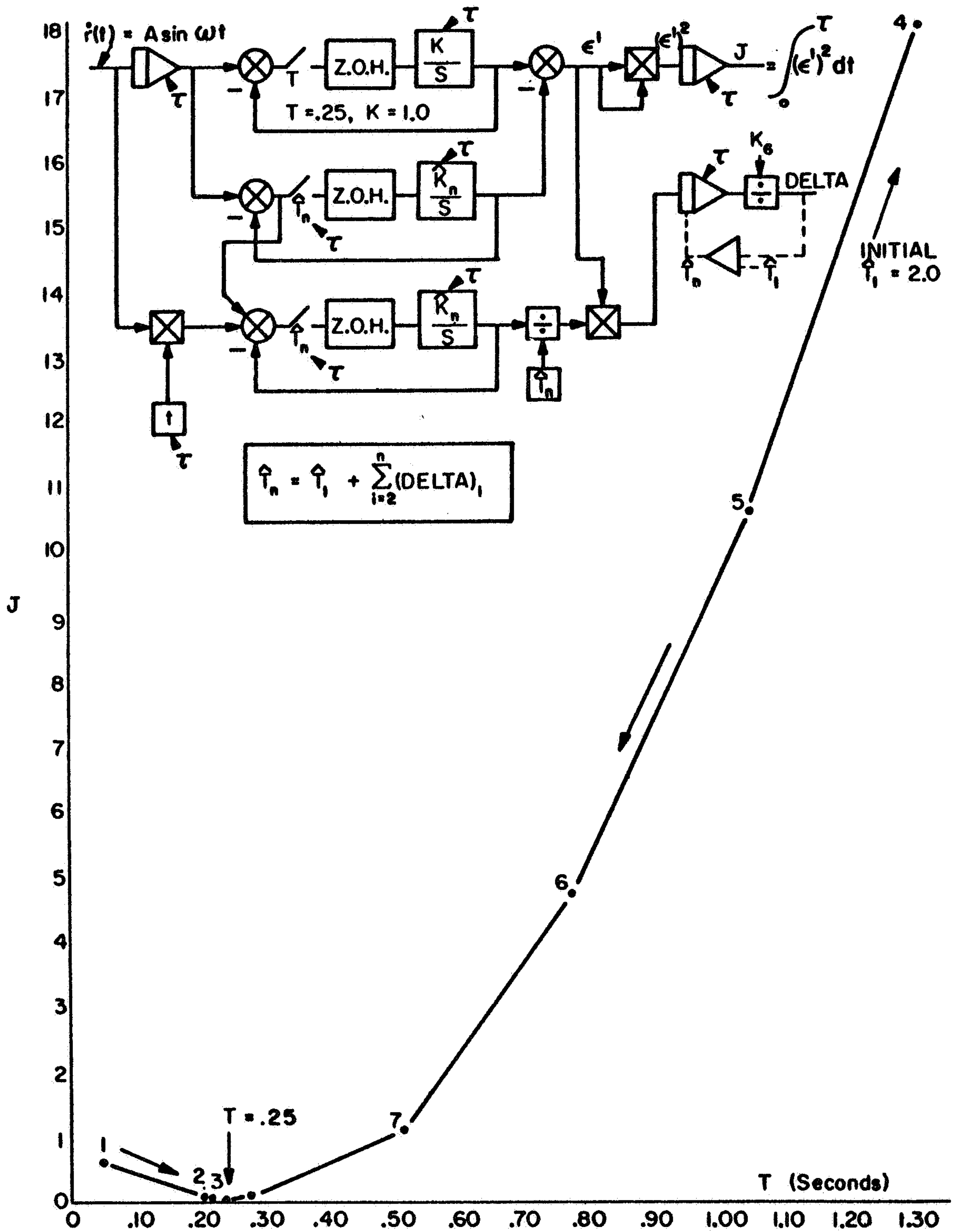
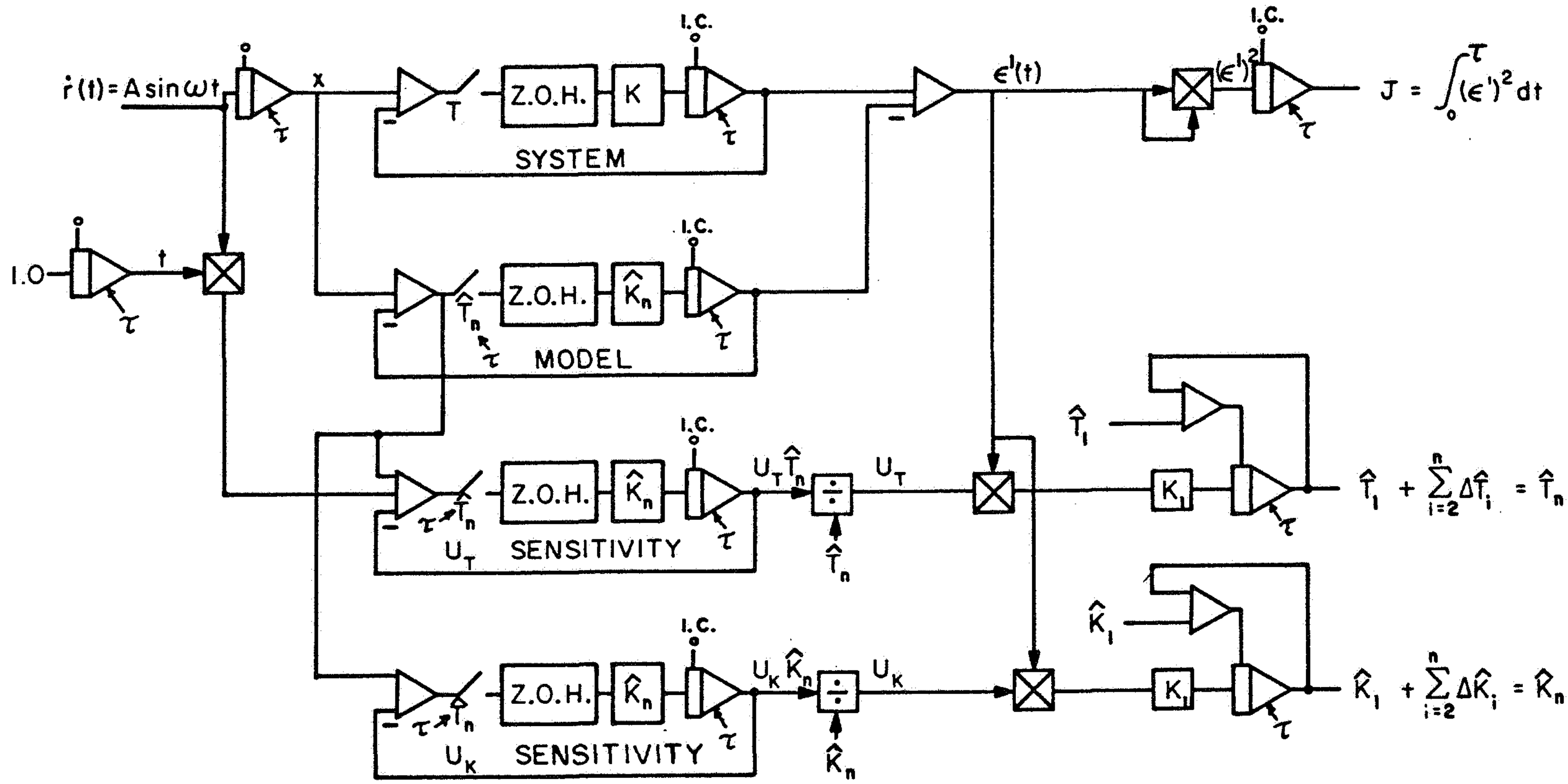


Figure 2.10 Gradient Search For Estimate Of T



Notes

1. All integrations cycles are repetitive over $[0, T_0]$ - the iteration time.
2. After each iteration T_a and K_3 are adjusted to new values. Procedure continues until J is minimized.
3. Steep descent is used, hence K is a constant.

Figure 2.11 Steep Descent Identification Of T And K

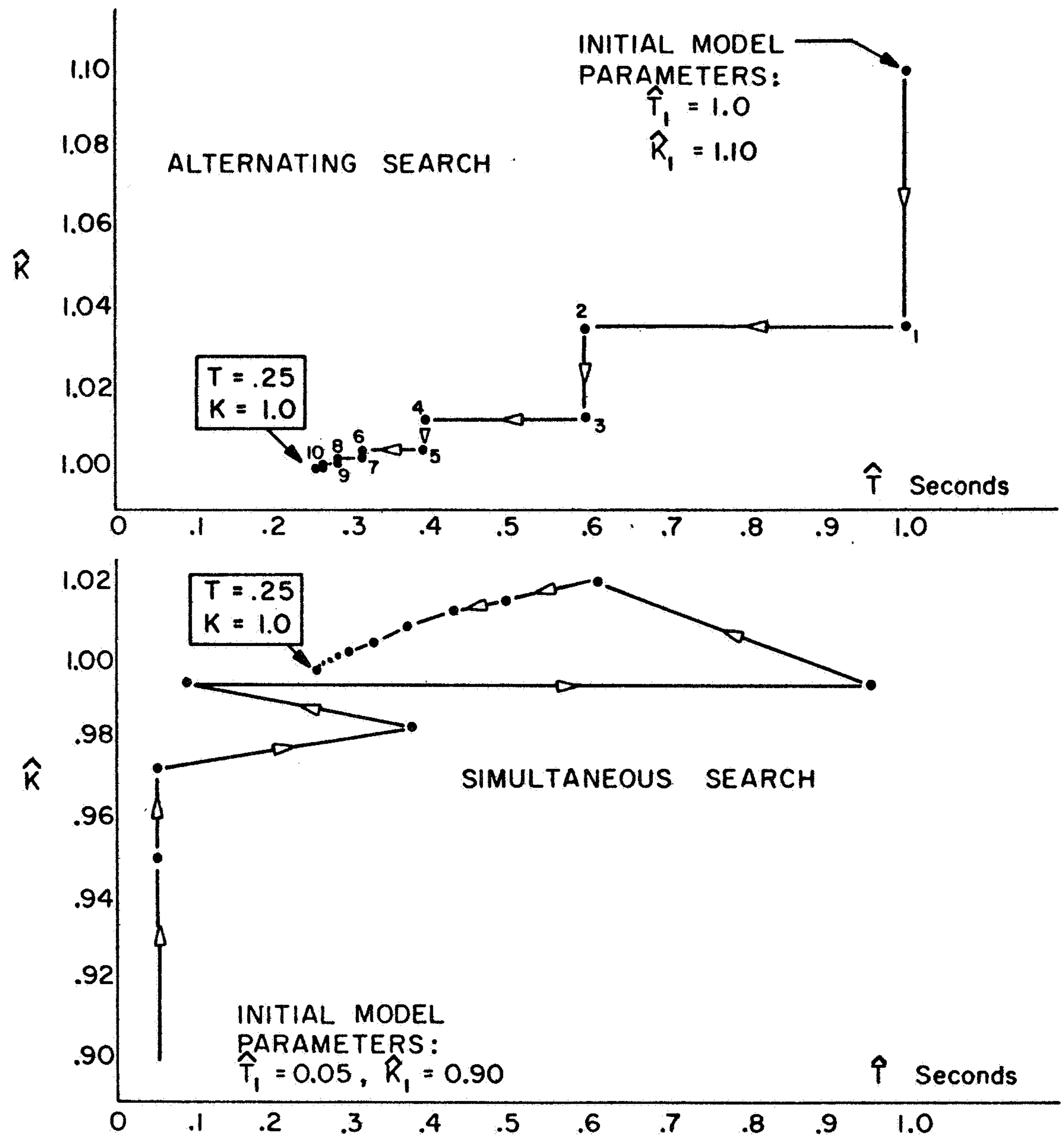


Figure 2.12 Gradient Search For Estimate Of Both Sampling Interval (T) And Gain (K) In First Order System By Means Of A First-Order Model

suitable approach to this problem would employ stochastic approximation. This is discussed in Chapter 3.

2.6 Summary Of Results Of Noise-Free Simulations

This section summarizes the respective advantages of programmed search and iterative gradient search. Generally speaking, the programmed search is to be preferred to the sensitivity equation formulation of the gradient search for the parameter estimates. This is because one does not, in general, know the exact form of the system well enough ahead of time to make it worth the extra effort necessary to mechanize the gradient search sensitivity difference equations. In addition, the sensitivity method requires the mechanization of one additional model circuit for each estimated parameter. This requirement is obviated, however, if the approximation to the gradient is used, as given by (2.26). In this case, the programmed search and gradient method are probably on a par as far as equipment and programming time are concerned.

Iterative gradient search is useful also when optimizing the parameters of a particular model. This situation is typical of the adaptive control problem.

In the next chapter we will present a discussion of stochastic approximation, a technique which is gradient-like in essence, but can be used to treat modeling situations where the observations are noisy.

CHAPTER 3

STOCHASTIC APPROXIMATION AND SAMPLED-DATA SYSTEM PARAMETER ESTIMATION

3.1 Introduction

Stochastic approximation is a recursive estimation procedure which can be applied to the problem of either (1) finding the parameter which causes a regression function to take on some preassigned value, or (2) finding the value of a parameter which maximizes (minimizes) the regression function. That is, suppose for every real valued parameter x, the observed random variable $Y = Y(x)$, denoting the value of a response to an experiment carried out at a controlled parameter level x, has the conditional distribution function $H(y|x)$, defined by [40, 41, 88][1]

$$H(y|x) = \Pr(Y(x) \leq y) \tag{3.1}$$

and the regression function, defined [88] as the conditional expectation of Y for the given x, written as

$$M(x) = \int_{-\infty}^{\infty} y \, d\, H\,(y|x) \tag{3.2}$$

---

[1] The notation used herein is that which is standard for the stochastic approximation literature. It is more concise than the usual notation found in mathematic statistics texts, as for example, Cramer [88]. In the sequel, we will carefully define all terms as they arise.

where the regression function is related to the observation $Y(x)$ by

$$Y(x) = M(x) + n, \tag{3.3}$$

where n represents a stationary random process which zero mean and finite variance, and where neither the exact nature of $H(y|x)$ nor $M(x)$ need be known [40,41]. For the present, $Y(x)$, $M(x)$, and x will be taken as scalars. In the statistics literature, the two above problems are called the (1) Robbins-Monro problem, and (2) Kiefer-Wolfowitz problem.

To be more explicit, in the Robbins-Monro problem, the regression function $M(x)$ is assumed to be an unknown monotone function of x. It is desired to find the particular value of parameter $x = \theta$ which causes $M(x)$ to take on an assigned constant value: $M(x) = \alpha$, where α is chosen.

In the Kiefer-Wolfowitz problem it is assumed that $M(x)$ has a unique maximum (minimum) at $x = \theta$ and is strictly increasing (decreasing) for $x < \theta$, and strictly decreasing for $x > \theta$.

The procedures used to solve the two problems are concerned with making successive experiments at parameter levels $x_1, x_2, \ldots$ in such a way that x_n tends to θ in some probability sense. In order of increasing strength, there are three types of convergence: convergence in probability, convergence in mean-square, and convergence with probability one. The latter is also referred to as convergence almost surely. These will be discussed in the sequel.

While the restrictions and details of the two problems are discussed below, it is pertinent here to remark that the advantage of stochastic approximation over the usual regression approach is that neither the conditional distribution function of the noisy observations $Y(x)$, here taken as $H(y|x)$, nor the underlying regression function $M(x)$ need be known. Thus, it is called a non-parametric method.

Stochastic approximation can be applied to any problem that can be formulated as some form of regression problem in which repeated observations are made. To be specific, we will consider the problem of estimating the parameters of an unknown sampled-data system when using the model-matching technique. Reference Figure 3.4. The cost function is the integral of the weighted error-squared, and the regression function is the cost function when the noise $n_1(t)$ is zero. We will use successive observations of the cost function and will adjust the model parameters as a function of the observations by means of a stochastic approximation algorithm of the Kiefer-Wolfowitz type. The aim, of course, will be to minimize the mean-square error between system and model over some allowable set of parameters. In general, sequential observations of the system behavior (cost function in our case) are used. However, it is also possible to use the same system input and output time histories repeatedly, meanwhile adjusting the model parameters by the stochastic approximation algorithm. In addition to parameter estimation, stochastic approximation can be applied to problems of prediction and data filtering [19, 20]. In the following short survey, we first discuss the Robbins-Monro and Kiefer-Wolfowitz procedures. This is followed by a discussion of the application of the Kiefer-Wolfowitz procedure to the modeling problem. Then the mean-square convergence of an extension of the Kiefer-Wolfowitz procedure is proved for the estimation configuration of Figure 3.4.

3.2 Survey Of Stochastic Approximation Methods

The following is a concise survey of stochastic approximation methods. Earlier surveys were given by Derman [40], Wilde [48], Loginov [59], Gardner [79], and Sakrison [19]. The latter two, in particular, have a number of engineering applications. The present survey includes recent results not included in the earlier surveys.

3.2.1 The Robbins-Monro Method

The Robbins-Monro procedure was the first stochastic approximation method [41]. Let (3.1), (3.2), (3.3) hold. It is desired to find the root $x = \theta$ such that, for a given α

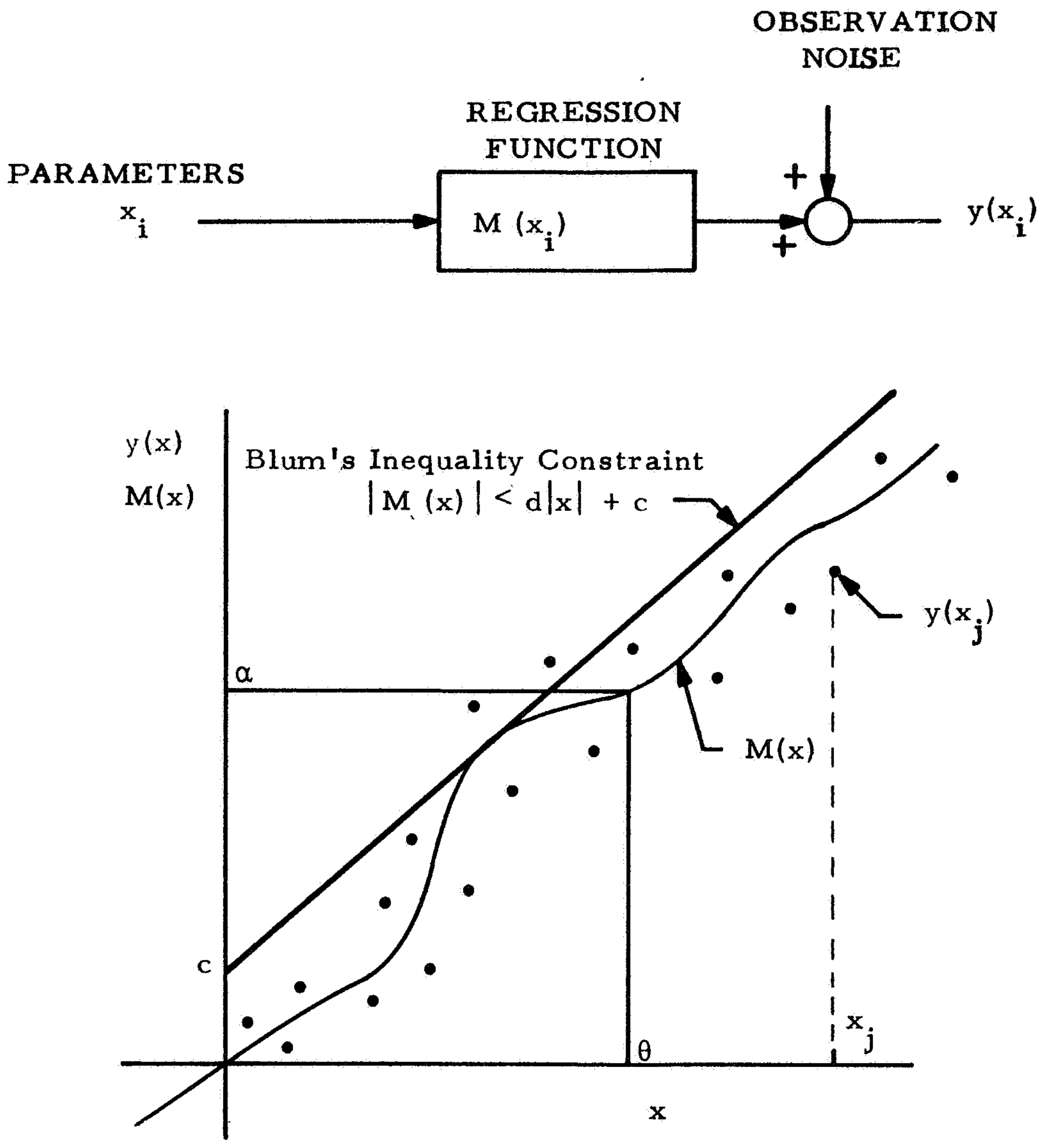
$$M(\theta) = \alpha \tag{3.4}$$

The procedure for finding the root $x = \theta$ is given by the following theorem.[1]

<u>Theorem</u> (Robbins-Monro [41]): Assume that for each x there

---

[1] Reference Figure 3.1.



Robbins-Monro Problem:

Given α and observations $\{y(x)\}$, solve for $x = \theta$ such that $E\{y(x)\} = M(\theta) = \alpha$.

Solution:

If conditions of 3.2.1 are satisfied $x_{n+1} = x_n + a_n(\alpha - y(x_n))$

Figure 3.1 The Robbins-Monro Problem

corresponds a random variable $Y=Y(x)$ with distribution function $H(y) = Pr(Y(x) \mid y)$; and that there exists a positive constant C such that for all x

$$Pr(|Y(x)| \leq C) = \int_{-C}^{C} d\,H(y|x) = 1 \tag{3.5}$$

I.e., $Y(x)$ is bounded with probability one. Assume that exist finite constants α and δ such that

$$M(x) \leq \alpha - \delta \qquad \text{for} \quad x < \theta, \tag{3.6}$$

and

$$M(x) \geq \alpha + \delta \qquad \text{for} \quad x > \theta, \tag{3.7}$$

where $\delta > 0$.

(Note that $M(x)$ need not equal α, nor must $M(x)$ be continuous).

Let $\{a_n\}$ be a fixed sequence of positive constants such that

$$0 < \sum_{n=1}^{\infty} a_n^2 < \infty, \tag{3.8a}$$

and

$$\sum_{n=1}^{\infty} a_n = \infty. \tag{3.8b}$$

(For example $a_n = 1/n$, $n = (1, 2, \ldots.)$).

Take x_1 to be an arbitrary constant and define a (nonstationary) Markov chain $\{x_n\}$ by

$$x_{n+1} = x_n + a_n(\alpha - y_n), \tag{3.9}$$

where y_n is a random variable[1] with conditional distribution function

$$H(y|x_n) = Pr(y_n \leq y|x_n). \tag{3.10}$$

Then

$$\lim_{n \to \infty} E(x_n - \theta)^2 = 0 \tag{3.11}$$

That is, x_n converges to θ in mean square. This also implies convergence in probability [89].

Wolfowitz [42] next considered the problem. He showed that x_n converges to θ in probability under weaker conditions on $Y(x)$. He replaced condition (3.5) with the requirements (on the measurement noise $(y - M(x))$)

$$\sigma_x^2 = \int_{-\infty}^{\infty} (y - M(x))^2 \, dH(y|x) < \infty. \tag{3.12}$$

He also required a bound on the regression function so that $M(x) < \infty$, where $M(x)$ is defined by (3.2). Blum [43] then weakened the above conditions. His requirements are:

---

[1] Using (3.3), we will define y_n as the random variable

$$y_n = Y(x_n) = M(x_n) + n \tag{3.3a}$$

where x_n is the random variable defined by (3.9).

A) $|M(x)| \leq c + d\,|x|$ for some constants c and d such that $c \geq 0$ and $d \geq 0$. (3.13)

B) $$\sigma_x^2 = \int_{-\infty}^{\infty} (y - M(x))^2 \, dH(y|x) \leq \sigma^2 < \infty. \tag{3.14}$$

C) $$M(x) < \alpha \quad \text{for} \quad x < \theta, \tag{3.15}$$

$$M(x) > \alpha \quad \text{for} \quad x > \theta. \tag{3.16}$$

D) $$\inf_{\delta_1 \leq |x-\theta| \leq \delta_2} |M(x) - \alpha| > 0 \tag{3.17}$$

for every pair of numbers (δ_1, δ_2) where $0 < \delta_1 < \delta_2 < \infty$.

E) $$0 < \sum_{n=1}^{\infty} a_n^2 < \infty, \tag{3.8a}$$

$$\sum_{n=1}^{\infty} a_n = \infty. \tag{3.8b}$$

(For example, $a_n = A/n$ where A is a positive constant.)

Then the Robbins-Monro algorithm (3.9) converges to θ with probability 1, i.e.,

$$\Pr\left(\lim_{n \to \infty} x_n = \theta \right) = 1 \tag{3.18}$$

Subsequently, Dvoretzky [47] showed that under Blum's condition x_n also converges in the mean-square, i.e.,

$$\lim_{n \to \infty} E(x_n - \theta)^2 = 0 \tag{3.11}$$

Thus, both Blum and Dvoretzky obtained weaker conditions for a stronger form of convergence than Robbins and Monro. The Robbins-Monro problem is illustrated in Figure 3.1.

3.2.2 The Kiefer-Wolfowitz Method

By the Robbins Monro method one can obtain the roots (x_i) for each given α_i of the unknown regression function $M(x_i) = \alpha_i$. Following this work, Kiefer and Wolfowitz [44] gave a procedure for finding the value of x which maximizes the unknown regression function $M(x)$.[1] The main restriction on $M(x)$ is that it must have a unique maximum. (By suitable modifications the following theorems can also be used to express conditions for convergence to the minimum of the unknown regression function $M(x)$).

Theorem (Kiefer-Wolfowitz [44]): Let $M(x)$ be an unknown regression which has its (unique) maximum at the unknown point $x = \theta$, and let $H(y|x)$ be a family of conditional distribution functions which depend on the parameter x, i.e.,

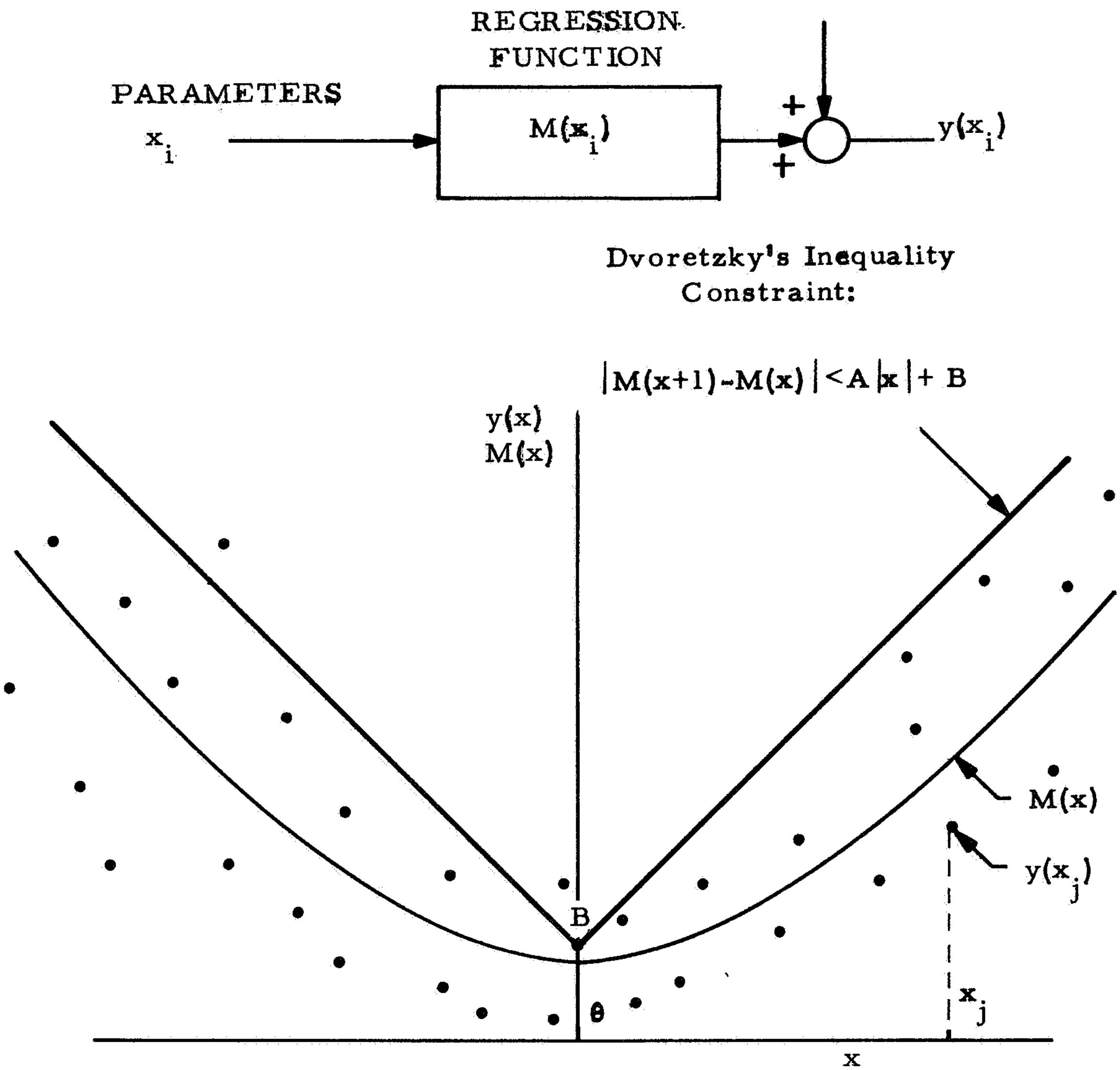
$$H(y|x) = \Pr(Y(x) \leq y). \tag{3.19}$$

Let

$$M(x) = \int_{-\infty}^{\infty} y \, dH(y|x). \tag{3.20}$$

---

[1]Reference Figure 3.2.



Kiefer-Wolfowitz Problem: Given the noisy observations $\{y(x)\}$, find $x = \theta$ which minimizes $M(x)$.

Solution: If the conditions of 3.2.2. are satisfied, then take

$$x_{n+1} = x_n + a_n \frac{(y(x_n - c_n) - y(x_n + c_n))}{c_n} .$$

Figure 3.2 The Kiefer-Wolfowitz Problem

Assume

A) $$\sigma_x^2 = \int_{-\infty}^{\infty} (y - M(x))^2 \, dH(y|x) \leq \sigma^2 < \infty \qquad (3.21)$$

Assume the following regularity condition on M(x):

B) (1) There exist positive β and B such that for distinct value of x given by x' and x''

$|x' - \theta| + |x'' - \theta| < \beta$ implies

$$|M(x') - M(x'')| < B|x' - x''| \qquad (3.22)$$

(2) There exist positive p and R such that

$|x' - x''| < p$ implies

$$|M(x') - M(x'')| < R \qquad (3.23)$$

(3) For every $\delta > 0$ there exists a positive $\pi(\delta)$ such that

$|x - \theta| > \delta$ implies

$$\inf_{\delta/2 > \epsilon > 0} \frac{|M(x+\epsilon) - M(x-\epsilon)|}{\epsilon} > \pi(\delta). \qquad (3.24)$$

C) $$\sum_{n=1}^{\infty} a_n = \infty \qquad (3.25a)$$

$$\sum_{n=1}^{\infty} a_n c_n < \infty \qquad (3.25b)$$

$$\sum_{n=1}^{\infty} (a_n/c_n)^2 < \infty \qquad (3.25c)$$

$$\lim_{n\to\infty} c_n = 0 \qquad (3.25d)$$

(For example: $a_n = A/n$, $c_n = C/n^{\frac{1}{3}}$, where A and C are positive constants, and $n = 1, 2, \ldots\ldots$).[1]

D) Take

$$x_{n+1} = x_n + a_n\left(\frac{y_{2n+1} - y_{2n-1}}{c_n}\right) \tag{3.26}$$

where y_{2n+1} and y_{2n-1} are independent random variables with respective conditional distribution functions $H(y|x_n + c_n)$ and $H(y|x_n - c_n)$. That is, using (3.3a), define y_{2n+1} as the observation of the random variable $Y(x_n + c_n)$, and define y_{2n-1} as the observation of the random variable $Y(x_n - c_n)$.[2] Then

$$\lim_{n\to\infty} P\left[|x_n - \theta| \geq \epsilon\right] = 0, \tag{3.27}$$

i.e., x_n converges to θ in probability.

---

[1]See the Appendix for a discussion of these sequences.

[2]Using (3.3a), we define the observed random variables

$$Y(x_n + c_n) = M(x_n + c_n) + n \tag{3.3c}$$

and

$$Y(x_n - c_n) = M(x_n - c_n) + n \tag{3.3d}$$

Departing slightly from the notation of Kiefer-Wolfowitz, we will henceforth denote for conciseness

$$y_{2n+1} = Y(x_n + c_n), \tag{3.3e}$$

and

$$y_{2n-1} = Y(x_n - c_n). \tag{3.3f}$$

The regularity conditions on the regression function $M(x)$ are explained as follows: B(1) assures that the magnitude of the slope of $M(x)$ is small near the maximizing point θ; B(2) prevents the slope of $M(x)$ being too large for any point x; B(3) assures the slope is not zero whenever $x \neq \theta$ thus eliminating the possibility of flat spots in $M(x)$.

Blum [49] then eliminated the need for conditions (3.22) and (3.25b) in proving

$$P\left\{\lim_{n\to\infty} x_n = \theta\right\} = 1, \tag{3.28}$$

i.e., convergence of equation (3.26) with probability one. However, up to this point important regression functions such as $M(x) = e^{-x^2}$, or $M(x) = -x^2$, were ruled out since they do not satisfy (3.22) and (3.23) for $x \geq 0$. Derman [45] considered functions whose difference quotients lie between two straight lines with positive slopes. Functions like $M(x) = -x^2$, for $x \geq 0$, satisfy these conditions. He showed convergence of x_n to θ in probability. Finally Burkholder [46] and Dvoretzky [47] obtained the weakest set of conditions which allow us to use stochastic approximation for regression functions such as $M(x) = e^{-x^2}$. Burkholder proved probability one convergence and Dvoretzky proved both mean square and probability one convergence. In Dvoretzky's form these conditions are (assuming, without loss of generality, that $\theta = 0$ and that we use the algorithm for x_{n+1} given by (3.26)):

A) $|M(x+1) - M(x)| < A|x| + B < \infty$ (3.29)

for all x and suitable A, B

B) $\sup_{1/k<x-\theta<k} \overline{D}M(x) < 0; \quad \inf_{1/k<\theta-x<k} \underline{D}M(x) > 0$ (3.30)

C) $$\sigma_x^2 = \int_{-\infty}^{\infty} (y - M(x))^2 \, dH(y|x) \le \sigma^2 < \infty \tag{3.31}$$

D) The sequences of (3.25a), (3.25c), (3.25d).

E) $$M(x) = \int_{-\infty}^{\infty} y \, dH(y|x) \tag{3.32}$$

(In (B) above $\overline{D}M(x)$ and $\underline{D}M(x)$ denote the upper and lower (Dini) derivatives [58] of M(x) at x and are given by

$$\overline{D}M(x) = \overline{\lim_{0\neq h\to 0}} \left(\frac{M(x+h) - M(x)}{h} \right) \tag{3.33}$$

and

$$\underline{D}M(x) = \underline{\lim_{0\neq h\to 0}} \left(\frac{M(x+h) - M(x)}{h} \right) \tag{3.34}$$

Note that the Kiefer-Wolfowitz procedure (3.26) is simply an approximate gradient search method. In fact, Loginov [59] points out that it is simply a stochastic version of an algorithm originally given by Germansky [60]. It differs from the deterministic gradient procedures in that the multiplier a_n is decreased with n rather than being held constant or increased. Also, the size of δx over which the gradient is calculated decreases with n according to the behavior of c_n. The Kiefer-Wolfowitz minimization problem is shown in Figure 3.2.

Dvoretzky [47] also considered a more general stochastic approximation approach, encompassing both the Robbins-Monro process, the Kiefer-Wolfowitz process, and others. In this he partitioned the stochastic approximation algorithm into a random part and a deterministic part, and obtained broad convergence requirements on the two parts. He obtained both probability one convergence and mean-square convergence for this process.

Multidimensional extensions of the Robbins-Monro and Kiefer-Wolfowitz processes were made by Blum [49]. However, for the latter process he required that $M(x)$ have continuous first and second derivatives. Furthermore, Blum's procedure develops a one-sided approximation to the gradient rather than the two-sided approach of equation (3.26). Sacks [50] stated a theorem for probability one convergence of a multidimensional Kiefer-Wolfowitz procedure. Subsequently, Derman and Sacks [51] proved the probability one convergence of the Kiefer-Wolfowitz procedure by providing a multidimensional extension and a corresponding probability one convergence proof of Dvoretzky's theorem.

Later, Venter [52] obtained both mean square and probability one convergence for a multidimensional Dvoretzky theorem and thus, by implication, provided a basis for the mean-square convergence of the multidimensional Kiefer-Wolfowitz process.

While the Dvoretzky procedure is elegant, it beclouds the simplicity of the more direct approach of the Kiefer-Wolfowitz procedure. Consequently, in subsequent work the Kiefer-Wolfowitz approach is used directly. Another reason for doing this is that Dvoretzky's formulation and the multiple parameter extension thereof when used for model matching are best suited to the estimation problem shown in Figure 1.3 when only noise $n_2(t)$ exists. In problems of system modeling, however, the presence of noise $n_2(t)$ is usually of small concern while noise $n_1(t)$ is very important. Therefore, the configuration to be analyzed will treat only the case where noise $n_1(t)$ is present. It remains to be proved that the Kiefer-Wolfowitz procedure applied to this case as well.

The question of the size of the estimation error after $\underline{k}$ iteration steps has been considered by Chung [55], Derman [45], Sacks [50], and Dupac [56]. Chung showed convergence of the parameter estimates for the Robbins-Monro procedure to a normal distribution with mean zero. Furthermore, he gave expressions for the upper bound on the absolute moments of x_n

$$\beta_n^{(r)} = E\left[|x_n - \theta|^r\right] \tag{3.35}$$

for all $\underline{r}$. However, his expressions can be evaluated only when the bound (σ^2) on the noise variance

$$\sigma_x^2 = \int_{\infty}^{\infty} (y - M(x))^2 \, dH(y|x) \leq \sigma^2 \tag{3.36}$$

is known.

Derman [57] obtained similar results for the Kiefer-Wolfowitz procedure.

The question of an optimal sequence a_n or a_n/c_n to minimize the variance $E(x_n - \theta)^2$ after any fixed number of iteration steps of either the R-M procedure or the K-W procedure is of interest. Dvoretzky [47] solved this problem for the R-M procedure. Dupac [56] solved it for the K-W procedure. In both cases their work is for the scalar formulation. Sakrison [65] extended Dupac's analysis to the multidimensional K-W procedure.

For the scalar Robbins-Monro procedure Dvoretzky assumed

A) $$\sigma_x^2 = \int_{-\infty}^{\infty} (y - M(x))^2 \, dH(y|x) \leq \sigma^2 < \infty. \tag{3.37}$$

B) There exist constants A and B such that

$$0 < A \leq \frac{M(x) - \theta}{x - \theta} \leq B < \infty. \tag{3.38}$$

C) It is assumed that a constant $c \geq 0$ exists such that

$$|x_n - \theta| \leq c \leq \sqrt{\frac{2\sigma^2}{A(B - A)}} \quad . \tag{3.39}$$

Then the sequence

$$a_n = \frac{Ac^2}{\sigma^2 + n A^2} \tag{3.40}$$

is optimal for the Robbins-Monro procddure and the variance of the estimates is bounded with the bound given by

$$E(x_n - \theta)^2 \leq \frac{\sigma^2 c^2}{\sigma^2 + (n-1)\, A^2 c^2} \tag{3.41}$$

The theorem of Dupac [56] which we will use as a reference basis in proving convergence of the K-W stochastic approximation procedure for the system modeling configuration is stated as follows:

Assume

A) $M(x)$ is increasing for $x < 0$, and is decreasing for $x > 0$, where

$$M(x) = \int_{-\infty}^{\infty} y \, dH(y|x) \tag{3.42}$$

B) For every x

$$\sigma_x^2 = \int_{-\infty}^{\infty} (y - M(x))^2 \, dH(y|x) \leq \sigma^2 < \infty \tag{3.43}$$

C) There exist constants $K_0 > 0$, $K_1 > 0$, such that

$$K_0 |x - \theta| \leq \left| \frac{dM(x)}{dx} \right| \leq K_1 |x - \theta| \tag{3.44}$$

Let a_n, c_n be positive sequences of constants such that

$$\lim_{n \to \infty} c_n = 0, \quad \sum_{n=1}^{\infty} a_n = \infty, \quad \sum_{n=1}^{\infty} a_n c_n < \infty, \quad \sum_{n=1}^{\infty} \left(\frac{a_n}{c_n}\right)^2 < \infty. \tag{3.25}$$

Take

$$x_{n+1} = x_n + a_n\left(\frac{y_{2n-1} - y_{2n+1}}{c_n}\right) \tag{3.26}$$

where y_{2n+1} and y_{2n-1} are independently distributed random variables with conditional distribution functions $H(y|x_n + c_n)$ and $H(y|x_n - c_n)$.[1] Then x_n converges to θ in mean square. Furthermore, for sequences of the type

$$a_n = \frac{A}{n^{\alpha}}, \qquad c_n = \frac{C}{n^{\gamma}} \tag{3.45}$$

where $\alpha = 1$ implies $A > \frac{1}{4K_0}$, the choice $\alpha = 1$, $\gamma = 1/6$ insures that

$$E(x_n - \theta)^2 = O(n^{-1/2}), \tag{3.46}$$

where $f(n) = O(g(n))$ means $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = K < \infty$. (K can be zero).

Any other choice of α and γ leads to a worse result. If, in addition, it is assumed that

$$\text{D)} \quad \left|\frac{d^3 M(x)}{dx^3}\right| < \infty \tag{3.47}$$

for $\underline{x}$ in some neighborhood of θ, then the choice $\alpha = 1$, $\gamma = 1/6$ insures that

---

[1]See (3.3e) - (3.3f) for explicit expressions for y_{2n+1} and y_{2n-1}.

$$E(x_n - \theta)^2 = O(n^{-2/3}) \tag{3.48}$$

and this choice is optimal in the same sense.

Sakrison [65] also obtained the same results for the multi-dimensional Kiefer-Wolfowitz procedure. Refer to the Appendix for a discussion for the properties of a_n and c_n.

3.3 Stochastic Approximation Applied to the System Modeling Problem

Stochastic approximation has been applied to the system modeling problem by Sakrison [18, 19, 65], Kirvaitis [24], Holmes [25] and others. Sakrison extended Dupac's work on optimal sequences a_n and c_n to the multiparameter case and treated such regression functions as error squared, magnitude error, and error to fourth power. He studied estimation of parameters of nonlinear systems and gave an example of the design of a linear prediction filter where the gain multipliers of k linearly independent stable, linear transfer functions were chosen by stochastic approximation. Sakrison's problem is illustrated by Figure 3.3.

Kirvaitis estimated the parameters of both linear and nonlinear differential equations. Both Sakrison and Kirvaitis required that the noise components have bounded variance and also that they be bounded in magniture. Also, they required that the system parameters be confined to a compact convex set.

Holmes represented the unknown nonlinear system as an analytic function expanded in a Volterra series in the parameter x

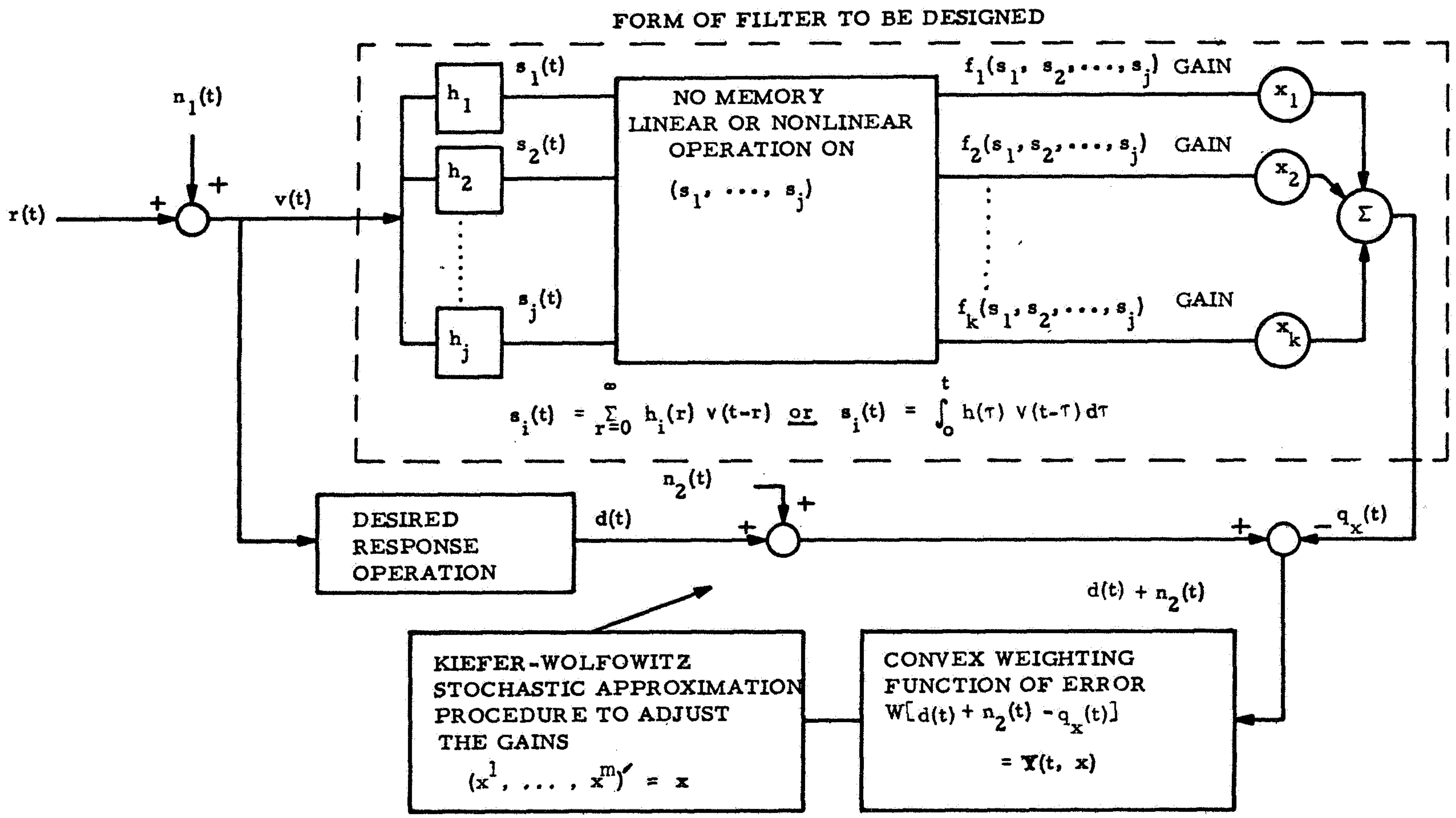


Figure 3.3 Sakrison's Problem

which he then estimated by stochastic approximation. He furnished estimates of both a linear kernel function and a nonlinear kernel function of a nonlinear stationary discrete-time control system. He required that all noise sequences have bounded variances and that the system parameters belong to a bounded convex set.

3.4 Stochastic Approximation Applied To Estimation Of Parameters Of Nonlinear Sampled-Data Systems With Noisy Observations

3.4.1 Introduction

Again consider the problem of Section 1.4. This problem is to estimate all the parameters of a sampled-data system including the sampling interval. The sampled-data system consists of a sampler, a zero-order data hold, and continuous dynamics. The sampled-data system, and corresponding sampled-data model are illustrated in Figure 3.4. Note that while the input to the sampled-data system and sampled-data model is scalar, the observed signal is taken as the noise-corrupted state vector. Later, in the simulation work, the observations will be limited to the scalar output of the sampled data system. This will be done because in a number of practically important problems the observations are limited to the scalar output. The same limitation is necessary for simulations in order that they yield a basis for later modeling work with real data.

In the following development no typographical distinction will be made between vectors and scalars, although scalar components of a vector will be indicated by superscripts. For example,

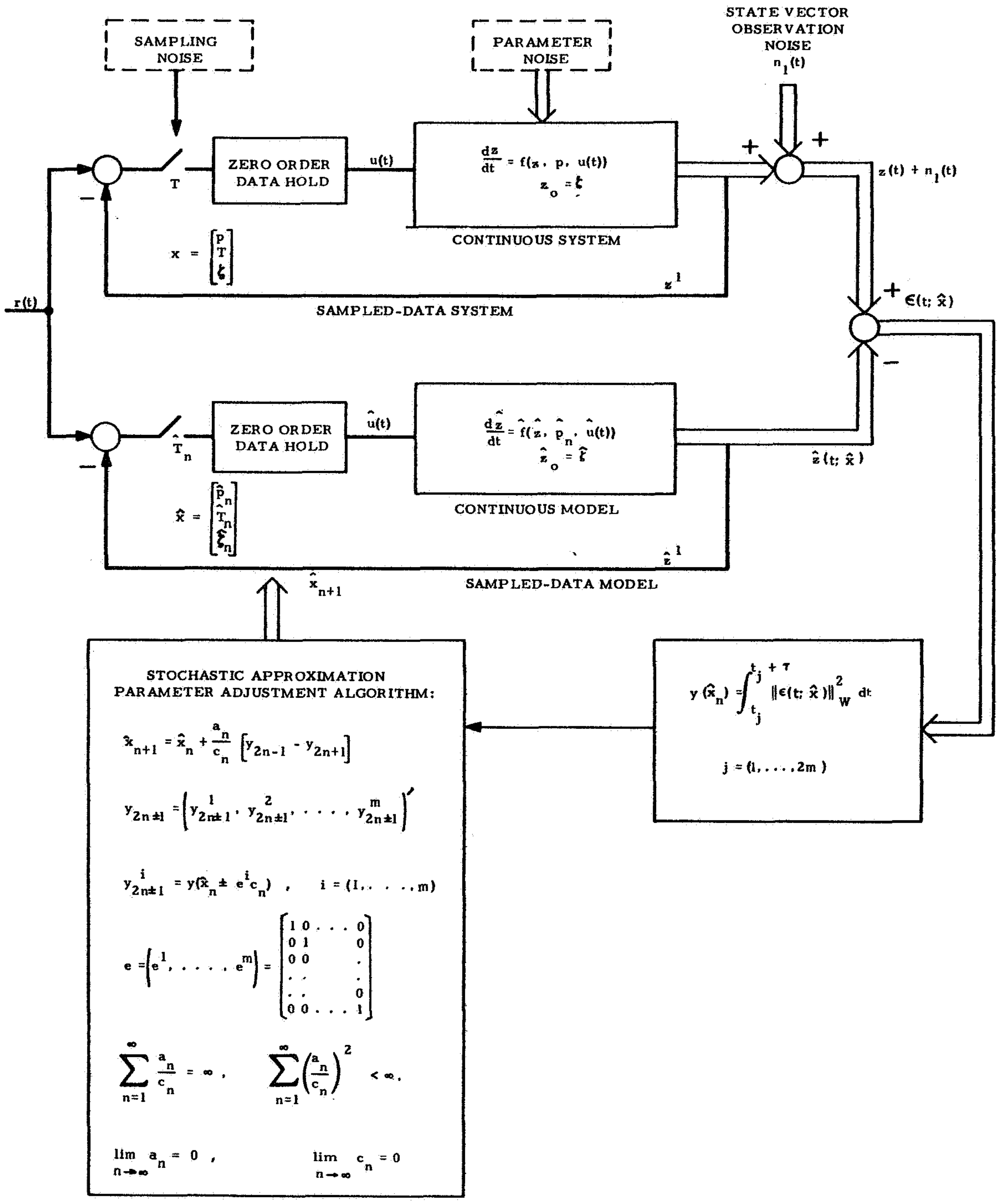


Figure 3.4 General Parameter Estimation Configuration Using Stochastic Approximation

$z = (z^1, z^2, \ldots., z^n)'$ denotes the relations between a vector z and its components z^i. The symbol $(')$ indicates the transpose of a vector. The vectors belong to Euclidean vector spaces and the Euclidean norm

$$\|z\| = \left(\sum_{i=1}^{n}(z^i)^2\right)^{1/2} \tag{3.49}$$

will be used for norms of vectors. The norm for $n \times n$ matrices A is defined by

$$\|A\| = \sum_{i,j=1}^{n}|a_{ij}| \tag{3.50}$$

All statistical averages $E(\cdot)$ are ensemble averages unless otherwise noted. The subscript k denotes the k^{th} iteration so that $z_k = (z_k^1, z_k^2, \ldots., z_k^n)'$ indicates the vector z and its components at the k^{th} iteration. We will also use the symbol 0 to denote both the scalar zero and the vector zero.

Referring to Figure 3.4, the continuous dynamic system is assumed to be given by

$$\frac{dz}{dt} = f(z, p, u(t)), \qquad z(t=0) = \zeta \tag{3.51}$$

where the state vector z and the dynamic system vector function f are both n vectors, p is an h vector of constant parameters, and u(t) is an r vector of controls. (In this case r = 1).[1]

Corresponding to the continuous dynamic system there is a continuous dynamic model

$$\frac{d\hat{z}}{dt} = \hat{f}(\hat{z}, \hat{p}, \hat{u}(t)), \qquad \hat{z}(t=0) = \hat{\zeta} \tag{3.52}$$

which has vectors of the same dimensions as the continuous system. We assume the form of the system and model to be the same. Hereafter, (3.51) will be called the continuous system to distinguish it from the sampled-data system. Likewise, (3.52) will be called the continuous model to distinguish it from the sampled-data model.

Define the constant parameter vector of the sampled-data system by the m dimensional vector

$$x = (p, T, \zeta)' \tag{3.53}$$

This vector is not, in general, completely known. In fact, it may be completely unknown.

Define the parameter vector of the sampled data model by the m dimensional vector

$$\hat{x} = (\hat{p}, \hat{T}, \hat{\zeta})' \tag{3.54}$$

---

[1]Throughout, we will use the convention, established in Chapter 1, and used in Chapter 2, of indicating the solution of (3.51) by either $z(t; p, \zeta, r(t))$, $z(t; p, \zeta)$, or $z(t)$ depending on whether we suppress the dependence on parameters, initial conditions, and control function. The same comment also applies to the solution of (3.52).

This vector is adjustable. As in Chapter 2, $m = (h+l+n) \leq 2n+1$. It will be held constant over an iteration interval of length τ where $\tau \gg \hat{T}_n$. This interval will also be indicated by $[t_n, t_n + \tau]$, where n indicates the iteration number $(n = 0, 1, 2, \ldots\ldots)$. Indicate by $\hat{x}_n$ the n^{th} iteration of the parameter vector of the sampled-data model. Explicitly, this is

$$\hat{x}_n = (\hat{p}_n, \hat{T}_n, \hat{\zeta}_n)' \tag{3.55}$$

At the end of an iteration interval, the stochastic approximation algorithm, to be discussed, will be used to increment the components of $\hat{x}_n$. The new parameter vector is indicated by $\hat{x}_{n+1}$.

Define the observation of the sampled-data system by the n dimensional vector

$$v(t; x, r(t)) = z(t; x, r(t)) + n_1(t) \tag{3.56}$$

where $n_1(t)$ is an n dimensional vector of observation noise with properties to be discussed subsequently. Note that $v(t; x, r(t))$ is a random vector. Define

$$\epsilon(t; x, \hat{x}, r(t)) = v(t; x, r(t)) - \hat{z}(t; \hat{x}, r(t)) \tag{3.57}$$

as the error between observed sampled-data system and sampled-data model. This is an n dimensional random vector. Note that when the system is not completely observable, then some components of $v(\cdot)$ will be zero. In this case, corresponding components of $\hat{z}(\cdot)$ and $\epsilon(\cdot)$ would also be set to zero. In effect, the dimension of the vectors defined by (3.56) and (3.57) would be accordingly reduced. This would be done by indicating explicitly the observable components of the state and error vectors.

Define the cost function by the integral norm-squared error function

$$J(t_n+\tau;t_n,x,\hat{x},r(t)) = \int_{t_n}^{t_n+\tau} (\epsilon(t;x,\hat{x},r(t)))' W\epsilon(t;x,\hat{x},r(t)))dt \quad (3.58)$$

where W is a diagonal weighting matrix with positive terms, and is hence positive definite. Note that $J(\cdot)$ is a scalar random variable. As before, τ is the (constant) iteration interval.

The Keifer-Wolfowitz stochastic approximation procedure for obtaining estimates $\hat{x}_n$ of the sampled-data system parameter vector x will now be described. We choose the sequences of positive numbers $\{a_n\}$ and $\{c_n\}$ which have the properties[1]

---

[1]We can show that the sequences a_n and c_n with properties described by (3.59) and (3.60) also satisfy the original K-W conditions (3.25). We have only to show that $\sum_{n=1}^{\infty} a_n c_n < \infty$ and that $\sum_{n=1}^{\infty} a_n = \infty$. But from the analysis given in the Appendix we can write

$$\sum_{n=1}^{\infty} a_n c_n = \sum_{n=1}^{\infty} AC/n^{1/6} < \infty$$

and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} A/n = \infty$$

Hence (3.59) and (3.60) imply (3.25).

$$\lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} c_n = 0$$

$$\sum_{n=1}^{\infty} \frac{a_n}{c_n} = \infty, \qquad \sum_{n=1}^{\infty} \left(\frac{a_n}{c_n}\right)^2 < \infty, \tag{3.59}$$

Specifically, we will follow the work of Dupac [56] and Sakrison [65] in choosing

$$a_n = A/n, \qquad \text{and} \qquad c_n = C/n^{1/6} \tag{3.60}$$

for optimal convergence properties of the Kiefer-Wolfowitz algorithm. In (3.60) A and C are positive constants, and $n \in [1, 2, \ldots\ldots\ldots)$ is the iteration number.

Define by e the $m x m$ matrix of m dimensional natural basis vectors

$$e = (e^1, e^2, \ldots, e^m) = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 1 & & & & \cdot & \cdot \\ 0 & 0 & & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & & & & 0 & \cdot \\ \cdot & \cdot & & & & 1 & 0 \\ 0 & 0 & & & & 0 & 1 \end{bmatrix} \tag{3.61}$$

Define the $2m$ perturbations of the m dimensional model parameter vector by

$$\hat{x}_n(+i) = \hat{x}_n + e^i c_n \qquad (i = 1, 2, \ldots, m) \tag{3.62}$$

and

$$\hat{x}_n(-i) = \hat{x}_n - e^i c_n \qquad (i = 1, 2, \ldots, m) \tag{3.63}$$

Note that only one scalar component of $\hat{x}_n$ is perturbed for each value of the index i.

We now use (3.58) and define the scalar random variables resulting from employing the perturbed parameter vectors (3.62) and (3.63). These are the 2m scalar cost functions, which we define by

$$y^1_{2n+1} = \int_{t_n}^{t_n+\tau} \| \epsilon (t; x, (\hat{x}_n + e^1 c_n), r(t)) \|^2_W \, dt, \tag{3.64}$$

$$y^1_{2n-1} = \int_{t_n+\tau}^{t_n+2\tau} \| \epsilon (t; x, (\hat{x}_n - e^1 c_n), r(t)) \|^2_W \, dt, \tag{3.65}$$

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}$$

$$y^i_{2n+1} = \int_{t_n+2(i-1)\tau}^{t_n+(2i-1)\tau} \| \epsilon (t; x, (\hat{x}_n + e^i c_n), r(t)) \|^2_W \, dt, \tag{3.66}$$

$$y^i_{2n-1} = \int_{t_n+(2i-1)\tau}^{t_n+2i\tau} \| \epsilon (t; x, (\hat{x}_n - e^i c_n), r(t)) \|^2_W \, dt, \tag{3.67}$$

$$(i = 2, 3, \ldots, m)$$

where the integrands are quadratic forms with the weighting matrix W. Note, by referring to Figure 3.4, that the y^i_{2n+1} and y^i_{2n-1} are observed random variables. Also note that one complete set of iterations is obtained in $2m\tau$ seconds. Successive time histories of $z(t; x, r(t))$ and $\hat{z}(t;(\hat{x}_n \pm e^i c_n, r(t)), (i = 1, 2, \ldots, m)$ are used in the above procedure; hence, it is suited to real time estimation problems. However, it is also possible to use the same time history of $z(t; x, r(t))$ repeatedly, while generating the successive model state vectors $\hat{z}(t; (\hat{x}_n \pm e^i c_n), r(t)), (i = 1, 2, \ldots, m)$. Naturally, in the latter case, we would use the input $r(t)$ corresponding to the particular $z(t; x, r(t))$ which we are using. The convergence theorem, to be discussed, will work for either procedure.

Using the set of $2m$ scalar cost functions given by (3.64)-(3.67), construct the m dimensional random vector defined by

$$(y_{2n-1} - y_{2n+1}) = \begin{bmatrix} y_{2n-1}^1 - y_{2n+1}^1 \\ y_{2n-1}^2 - y_{2n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{2n-1}^m - y_{2n+1}^m \end{bmatrix} \tag{3.68}$$

Notice that each component of this vector is an observed random process.

Now define the stochastic approximation algorithm which will be used for successive estimates $\hat{x}_n$ of the m dimensional parameter vector x of the sampled-data system. These estimates are defined by

$$\hat{x}_{n+1} = \hat{x}_n + a_n(y_{2n-1} - y_{2n+1})/c_n \qquad (3.69)$$

where $\hat{x}_1$ is a chosen m dimensional vector having finite components. Notice that all iterations of (3.69) yield random vectors $\hat{x}_{n+1}$ since (3.68) is a random vector. Since this algorithm has the same form as the well-established Kiefer-Wolfowitz algorithm, (3.26), it will subsequently be referred to as such. We will subsequently state and prove a theorem for mean square convergence of $\hat{x}_n$ to x; written as

$$\lim_{n \to \infty} E\left[\|\hat{x}_n - x\|^2\right] = 0 \qquad (3.70)$$

At this point, it is interesting to compare (3.69) to the algorithm for the usual steep descent gradient search, given by (2.30). Clearly, the positive number a_n corresponds to the positive gain K_n, and the random vector $(y_{2n-1} - y_{2n+1})/c_n$ can be regarded as an approximation of the gradient vector $\nabla_{\hat{x}}[J(\tau; x, \hat{x}, r(t))]$.

An assumption of a unique minimum of $J(\cdot)$, given by (3.58), is required in order to prove convergence of the K-W procedure (3.69) to the vector $\hat{x}_n = x$, where x is the parameter vector of the sampled-data system, and $\hat{x}_n$ is the n^{th} iteration of the parameter vector $\hat{x}$ of the sampled-data model. In practice, a quick scan of $\hat{x}$ over the space of possible parameter vectors may give some idea of local minima of (3.58). Then the K-W stochastic approximation procedure of (3.69) can be employed.

From (3.69), and recalling (3.62) to (3.67), it is now clear that $(y_{2n-1} - y_{2n+1})/c_n$ is a random vector conditioned on the sequence of random vectors $\{\hat{x}_n, \hat{x}_{n-1}, \ldots, \hat{x}_1\}$. For conciseness, we will usually indicate this sequence by $\{\hat{x}_n\}$. Thus, we will describe y_{2n-1} and y_{2n+1} as statistically independent random vectors with respective conditional distribution functions $H(y|\hat{x}_n - c_n)$ and $H(y|\hat{x}_n + c_n)$.

Now, using (3.64) to (3.67), we define the vector-valued deterministic regression functions underlying the random vectors y_{2n-1} and y_{2n+1} by the m dimensional vectors

$$M_{2n-1} = (y_{2n-1} | n_1(t) = 0), \tag{3.71}$$

and

$$M_{2n+1} = (y_{2n+1} | n_1(t) = 0). \tag{3.72}$$

Assuming that the noise vector $n_1(t)$ is a stationary finite variance random process with components having zero mean, i.e.,

$$E\left[n^i_1(t)\right] = 0, \quad (i = 1, \ldots, n) \tag{3.73}$$

and that the noise is not correlated with either $z(t; x, r(t))$ or $\hat{z}(t; \hat{x}, r(t))$, so that

$$\left\| E\left[n_1(t_1)\, z'(t_2, x, r(t))\right] \right\| = \left\| E\left[n_1(t_1)\, \hat{z}'(t_2, \hat{x}, r(t))\right] \right\| = 0 \tag{3.74}$$

for t_1 and t_2 belonging to $[t_n, t_n + 2i\tau]$, $(i = 1, 2, \ldots, m)$, and $t_n \in [0, \infty)$, then it will later be shown that

$$E\left[(y_{2n-1} - y_{2n+1})|\hat{x}_n\right] = M_{2n-1} - M_{2n+1} \tag{3.75}$$

where the dependence of y_{2n-1} and y_{2n+1} on the sequence $\hat{x}_n$ is clear from (3.64) to (3.67). Thus, our definition of M_{2n+1} and M_{2n-1} as regression functions satisfies the usual statistical definition that the regression function is the conditional expectation of y_{2n+1} and y_{2n-1} for the given $\hat{x}_n$ [88].

Another requirement that we will place on the noise vector $n_1(t)$ is motivated from consideration of (3.56) and (3.69). Notice that the parameter estimates $\hat{x}_n$ are generated as functions of the noisy observations $v(t; x, r(t))$ of the sampled data system. Recall that the proof of the existence theorem for differential equations, stated in Chapter 2, required that the parameters lie in closed balls. One way of conforming with this requirement, is to require (1) that the components of the first estimated of these parameters, given by $\hat{x}_1$, must lie in a closed ball, and (2) that components of subsequent estimates $\hat{x}_n$ must also lie in a closed ball. From a consideration of (3.58) and (3.69), it is clear that in order to satisy the latter requirement, we should place a magnitude bound on the components of the observation noise vector [24]. This will then assure that the components of the resulting parameter estimate vector $\hat{x}_{n+1}$, as obtained from (3.69), will be bounded. This restriction is expressed by requiring that a constant $C < \infty$ must exist such that

$$\Pr\left\{\|n_1(t)\| \leq C\right\} = 1. \tag{3.76}$$

Requirement (3.76), together with (1) above, insures that the components of all parameter vector estimates $\hat{x}_n$ will lie within a closed ball. Closed balls are convex [67], are bounded and hence the set of points within the closed ball is compact [33]. Equivalently, by requiring that the components of $\hat{x}_1$ lie in a convex compact set, together with (3.76), would insure the above boundedness of the components of $\hat{x}_n$.

3.4.2 Mathematical Basis And Mean Square Convergence Proof

The purpose of this section is to prove mean-square convergence of (3.59) to the parameter vector x, where x is the system parameter vector given by (3.53). In the sequel, this fixed vector x will be denoted by θ. We will first state several supporting theorems from differential equations, so as to provide an analytical basis for the convergence proof.

Reference has already been made to the work of Dupac [56] in proving mean-square convergence of the scalar parameter K-W procedure. Sakrison [65] and Kirvaitis [24] followed with similar proofs for the vector parameter case. However, Kirvaitis imposed a number of restrictions on the vector $E((y_{2n-1} - y_{2n+1})|\hat{x}_n)$. In this work we achieve the same result, more fundamentally, by placing differentiability restrictions on the continuous model $\hat{f}(\cdot)$ and of course on $f(\cdot)$ also. Thus, in general the approach taken here is to treat the entire estimation configuration of Figure 3.4

and in so doing place restrictions on $f(\cdot)$ and $\hat{f}(\cdot)$ which then guarantee the desired behavior of $E((y_{2n-1} - y_{2n+1}) \mid \hat{x}_n)$.

3.4.2.1 Theorems From Differential Equations

The following existence and uniqueness theorem for ordinary differential equations with controls is well-known [33]. We here paraphrase it in terms of the differential equations of the continuous model since ultimately we will want sufficient conditions under which the first partial derivatives with respect to parameters of the solution of this differential equation are continuous and bounded functions on a compact set. Real variables are assumed throughout. Reference Figure 3.4.

Theorem 3.1 [33]. Let functions $\hat{f}^i$ given by

$$\frac{d\hat{z}^i}{dt} = \hat{f}^i(\hat{z}, \hat{u}(t), t); \qquad \hat{z}^i(t=0) = \hat{\zeta}^i \tag{3.52}$$

$$(i = 1, 2, \ldots, n)$$

together with the partial derivatives $\partial\hat{f}^i/\partial\hat{z}^g$ $(i, g = 1, 2, \ldots, n)$ exist and be continuous functions from the cross product of open sets in E^{n+r+1} (given by $\hat{Z}^n \times \hat{U}^r \times (T_1, T_2)$) into E^1. Let $\hat{u}(t)$ be a vector of piecewise continuous functions from (t_1, t_2) into U^r, where the vector of values of $u(t)$ will be denoted by $(u^1, u^2, \ldots, u^r)'$. Then there exists a function $\hat{\psi}$ from an interval $(t_1, t_2) \subset (T_1, T_2)$ containing t_0 into $\hat{Z}^n$ with components $\hat{\psi}^i$ $(i = 1, 2, \ldots, n)$ such that $\hat{\psi}$ is a continuous function on (t_1, t_2), $\hat{\psi}(t=0) = \hat{\zeta}$, and $\hat{\psi}^i$ is a solution of

$$\frac{d\hat{\psi}^i}{dt} = \hat{f}^i(\hat{\psi}, \hat{u}(t), t); \qquad (i = 1, 2, \ldots, n) \tag{3.77}$$

for all but a countable set of points in (t_1, t_2). Furthermore, the solution $\hat{\Psi}$ is unique for the given $\hat{\zeta}$ and $\hat{u}(t)$ data.

Remark 1: While the above sets in E^{n+r+1} are open, the proof requires that (z, u, t) lie in corresponding compact convex (closed spheres) subsets of the open sets Z^n, U^r, and (T_1, T_2) respectively.

We next consider the case where $\hat{f}$ contains an h dimensional constant parameter vector $\hat{p}$. For reasons mentioned above, we desire that the solutions $\hat{\psi}^i$ $(i = 1, 2, \ldots, n)$, which will later be written informally as $\hat{z}^i$, be differentiable with respect to each $\hat{p}^j$ $(j = 1, 2, \ldots, h)$, and that these derivatives exist and be continuous functions on open sets (and hence bounded on a compact subset [67]). A theorem for this case is also well known [32, 80] and is here paraphrased in terms of the variables of the continuous model.

Theorem 3.2 [32,80]: Let the functions $\hat{f}^i$, given by

$$\frac{d\hat{z}^i}{dt} = \hat{f}^i(\hat{z}, \hat{p}, t), \quad \hat{z}^i(t{=}0) = \hat{\zeta}^i, \quad (i = 1, 2, \ldots, n), \tag{3.78}$$

together with the partial derivatives $\partial\hat{f}^i/\partial\hat{z}^g$ and $\partial\hat{f}^i/\partial\hat{p}^j$ $(i,g = 1, 2, \ldots, n)$, $(j = 1, 2, \ldots, h)$ exist and be continuous functions from a cross-product of open sets in E^{n+h+1} (given by $\hat{Z}^n \times \hat{P}^h \times (T_1, T_2)$) into an open set R in E^1, and let $\hat{f}$ satisfy a Lipschitz condition in $\hat{z}$ uniformly on $\hat{Z}^n \times \hat{P}^h \times (T_1, T_2)$. Then there exists a solution $\hat{\psi}$ from an interval $(t_1, t_2) \subset (T_1, T_2)$ containing t_0 into $\hat{Z}^n$ with components $(\hat{\psi}^1, \hat{\psi}^2, \ldots, \hat{\psi}^n)$ such that the $\hat{\psi}^i$ $(i = 1, 2, \ldots, n)$ are jointly continuous functions of $\hat{z}$, $\hat{p}$, and t, the $\partial\hat{\psi}^i/\partial\hat{p}^j$ and $\partial\hat{\psi}^i/\partial\hat{z}^g$ exist and are jointly continuous in t, $\hat{z}$, and $\hat{p}$, and the $\hat{\psi}^i$ are solutions of

$$\frac{d\hat{\psi}^i}{dt} = \hat{f}^i(\hat{\psi}, \hat{p}, t), \qquad (i = 1, 2, \ldots, n) \tag{3.79}$$

for all but a countable set of points t in (t_1, t_2), and $\hat{\psi}(t=0) = \hat{\zeta}$. Furthermore, the solutions $\hat{\psi}^i$ are unique for the given $\hat{\zeta}$ and $\hat{p}$ data.

<u>Remark 1:</u> The existence and continuity of the $\partial\hat{f}^i/\partial\hat{z}^g$ on a compact subset of $\hat{Z}^n \times \hat{P}^h \times (T_1, T_2)$ is a stronger sufficient condition than the Lipschitz condition for the uniqueness of the solution $\hat{\psi}$; see Theorem 3.1, Remark 1. Hence the requirement of the Lipschitz condition can here be omitted. In fact, the existence and continuity of the $\partial\hat{f}^i/\partial\hat{z}^g$ (on a compact subset of $\hat{Z}^n \times \hat{P}^h \times (T_1, T_2)$) imply the above Lipschitz condition [73]. Note that the existence and continuity of $\partial\hat{f}^i/\partial\hat{z}^g$ implies the existence and continuity of $\partial\hat{f}^i/\partial\hat{\zeta}^g$ [80].

<u>Remark 2:</u> Consider the system

$$\frac{d\hat{z}^i}{dt} = \hat{f}^i(\hat{z}, \hat{p}, \hat{u}(t), t); \quad \hat{z}(t=0) = \hat{\zeta}, \quad (i = 1, 2, \ldots, n) \tag{3.80}$$

Let the $\hat{f}^i$, $\partial\hat{f}^i/\partial\hat{z}^g$, and $\partial\hat{f}^i/\partial\hat{p}^j$, $(i, g = 1, 2, \ldots, n)$, $(j = 1, 2, \ldots, h)$ exist and be continuous functions from the cross-product of open set in $E^{n+h+r+1}$ (given by $\hat{Z}^n \times \hat{P}^h \times \hat{U}^r \times (T_1, T_2)$) into an open set R in E^1. Let $\hat{\zeta}$ belong to $\hat{Z}^n$, t_0 belong to (T_1, T_2), $\hat{p}$ belong to $\hat{P}^h$, and let $\hat{u}(t)$ be a piecewise continuous function from $(t_1, t_2) \subset (T_1, T_2)$ into $\hat{U}^r$ (i.e., let $\hat{u}(t)$ take its vector of values $(\hat{u}^1, \hat{u}^2, \ldots, \hat{u}^r)'$ in $\hat{U}^r$). Then, the hypotheses of Theorem 3.1 and Theorem 3.2 (with Remark 1) are satisfied and there exists a solution $\hat{\psi}$ from an interval $(t_1, t_2) \subset (T_1, T_2)$ containing t_0 into $\hat{Z}^n$ with components $(\hat{\psi}^1, \hat{\psi}^2, \ldots, \hat{\psi}^n)$ such that the $\hat{\psi}^i$, $\partial\hat{\psi}^i/\partial\hat{p}^j$, and $\partial\hat{\psi}^i/\partial\hat{z}^g$ are continuous functions of $(\hat{z}, \hat{p}, \hat{u}, t)$, and the $\hat{\psi}^i$ $(i = 1, 2, \ldots, n)$ are solutions of

$$\frac{\partial\hat{\psi}^i}{\partial t} = f(\hat{\psi}, \hat{p}, \hat{u}(t), t), \quad (i = 1, 2, \ldots, n), \tag{3.81}$$

on all but a countable set of points t in (t_1, t_2), and $\hat{\psi}(t{=}0) = \hat{\zeta}$. Furthermore, the solutions $\hat{\psi}^i$ are unique for the given $\hat{\zeta}$, $\hat{p}$, and $\hat{u}(t)$ data. As in Remark 1, the existence and continuity of $\partial\hat{f}^i/\partial\hat{z}^g$ imply the existence and continuity of $\partial\hat{f}^i/\partial\hat{\zeta}^g$.

We next incorporate $\hat{f}(\cdot)$ and $f(\cdot)$ in the feedback configuration of the parameter estimation scheme of Figure 3.4. Recall that the parameter vector of the sampled-data model is given by $\hat{x} = (\hat{p}, \hat{T}, \hat{\zeta})'$, and the vector $(y_{2n-1} - y_{2n+1})$ is defined by (3.68). The boundedness of the vector of partial derivatives $\frac{\partial E}{\partial \hat{x}}((y_{2n-1} - y_{2n+1})|\hat{x}_n)$ is an important requirement for our subsequent mean-square convergence proof of the K-W parameter estimation algorithm for the model-matching configuration of Figure 3.4. The following theorem states sufficient conditions such that the components of the vectors of partials

$\frac{\partial E}{\partial \hat{p}}((y_{2n-1} - y_{2n+1})|\hat{x}_n)$ and $\frac{\partial E}{\partial \hat{\zeta}}((y_{2n-1} - y_{2n+1})|\hat{x}_n)$ are bounded. The boundedness of the remaining partial derivative,

$\frac{\partial E}{\partial \hat{T}}((y_{2-1} - y_{2n+1})|\hat{x}_n)$ will be discussed in the sequel.

Theorem 3.3: Let the assumption on noise $n_1(t)$ given by (3.73) and (3.74) hold. Let the continuous system and continuous model of the model-matching parameter estimation scheme of Figure 3.4 be of identical form, and let the continuous model be given by

$$\frac{d\hat{z}^i}{dt} = \hat{f}(\hat{z}, \hat{p}, \hat{u}(t), t), \quad \hat{z}(t=0) = \hat{\zeta}, \quad (i = 1, 2, .., n) \tag{3.82}$$

where all notation is as in Theorems 3.1 and 3.2. Let the $\hat{f}$, $\partial \hat{f}^i/\partial \hat{z}^g$, and $\partial \hat{f}^i/\partial \hat{p}^j$ $(i, g = 1, 2, \ldots, n)$, $(j = 1, 2, \ldots, h)$ exist and be continuous functions from the cross product of open sets in $E^{n+h+r+1}$ given by $\hat{Z}^n \times \hat{P}^h \times \hat{U}^r \times (T_1, T_2)$ into an open set R of E^1, where $\hat{\zeta}$ belongs to $\hat{Z}^n$, $\hat{p}$ belongs to $\hat{P}^h$, and $\hat{u}(t)$ is a vector of piecewise continuous functions taking its vector of values $\hat{u}$ in $\hat{U}^r$. Specifically, let $\hat{u}(t)$, as obtained from the zero-order data hold of the sampled-data model of Figure 3.4, be given by

$$\hat{u}(t) = r(k_2\hat{T}) - \hat{z}^1(k_2\hat{T}) \tag{3.83}$$

where t: $k_2\hat{T} \le t < (k_2 + 1)\hat{T}$

and where $\hat{z}^1$ is the output components of the state vector of the sampled-data model $\hat{z}$ as defined below. Then the vector (3.75)

$$E((y_{2n-1} - y_{2n+1})|\hat{x}_n) = M_{2n-1} - M_{2n+1} \qquad (3.75)$$

is differentiable with respect to the model parameter vector $\hat{p}$ and the initial condition vector $\hat{\zeta}$, and the components of the vector derivative are continuous in $(\hat{z}, \hat{p}, \hat{u}, t)$ and are bounded when $(\hat{z}, \hat{p}, \hat{u}, t)$ belongs to a compact subset of $\hat{Z}^n \times \hat{P}^h \times \hat{U}^1 \times (T_1, T_2)$ in E^{n+h+2}.

Proof: The hypothesis is the same as that of Theorem 3.2 with Remarks 1, 2. Hence the solution $\hat{\psi}$ is unique, and the $\hat{\psi}^i$, $\partial\hat{\psi}^i/\partial\hat{z}^g$, and $\partial\hat{\psi}^i/\partial\hat{p}^j$ $(i, g = 1, 2, \ldots, n)$, $(j = 1, 2, \ldots, h)$ are continuous functions of $(\hat{z}, \hat{p}, \hat{u}, t)$ and the $\partial\hat{f}^i/\partial\hat{\zeta}^g$ are continuous functions of $(\hat{\zeta}, \hat{p}, \hat{u}, t)$. In particular, if $(\hat{\zeta}, \hat{p}, \hat{u}, t)$ is constrained to a compact subset of $Z^n \times P^h \times U^1 \times (T_1, T_2)$ then the continuous mappings $\partial\hat{\psi}^i/\partial\hat{\zeta}^g$ and $\partial\hat{\psi}^i/\partial\hat{p}^j$ are compact, and hence are bounded [67].

Hence, from (3.68) and (3.75), introducing appropriate notation and subscripts, representing $\hat{\psi}$ by $\hat{z}$ for notational convenience, and writing $\hat{q} = (\hat{p}, \hat{\zeta})$, we can express the $(h+n)m$ dimensional gradient vector

$$\frac{\partial E}{\partial \hat{q}}((y_{2n-1} - y_{2n+1})|\hat{x}_n) = \frac{\partial}{\partial \hat{q}}(M_{2n-1} - M_{2n+1})$$

$$= -2 \cdot \begin{bmatrix} \displaystyle\int_{t_n}^{t_n+\tau} \frac{\partial \hat{z}'}{\partial \hat{q}}(t;(\hat{x}_n - e^1 c_n), r(t)) W \left[z(t;x,r(t)) - \hat{z}(t;(\hat{x}_n - e^1 c_n, r(t)) \right] dt \\ \vdots \\ \displaystyle\int_{t_n+2(m-1)\tau}^{t_n+2m\tau} \frac{\partial \hat{z}'}{\partial \hat{q}}(t;\hat{x}_n - e^m c_n), r(t)) W \left[z(t;x,r(t)) - \hat{z}(t(\hat{x}_n - e^m c_n, r(t)) \right] dt \end{bmatrix}$$

$$+2 \cdot \begin{bmatrix} \displaystyle\int_{t_n}^{t_n+\tau} \frac{\partial \hat{z}'}{\partial \hat{q}}(t;(\hat{x}_n + e^1 c_n), r(t)) W \left[z(t;x,r(t)) - \hat{z}(t;\hat{x}_n + e^1 c_n), r(t)) \right] dt \\ \vdots \\ \displaystyle\int_{t_n+2(m-1)\tau}^{t_n+2m\tau} \frac{\partial \hat{z}'}{\partial \hat{q}}(t;\hat{x}_n + e^m c_n), r(t)) W \left[z(t;x,r(t)) - \hat{z}(t;(\hat{x}_n + e^m c_n), r(t)) \right] dt \end{bmatrix} \tag{3.84}$$

where $\partial/\partial\hat{q}$ is regarded as an (h+n) dimensional column vector. Because each component of this gradient vector is the definite integral (of a bounded function defined on a compact set) it is hence a continuous function defined on the above compact set. Hence it is also bounded [67].

Remark 1: Since the components of (3.84) are bounded then $\left\| \frac{\partial E}{\partial \hat{q}} (y_{2n-1} - y_{2n+1}) | \hat{x}_n) \right\|$ is also bounded for the assumed conditions on the noise $n_1(t)$). Then there exist constants $0 \leq K_0 \leq K_1 < \infty$ such that

$$K_0 \| \hat{x}_n - \theta \| \leq \left\| \frac{\partial E}{\partial \hat{q}} ((y_{2n-1} - y_{2n+1}) | \hat{x}_n) \right\| \leq K_1 \| \hat{x}_n - \theta \| \qquad (3.85)$$

where θ is the true vector of parameters of the sampled-data system as given by (3.53) and $\hat{q}_n = (\hat{p}_n, \hat{\zeta}_n)'$.

Remark 2: By the above treatment, we have established the boundedness of components of the vectors $\frac{\partial E}{\partial \hat{p}} ((y_{2n-1} - y_{2n+1}) | \hat{x}_n)$ and $\frac{\partial E}{\partial \hat{\zeta}} ((y_{2n-1} - y_{2n+1}) | \hat{x}_n)$. The remaining vector of $\frac{\partial E}{\partial \hat{x}} ((y_{2n-1} - y_{2n+1}) | \hat{x}_n)$ is $\frac{\partial E}{\partial \hat{T}} ((y_{2n-1} - y_{2n+1}) | \hat{x}_n)$. The treatment for this vector is slightly more involved. The most convenient approach is to use (3.58) and determine whether $\frac{\partial E J}{\partial \hat{T}}(t_n + \tau; t_n, x, \hat{x}_n, r(t))$ is bounded for values of $\hat{T}_n$ selected from the possible range of sampling intervals. We can use the approximation for the partial derivative given by (2.26). Hence using (3.58), and for notational simplicity suppressing all but the significant parameters, an approximation to the partial derivative is

$$\frac{\partial E(J(\hat{x}_n))}{\partial \hat{T}} \simeq \frac{E\left[J(\hat{q}_n, \hat{T}_n + \Delta\hat{T}) - J(\hat{q}_n, \hat{T}_n)\right]}{\Delta \hat{T}} \qquad (3.86)$$

Using the above assumption on the noise $n_1(t)$ and (3.86) the approximation to the vector is obtained by differentiating (3.75) to obtain

$$\frac{\partial E}{\partial \hat{T}}\{(y_{2n-1} - y_{2n+1}) | \hat{x}_n\} = \frac{\partial}{\partial \hat{T}}(M_{2n-1} - M_{2n+1}), \tag{3.87}$$

An approximation to the statistical expectation required in (3.87) can be computed by time averaging by using (3.58) as follows:

$$\frac{\partial}{\partial \hat{T}}(M_{2n-1} - M_{2n+1}) \simeq \frac{E}{\Delta \hat{T}} \left\{ \begin{matrix} \begin{bmatrix} J((\hat{x}_n - e^1 c_n), \hat{T}_n + \Delta\hat{T}) - J((\hat{x}_n - e^1 c_n), \hat{T}_n) \\ -(J((\hat{x}_n + e^1 c_n), \hat{T}_n + \Delta\hat{T}) - J((\hat{x}_n + e^1 c_n), \hat{T}_n)) \end{bmatrix} \\ \vdots \\ \begin{bmatrix} J((\hat{x}_n - e^j c_n), \hat{T}_n + \Delta\hat{T}) - J((\hat{x}_n - e^j c_n), \hat{T}_n) \\ -(J((\hat{x}_n + e^j c_n), \hat{T}_n + \Delta\hat{T}) - J((\hat{x}_n + e^j c_n), \hat{T}_n)) \end{bmatrix} \\ \vdots \\ \begin{bmatrix} J((\hat{x}_n - e^m c_n), \hat{T}_n + \Delta\hat{T}) - J((\hat{x}_n - e^m c_n, \hat{T}_n) \\ -(J((\hat{x}_n + e^m c_n), \hat{T}_n + \Delta\hat{T}) - J((\hat{x}_n + e^m c_n, \hat{T}_n)) \end{bmatrix} \end{matrix} \right\}$$

$$j = (2, 3, \ldots, (m-1)) \tag{3.88}$$

where $E(\bullet)$ is here defined as the time average.

In the sequel, we will proceed on the basis of the assumption that every component of the right side of (3.88) is bounded for each selected value of $\hat{T}_n$ when $\hat{T}_n$ is allowed to vary over the range of possible values that $\hat{T}_n$ can assume. Hence, we will have

bounds on all of the components of $\frac{\partial E}{\partial \hat{x}}((y_{2n-1} - y_{2n+1}) | \hat{x}_n)$.
Thus by referencing (3.85), we can write

$$K_0 \| \hat{x}_n - \theta \| \leq \| \frac{\partial E}{\partial \hat{x}}((y_{2n-1} - y_{2n+1}) | \hat{x}_n \| \leq K_1 \quad \| \hat{x}_n - \theta \| \tag{3.89}$$

3.4.2.2 Convergence Proof of K-W Procedure For Parameter Estimation By Model-Matching

The following summarizes the above assumptions and presents the proof of mean-square convergence of the K-W procedure (3.69) for the modeling configuration of Figure 3.4.

Theorem 3.4: Let there exist a parameter vector θ for which a unique minimum of the cost function of (3.58) exists (when $n_1(t)$ is zero). Let $f(\cdot)$ and $\hat{f}(\cdot)$ be of identical form and satisfy the hypotheses of Theorems 3.1, 3.2, and 3.3, and the assumption in connection with (3.88), as well as the following hypothesis:

A) Assume that the observation noise $n_1(t)$ is stationary and has the properties

1) $$\| E\left\{n_1(t)\right\} \| = 0 \tag{3.73}$$

2) $$\left\| \begin{bmatrix} u^{11}, & \ldots & \ldots & u^{1n} \\ \vdots & & & \vdots \\ \cdots & & u^{ij} & \cdots \\ \vdots & & & \vdots \\ u^{n1}, & \ldots & \ldots & u^{nn} \end{bmatrix} \right\| = \sigma^2_{n_1 n_1} < \infty \tag{3.90}$$

where $u^{ij} = E(n_1^i(t_1) n_1^j(t_2))$, $(i, j = 1, 2, \ldots, n)$.

3) $$\left[\Pr \left| n_1^i(t) \right| \leq C\right] = 1, (i, j = 1, 2, \ldots, n) \tag{3.76}$$

4) $\| E(n_1(t_1) z'(t_2;x) \| = \| E(n_1(t_1) \hat{z}'(t_2;\hat{x}). = 0$ (3.74)

B) Use the Kiefer-Wolfowitz procedure

$$\hat{x}_{n+1} = \hat{x}_n + \frac{a_n}{c_n} (y_{2n-1} - y_{2n+1}) \tag{3.69}$$

to estimate the true parameter vector $\theta = x$ of the sampled-data system, and assume that $\hat{x}$ and θ belong to a compact set in E^m, where $m \leq (2n+1)$, and where x and $\hat{x}$ are given by (3.53) and (3.54) respectively.

C) Assume that the sequences $\{a_n\}$ and $\{c_n\}$ will have the properties

$$1)\quad \sum_{n=1}^{\infty} \frac{a_n}{c_n} = \infty, \qquad 2)\quad \sum_{n=1}^{\infty} \left(\frac{a_n}{c_n}\right)^2 < \infty \tag{3.59}$$

$$3)\quad \lim_{n \to \infty} c_n = \lim_{n \to \infty} a_n = 0,$$

Specifically, $\{a_n\}$ and $\{c_n\}$ will be given by (3.60).

D) Assume that the components of the random vectors y_{2n-1} and y_{2n+1} are given by (3.64) to (3.67) and that these y^i_{2n-1} and y^i_{2n+1} $(i = 1, 2, \ldots, m)$ are statistically independent with probability distribution functions $H(y|\hat{x}_n - e^i c_n)$ and $H(y|\hat{x}_n + e^i c_n)$, $(i = 1, 2, \ldots, m)$ respectively.

E) Assume

$$E\left[\|\hat{x}_1 - \theta\|^2\right] < K < \infty \tag{3.91}$$

where $\hat{x}_1$ is chosen as the first approximation to $x = \theta$.

Then the K-W procedure of (3.69) converges to θ in mean-square. Moreover, the estimate is asympototically unbiased, i.e., $\lim_{n\to\infty} E(\hat{x}_n) = \theta$.

Proof: Using (3.69), take the inner product of the error in parameter estimation $(\hat{x}_{n+1} - \theta)$ with itself.

$$\| \hat{x}_{n+1} - \theta \|^2 = \| \hat{x}_n - \theta \|^2 + 2\left(\hat{x}_n - \theta, \frac{a_n}{c_n}(y_{2n-1} - y_{2n+1})\right) + \left(\frac{a_n}{c_n}\right)^2 \| y_{2n-1} - y_{2n+1} \|^2 \tag{3.92}$$

Recalling that $y_{2n\pm1}$ is a vector of random variables conditioned on the random parameter sequence $\left\{\hat{x}_n, \hat{x}_{n-1}, \ldots, \hat{x}_1\right\}$, which will here be written as either $\left\{\hat{x}_n\right\}$ or $\hat{x}_n$, we can write the expectation [69] of the left side of (3.92) as

$$E\left[\| \hat{x}_{n+1} - \theta \|^2\right] = E\left[E\left[\|\hat{x}_{n+1} - \theta \|^2 | \hat{x}_n\right]\right] \tag{3.93}$$

Next, take the conditional expectation of (3.92)

$$E\left[\| \hat{x}_{n+1} - \theta \|^2 | \hat{x}_n\right]$$

$$= \|\hat{x}_n - \theta\|^2 + 2\left((\hat{x}_n - \theta), \frac{a_n}{c_n} E\left((y_{2n-1} - y_{2n+1}) | \hat{x}_n\right)\right) + \left(\frac{a_n}{c_n}\right)^2 E\left[\|y_{2n-1} - y_{2n+1}\|^2 | \hat{x}_n\right] \tag{3.94}$$

To treat the right side of (3.94), note that for the second term

$$E((y_{2n-1} - y_{2n+1})|\hat{x}_n) = E\left[\begin{array}{c|c} y^1_{2n-1} - y^1_{2n+1} & \\ \vdots & \hat{x}_n \\ y^m_{2n-1} - y^m_{2n+1} & \end{array}\right]. \tag{3.95}$$

From (3.64) to (3.67) the components of (3.95) are:

$$E(y^i_{2n+1}|\hat{x}_n) = E\int_{t_n+2(i-1)\tau}^{t_n+(2i-1)\tau} \sum_{j=1}^{n} w^j(\epsilon^j(t;x,(\hat{x}_n+e^i c_n),r(t)))^2\, dt, \tag{3.96}$$

$$(i = 1,\ 2,\ \ldots,\ m)$$

and

$$E(y^i_{2n-1}|\hat{x}_n) = E\int_{t_n+(2i-1)\tau}^{t_n+2i\tau} \sum_{j=1}^{n} w^j(\epsilon(t;x,(\hat{x}_n-e^i c_n),r(t)))^2\, dt \tag{3.97}$$

$$(i = 1,\ 2,\ \ldots,\ m)$$

Using Assumption (A) and (3.96) and (3.97), (3.95) reduces to

$$E((y_{2n-1} - y_{2n+1})|\hat{x}_n) = M_{2n-1} - M_{2n+1}. \tag{3.98}$$

where M_{2n-1} and M_{2n+1} are defined by (3.71) and (3.72).

From (3.58), we see that the integrand of $J(\cdot)$ is a quadratic form, thus $J(\cdot)$ is at least locally convex in $\hat{x}$ for $\hat{x}$ near θ. Hence, if $\hat{x}_n \neq \theta$, the inner product of vectors

$$((\hat{x}_n - \theta), E((y_{2n-1} - y_{2n+1})|\hat{x}_n)) < 0. \tag{3.99}$$

Consequently, for some constant $K_0 > 0$ and $\hat{x}_n \neq \theta$

$$\left((\hat{x}_n - \theta), E\left((y_{2n-1} - y_{2n+1})\middle|\hat{x}_n\right)\right) < -K_0 \|\hat{x}_n - \theta\|^2. \qquad (3.100)$$

The third term of (3.94) is treated by noting that the definition of the conditional covariance [69] of a random vector y, conditioned on a parameter vector x, is given by

$$\begin{aligned} \operatorname{cov}\left[y|x\right] &= E\left[(y - E(y|x))\,(y - E(y|x))' \,|\, x\right] \\ &= E\,(y\,y'|x) - E(y|x)\,(E(y|x))' \end{aligned} \qquad (3.101)$$

Therefore,

$$E(y\,y'|x) = E(y|x)(E(y|x))' + \operatorname{cov}(y|x) \qquad (3.102)$$

The trace of (3.102) is

$$\operatorname{tr}\left[E(y\,y'|x)\right] = \operatorname{tr}\left[E(y|x)(E(y|x))'\right] + \operatorname{tr}\left[\operatorname{cov}(y|x)\right] \qquad (3.103)$$

Hence, for y an m vector, (3.103) reduces to

$$E\left[\|y\|^2|x\right] = \|E(y|x)\|^2 + \sum_{i=1}^{m} \sigma^2(y^i|x) \qquad (3.104)$$

where $\sigma^2(y^i|x)$ is the scalar variance of the random variable y^i conditioned on the vector x. Applying this result to the third term of (3.94)

$$E\left[\|y_{2n-1} - y_{2n+1}\|^2 | \hat{x}_n\right]$$

$$= \|E((y_{2n-1} - y_{2n+1}) | \hat{x}_n)\|^2 + \sum_{i=1}^{m} \sigma^2 \left[(y^i_{2n-1} - y_{2n+1}) | \hat{x}_n\right] \tag{3.105}$$

Using (3.98), (3.105) reduces to

$$E\left[\|y_{2n-1} - y_{2n+1}\|^2 | \hat{x}_n\right]$$

$$= \|M_{2n-1} - M_{2n+1}\|^2 + \sum_{i=1}^{m} \sigma^2 \left[(y^i_{2n-1} - y^i_{2n+1}) | \hat{x}_n\right] \tag{3.106}$$

where $\sigma^2 [\cdot]$ represents the variance of $[\cdot]$.

From assumption (A2), the terms of the noise covariance matrix are bounded. Hence, the terms of the covariance of the mappings of the noise (3.66) and (3.67) are also bounded. Consequently,

$$\sum_{i=1}^{m} \sigma^2 \left[(y^i_{2n-1} - y^i_{2n+1}) | \hat{x}_n\right] \le \sum_{i=1}^{m} k_i \sigma^2_{n_1 n_1} \le \sigma^2 < \infty \tag{3.107}$$

where the constants $0 \le k_i < \infty$, $(i = 1, 2, \ldots, m)$.

From Theorem 3.3, $M_{2n\pm1}$ is differentiable, hence we can approximate M_{2n-1} and M_{2n+1} by the first terms of a Taylor's series expansion about $\hat{x}_n$

$$M_{2n-1} \simeq M(\hat{x}_n) - \frac{\partial M}{\partial \hat{x}} (\hat{x}_n)\, d\, c_n \tag{3.108a}$$

$$M_{2n+1} \simeq M(\hat{x}_n) + \frac{\partial M}{\partial \hat{x}} (\hat{x}_n)\, d\, c_n \tag{3.108b}$$

where (dc_n) is an m dimensional vector $(c_n, c_n, \ldots, c_n)'$, and where by using (3.68), we define

$$M(\hat{x}_n) = E\,(y_{2n\pm1} | c_n = 0). \tag{3.109}$$

Recalling (3.89) and using (3.108a) and (3.108b)

$$\| M_{2n-1} - M_{2n+1} \|^2 \simeq \| 2 \frac{\partial M(\hat{x}_n)}{\partial \hat{x}} dc_n \|^2 \le K_1 \| \hat{x}_n - \theta \|^2 . \tag{3.110}$$

Using (3.100), (3.106), (3.107) and (3.110) in (3.94), and taking expectations of both sides

$$E\left\{E\left[\| \hat{x}_{n+1} - \theta \|^2 \; \hat{x}_n\right]\right\} \le E\left[\| \hat{x}_n - \theta \|^2\right] - 2 \frac{a_n}{c_n} K_0 E\left[\| \hat{x}_n - \theta \|^2\right]$$
$$+ \left(\frac{a_n}{c_n}\right)^2 E\left[K_1^2 \| \hat{x}_n - \theta \|^2 + \sigma^2\right] \tag{3.111}$$

By using (3.93), (3.111) reduces to

$$E\left[\| \hat{x}_{n+1} - \theta \|^2\right] \le E\left[\| \hat{x}_n - \theta \|^2\right]\left[1 - 2\frac{a_n}{c_n} K_0 + K_1^2 \left(\frac{a_n}{c_n}\right)^2\right] + \left(\frac{a_n}{c_n}\right)^2 \sigma^2 \tag{3.112}$$

From (3.89) we are free to take $K_0 = K_1$ so that

$$E\left[\| \hat{x}_{n+1} - \theta \|^2\right] \le E\left[\| \hat{x}_n - \theta \|^2\right]\left[1 - K_1 \frac{a_n}{c_n}\right]^2 + \left(\frac{a_n}{c_n}\right)^2 \sigma^2 \tag{3.113}$$

Define $E \| \hat{x}_n - \theta \|^2 = b_n$, and iterate (3.113) to obtain

$$b_{n+1} \le b_1 \prod_{i=1}^{n}\left(1-K_1\frac{a_i}{c_i}\right)^2 + \sigma^2\left[\sum_{i=1}^{n-1}\left(\frac{a_i}{c_i}\right)^2\prod_{k=i+1}^{n}\left(1-\frac{a_k}{c_k}K_1\right)^2+\left(\frac{a_n}{c_n}\right)^2\right]$$

where $(n = 1, 2, \ldots,)$. (3.114)

It is shown in the Appendix that $\sum_{n=1}^{\infty}\left(\frac{a_n}{c_n}\right)^2 < \infty$ implies

$$\lim_{n\to\infty}\left(\frac{a_n}{c_n}\right)^2 = 0, \tag{3.115}$$

and

$$\lim_{n\to\infty}\left(\frac{a_n}{c_n}\right) = 0. \tag{3.116}$$

Hence, there is a (finite) n_o such that

$$\left(1 - K_1\frac{a_n}{c_n}\right)^2 \le \left(1 - K_1\frac{a_n}{c_n}\right) \quad \text{for } n \ge n_o. \tag{3.117}$$

Rewriting (3.114) in view of (3.117)

$$\begin{aligned} b_{n+1} \le{} & b_1 \prod_{i=1}^{n_o-1}\left(1-\frac{a_i}{c_i}K_1\right)^2 \prod_{j=n_o}^{n}\left(1-\frac{a_j}{c_j}K_1\right)^2 \\ & + \sigma^2 \sum_{i=1}^{n_o-1}\left(\frac{a_i}{c_i}\right)^2 \prod_{k=i+1}^{n_o-1}\left(1-\frac{a_k}{c_k}K_1\right)^2 \prod_{j=n_o}^{n}\left(1-\frac{a_j}{c_j}K_1\right)^2 \\ & + \sigma^2 \sum_{i=n_o}^{n-1}\left(\frac{a_i}{c_i}\right)^2 \prod_{k=i+1}^{n}\left(1-\frac{a_k}{c_k}K_1\right)^2 + \sigma^2\left(\frac{a_n}{c_n}\right)^2 \end{aligned} \tag{3.118}$$

This can also be written

$$b_{n+1} \le b_{n_o} \prod_{j=n_o}^{n} \left(1 - \frac{a_j}{c_j} K_1\right)^2 + \sigma^2 \sum_{i=1}^{n_o-1} \left(\frac{a_i}{c_i}\right)^2 K_4 \prod_{j=n_o}^{n} \left(1 - \frac{a_j}{c_j} K_1\right)^2$$

$$+ \sigma^2 \sum_{i=n_o}^{n-1} \left(\frac{a_i}{c_i}\right)^2 \prod_{k=i+1}^{n} \left(1 - \frac{a_k}{c_k} K_1\right)^2 + \sigma^2 \left(\frac{a_n}{c_n}\right)^2 \quad (3.119)$$

where, from Assumptions (C) and (E),

$$b_{n_o} \triangleq b_1 \prod_{i=1}^{n_o-1} \left(1 - \frac{a_i}{c_i} K_1\right)^2 \le K_3 < \infty \quad (3.120)$$

and where (since n_o is fixed and $0 \le n_o < \infty$, and using Assumption C(1,2) and the fact that $K_1 < \infty$) we can bound the partial product in (3.118) to obtain

$$\prod_{k=i+1}^{n_o-1} \left(1 - \frac{a_k}{c_k} K_1\right)^2 \le K_4 < \infty. \quad (3.121)$$

$$(i = 1, 2, \ldots, n_o-1)$$

Now from (3.115), for the last term of (3.119) we have

$$\lim_{n \to \infty} \sigma^2 \left(\frac{a_n}{c_n}\right)^2 = 0. \quad (3.122)$$

Using (3.117), for the first term of (3.119) we have

$$b_{n_o} \prod_{j=n_o}^{\infty} \left(1 - \frac{a_j}{c_j} K_1\right)^2 \le b_{n_o} \prod_{j=n_o}^{\infty} \left(1 - \frac{a_j}{c_j} K_1\right) \quad (3.123)$$

Next, use the inequality [71]

$$(1 - \frac{a_j}{c_j} K_1) \le e^{-\frac{a_j}{c_j} K_1}, \tag{3.124}$$

which is true for all $\frac{a_j}{c_j} K_1$.

Using (3.120) and (3.124), along with Assumption (C1), (3.123) can be written, in the limit, as

$$b_{n_o} \prod_{j=n_o}^{\infty} (1 - \frac{a_j}{c_j} K_1)^2 \le b_{n_o} \exp\left(- \sum_{j=n_o}^{\infty} \frac{a_j}{c_j} K_1\right) = 0 \tag{3.125}$$

for $n_o < \infty$.

Following Dupac [56], we next use Kronecker's Theorem [71, 72] to show the convergence of the summation terms of (3.119). This theorem is here paraphrased in terms of the notation of (3.119).

Theorem [71] If $\sum_{n=1}^{\infty} \left(\frac{a_n}{c_n}\right)^2$ is a convergent series of arbitrary terms and if $(P_1, P_2, \ldots)$ denotes an arbitrary monotone increasing sequence of positive numbers tending to $+\infty$, then the ratio

$$\frac{P_1 \left(\frac{a_1}{c_1}\right)^2 + P_2 \left(\frac{a_2}{c_2}\right)^2 + \ldots + P_n \left(\frac{a_n}{c_n}\right)^2}{P_n} \to 0 \tag{3.126}$$

To use this result in connection with (3.119) note, from Assumption (C2), that $\sum_{n=1}^{\infty} \left(\frac{a_n}{c_n}\right)^2 < \infty$ and consequently $\lim_{n \to \infty} \sum_{i=n_o}^{n-1} \left(\frac{a_i}{c_i}\right)^2 < \infty$ as well. Next, define (3.127)

$$P_j = \frac{1}{\prod_{j=i+1}^{n} (1 - \frac{a_i}{c_j} K_1)^2} \tag{3.128}$$

where i is any integer $i \in [1, n]$ and where, from (3.125)

$$\lim_{n \to \infty} P_j = \infty \tag{3.129}$$

Also, for example,

$$P_2 = \frac{1}{\prod_{i=2}^{n} (1 - \frac{a_i}{c_i} K_1)^2} < P_3 \tag{3.130}$$

which establishes the monotonicity of the sequence.

Next, write out terms of the last summation of (3.119)

$$\sum_{i=n_o}^{n-1} \left(\frac{a_i}{c_i}\right)^2 \prod_{k=i+1}^{n} \left(1 - \frac{a_k}{c_k} K_1\right)^2 \tag{3.131}$$

$$= \left(\frac{a_{n_o}}{c_{n_o}}\right)^2 \prod_{k=n_o+1}^{n} \left(1 - \frac{a_k}{c_k} K_1\right)^2 + \left(\frac{a_{n_o+1}}{c_{n_o+1}}\right)^2 \prod_{k=n_o+2}^{n} \left(1 - \frac{a_k}{c_k} K_1\right)^2$$

$$+ \ldots + \left(\frac{a_{n-2}}{c_{n-2}}\right)^2 \prod_{k=n-1}^{n} \left(1 - \frac{a_k}{c_k} K_1\right)^2 + \left(\frac{a_{n-1}}{c_{n-1}}\right)^2 \left(1 - \frac{a_n}{c_n} K_1\right)^2$$

and multiply and divide by $1 \Big/ \prod_{k=n_o+1}^{n} \left(1 - \frac{a_k}{c_k} K_1\right)^2$ and apply Kronecker's Theorem (3.126), with the result:

$$\lim_{n\to\infty} \sum_{i=n_o}^{n-1} \left(\frac{a_i}{c_i}\right)^2 \prod_{k=i+1}^{n} \left(1 - \frac{a_k}{c_k} K_1\right)^2 \tag{3.132}$$

$$= \lim_{n\to\infty} \left[\frac{\left(\frac{a_{n_o}}{c_{n_o}}\right)^2 + \left(\frac{a_{n_o+1}}{c_{n_o+1}}\right)^2 \frac{1}{\left(1 - \frac{a_{n_o+1}}{c_{n_o+1}} K_1\right)} + \left(\frac{a_{n_o+2}}{c_{n_o+2}}\right)^2 \frac{1}{\prod_{k=n_o+1}^{n_o+2} \left(1 - \frac{a_k}{c_k} K_1\right)^2} + \cdots + \left(\frac{a_{n-1}}{c_{n-1}}\right)^2 \frac{1}{\prod_{k=n_o+1}^{n-1} \left(1 - \frac{a_k}{c_k} K_1\right)^2}}{\frac{1}{\prod_{k=n_o+1}^{n} \left(1 - \frac{a_k}{c_k} K_1\right)^2}} \right]$$

$$= 0.$$

The convergence of the remaining terms (involving K_4) of (3.119) follows because each term is bounded by a corresponding term from (3.131). Thus, from (3.122), (3.125) and (3.132) we conclude

$$\lim_{n\to\infty} E\left[\|\hat{x}_{n+1} - \theta\|\right]^2 = \lim_{n\to\infty} b_{n+1} = 0 \tag{3.133}$$

which is the desired mean-square convergence.

Remark 1: Our derived equations (3.98), (3.100), (3.106), and (3.110) are essentially the same as several assumptions Kirvaitis [24] made regarding the behavior of the estimation system. In his dissertation, these assumptions are given by his equations (2.25), (2.23), (2.24), and (2.22) respectively.

Remark 2: Mean-square convergence implies convergence is probability [88,89]. This is written

$$\lim_{n\to\infty} \Pr\left\{\|\hat{x}_n - \theta\| > \epsilon\right\} = 0 \tag{3.134}$$

An estimate $\hat{x}_n$ with this property is terms a consistent estimate [88].

Remark 3: We can show that the parameter estimate is asymptotically unbiased by expanding the left side of (3.133)

$$\lim_{n\to\infty} E\|x_n - \theta\|^2 = \lim_{n\to\infty} E\|(\hat{x}_n - E(\hat{x}_n)) - (\theta - E(\hat{x}_n))\|^2$$

$$= \lim_{n\to\infty} \left\{E\|\hat{x}_n - E(\hat{x}_n)\|^2 - 2E((\hat{x}_n - E(\hat{x}_n)), (\theta - E(\hat{x}_n))) + E\|\theta - E(\hat{x}_n)\|^2\right\}$$

$$= \lim_{n\to\infty} \left\{\text{tr cov }(\hat{x}_n) + E\|\theta - E(\hat{x}_n)\|^2\right\} \tag{3.135}$$

Now (3.135) is composed of two non-negative terms. Hence, in view of (3.133), both of these terms are zero when mean-square convergence occurs. The term

$$\theta - E(\hat{x}_n) \tag{3.136}$$

is commonly called the bias of the estimator [115]. Clearly, mean-square convergence implies that the estimate $\hat{x}_n$ (of the parameter vector θ) obtained from (3.69) is asymptotically unbiased as $n\to\infty$, i.e.,

$$\lim_{n\to\infty} E(\hat{x}_n) - \theta = 0 \tag{3.137}$$

Remark 4: Note that no knowledge of the statistical conditional distribution functions $H(y|x_n - c_n)$ and $H(y|x_n + c_n)$ was required.

CHAPTER 4

SIMULATION STUDIES

4.1 Introduction

Simulation studies were undertaken to demonstrate the application of stochastic approximation to the estimation of dynamic system parameters when it could be assumed that a model which exactly matched the form of the system was known a priori. Reference Figure 3.4. In preparation for the studies involving the human operator, to be reported in Chapter 5, only the scalar output of both model and system were used in generating the cost function. Various levels of scalar observation noise $n_1(t)$ were introduced and, in addition, parameter noises were also introduced in some cases so as to study the effects on parameter estimates of the random behavior of all of the modeled parameters, including the sampling interval.

Simulations were performed on the IBM 360-44 digital computer. The IBM-supplied continuous system modeling program (CSMP), which was originally designed for the IBM-1130 digital computer, was modified for usage on the IBM-360-44. Various special control subroutines were developed so that the basic CSMP program could be used iteratively in parameter estimation. All simulations were performed by means of this special CSMP program. For example, Subroutine 1, described in the Appendix, is the main control program for the stochastic approximation algorithm and iteration procedure. It implements the K-W algorithm, (3.69). Other special subroutines will be referred to in the sequel. Listings for representative programs are given in the Appendix.

Parameter noises and observation noise were obtained from digital white noise generators designed to yield numerical sequences approximately uniformly distributed between -1 and +1. The generators could be called through the CSMP program. The basic noise sequence generator, in Fortran notation, is typically represented by

```
        IR   = 7243
1       IR   = 259*IR
        C(I) = FLOAT(IR)*2.0**(-31.0)
        GO TO 1                                  (4.1)
```

where IR in an odd integer (ordinarily specified internally in the program) and where C(I) denotes the output of the simulation noise sequence generator whose number is given by I. For a 32 bit digital computer, this sequence generator will produce 2^{30} terms before repeating [74]. Hence, for our purposes, the sequences are random because we will deal with sequences in the order of 2^{11} terms or less.[1] In the sequel, these approximately uniformly distributed noise generators will be represented by the equation

---

[1]For the CSMP program, the generator of (4.1) outputs two members of the random sequence during each integration interval (0.01 second). The iteration interval was 10.0 seconds or less. Hence, no more than 2000 members of the random sequence were required during a particular iteration.

$$n(t) = k_1 \left[-1, +1\right] + k_2 \tag{4.2}$$

where k_1 is the maximum amplitude of the noise sequence numbers and k_2 is the desired mean value.

Both linear and nonlinear systems were modeled. All notation on simulation diagrams corresponds to conventional analog computer usage.

Generally, convergence time of the parameter estimates depended on the level of the parameter noise present. For cases where only zero-mean observation noise was present, convergence of the model parameters to the true values of the system parameters occurred. When observation noise did not have zero mean, it was found to induce a slight biasing of the parameter estimates proportional to the mean value of the observation noise. This is attributed to the fact that Assumption (A) of Chapter 3 was not then satisfied. The presence of parameter noises (also described by 4.2)) caused small biases to occur in parameter estimates.

A different effect on parameter estimation resulted if the input signal to both system and model did not have zero mean value: The convergence rate of the sampling interval estimate was very much reduced. This was true whether or not observation noise and/or parameter noise was present. Therefore, when dealing with actual time history sequences, as is done in the next chapter, care must be taken to insure that the iteration time (τ) is chosen such that the input signal has zero mean value.

In summary, the simulation results are as follows:

a) The sampling interval and gain of a first order linear closed loop sampled-data system were accurately estimated in the presence of various levels of additive observation noise.

b) The sampling interval, gain, and time constant of a second order linear closed-loop sampled-data system were accurately estimated in the presence of various levels of additive observation noise.

c) Good, but less accurate, estimates of the above parameters were obtained when randomness was introduced into each parameter. When the ratio of the maximum random deviation of the parameter to its constant nominal value was as high as unity, estimation accuracies were still 90% or better.

d) Good estimates were also obtained in the presence of both random parameters and additive output observation noise.

e) The presence of a d.c. term in the input signal had the effect of introducing a slight bias into parameter estimates which depended on the size of the d.c. component.

4.2 Simulation Examples

4.2.1 Example 1: Linear First Order Continuous System And Model

Referring to Figure 3.4, the continuous system and continuous model are given by the linear differential equations

$$\dot{z}^1 = Ku(t) \qquad z_o^1 = 0$$

and

$$\dot{\hat{z}}^1 = \hat{K}\hat{u}(t) \qquad \hat{z}_o^1 = 0 \tag{4.3}$$

where z, $\hat{z}$, K, $\hat{K}$, u, and $\hat{u}$ are scalars. The cost function is given by (3.58). The complete sampled-data system parameter vector is the two dimensional vector

$$x = \begin{bmatrix} K \\ T \end{bmatrix} \tag{4.4}$$

and the sampled-data model parameter vector is the two dimensional vector

$$\hat{x} = \begin{bmatrix} \hat{K} \\ \hat{T} \end{bmatrix} \tag{4.5}$$

From the basic fact that for a closed-loop sampled-data system instability occurs if either, or both, T and K are too large [73], the initial estimates $\hat{T}_1$ and $\hat{K}_1$ were selected so that the closed-loop model was stable. Since all variables are scalar, and taking $w_1 = 1.0$ in (3.58), the cost function is written

$$J(t_n+\tau;t_n,x,\hat{x},r(t)) = \int_{t_n}^{t_n+\tau} (z^1(t;x,r(t))+n_1(t)-\hat{z}^1(t;\hat{x},r(t)))^2 dt \tag{4.6}$$

The K-W procedure is given by the algorithm (3.69)

$$\hat{x}_{n+1} = \hat{x}_n + a_n(y_{2n-1} - y_{2n+1})/c_n \tag{4.7}$$

where the $y_{2n\pm1}$ are defined by (3.64) to (3.67) with $m = 2$, and where a_n and c_n are given by (3.60).

The driving signal consisted of either a single low frequency sine wave or a random signal. The sinusoid was

$$r(t) = 20.0 \sin(.63t) \tag{4.8}$$

where $\omega_c = .63$ was chosen as representative of the low frequency content of human operator test signals [27]. The iteration interval was chosen such that $r(t)$ would have mean value of zero.

The random signal was given by

$$r_n(t) = n_0(t) + k_0 \tag{4.9}$$

where k_0 is a constant selected, in general, to remove the inherent bias of $n_0(t)$, and $n_0(t)$ is the output of a second order filter

$$F(s) = \frac{K_f \, \omega_c^{\,2}}{s^2 + 2\zeta s\omega_c + (\omega_c)^2} \tag{4.10}$$

when it is driven by the uniformly distributed zero mean white noise sequence generator of (4.2). The gain K_f was chosen such that the relative energy of the signal $r_n(t)$ would be the same as that of $r(t)$, i.e., so that over the particular iteration interval τ

$$\int_0^{\tau} (20 \sin(.63t))^2 dt = \int_0^{\tau} (r_n(t))^2 dt \tag{4.11}$$

In (4.10), the cutoff frequence $\omega_c^{\,2} = .63$ was chosen to agree with the approximate bandpass of the drive signal used with the human operator experiments [27] which will bereported in Chapter 5. In (4.9), the value of k_0 depends on the iteration interval τ and is given by

$$k_0 = \int_0^T n_0(t)\, dt \tag{4.12}$$

For τ = 4.0 seconds, $k_0 \simeq 10.8$ for the filter of (4.10), when $\zeta = .49$ and $\omega_c^2 = .63$. However, in the following studies, we will not always use this value of k_0; rather, we will study the effect on parameter estimates due to using driving signals which have varying levels of bias. The entire low-pass noise filter set-up is shown in Figure 4.1a.

In this simulation a random component of the system gain was also generated by means of the set-up shown in Figure 4.1b.

Figure 4.3 shows the simulation results for the cases where (zero-mean) observation noise ($n_1(t) = [-1, +1]$) is absent in one case and present in the other. When observation noise of this size was present, it did not induce any apparent bias in parameter estimates.

Figure 4.4 shows the effect of adding a large uniformly distributed white noise component to the gain parameter K so that the resultant system gain was

$$K_n = 5.0 + 5.0\left[-1, +1\right]. \tag{4.13}$$

The zero-mean observation noise is $[-1, +1]$ and the sinusoidal drive to the estimator is given by (4.8). Clearly, very little

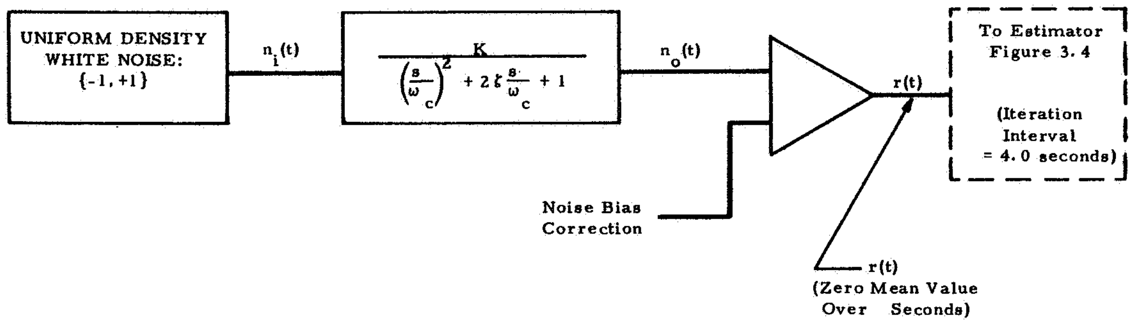


Figure 4.1a Random Drive Set-Up

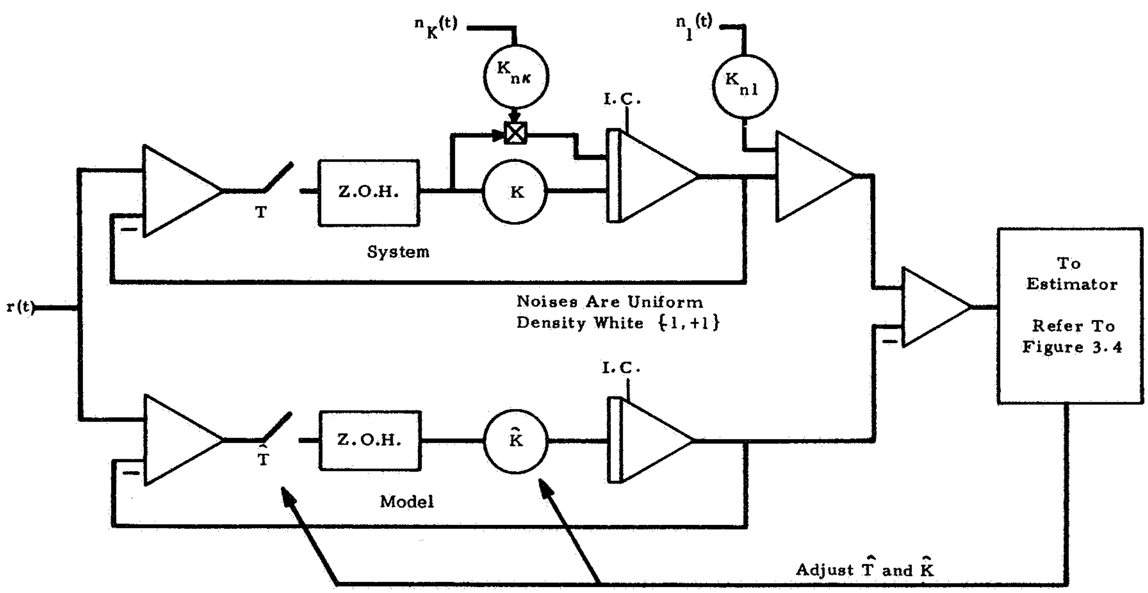


Figure 4.1b Simulation Set-Up For Estimating Noisy Gain and Deterministic Sampling Interval

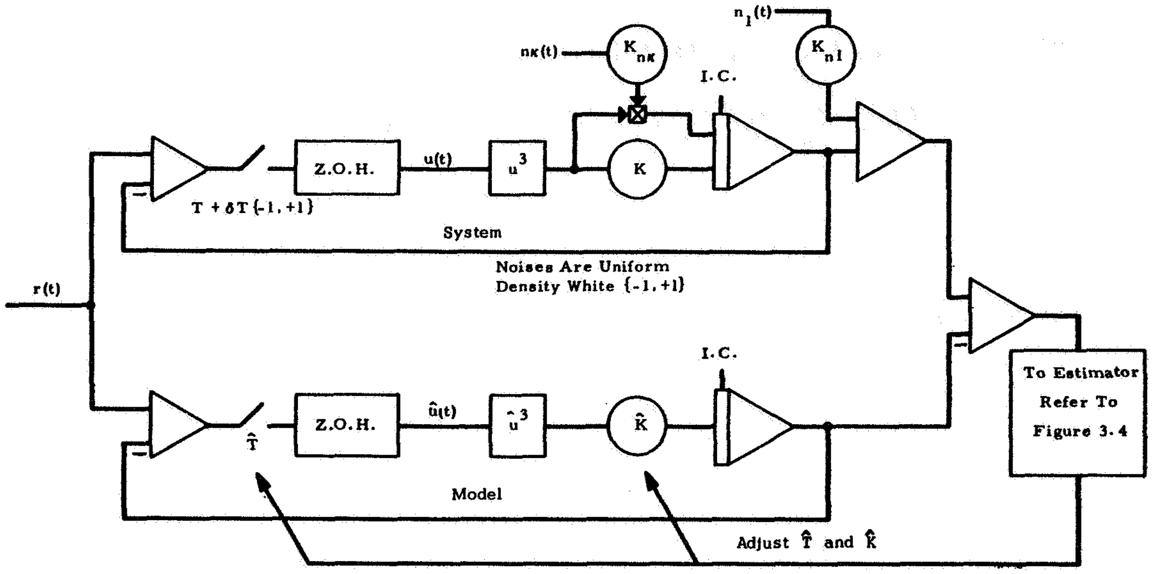


Figure 4.2a Simulation Set-Up For Estimating Noisy Sampling and Noisy Gain. First Order Nonlinear System and Model.

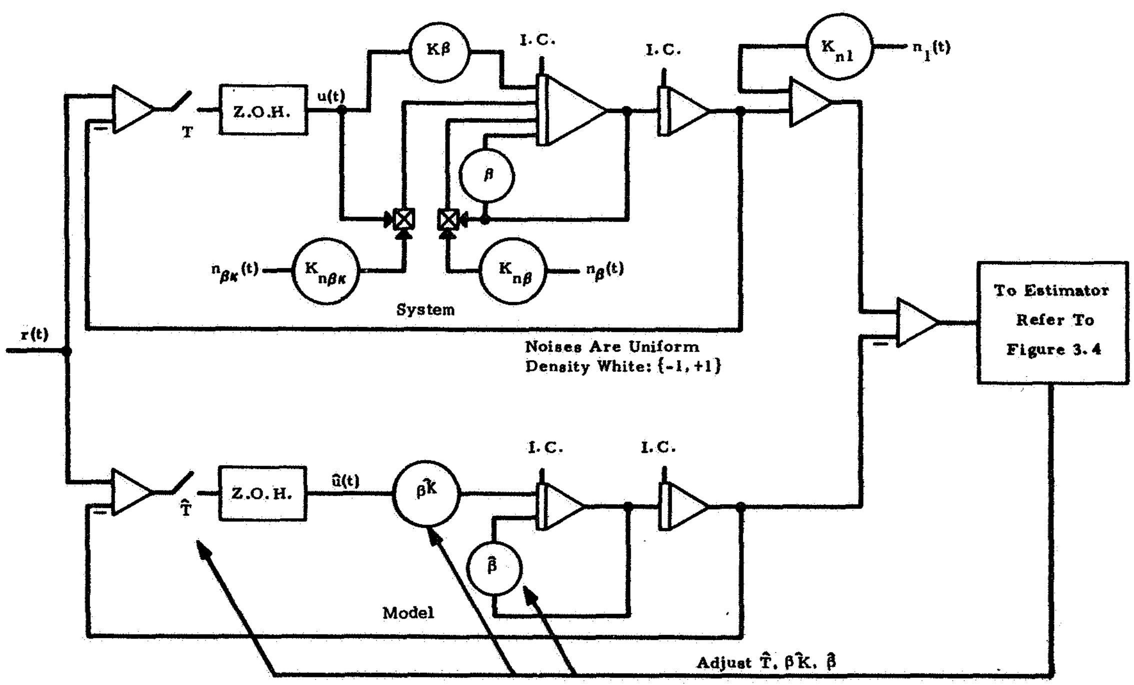


Figure 4.2b Simulation Set-Up For Estimating Noisy Gain and Time Constant. Second Order System and Model.

biasing of parameter estimates is induced in this case by the combination of large random gain component and small observation noise.

Figure 4.5 is for noise-free observations and parameters. However, in this case the random drive function given by (4.9) is used with k_0 chosen such that $r_n(t)$ has zero-mean over the iteration interval (τ = 4.0 seconds). That is, $k_0 = -10.8394$. Again, there is no resulting bias in parameter estimates.

Figure 4.6 shows estimation results for the same drive function but for the case of noisy observations and noisy gain. Observation noise (4.2) was used with and without a bias term (k_2). In the former case

$$n_1(t) \;=\; 1.0\left[-1,\ +1\right] + 1.0 \tag{4.14}$$

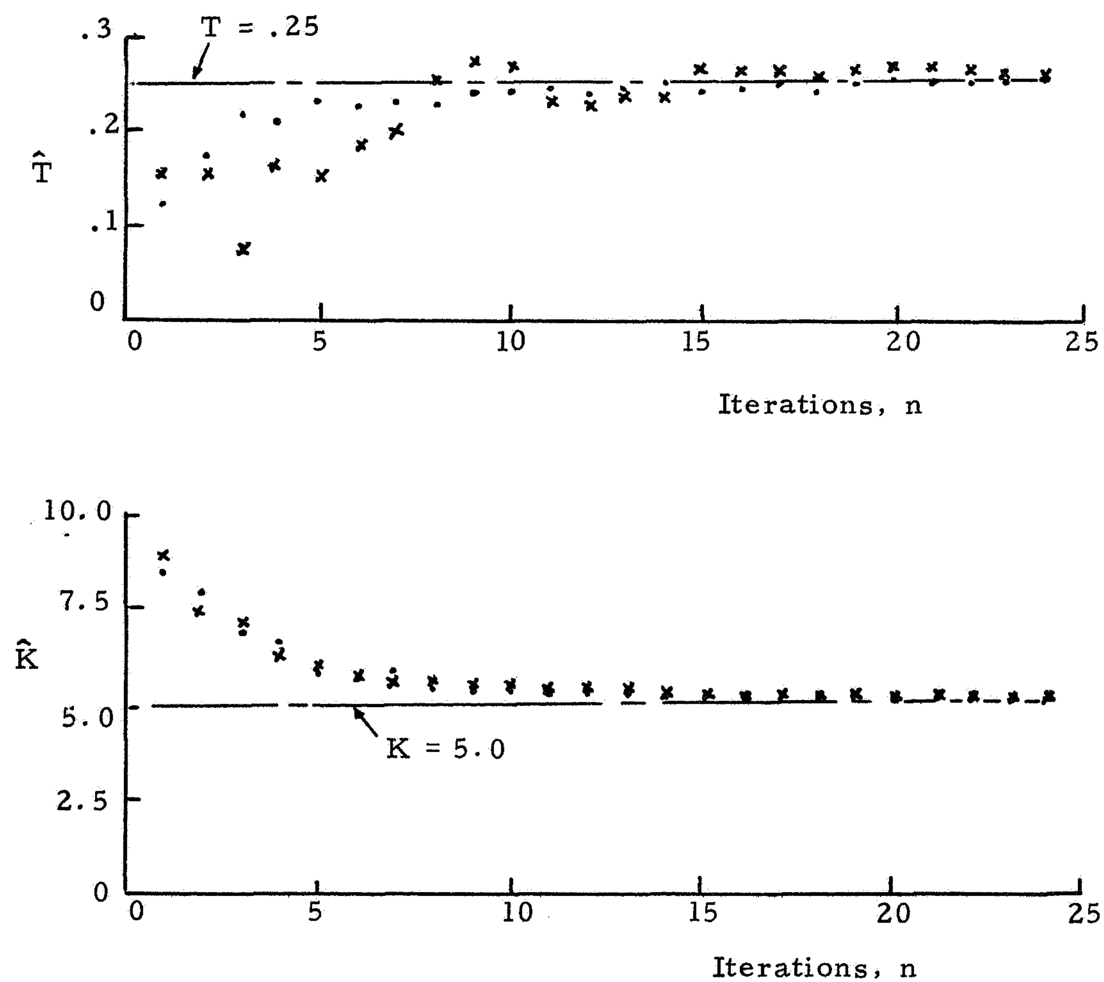
The estimation result is given by the dot sequence. Asymptotic parameter estimates are: $\hat{T} = .225$, $\hat{K} = 5.41$. In the latter case, the observation noise is

$$n_1(t) \;=\; 1.0\left[-1,\ +1\right] \tag{4.15}$$

The estimation result is given by the cross sequence, with final values of parameter estimates: $\hat{T} = .236$, $\hat{K} = 5.02$. In both cases, the noisy gain was given by

$$K_n \;=\; 5.0 + 0.5\left[-1,\ +1\right] \tag{4.16}$$

Clearly, the estimation errors are larger when the observation noise is biased than when it is not.



First Order Continuous System

$\dot{z} = K\,u(t)\,,\quad z_o = 0$

(Refer to Fig. 3.4 for Estimator Configuration).

Parameters (Noise-Free);

$K = 5.0$

$T = .25$

Observation Noise:

x - $n_1(t) = 0.0$

● - $n_1(t) = 1.0[-1,+1]$

Drive:

$r(t) = 20.0 \sin(.63t)$

Gains:

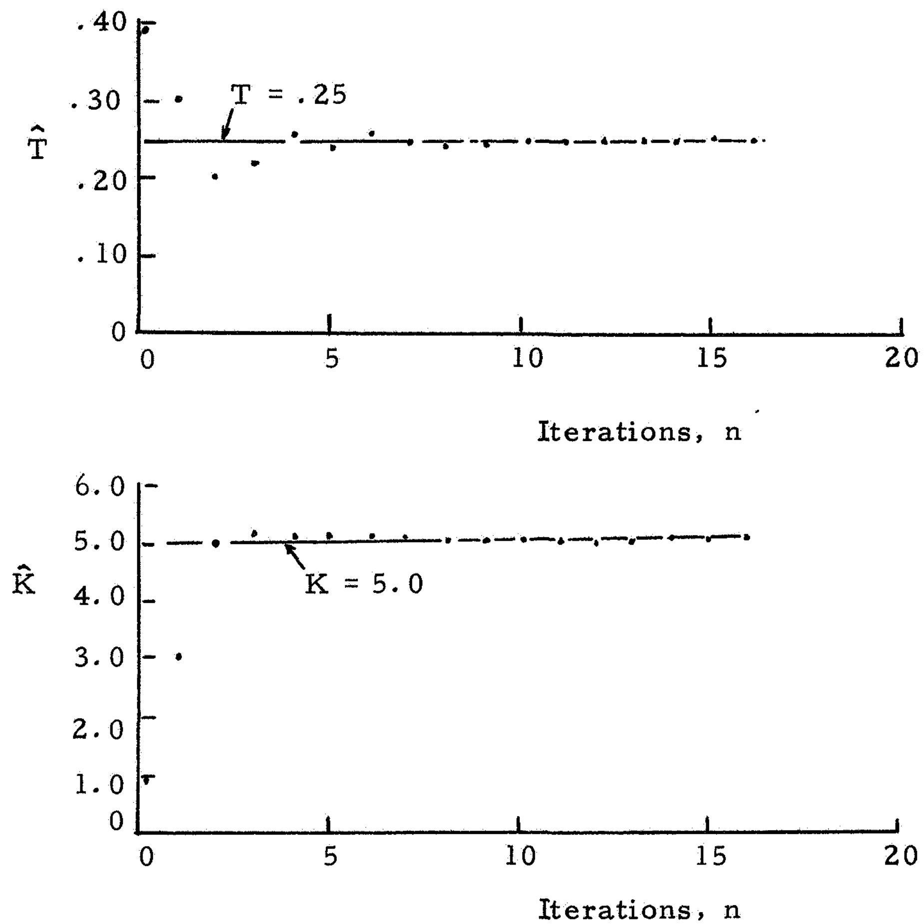
$\hat{T}$: $a_n^1 = 0.05n^{-1}$, $c_n^1 = 0.1n^{-1/6}$

$\hat{K}$: $a_n^2 = 0.05n^{-1}$, $c_n^2 = 0.1n^{-1/6}$

Iteration time: 10 seconds.

(Ref. Runs 2-27-1-1, 2, 9-13-1-1)

Figure 4.3 Estimation Of T And K In First Order Linear Sampled-Data System With And Without Observation Noise. Sinusoidal Drive.



First Order Linear Continuous Systems

$\dot{z} = Ku(t), \quad z_o = 0$

(Ref. Fig. 3.4 for Estimator Configuration)

Parameters (noisy):

$K = 5.0 + 5.0[-1, +1]$

$T = 0.25$

Observation Noise:

$n_1(t) = 1.0[-1, +1]$

Drive:

$r(t) = 20.0 \sin(.63t)$

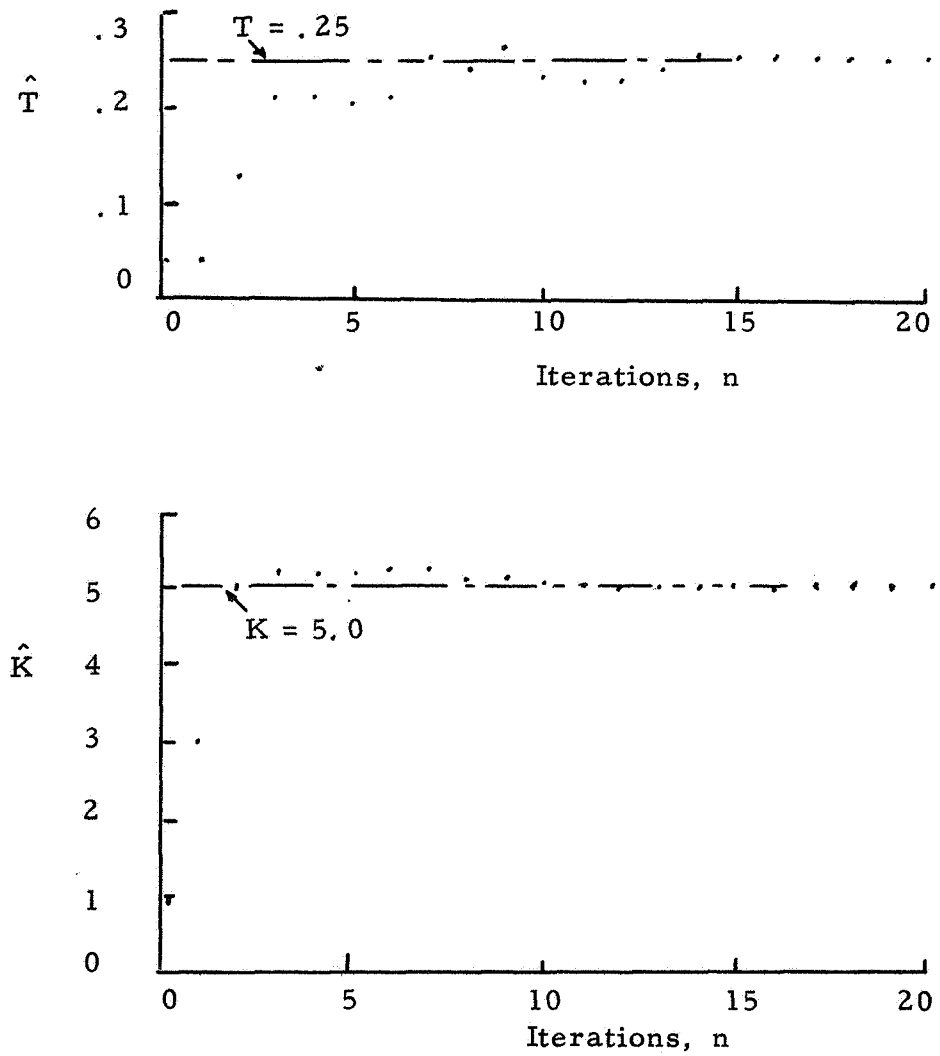
Gains:

$\hat{T}: a_n^1 = 0.05n^{-1}, \; c_n^1 = 0.01n^{-1/6}$

$\hat{K}: a_n^2 = 0.05n^{-1}, \; c_n^2 = 0.1n^{-1/6}$

(Runs 3-11-1-1, 2, 3)

Figure 4.4 Estimation of T and K in First Order Linear Sampled-Data System. Observation Noise and Random Gain.



First Order Linear Continuous System

$\dot{z} = Ku(t), \; z_o = 0$

(Ref. Fig. 3.4 for Estimator)

Parameters (Noise Free):

$K = 5.0$

$T = 0.25$

Observation Noise:

$n_1(t) = 0$

Drive (Random with Bias Correction):

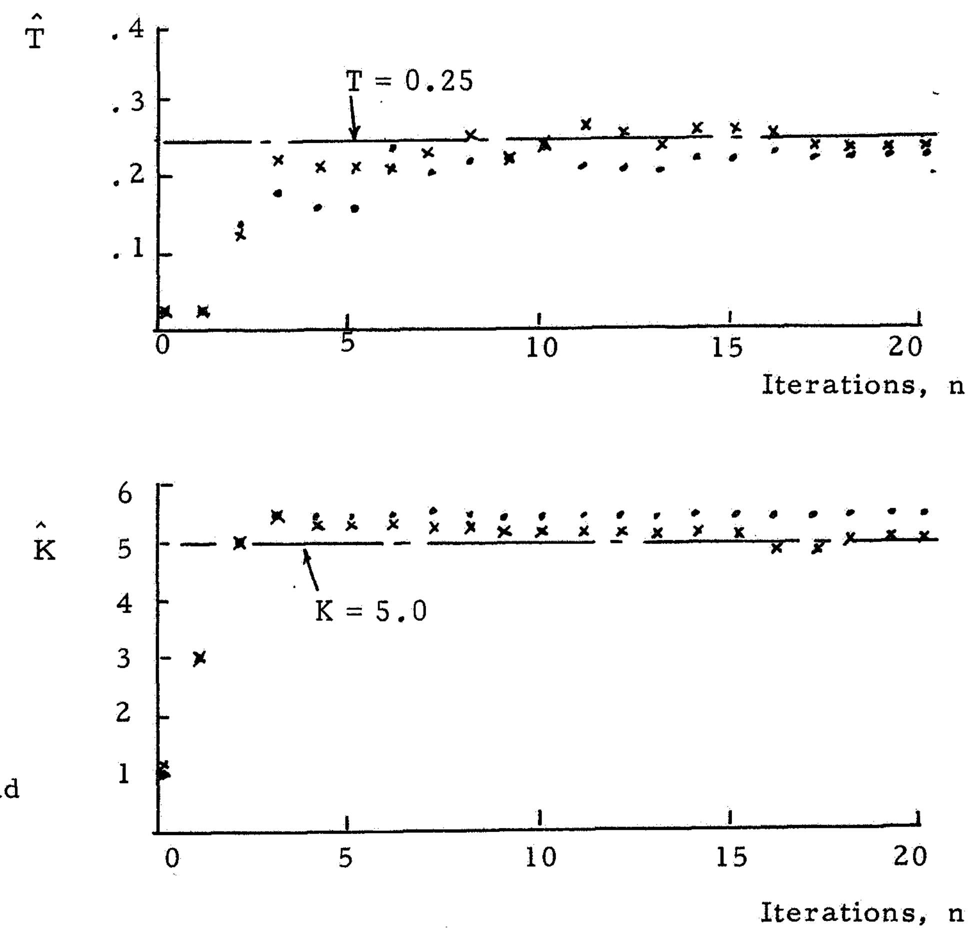
$r(t) = n_o(t) - 10.8394$

Gains:

$\hat{T}$: $a_n^1 = 0.005n^{-1}, \; c_n^1 = 0.01n^{-1/6}$

(Ref. Run 4-9-2-1)

Figure 4.5 Estimation of T and K In First Order Linear Sampled-Data System-Noise Free Case.



First Order Linear Continuous System.

$\dot{z} = Ku(t), \; z_o = 0$

(Ref. Fig. 3.4 for Estimator Configuration)

Parameters (Noisy):

$K = 5.0 + 0.5[-1, +1]$

$T = 0.25$

Observation Noise:

• $n_1(t) = 1.0[-1, +1] + 1.0$

x $n_1(t) = 1.0[-1, +1]$

Drive(Random with Bias Correction):

$r(t) = n_o(t) - 10.8394$

Gains:

$\hat{T}$: $a_n^1 = 0.005 n^{-1}$,

$c_n^1 = 0.01 n^{-1/6}$

$\hat{K}$: $a_n^2 = 0.005 n^{-1}$,

$c_n^2 = 0.1 n^{-1/6}$

(Runs 4-10-2-1 and 4-9-3-1)

Figure 4.6 Estimation of T and K in First Order Linear Sampled-Data System when K is Random and when Observation Noise has Bias.

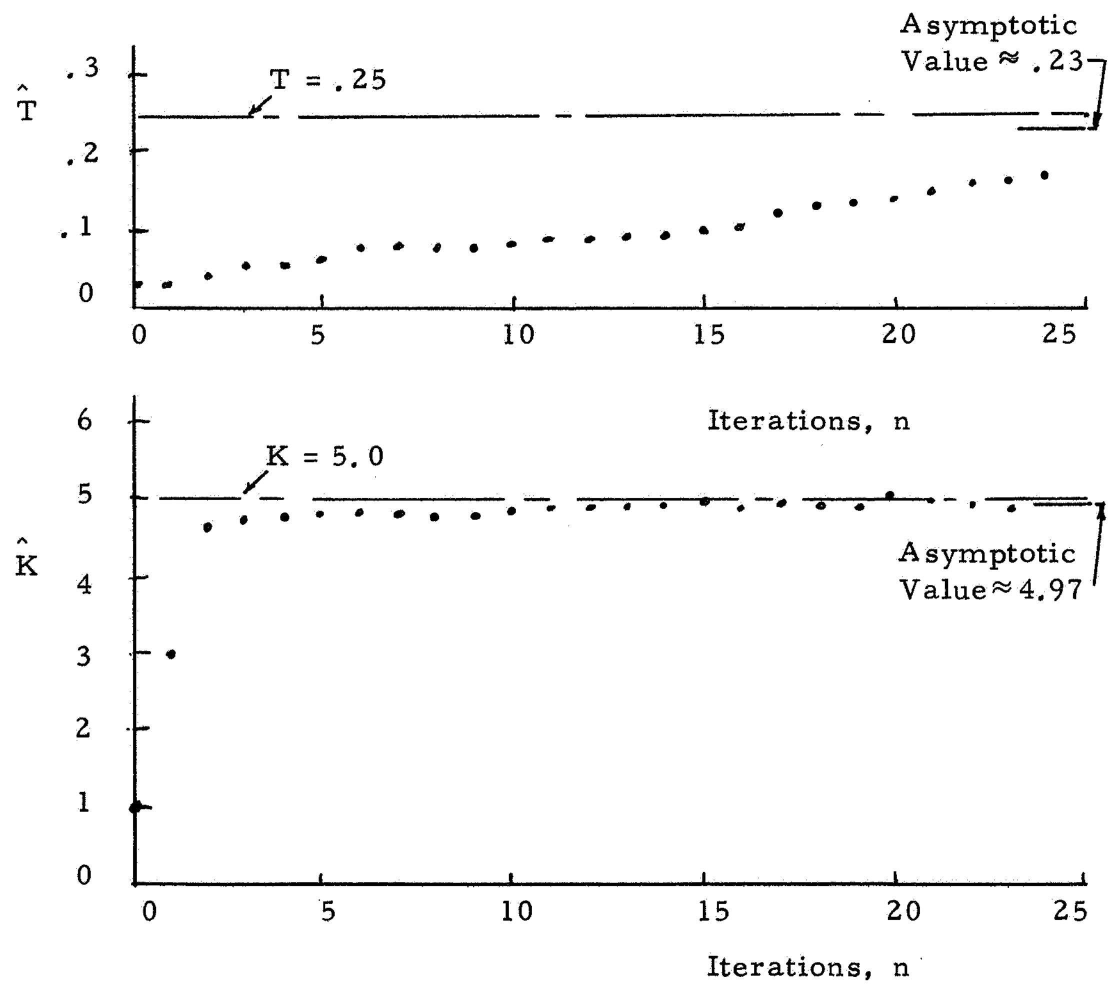


Figure 4.7 Estimation of T and K in First Order Linear Sampled-Data System when Driving Signal has Large Bias.

First Order Linear Continuous System

$\dot{z} = K\,u(t), \quad z_o = 0$

(Ref. Fig. 3.4 for Estimator Configuration).

Parameters (Noise Free):

$K = 5.0$

$T = 0.25$

Observation Noise:

$n_1(t) = 0$

Drive (Random Without Bias Correction):

$r(t) = n_o(t)$

Gains:

$\hat{T}$: $a_n^1 = 0.005 n^{-1}, c_n^1 = 0.01 n^{-1/6}$

$\hat{K}$: $a_n^2 = 0.005 n^{-1}, c_n^2 = 0.1 n^{-1/6}$

(Ref. Run 4-17-1-1; for asymptotic values: 4-17-2-1).

Figure 4.7 illustrates the effect of using a driving signal (4.9) with non-zero mean value. Observation noise and parameter noise are zero. Referring to Figure 4.5 for comparison, the main result is to reduce the convergence rate of $\hat{T}$. Additionally, the asymptotic value of $\hat{T}$ is now biased: $\hat{T} = .23$. However, neither the final value of $\hat{K}$ nor its convergence rate were affected substantially. Hence, we conclude in this case that only $\hat{T}$ is particularly sensitive to bias of the driving signal.

4.2.2 <u>Example 2:</u> <u>Nonlinear First Order Continuous System And Model</u>

Again, referring to the nomenclature of Figure 3.4, the continuous system and model are described by the nonlinear differential equations

$$\dot{z}^1 = K(u(t))^3, \qquad z_0^1 = 0 \tag{4.17}$$

and

$$\dot{\hat{z}}^1 = K(\hat{u}(t))^3, \qquad \hat{z}_0^1 = 0 \tag{4.18}$$

where z, $\hat{z}$, u, and $\hat{u}$ are scalars. The cost function of (3.58) is again used. The sampled-data system parameter vector and model parameter vectors are

$$x = \begin{bmatrix} K \\ T \end{bmatrix} \tag{4.19}$$

and

$$\hat{x} = \begin{bmatrix} \hat{K} \\ \hat{T} \end{bmatrix} \tag{4.20}$$

respectively.

The drive signal is the random function given by (4.9). Figure 4.2a is a schematic of this simulation. Any, or all, of the noises shown there could be used in combination to furnish a very complete simulation of a nonlinear system with noisy parameters and observations.

Figure 4.8 gives estimation results for Example 2 for the case where the random drive signal (4.9) has zero mean value, the parameters are noise-free, and where the observations are both noise-free and noisy. There is a slight bias in the parameter estimates for the latter case.

Figure 4.9 shows estimation results for the case where the observation noise is zero and the random drive signal does not have zero mean value over the iteration interval. A slight bias is induced in the estimate of T: $\hat{T} \simeq .226$ (10% error).

Figure 4.10 shows estimation results where the (zero-mean) observation noise is ten times larger than in Example 1, so that

$$n_1(t) = 10 \left[-1, +1\right] \tag{4.21}$$

The gain is also noisy with maximum excursion of random component equal to nominal gain, i.e.,

$$K = 0.025 + 0.025 \left[-1, +1\right] \tag{4.22}$$

The random drive signal has been bias corrected. Despite the fact that the observation noise is larger than in previous experiments, reference, for example, Figure 4.8, and considering the presence of the large random gain component, a comparison of Figure 4.10 to Figure 4.8 indicates only a slight difference in parameter estimates.

Figure 4.11 is for the same set of system conditions as Figure 4.10 with the addition of a large white uniformly distributed zero-mean random component to the system sampling interval by means of subroutine Sub 2 (described in the Appendix). The random parameters are

$$T = 0.25 + 0.25\left[-1, +1\right], \tag{4.23}$$

and

$$K = 0.025 + 0.025\left[-1, +1\right]. \tag{4.24}$$

The observation noise is also large:

$$n_1(t) = 10\left[-1, +1\right]. \tag{4.25}$$

From a comparison of Figure 4.11 and Figure 4.10 it is clear that the addition of the random sampling component induced some error into estimation of the sampling interval. An experiment, not reported in detail here, indicated that the random component of the sampling interval had a bias of approximately -0.015 when the mean of (4.23) was checked for $\tau_i = 4.0$ seconds. Hence the mean value of the system sampling (over the 4.0 second iteration interval) was: $\bar{T} = 0.235$. The estimates $\hat{T}$ are asymptotic to $\hat{T} \approx 0.262$; hence the bias error in $\hat{T}$ is in the order of 10%.

4.2.3 Example 3: Second Order Linear Continuous System and Model

Again, referring to Figure 3.4, the system equations are

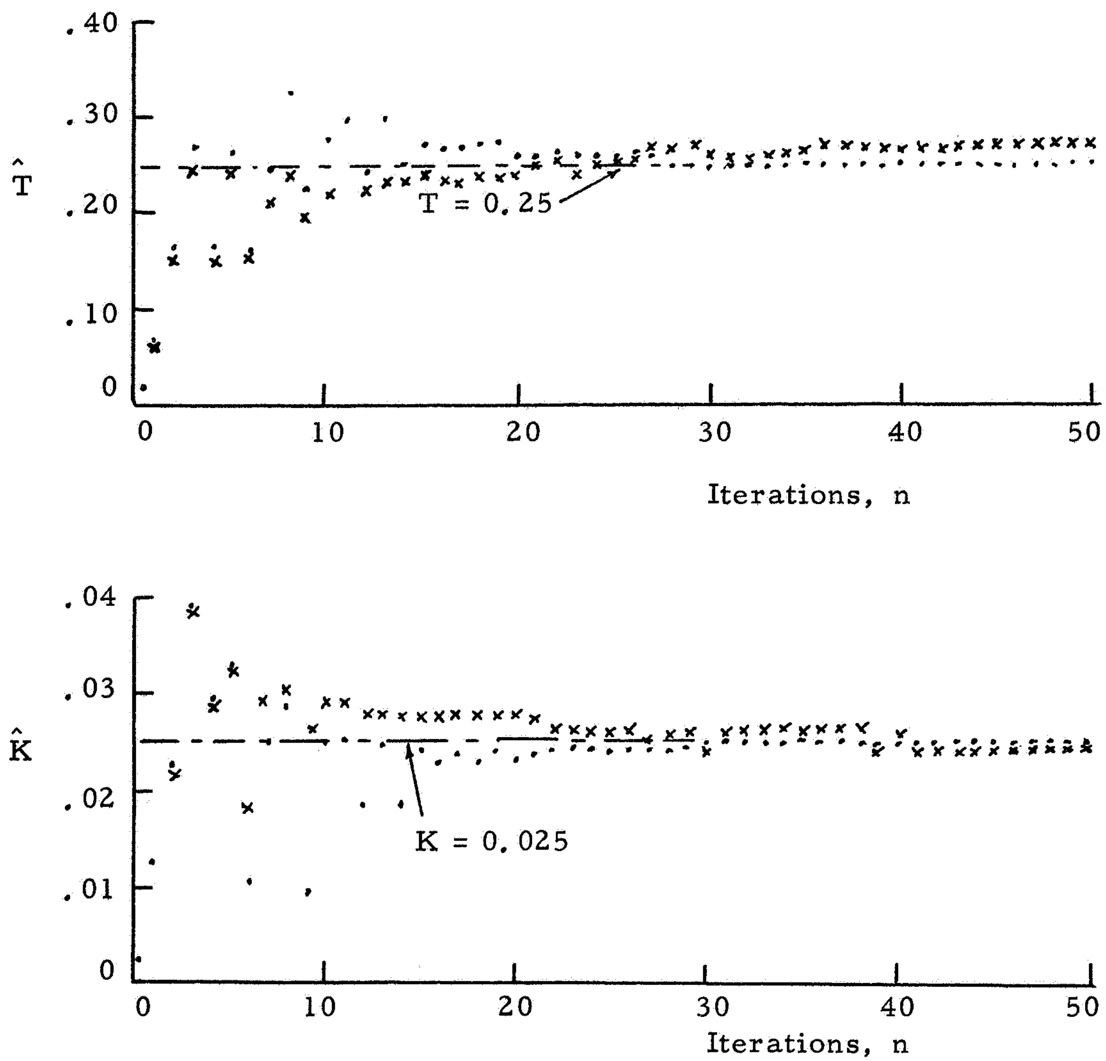


Fig. 4.8 Estimation of T and K in First Order Nonlinear Sampled-Data System.

First Order Nonlinear Continuous System

$$\dot{z} = K(u(t))^3, \quad z_o = 0$$

(Ref. Fig. 3.4 for Estimator Configuration).

Parameters (Noise-Free):

$K = 0.025$

$T = 0.25$

Observation Noise: $\bullet$ - $n_1(t) = 0$

x - $n_1(t) = 10[-1, +1]$

Drive (Random with Bias Correction):

$$r(t) = n_o(t) - 10.8394$$

Gains:

T: $a_n^1 = .0025n^{-1}$, $c_n^1 = 0.01n^{-1/6}$

K: $a_n^2 = .0025n^{-1}$, $c_n^2 = 0.01n^{-1/6}$

(Ref. Runs 5-8-1-1, 10-6-1-1)

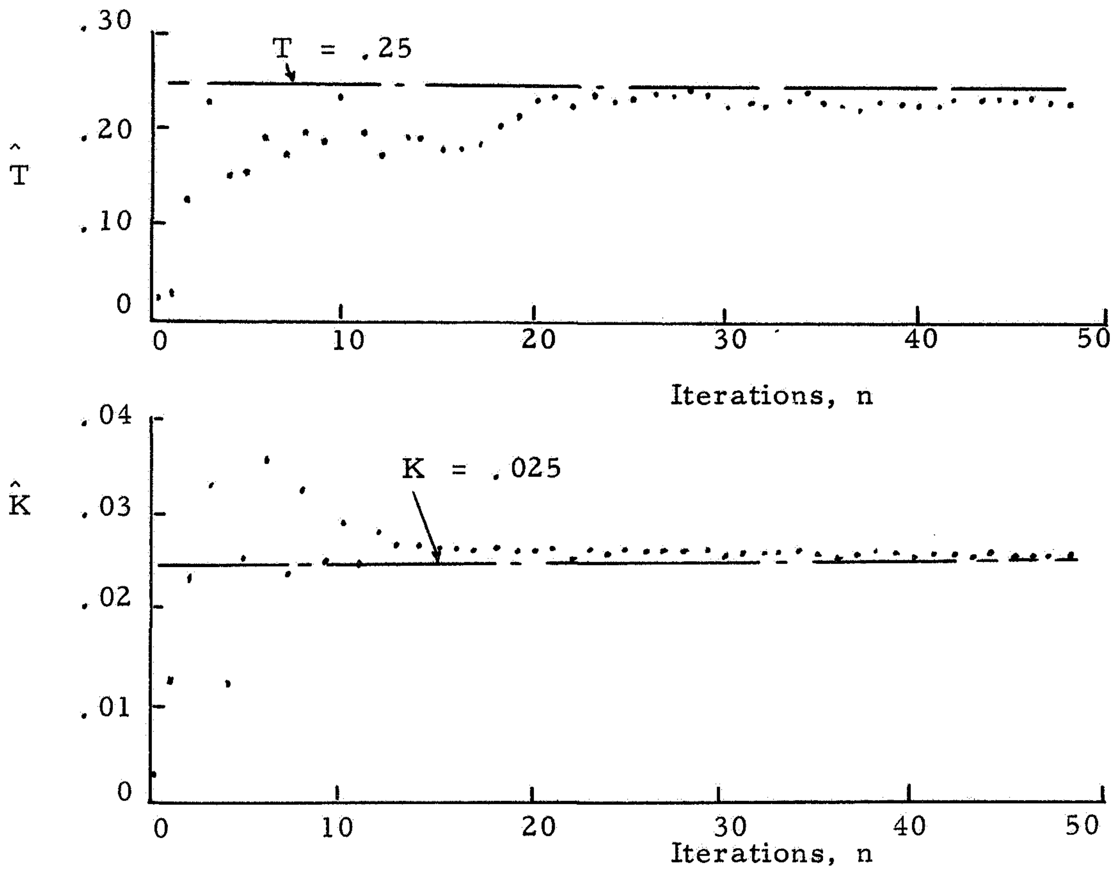


Figure 4.9 Estimation of T and K In First Order Nonlinear Sampled-Data System, Biased Drive Case, Noise-Free Case.

First Order Nonlinear Continuous System

$$\dot{z} = K(u(t))^3, \quad z_o = 0$$

(Ref. Fig. 3.4 For Estimation Configuration)

Parameters (Noise-Free):

$K = 0.0250$

$T = 0.25$

Observation Noise:

$n_1(t) = 0$

Drive (Random Without Bias Correction):

$r(t) = n_o(t)$

Gains:

$\hat{T}$: $a_n^1 = .0025\, n^{-1}$

$c_n^1 = 0.01\, n^{-1/6}$

$\hat{K}$: $a_n^2 = .0025\, n^{-1}$

$c_n^2 = 0.01\, n^{-1/6}$

(Ref. Run 5-6-1-1)

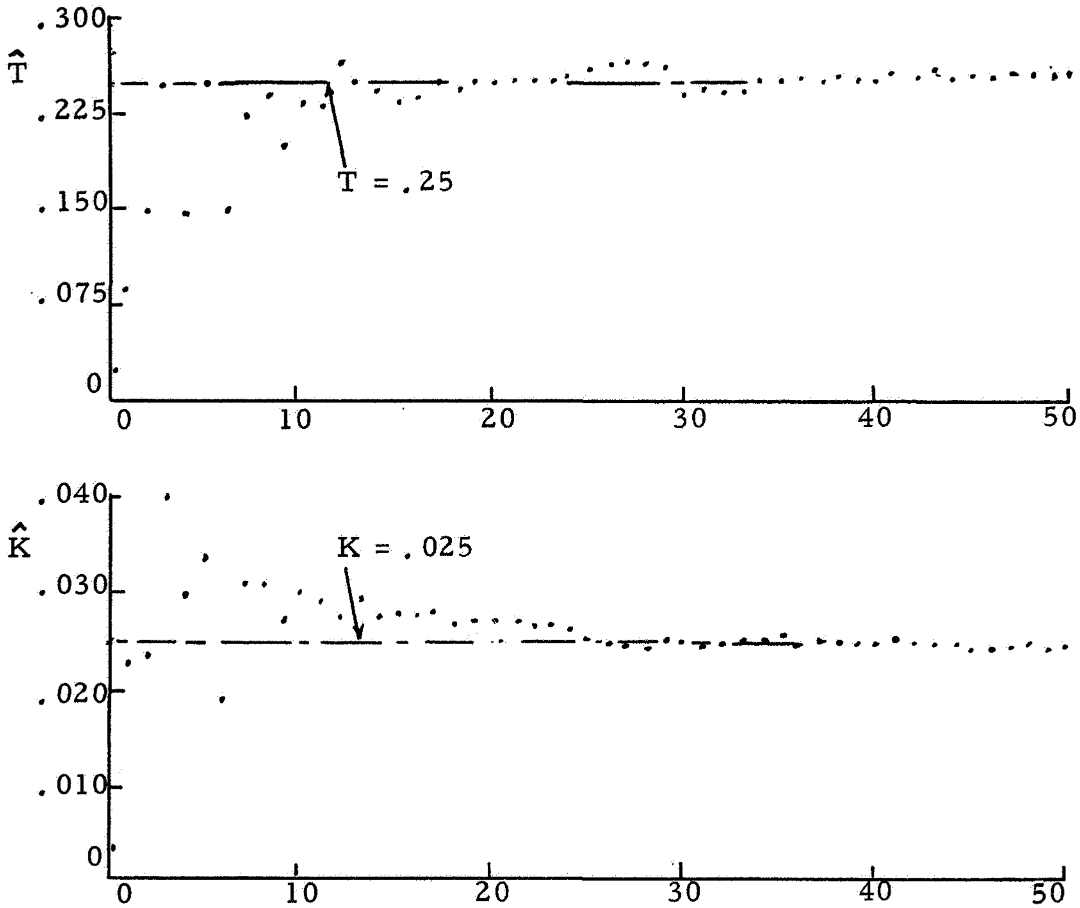


Figure 4.10 Estimation of T and K in First Order Nonlinear Sampled-Data System - Noisy Gain, Noisy Observations.

First Order Nonlinear Continuous System

$\dot{z} = K(u(t))^3, \; z_o = 0$

(Ref. Fig. 3.4 For Estimator Configuration):

Parameter(Noisy):

$K = 0.025 + 0.025[-1, +1]$

$T = 0.250$

Observation Noise:

$n_1(t) = 10.0[-1, +1]$

Drive (Random with Bias Correction):

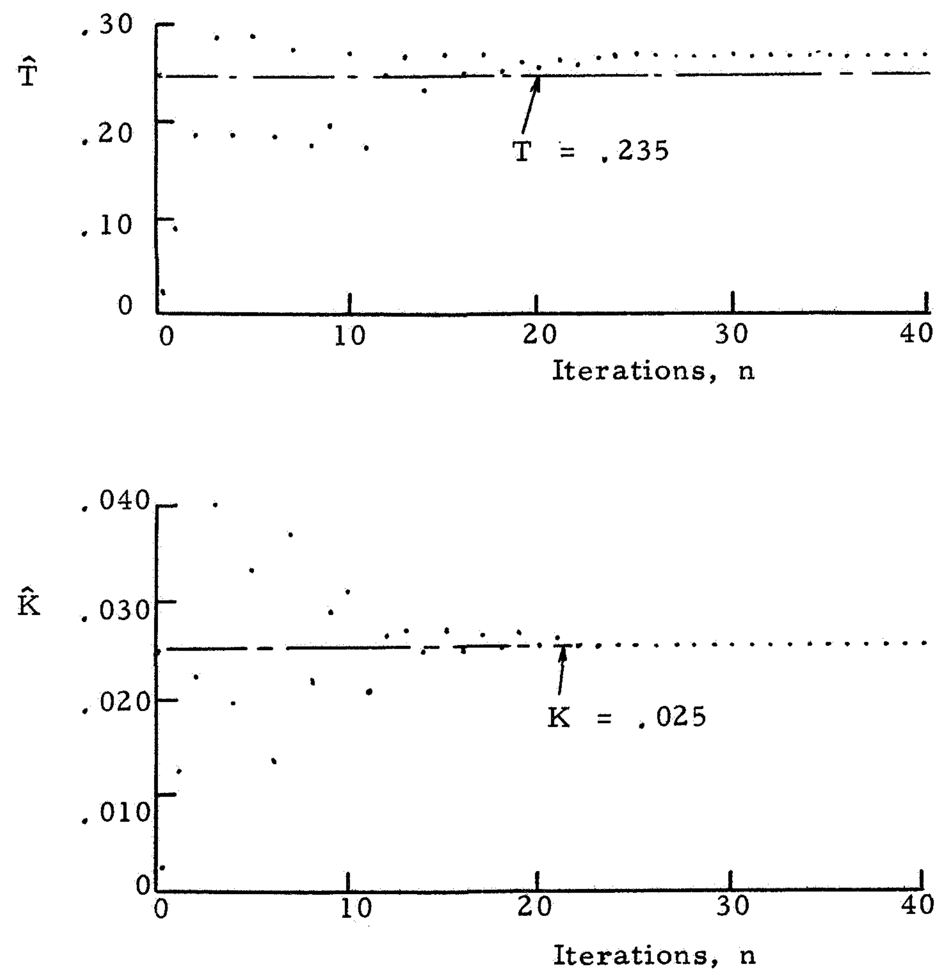
$r(t) = n_0(t) - 10.8394$

Gains:

$\hat{T}$: $a_n^1 = .0025\,n^{-1}, c_n^{-1} = .01\,n^{-1/6}$

$\hat{K}$: $a_n^2 = .0025\,n^{-1}, c_n^2 = .01\,n^{-1/6}$

(Ref. Runs 5-12-1-1, 2)



First Order Nonlinear Continuous System:

$$\dot{z} = K(u(t))^3, \quad z_o = 0$$

(Ref. Fig. 3.4 For Estimator Configuration)

Parameter (Noisy):

$K = .025 + .025[-1, +1]$

$T = .25 + .25[-1, +1]$

Observation Noise:

$n_1(t) = 10[-1, +1]$

Drive (Random with Bias Correction):

$r(t) = n_o(t) - 10.8394$

Gains:

$\hat{T}$: $a_n^1 = .0025n^{-1}, c_n^1 = .01n^{-1/6}$

$\hat{K}$: $a_n^2 = .0025n^{-1}, c_n^2 = .01n^{-1/6}$

(Ref. Run 6-9-1-1)

Figure 4.11 Estimation of T and K in First Order Nonlinear Sampled-Data System - Noisy Gain, Noisy Sampling, and Noisy Observations.

$$\dot{z}^1 = z^2 \qquad z_0^1 = 0 \tag{4.26}$$

$$\dot{z}^2 = -\beta z^2 + K\beta u(t), \qquad z_0^2 = 0 \tag{4.27}$$

and the model equations correspond. Here β is the time constant and $K\beta$ is the effective gain. The foregoing remarks concerning cost function apply here as well. The system model vector is

$$x = \begin{bmatrix} K\beta \\ \beta \\ T \end{bmatrix} \tag{4.28}$$

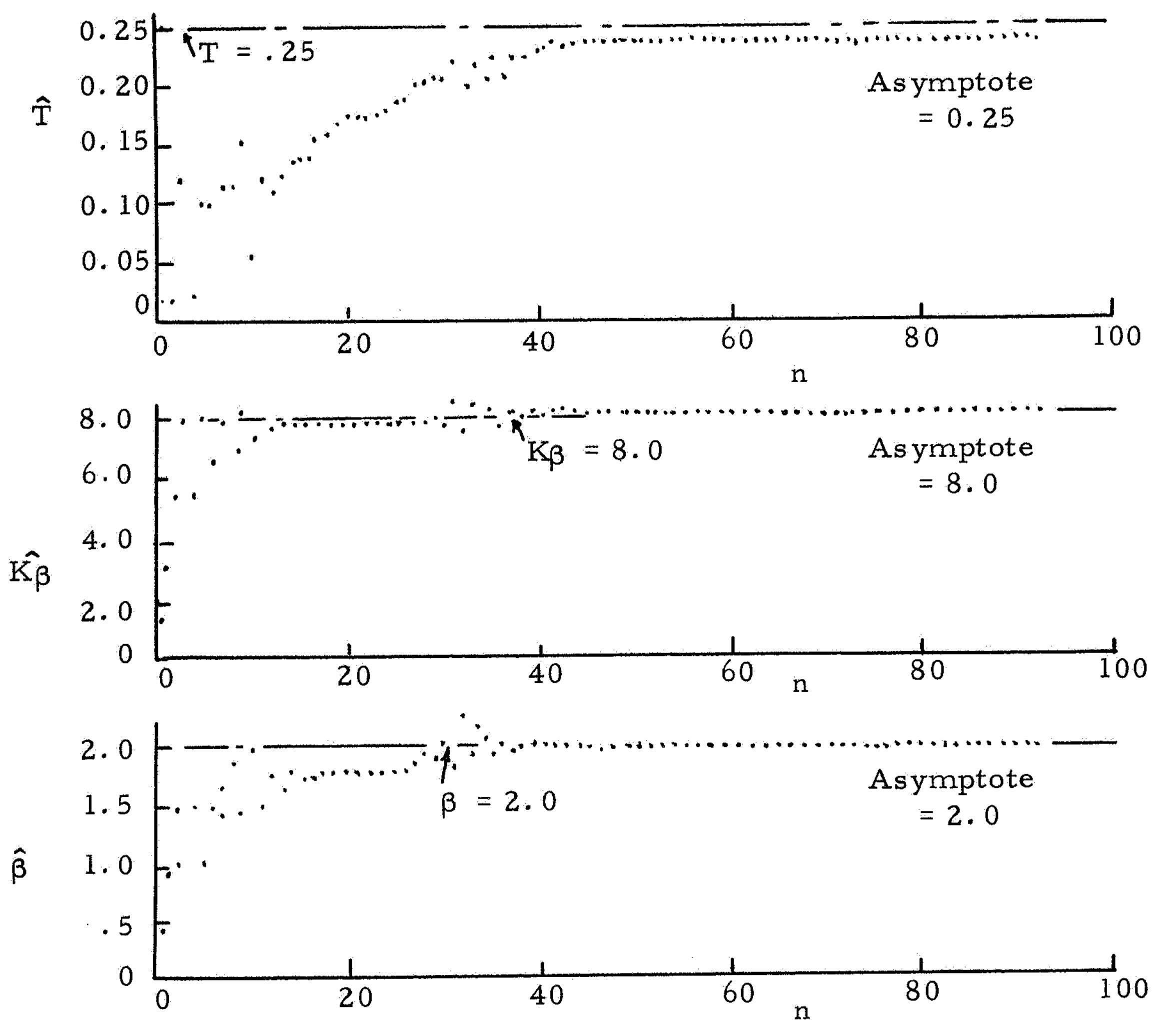
The model parameter vector is

$$\hat{x} = \begin{bmatrix} \widehat{K\beta} \\ \hat{\beta} \\ \hat{T} \end{bmatrix} \tag{4.29}$$

Figure 4.2b shows a schematic of the simulation. In some simulations, random components were added to both $K\beta$ and β. In contrast to Example 2, T was always deterministic.

Figure 4.12 shows estimation results for the completely noise-free case. Note in comparison to the first-order systems of Examples 1 and 2, that the increased system complexity induced a slower convergence rate of the estimates. However, the asymptotic values are unbiased.

Figure 4.13 shows the estimation results for the noisy parameter and noise observation case. The noisy system parameters are



Second Order Linear Continuous System:

$$\dot{z}_1 = z_2, \; z_{10} = 0$$
$$\dot{z}_2 = -\beta z_2 + K\beta u(t), \; z_{20} = 0$$

Parameters (Noise-Free);

$K\beta = 8.0$
$\beta = 2.0$
$T = 0.25$

Observation Noise:

$n_1(t) = 0$

Drive (Random with Bias Correction):

$r(t) = n_o(t) - 10.8394$

Gains:

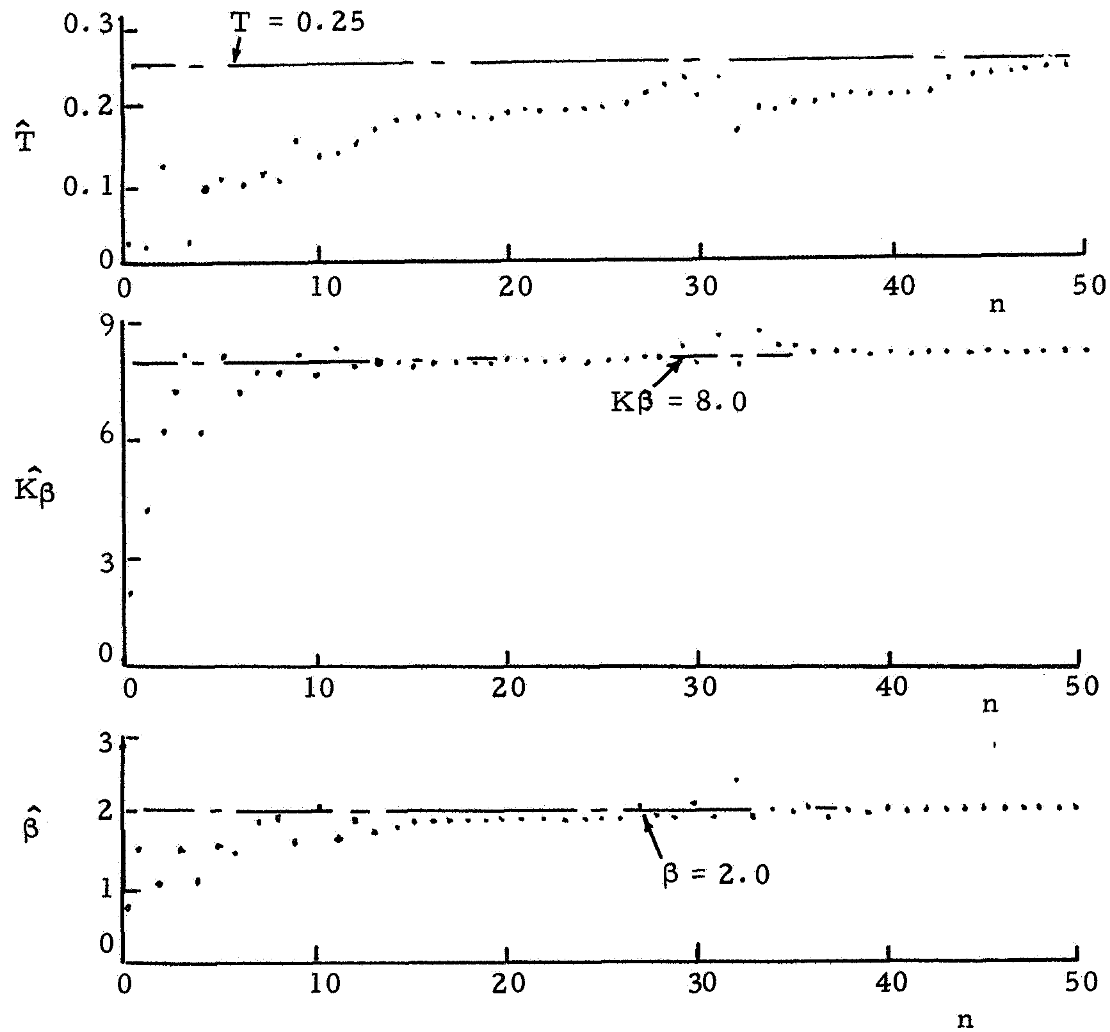
$\hat{T} : a_n^1 = .005n^{-1}, \; c_n^1 = .01n^{-1/6}$

$\hat{K\beta} : a_n^2 = .025n^{-1}, \; c_n^2 = .3n^{-1/6}$

$\hat{\beta} : a_n^3 = .005n^{-1}, \; c_n^3 = .3n^{-1/6}$

(Run 4-11-2-1,2,3,4, 5,6; 4-13-10-1)

Figure 4.12 Estimation of T, Kβ, β In Linear Second Order Sampled-Data System. Noise-Free Parameters and Noise-Free Observations.



Second Order Linear Continuous System

$$\dot{z}_1 = z_2, \quad z_{10} = 0$$

$$\dot{z}_2 = -\beta z_2 + K\beta u(t), \quad z_{20} = 0$$

Parameters (Noisy)

$K\beta = 8.0 + 0.8[-1,+1]$

$\beta \ = 2.0 + 0.2[-1,+1]$

$T \ = 0.25$

Observation Noise

$n_1(t) = 1.0[-1,+1]$

Drive (Random with Bias Correction)

$r(t) = n_o(t) - 10.8394$

Gains:

$\hat{T} : a_n^1 = .005n^{-1}, \quad c_n^1 = .01n^{-1/6}$

$\hat{K\beta} : a_n^2 = .025n^{-1}, \quad c_n^2 = .3n^{-1/6}$

$\hat{\beta} : a_n^3 = .005n^{-1}, \quad c_n^3 = .3n^{-1/6}$

(Ref. Run 4-11-3-1)

Figure 4.13 Estimation of T, Kβ, And β In Linear Second-Order Sampled-Data System. Noisy Parameters and Noisy Observations.

$$K\beta = 8.0 + 0.8\left[-1, +1\right] \tag{4.30}$$

and

$$\beta = 2.0 + 0.2\left[-1, +1\right] \tag{4.31}$$

The observation noise is $[-1, +1]$. In this case a slight bias was induced in the asymptotic values of parameter estimates and is imputed to the presence of the moderately large random components of $K\beta$ and β.

4.3 Conclusions From Simulation Studies

The simulations have demonstrated the convergence properties which were analytically predicted in Chapter 3, i.e., that unbiased estimates are obtained when the observation noise has zero mean-value and is uncorrelated with both system and model outputs. Parameter estimate biases are introduced by the presence of a non-zero mean in the observation noise, and the estimation errors are proportional to the noise bias.

When parameter noise is introduced, even when it is relatively large, the effect on obtaining estimates of the mean value of the parameter is quite small. Therefore, through these simulation studies we may proceed with some hope of obtaining reliable estimates of human operator parameters in view of the probably stochastic nature of the human operator's parameters.

The effect of the bias in the input signal is to induce a very slow convergence rate in the estimate $\hat{T}$ of the sampling interval T.

However, the convergence rate of other parameters is not seriously affected. The asymptotic estimate $\hat{T}$ of sampling interval T was, however, not seriously biased.

CHAPTER 5

RESULTS OF MODELING EXPERIMENTS USING ACTUAL PLANT DATA

5.1 Introduction

In this chapter we will apply the Kiefer-Wolfowitz stochastic approximation procedure described in Chapter 3 and simulated in Chapter 4 to the problem of estimating the parameters of a plant. Actual operating data of plant input and plant output are used. The particular problem chosen is concerned with estimating the parameters of a human operator model from discretized data obtained from a control situation involving a human operator while he is operating a dynamic load in the closed-loop feedback configuration of Figure 5.1.

Prior estimates of both the model form and model parameters of the human operator have been given by several authors: McRuer et al[27] used the spectral analysis approach and developed linear models. Adams[75] and Bekey et al [76] used continuous parameter tracking methods for finding the parameters of a linear second order model. Elkind [77] applied regression analysis using orthonormal filters and obtained linear models. Brainin [78] estimated statistical moments of the parameters of a simple linear model of the human operator by analog computer solution of the Fokker-Planck partial differential equations for the moments when the random parameter component was assumed to be white gaussian. Holmes [25] used stochastic approximation to solve for a Volterra expansion representation of the generally nonlinear human operator.

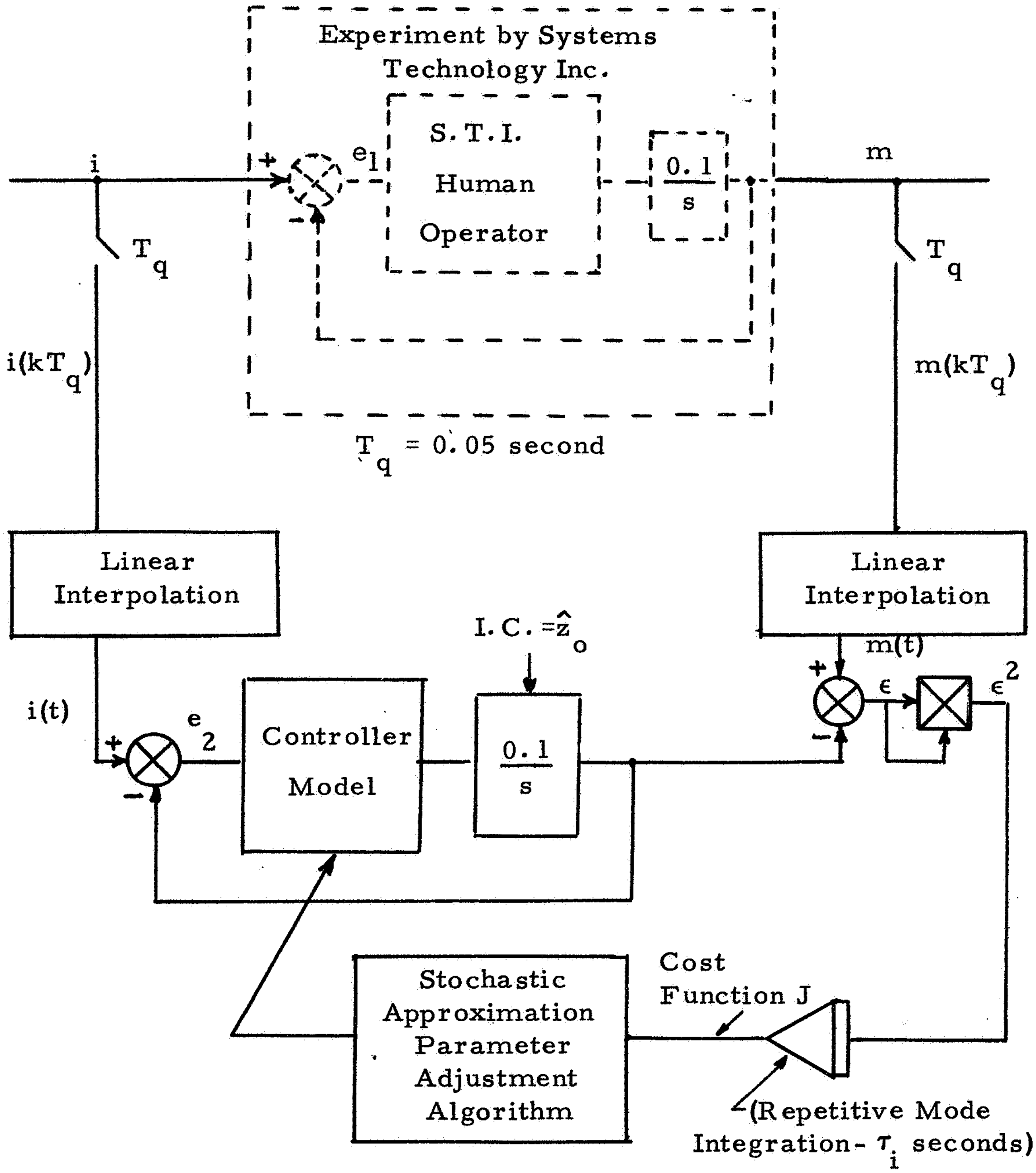


Figure 5.1: Configuration of the Experimental Determination of the Dynamic Characteristics of the S.T.I. Human Operator.

In particular the models and parameter estimates given by McRuer will be used here as a basis for determining the relative advantages of stochastic approximation in comparison with some of the other parameter estimation models. The parameters which are to be estimated in this study depend on the particular model chosen. Candidate models include: (1) sampler, data-hold, and gain, (2) transport delay and gain, (3) sampler, data-hold, and gain, (4) transport delay, gain, and lead-lag filter.

Data from actual human operator experiments were obtained from Systems Technology, Incorporated, Hawthorne, California. Data for the four variables shown in Figure 5.1 were supplied in discretized form for coincident sampling time points spaced 0.05 second apart.

5.2 System Technology Incorporated Test Data and Models

The data used for our human operator modeling studies were obtained from Systems Technology, Incorporated (S.T.I.). The results of their human operator experiments are summarized in Table 5.1. Table 5.2 furnishes the particular form of human operator model (Y_p) derived by Systems Technology, Incorporated to correspond to a particular controlled load (Y_c). The tables are to be used together to provide a complete description of a model. For example, for the controlled load dynamics 0.1/s, the first approximation model is

$$Y_p = K_p \frac{(T_L s + 1)}{(T_I s + 1)} e^{-\tau s} = 31 e^{-.27s} \quad . \tag{5.1}$$

S.T.I. Run Number	Y_c	τ (sec)	Parameters of Y_p		Functions of Y_pY_c	
			T_L(sec)	T_I(sec)	$\omega_c = K_pK_c$	$\phi m(°)$*
671129-09	$0.1/s$	0.270	0	0	3.1	44
-01	$1/s(s+2)$	0.264	0.5	0	4.2	24
-03	$1/s(s+4)$	0.250	0.25	0	4.2	6
-05	$0.1/s^2$	0.333	>1	0	1.5	40
-07	$0.1/s(s+1)$	0.384	1	0	2.8	12
-11	$1/s^2$	0.330	>1	0	4.0	11
-15	$1/s^2$	0.345	>1	0	3.3	20
*crossover phase when $Y_pY_c = 1.0$						

Table 5.1 S.T.I. Experiments And Results

Controlled Load Dynamics (Y_c)	Human Operator (First Approximation Model) (Y_p)
$\frac{K_c}{s}$	$K_p \frac{(T_L s+1)e^{-\tau s}}{(T_I s+1)}$
$\frac{K_c}{s(s+\beta)}$	$K_p\ (s+\frac{1}{T_L})e^{-\tau s}$
$\frac{K_c}{s^2}$	$K_p\ (s+\frac{1}{T_L})e^{-\tau s}$

Table 5.2 Correspondence Between Loads and Human Operator Models

S.T.I. has derived four models in order of increasing accuracy: the crossover model, the first approximation model, the second approximation model, and the precision model. They are tabulated in Reference 27. It should be noted that great care was exercised by the experimenters to insure that the input signal was random appearing and Gaussian in character.

5.3 Other Current Models

According to other recent work [26,28], the human operator is currently thought to exhibit an ability to adapt to sudden changes in almost any portion of the overall controlled system. However, discussion of models with such adaptation is unnecessary from our point of view: we confine our investigation to the estimation of sampling intervals and use data from the human operator experiments because it is available and because it presents an important problem in modeling a noisy, nonlinear system where there is reason to suspect that sampling may occur.

5.4 Procedure For Modeling Plant Data By Stochastic Approximation

The data for two of the four signal points of the human operator compensatory tracking problem of Figure 5.1 were used in the modeling studies. The studies were restricted to using the data for the load $Y_c = 0.1/s$. In order that the results of this study realistically represent the most difficult modeling situation, only the scalar input and scalar output variables $i(kT_q)$ and $m(kT_q)$ were used. The S.T.I.

notation will be used when we are dealing with data derived from the S.T.I. experiments.

Details of the various digital programs used in the modeling study are given in the Appendix. This section is limited to explaining the various modeling procedures.

Figure 5.1 shows a schematic diagram applicable to the various modeling studies. A special CSMP program module replacing module CSMM was written to read data cards as well as to perform the functions of module CSMM.

5.4.1 Special Subroutines

Because the data $i(kT_q)$ and $m(kT_q)$ were in discrete form, linear interpolation was used to obtain additional data points. The new sequences are defined here as $i(t)$ and $m(t)$. This was performed by a special CSMP subroutine. Special subroutines were also necessary for iterative control of the stochastic approximation procedure and also to generate special functions. These subroutines are briefly summarized as follows:

a) Subroutine Sub 1: This is the basic subroutine which performs both the modeling and also the stochastic approximation iterative calculations.

b) Subroutine Sub 2: This subroutine performs the linear interpolation of the data $i(kT_q)$ and $m(kT_q)$ and outputs $i(t)$ and $m(t)$. Linear interpolation was performed twice in each numerical integration interval, and the integration intervals were not larger than 0.01 second.

c) Subroutine Sub 3: This generates the transport lag $e^{-s\tau}$ as required in the modeling.

5.4.2 Study Procedures

The sequence of experiments was directed at obtaining a simple optimal model of the unknown human operator from the candidate models of Table 5.3. Steps in the sequence were as follows:

(1) Use the S.T.I. first order approximation model and record the cost function obtained at the end of an iteration interval. Use this number as a standard of comparison for evaluating the relative merit of other human operator models.

(2) Adjust the parameters $\hat{T}$ and $\hat{K}_p$ by stochastic approximation to determine whether improvement in the model, as measured by the cost function,

$$J = \int_0^{T_i} (\epsilon(t))^2 \, dt \qquad (5.2)$$

could be achieved.

(3) Represent the human operator by the combination of gain $\hat{K}$ and sampler and zero-order data hold of period $\hat{T}$. Adjust $\hat{T}$ and $\hat{K}$ by stochastic approximation.

(4) Add linear lead-lag compensation $s/(s+\hat{\beta})$ to the sampled-data model of (3). Adjust the parameters $\hat{T}$, $\hat{K}$, and $\hat{\beta}$ by stochastic approximation.

Model of Human Operator Controller	Optimal Parameters	Minimum Cost J_{min}
(1) $\hat{K}_p e^{-.27s} = \hat{K}_p e^{-\hat{\tau}s}$ (see note 1)	$\hat{\tau} = .27$ second $\hat{K}_p = 31.0$	99,634
(2) $\hat{K}_p e^{-\hat{\tau}s}$ (see note 2)	$\hat{\tau} = .2351$ $\hat{K}_p = 28.613$	94,105
(3) $\underset{\hat{T}}{\diagup}\,\boxed{\text{Z.O.H.}}-\boxed{\hat{K}}-$ (see note 3)	$\hat{T} = .2577$ $\hat{K} = 26.07$	101,114
(4) $\underset{\hat{T}}{\diagup}\,\boxed{\text{Z.O.H.}}-\boxed{\frac{\hat{K}s}{s+\hat{\beta}}}-$ (see note 4)	$\hat{T} = .2604$ $\hat{K} = 26.40$ $\beta = 0.29$	89,075
(5) $e^{-\hat{\tau}s}\left(\frac{\hat{K}s}{s+\hat{\beta}}\right)$ (see note 5)	$\hat{\tau} = .2873$ $\hat{K} = 31.369$ $\beta = 0.5759$	62,034

Note 1: This is the S.T.I. Model.

Note 2: This is S.T.I. Model after parameter adjustment by stochastic approximation.

Note 3: This is the sampled-data model. The Z.O.H. refers to a zero order data hold.

Note 4: This is the sampled-data model with phase lead compensation.

Note 5: This is the S.T.I. Model improved by phase lead.

Note 6: Parameter values for models 2 through 5 were derived by means of stochastic approximation.

Table 5.3 A Comparison of Various Models of the Human Operator in the Tracking Task of Figure 5.1

(5) Determine the effect of adding the lead-lag compensator of (4) to the S.T.I. model. Adjust the parameters $\hat{\tau}$, $\hat{K}_p$ and $\hat{\beta}$ by stochastic approximation.

It will be noted that the above experiments are quite simple. However, this does not limit the generality of the method. The object here is to illustrate the application of stochastic approximation to the problem of estimating the parameters of a plant from actual operating data. If desired, the order and complexity of the candidate model could be increased as long as the cost function reflected a corresponding decrease after the application of the stochastic adjustment techniques.

5.4.3 Zero-Mean Compensation Of Input Signal

The adverse effect of a non-zero mean value of input signal on the convergence rate and bias of the estimate of the sampling interval was noted in Chapter 4. In order to obtain an input signal i(t) with mean value substantially close to zero, the running average of the sequence $i(kT_q)$ was obtained for each $k = 1,2,...$ Then the smallest k was selected for which the running average was substantially zero. This was termed k_o. The iteration interval $\mathcal{T}_i$ was then fixed at $\mathcal{T}_i = k_o T_q$.

For the data of Table 5.1, and for $Y_c = 0.1/s$, $\mathcal{T}_i = 29.4$ seconds. Naturally, the particular $i(kT_q)$ and $m(kT_q)$ sequences were fixed once $\mathcal{T}_i$ was chosen. These same sequences were then used for each iteration of the adjustment procedure. (The original S.T.I. data traces were 100 seconds in duration.)

5.4.4 Initial Conditions Of The Model

The printout of the selected time sequence $m(kT_q)$ from the card data indicated that $m(0) = 42.0$. Both $\hat{z}_{10} = 42.0$ as well as $\hat{z}_{10} = 0$ were tried as model initial conditions. The cost function was about 5% lower when the former was used; hence, this value was used for all modeling experiments. Actually, the initial conditions could also have been included in the parameter vector of the model. However, this would have substantially increased the computation time requirements for sequence convergence.

5.5 Results of Modeling Studies

Table 5.3 shows the various models of the human operator controller used in this sequence of experiments. The optimal values of the parameters are indicated, along with the resulting value of the cost function at the end of the particular stochastic approximation iterative search sequence. The cost function, Eq. (5.2), measures the fit of the model output to the tracking data. Specifically, the cost function was the integral squared error, where the error is between noisy system and model and $\mathcal{T}_i$ is the iteration interval. The adequacy of the different models can be compared by examining the values of the cost function for a sufficiently large number of data samples.

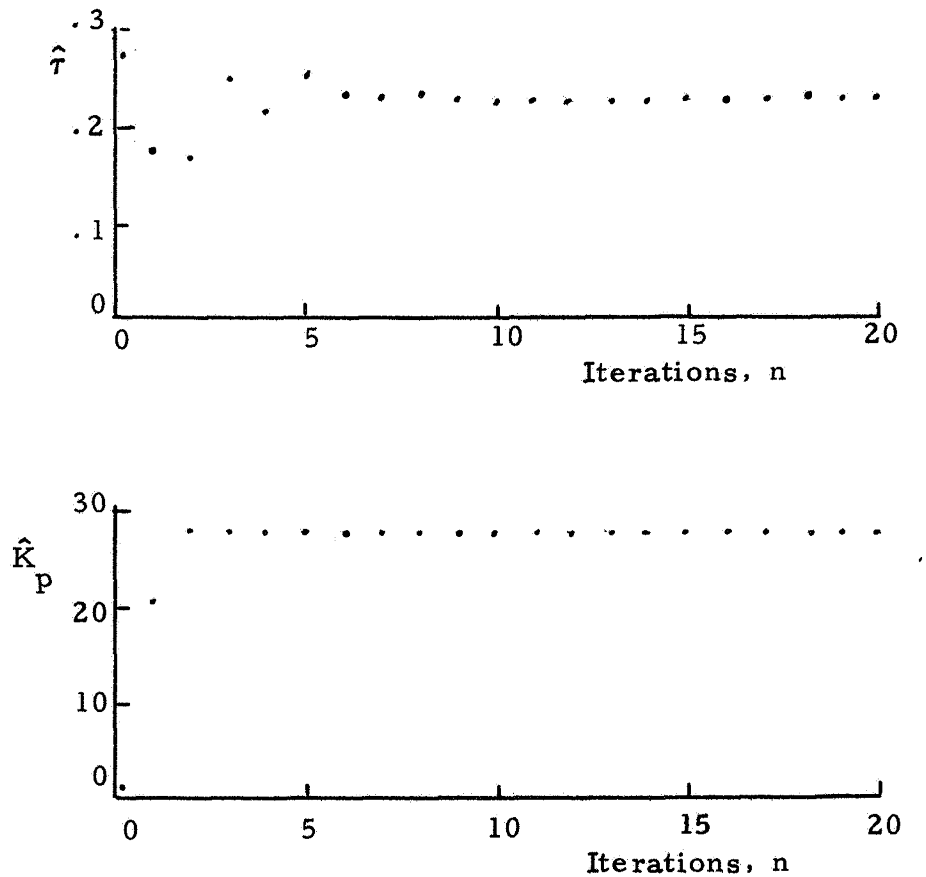
5.5.1 Discussion Of The Modeling Results

Figure 5.2 shows the results of stochastic approximation adjustment of the parameters $\hat{\mathcal{T}}$ and $\hat{K}_p$ of the S.T.I. transport lag model. Note that relatively stationary parameter values are achieved after only five iterations. The initial estimate of the parameter $\hat{K}_p$ was purposely chosen as very small so that large transient corrections would be induced in the estimation sequence for both $\hat{\tau}$ and $\hat{K}_p$ and thereby expose local minima in the cost function if the local minima existed. We conclude that local minima do not exist for the set of parameter vectors here calculated because the set of parameters which minimized the cost function has minimizing values which are close to those of the S.T.I. model. Furthermore, the cost function is smaller than that realized with the S.T.I. model for the data samples utilized.

Figure 5.3 shows the parameter estimates obtained when using the sampled-data model of the human operator controller. Qualitatively, the model appears to be poorer than the transport lag model as judged by both the larger value of the minimum cost function and the rougher appearance of the sequential parameter estimates. The minimum cost function is about 7% larger than that obtained with the transport lag model of Figure 5.2.

Figure 5.4 shows parameter estimates for the sampled-data model with first order linear lead-lag compensation. The sequence of the sequential estimates of sampling interval is smoother than that of Figure 5.3. The cost function is also about 6% lower than for the optimal transport lag model of Figure 5.2.

Finally, Figure 5.5 shows the transport lag model with lead-lag compensation. Clearly, this is a much better approximation than either of the sampled-data models as evidenced by the smooth iteration sequences and the fact that the cost function is about 30%



Controller Model

$$\hat{K}_p e^{-s\hat{\tau}}$$

Minimum Cost Function

At $\hat{\tau} = 0.2323$

$\hat{K}_p = 28.754$

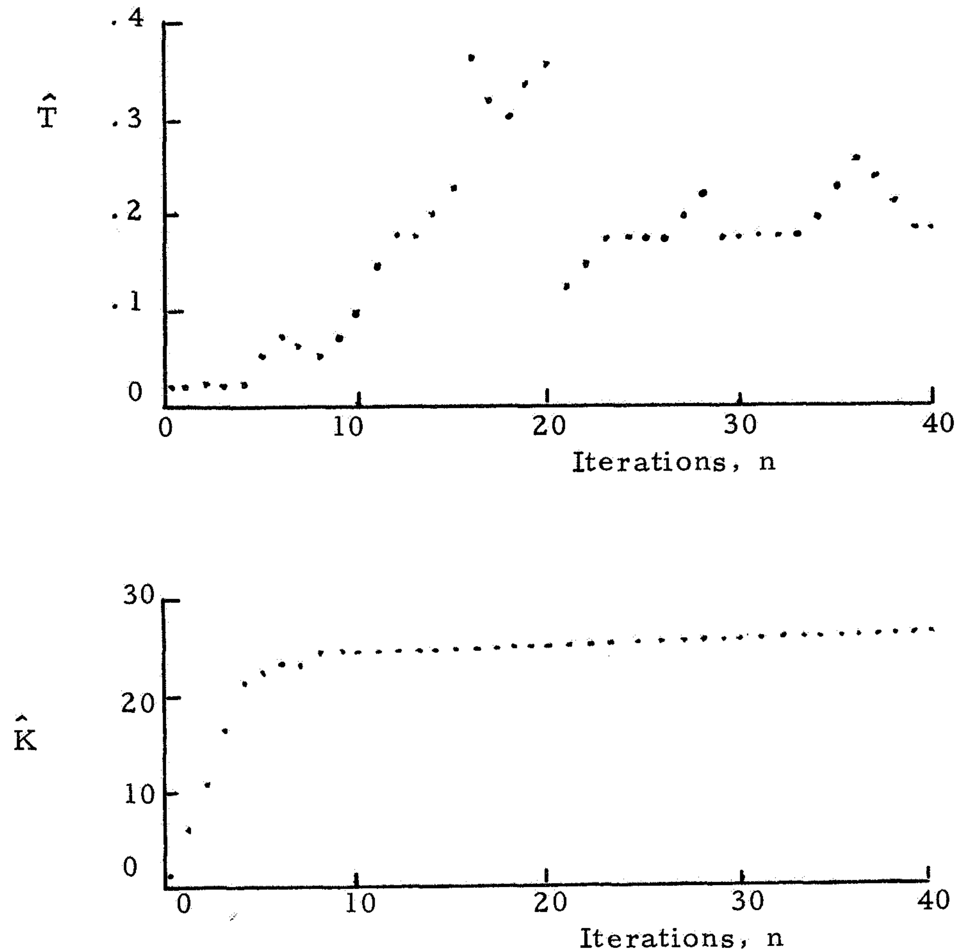
$n = 20$

$$J_{min} = \int_0^{\tau_i} \epsilon^2 dt = 94{,}087$$

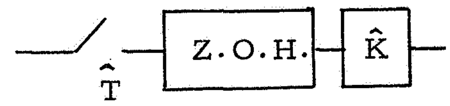
See Figure 5.1 For Estimator Configuration.

(Run 6-30-2-1)

Figure 5.2 Estimation of Parameters $\hat{\tau}$ and $\hat{K}_p$ By Stochastic Approximation.



Controller Model



Minimum Cost At

$\hat{T} = 0.2577$

$\hat{K} = 26.07$

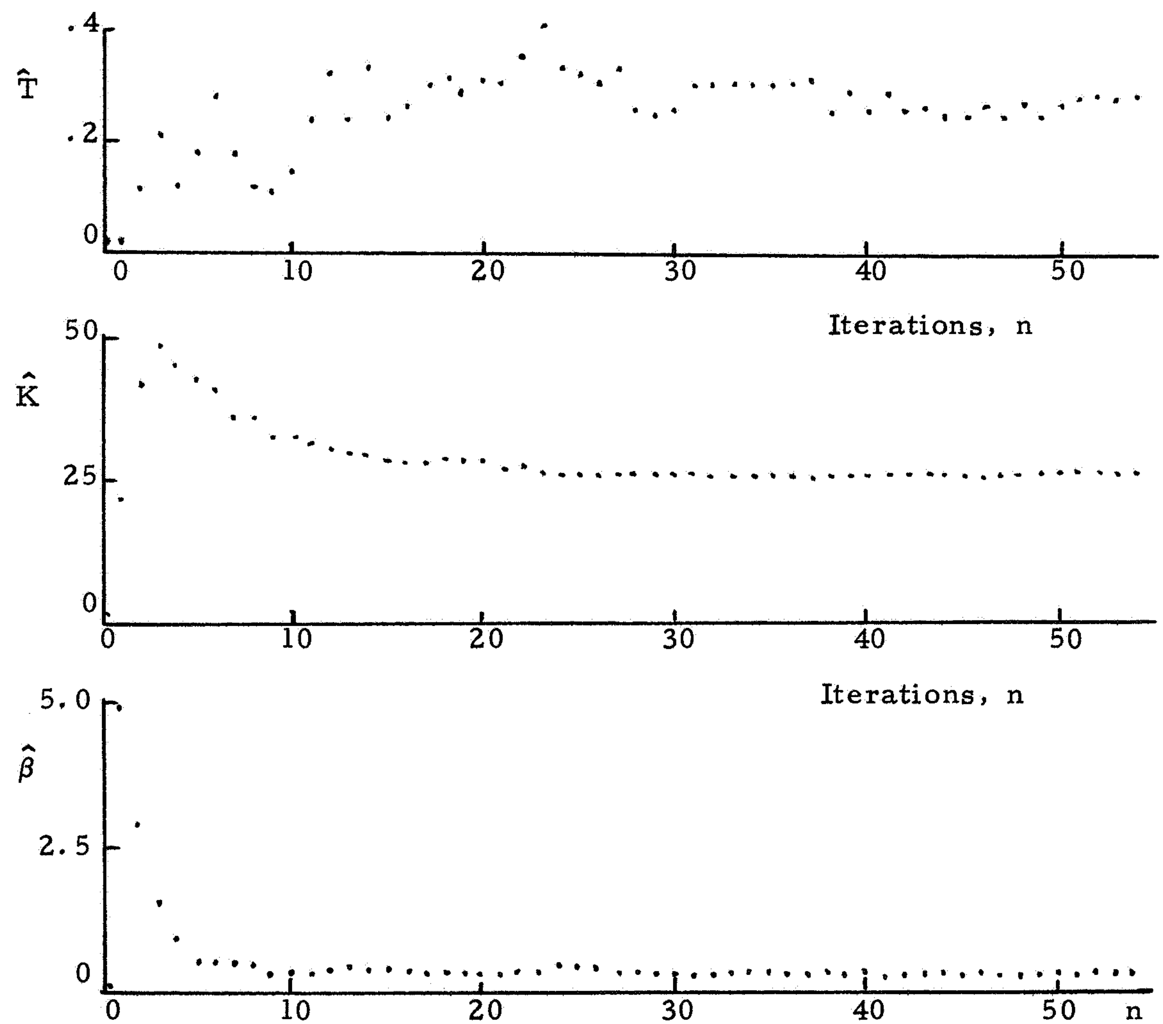
$n = 37$

$$J_{min} = \int_0^{T_i} \epsilon^2 \, dt = 101,114$$

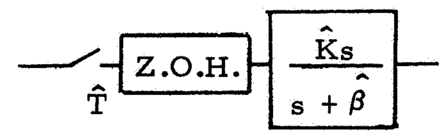
See Figure 5.1 For Estimator Configuration

(Ref. Run 7-1-1-1)

Figure 5.3 Estimation of Parameters $\hat{T}$ and $\hat{K}$ Using Stochastic Approximation.



Controller Model



Minimum Cost At

$\hat{T} = 0.2604$

$\hat{K} = 26.40$

$\hat{\beta} = 0.29$

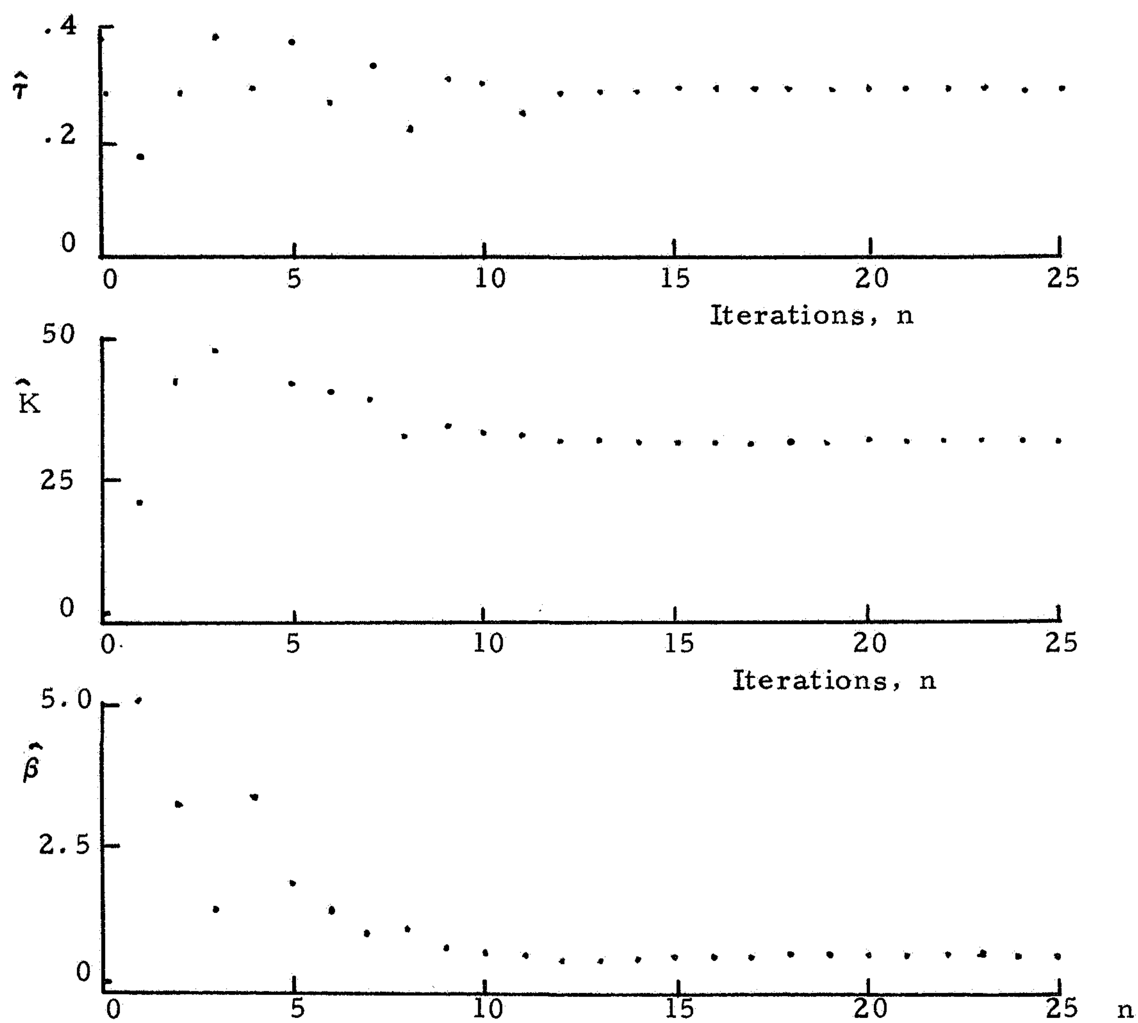
$n = 55$

$$J_{min} = \int_0^{T_i} \epsilon^2 \, dt = 89,075$$

See Figure 5.1 For Estimator Configuration

(Run 6-30-1-1)

Figure 5.4 Estimation of Parameters $\hat{T}$, $\hat{K}$, $\hat{\beta}$ By Stochastic Approximation



Controller Model

$$e^{-s\hat{\tau}}\left(\frac{\hat{K}s}{s+\hat{\beta}}\right)$$

Minimum Cost At

$\hat{\tau} = 0.2873$

$\hat{K} = 31.369$

$\hat{\beta} = 0.5759$

$$J_{min} = \int_0^{T_i} \epsilon^2 \, dt = 62,034$$

See Figure 5.1 For Estimator Configuration

(Run 7-1-2-1)

Figure 5.5 Estimation of $\hat{\tau}$, $\hat{K}$, And $\hat{\beta}$ By Stochastic Approximation.

smaller than for the better of the sampled-data models. Compared with the original S.T.I. model, the cost function is about 37% smaller, again, for the particular data samples here chosen.

5.6 Conclusions

Stochastic approximation has been applied successfully to problems in the modeling and estimation of parameters in a system of unknown order, unknown nonlinearities, and with possibly random parameters and with possibly noisy observations of system output. System input was a random function. In all cases linear models were used. These included both sampled-data models and transport lag models. Convergence of the parameter estimates occurred in every modeling situation, although convergence was smoother and quicker with the transport lag models than with the sampled-data models. Also, for models of the same complexity, the transport lag model yielded a smaller value of cost function than the sampled-data model.

So far as is known, this is the first study where estimates of the various parameters of linear transfer function models of unknown systems have been obtained by stochastic approximation from off-line operating data. By contrast, Sakrison obtained estimates of linear gains of nonlinear transfer functions comprising an optimal prediction filter. Holmes used off-line data to obtain an optimal Volterra series nonlinear representation of the human operator. Both used stochastic approximation to obtain their parameter estimates.

In our work, no difficulty in obtaining convergence was experienced when the complex human operator controller was represented by the relatively simple models. Furthermore, the optimal estimates of the parameters $\hat{\tau}$ and $\hat{K}_p$, estimated with the simple transport lag model, changed by only 24% and 8% respectively when the compensated transport lag model was used instead of the simple transport lag model.

From the results of the study it is concluded that the human operator controller is better represented by the transport lag model, with or without linear lead-lag compensation, than it is by a comparable sampled-data model.

While the results we have here obtained suggest that stochastic approximation may lead to a better model for the human operator than heretofore obtained by conventional spectral analysis methods, we cannot firm up such a conclusion until a sufficient amount of data has been used with the method. In this study, the data traces $i(kT_q)$ and $m(kT_q)$ which we used for modeling were of 29.4 seconds duration, and were chosen from the S.T.I. 240 second duration time traces [27]. The parameters of the S.T.I. model were based on data from the entire time interval, while we used a little over one-tenth of the data. It is quite possible that the parameters that S.T.I. obtained represent an average model, while our parameters represent the model for the particular subset of data which we used. Clearly, by applying stochastic approximation to time slices of the original data, e.g., 24 second subintervals of the original 240 second time trace, it should be possible to estimate the temporal behavior of time-varying parameters.

5.7 Recommendations For Subsequent Investigations

In Chapter 3 we proved mean-square convergence of parameter estimates of sampled-data systems for the estimation configuration of Figure 3.4 and for the stated restrictions on observation noise and dynamics of the continuous system. The parameters of the continuous system were assumed to be fixed. It is desirable to extend this work to the cases where the continuous system has either slowly-varying parameters, or random parameters, or both. In connection with the former, Dupac [104] has recently proved mean-square convergence of the estimates of the parameter which minimizes a regression function when that parameter varies by the multiplier $(1+1/n)$. Thus, the K-W estimator (3.69) would then be given by

$$\hat{x}_{n+1} = (1 + 1/n)\hat{x}_n + \frac{a_n}{c_n}(y_{2n-1} - y_{2n+1}) \qquad (5.3)$$

An approach to the analysis of conditions for the convergence of estimates obtained by stochastic approximation when a parameter has additive noise has been taken by de Figueiredo and Dyer [113].

In addition, work is needed to yield both insight and possibly some sort of convergence result for the general modeling case where the model is of lower order than the unknown system. Some work along this line has recently been reported by Mork [114].

APPENDIX I

ITERATIVE STEEP DESCENT METHODS

The various expressions for the K matrix of Table 2.1 have a common basis. Suppose it is desired to minimize a scalar function of several parameters say

$$J(\hat{x}) = J(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^k) \qquad (1)$$

where $\hat{x}$ is a k dimensional parameter vector, with components as indicated. Assuming that the third order partials exist and are bounded, J can be expanded in the Taylor series (to the second order term) about the j^{th} iteration of the parameter vector $\hat{x}$. For an increment $\Delta\hat{x}_j$ in the parameter vector, defined as the vector difference between the $(j+1)^{th}$ and the j^{th} iterations of the parameter vector, we have

$$\hat{x}_j = \hat{x}_{j+1} - \hat{x}_j \qquad (2)$$

The expansion of $J(\cdot)$ about the parameter vector $\hat{x}_j$ is then

$$J(\hat{x}_{j+1}) = J(\hat{x}_j) + \left[\nabla_{\hat{x}} J(\hat{x}_j)\right]' \Delta\hat{x}_j + 1/2(\Delta\hat{x}_j)' H_j \Delta\hat{x}_j + o(\Delta\hat{x}_j) \qquad (3)$$

where $o(\Delta\hat{x}_j)$ vanishes when $\|\Delta\hat{x}_j\|$ goes to zero, $\nabla_{\hat{x}} J(\hat{x}_j)$ indicates the gradient of J with respect to the vector $\hat{x}$ evaluated at $\hat{x}_j$, and H_j is the matrix

$$H_j = \nabla_{\hat{x}}\left[(\nabla_{\hat{x}} J(\hat{x}_j))'\right] \qquad (4)$$

Note that H_j depends on the vector $\hat{x}_j$, hence its components may be changed after each iteration.

The Newton-Raphson technique requires that we select the parameter perturbation vector which minimizes the right hand side of (3) with respect to $\Delta\hat{x}$. This is found by setting the gradient of (3) with respect to $\Delta\hat{x}_j$ to zero, so that

$$0 = \nabla_{\Delta\hat{x}}\left[(\Delta\hat{x}_j)' \nabla_{\hat{x}} J(\hat{x}_j) + 1/2(\Delta\hat{x}_j)' H_j\Delta\hat{x}_j\right] \tag{5}$$

This results in

$$\Delta\hat{x}_j = -H_j^{-1}\left[\nabla_{\hat{x}}J(\hat{x}_j)\right] \tag{6}$$

Hence, K_j in (2.30) is simply H_j^{-1}. Note, this is analogous, in the scalar case, to expanding the first derivative in a Taylor series and solving for the iteration which renders it zero.

Sometimes, instead of the above approach, a more limited Newton-Raphson approach is used. This is done as follows: Take only terms of the linear term in $\Delta\hat{x}_j$ in (3):

$$J(\hat{x}_{j+1}) = J(\hat{x}_j) + \left[\Delta\hat{x}_j\right]' \nabla_{\hat{x}}J(\hat{x}_j) \tag{7}$$

Choosing $\Delta\hat{x}_j$ such that movement is opposite to the gradient of J yields

$$\Delta\hat{x}_j = -k_1\nabla_{\hat{x}}J(\hat{x}_j) \tag{8}$$

where k_1 is a scalar. Substituting in (7)

$$J(\hat{x}_{j+1}) = J(\hat{x}_j) - k_1\left[\nabla_{\hat{x}}J(\hat{x}_j)\right]' \left[\nabla_{\hat{x}}J(\hat{x}_j)\right] \tag{9}$$

Setting (9) to zero yields k_1

$$k_1 = \frac{-J(\hat{x}_j)}{\|\nabla_{\hat{x}} J(\hat{x}_j\|^2} \tag{10}$$

Substituting into (8) give the incremental parameter vector

$$\Delta\hat{x}_j = \frac{-J(\hat{x}_j)\nabla_{\hat{x}} J(\hat{x}_j)}{\|\nabla_{\hat{x}} J(\hat{x}_j)\|^2} \tag{11}$$

Hence, K_j in (2.30) is $\dfrac{J(\hat{x}_j)}{\|\nabla_{\hat{x}} J(\hat{x}_j)\|^2}$. This is however, not included in Table 2.1 for the following reasons: This form of the Newton-Raphson method unfortunately yields an incremental parameter vector which becomes infinite if the criterion function J does not go to zero when the gradient $\nabla_{\hat{x}} J$ goes to zero. Such is not the case with (6). Hence, (11), by itself, is not much used in gradient work although the optimum gradient method does use it [90].

The <u>steep descent</u> method simply uses a matrix of constant positive multipliers for the K matrix. It is not necessarily updated. From (2) we have

$$\hat{x}_{j+1} = \hat{x}_j + \Delta\hat{x}_j. \tag{2}$$

Take

$$\Delta\hat{x}_j = -kI\nabla_{\hat{x}} J(\hat{x}_j) \tag{12}$$

where k is a positive constant and I is the kxk unit matrix.

Substituting into (2) yields

$$\hat{x}_{j+1} = \hat{x}_j - kI\nabla_{\hat{x}} J(\hat{x}_j) \tag{13}$$

Hence, K_j in (2.30) is simply kI.

This method, though simple, will not converge if k is chosen too large. On the other hand if k is small enough for convergence then more computer time may be used than with the Newton-Raphson method.

The Gauss-Newton method will be illustrated after the application of the Newton-Raphson method to the scalar integral cost function

$$J = \int_0^T e^2(t;\ \hat{x}_j)dt \tag{14}$$

where e is a scalar function of time and is dependent on the parameter vector $\hat{x}_j$.

The Newton-Raphson method applied to (14) yields the correction parameter vector

$$\begin{aligned}\Delta\hat{x}_j &= -H_j^{-1}\ \nabla_{\hat{x}} J(\hat{x}_j) \\ &= -H_j^{-1} \int_0^T \nabla_{\hat{x}} e^2(t;\ \hat{x}_j)\ dt \\ &= -2H_j^{-1} \int_0^T (\nabla_{\hat{x}} e'(t;\ \hat{x}_j))\, e(t;\ \hat{x}_j)\ dt \end{aligned} \tag{15}$$

But from (4), we wrote H_j as

$$H_j = \nabla_{\hat{x}}\left[(\nabla_{\hat{x}} J(\hat{x}_j))'\right] \tag{4}$$

Applying (4) to (14) and writing $e(t;\ \hat{x}_j)$ concisely

$$H_j = \nabla_{\hat{x}}\left[\left(\nabla_{\hat{x}} \int_0^T e^2(t;\ \hat{x}_j)\ dt\right)'\right]$$

$$= 2\int_0^T \left[e\nabla_{\hat{x}}((\nabla_{\hat{x}}(e))') + \nabla_{\hat{x}}(e)(\nabla_{\hat{x}}(e))'\right] dt \tag{16}$$

The use of (16) guarantees quadratic convergence of the gradient technique when J has a regular minimum [91].

The Gauss-Newton method uses the development leading to (16) but simplifies the computation of H by omitting the first term in the integrand [91, 92]. The multiplying matrix is then

$$H_j = 2\int_0^T \nabla_{\hat{x}}(e)\ (\nabla_{\hat{x}}(e))'\ dt \tag{17}$$

As shown in Chapter 2, the gradient terms in the integrand of (17) are simply the sensitivity functions as discussed in connection with (2.27) and (2.51). Hence, the gain matrix from (15) is

$$K_j = H_j^{-1} \tag{18}$$

Using (17)

$$K_j = \left[2\int_0^T \nabla_{\hat{x}}(e(t;\ \hat{x}_j))(\nabla_{\hat{x}}(e(t;\ \hat{x}_j)))'\ dt\right]^{-1} \tag{19}$$

If we consider a sampled-data system with sampling of period $\hat{T}$; then t is replaced by $k_2\hat{T}$, where $k_2 \epsilon [0, 1, 2, \ldots.)$.

$$K_j = \left[2\int_0^T \nabla_{\hat{x}}(e(k_2\hat{T};\hat{x}_j))(\nabla_{\hat{x}}(e(k_2\hat{T};\hat{x}_j)))' \, dt\right]^{-1}$$

$$= \left[2\int_0^T \sigma(k_2\hat{T})(\sigma(k_2\hat{T}))' \, dt\right]^{-1} \qquad (20)$$

where $\sigma(\cdot)$ is the vector solution of the sensitivity difference equation. (See Chapter 2.)

Finally, if we reduce K_j by means of a positive constant k, we obtain the modified Gauss-Newton method; for which

$$K_j = k\left[2\int_0^T \sigma(k_2\hat{T})(\sigma k_2\hat{T}))' \, dt\right]^{-1}. \qquad (21)$$

When (20) is used, the gradient procedure may not converge [91].

APPENDIX II

THE EQUATION FOR THE DERIVATIVE OF THE DRIVING FUNCTION

We desire the expression for the term $\frac{dr(nT)}{dT}$ which appears as one of the driving functions in the sampling interval sensitivity difference equations of Chapter 2. The analysis is restricted to sinusoidal (or cosinusoidal) inputs, but, even so, the results are quite general since any continuous input can be constructed from a Fourier series of sines and cosines. Additionally, a simple sine or cosine drive is still a satisfactory input test drive signal since it is sampled and held in each loop. Consequently, a succession of step functions is imposed on both of the continuous systems. The result is that all modes of each of the continuous systems are excited by the infinite frequency content of these signals.

The driving signal to each closed loop system is

$$r(t) = A \sin \omega t. \tag{1}$$

At the sampling instant $t = k_2\hat{T}$

$$r(nt) = A \sin \omega k_2 t \tag{2}$$

Likewise, the continuous derivative of the driving signal (at $t = k_2\hat{T}$) is

$$\dot{r}(t) = A\,\omega \cos \omega k_2 \hat{T}. \tag{3}$$

In deriving the sensitivity difference equation in $\hat{T}$ in Chapter 2, we were interested in the input signal to, and the output signal from the continuous dynamics. Consequently, it was convenient to express these signals at the sampling instants $t = k_2\hat{T}$ by means of a difference equations. The input signal to each continuous system was obtained from a data hold. Therefore, the reconstructed signal obtained from the (zero-order) hold can be written

$$r(k_2\hat{T}) = A \sin \omega(k_2-1)\hat{T} \tag{4}$$

and the reconstructed derivative of the output of the data hold is

$$\frac{dr(k_2\hat{T})}{d\hat{T}} = A\,(k_2-1)\,\omega \cos \omega(k_2-1)\hat{T} \tag{5}$$

Assuming $k_2 > 5$, (5) becomes

$$\frac{dr(k_2\hat{T})}{d\hat{T}} \approx A\,k_2\,\omega \cos \omega k_2\hat{T}. \tag{6}$$

This can also be written

$$\frac{dr(k_2\hat{T})}{d\hat{T}} \approx \frac{1}{\hat{T}}\left[t\,\dot{x}\,(t)\right]_{t=k_2\hat{T}} \tag{7}$$

The desired quantity for the purpose of generating sensitivity difference equations appears on the left side of (7). The right side of (7) shows how this derivative is constructed from the derivative of the input driving signals (1).

APPENDIX III

PROPERTIES OF SEQUENCES

The following properties of series [70] are used in the proof of mean-square convergence of the Kiefer-Wolfowitz procedure:

(1) If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \to \infty} b_n = 0$. Note that this is only a necessary condition.

<u>Proof:</u> Let $\sum_{n=1}^{N-1} b_n = S_{N-1}$ and $\sum_{n=1}^{N} b_n = S_N$.

Then $b_N = S_N - S_{N-1}$ But $\lim_{N \to \infty} \sum_{n=1}^{N} b_n = S < \infty$

and hence $\lim_{N \to \infty} \sum_{n=1}^{N-1} b_n = S$

Therefore, $\lim_{N \to \infty} b_N = \lim_{N \to \infty} \left[\sum_{n=1}^{N} b_n - \sum_{n=1}^{N-1} b_n \right] = S - S = 0.$

(2) For n = 1, 2, 3, ..., the p series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ has the properties that is converges (diverges) as $p > 1$, $(p \le 1)$, i.e., $\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$ if $p > 1$, and $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$ if $p \le 1$.

Problem: Using the above properties, determine the range of γ for which the following are true

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{c_n}\right)^2 < \infty, \quad \sum_{n=1}^{\infty}\left(\frac{a_n}{c_n}\right) = \infty, \quad \lim_{n\to\infty} c_n = 0, \quad \lim_{n\to\infty} a_n = 0 \tag{1}$$

where we assume $a_n = A/n$, $n = 1, 2, 3, 4, \ldots$, and $c_n = C/n^{\gamma}$, and where $A, C > 0$.

Solution: From the convergent p series, we have

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{c_n}\right)^2 = \sum\left(\frac{A}{C}\right)^2 \frac{1}{n^{2(1-\gamma)}} < \infty \text{ if } 2(1-\gamma) > 1, \text{ i.e., when}$$

$\gamma < 1/2$. Also, $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{c_n} = \sum_{n=1}^{\infty} \frac{A}{C}\frac{1}{n^{(1-\gamma)}} = \infty$ when $\frac{A}{C} > 0$ and

when $1-\gamma \le 1$, i.e., when $\gamma \ge 0$. In addition, if $c_n = C/n^{\gamma}$,

then $\lim_{n\to\infty} C/n^{\gamma} = 0$ if $\gamma > 0$. Also, note from (1) that

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{c_n}\right)^2 < \infty \text{ implies } \lim_{n\to\infty}\left(\frac{a_n}{c_n}\right)^2 = 0 \text{ which also implies}$$

$\lim_{n\to\infty} (a_n/c_n) = 0$.

Summary: The desired properties $\displaystyle\sum_{n=1}^{\infty}\left(\frac{a_n}{c_n}\right)^2$, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{a_n}{c_n}\right)$,

$\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} c_n = 0$, will obtain when $0 < \gamma < 1/2$.

APPENDIX IV

LISTINGS OF SIMULATION PROGRAMS

This appendix presents an example of the special CSMP computer subroutines and program used in the simulations of Chapter 4. It was selected because it illustrates all aspects of the simulation effort. Specifically, the listing is for the sampled-data feedback system with nonlinear first-order continuous dynamic system given by Example 2 of Chapter 4. Both the sampling interval T and the gain K have random components with excursions set equal to the nominal values. Simulation results for this set of listings are given in Figure 4.11. Also included are several iterations of the parameter vector of the sampled-data model: $\hat{x} = (\hat{T}, \hat{K})'$. The nominal values of the parameter vector of the sampled-data system are: $T = 0.235$, $K = 0.025$.

```
//NFAL      JOB ,111899
IA55I  155355
//SUB1    EXEC FORTRAN(BCD)

 FORTRAN IV     MODEL 44  PS        VERSION 3,  LEVEL 1   DATE  68353          USC/SSL        PAGE  0001

              SUBROUTINE    SUB1
C                THIS SUBROUTINE CALCULATES THE ITERATIVE KIEFER WULFOWITZ
C                STOCHASTIC APPROXIMATION ALGORITHM.
C                THE SYSTEM CONSIDERED HAS A CUBIC FUNCTION FOLLOWED BY AN
C                INTEGRATOR WITH AN UNKNOWN GAIN.   THE SAMPLING INTERVAL IS ALSO
C                UNKNOWN.   BOTH THE GAIN AND THE SAMPLING INTERVAL HAVE NOISY
C                COMPONENTS.    THE SAMPLING INTERVAL NOISE IS GENERATED BY SUB2.
C                WHICH FOLLOWS THIS SUBROUTINE.
              REAL  REALS(395)
              INTEGER  INTS(587)
              DIMENSION C(76),PAR1(75),MTRX2(75)
              DIMENSION DD(100)
              COMMON REALS, INTS
              COMMON DD
              EQUIVALENCE (INTS(76), MTRX2(1)), (REALS(2), C(1))
              EQUIVALENCE (INTS(376),   I    ), (REALS(79), DTS2 )
              EQUIVALENCE                        (REALS(81), PAR1(1))
              DIMENSION  MTRX3(75), PAR2(75), PAR3(75), MTRX4(75)
              EQUIVALENCE (INTS(151),  MTRX3(1)),(INTS(226),MTRX4(1))
              EQUIVALENCE (REALS(156), PAR2(1)) ,(REALS(231),PAR3(1))
              EQUIVALENCE (INTS(529) , TEST5 )
              EQUIVALENCE (DD(1),YM1), (DD(2),YP1),(DD(3),YM2), (DD(4),YP2)
              INTEGER  TEST5
              IF(C(76))1,1,2
            1 PAR1(I)=DTS2/2.0
              C(I)=1.0
            4 RETURN
            2 PAR1(I)=PAR1(I)+DTS2
                IF(PAR1(I)-PAR2(I))3,1,1
            3 C(I)=0.0
              IF(C(76)-4.0)4,5,5
            5 TEST5=6
              J=MTRX3(I)
              CN1=0.01*(PAR2(J)**(-.166))
               CN2=0.01*CN1
              N=PAR3(I)
              GO TO (20, 21, 22, 23, 24),N
C                PAR2(I) IS THE SAMPLING INTERVAL
           20 PAR2(I)=PAR2(I)-CN1
              PAR3(I)=2.0
              RETURN
           21 J=MTRX2(I)
              YM1=C(J)
              J=MTRX4(I)
              PAR2(I)=PAR2(I) + 2.0*CN1
              PAR3(I)=3.0
              RETURN
           22 J=MTRX2(I)
              YP1=C(J)
              PAR2(I)=PAR2(I)-CN1
              J=MTRX4(I)
C                PAR2(J) IS THE GAIN PARAMETER
              PAR2(J)=PAR2(J)-CN2
              PAR3(I)=4.0
              RETURN

           23 J=MTRX2(I)
              YM2=C(J)
              J=MTRX4(I)
              PAR2(J)=PAR2(J)+2.0*CN2
              PAR3(I)=5.0
              RETURN
           24 J=MTRX2(I)
              YP2=C(J)
              J=MTRX4(I)
              PAR2(J)=PAR2(J)-CN2
              P2=PAR2(I)
              P3=PAR2(J)
              JJ=MTRX3(I)
              AN=0.000005/PAR2(JJ)
              AL=(AN/CN1)
              D1=AL*(YM1-YP1)
              D2=AL*(YM2-YP2)
              P1=D1
              P4=D2
              IF(ABS(D1).LE.0.1)GO TO 9
              D1=0.1*D1/ABS(D1)
            9 CONTINUE
              IF((PAR2(I)+D1).LE.0.015)GO TO 10
              PAR2(I)=PAR2(I)+D1
           10 CONTINUE
              IF(ABS(D2).LE.0.025)GO TO 11
              D2=0.01*D2/ABS(D2)
           11 CONTINUE
              IF((PAR2(J)+D2).LE.0.002)GO TO 12
              PAR2(J)=PAR2(J)+D2
           12 CONTINUE
              PAR3(I)=1.0
              PAR2(JJ)=PAR2(JJ)+1.0
              WRITE(3,30)PAR2(JJ), YM1, YP1, P1,D1, P2, PAR2(I)
              WRITE(3,30)PAR2(JJ), YM2, YP2, P4, D2, P3, PAR2(J)
              RETURN
           30 FORMAT(1H1, 7F17.4)
              END
```

```
FORTRAN IV    MODEL 44  PS       VERSION 3,  LEVEL 1   DATE  68353            USC/SSL        PAGE  0001

 0001              SUBROUTINE SUB2
            C         THIS ELEMENT GENERATES A UNIT PULSE FOR TX.  THE UNIT
            C         PULSE IS C(I), AND IS GENERATED WHENEVER THE TIME SINCE
            C         THE LAST PULSE ( PAR1(I)  ) EQUALS OR EXCEEDS THE VALUE
            C         OF THE QUANTITY( PAR3(I)+0.1*PAR3(I)*C(J)  )
            C         C(J) IS THE OUTPUT OF THE JITTER NOISE GENERATOR .
 0002              REAL  REALS(395), NN, AVTX,  TIM
 0003              INTEGER  INTS(587)
 0004              DIMENSION C(76), PAR1(75), PAR2(75), PAR3(75)
 0005              DIMENSION MTRX2(75), MTRX3(75), MTRX4(75)
 0006              DIMENSION DD(100)
 0007              COMMON  REALS, INTS
 0008              COMMON DD
 0009              EQUIVALENCE(REALS(2), C(1))
 0010              EQUIVALENCE (REALS(79), DTS2)
 0011              EQUIVALENCE (REALS(81), PAR1(1))
 0012              EQUIVALENCE (REALS(156), PAR2(1))
 0013              EQUIVALENCE (REALS(231), PAR3(1))
 0014              EQUIVALENCE (INTS(76), MTRX2(1))
 0015              EQUIVALENCE (INTS(151), MTRX3(1))
 0016              EQUIVALENCE (INTS(226), MTRX4(1))
 0017              EQUIVALENCE (INTS(376),  I)
 0018              IF(C(76))1,1,2
 0019            1 PAR1(I)=DTS2/2.0
 0020              J=MTRX2(I)
 0021              DD(4)=PAR3(I)+PAR3(I)*C(J)
 0022              C(I)=1.0
 0023              GO TO 4
 0024            2 PAR1(I)=PAR1(I)+DTS2
 0025              IF(PAR1(I)-DD(4))3,1,1
 0026            3 C(I)=0.0
 0027            4 RETURN
 0028              END

//          EXEC RLNKEDT(MAP,SYS002)

LIST        PHASE      ROUT,ROOT,NOAUTO
LIST            INCLUDE    CSMM,R
LIST            INCLUDE    CSM9,R
LIST            INCLUDE    DATSW,R
LIST            INCLUDE    LOAD,R
LIST            INCLUDE    IBCOM#,R
LIST            INCLUDE FIOCS#,R
LIST            INCLUDE    USEROPT,R
LIST            INCLUDE    UNITAB#,R
LIST            INCLUDE    SQRT,R
LIST            INCLUDE    FRXPR#,R
LIST            INCLUDE    ALOG,R
LIST            INCLUDE    EXP,R
LIST       PHASE      SORT,*,NOAUTO
LIST            INCLUDE    CSM0,R
LIST            INCLUDE    CSM1,R
LIST            INCLUDE    CSM2,R
LIST            INCLUDE    CSM3,R
LIST            INCLUDE    CSM4,R
LIST            INCLUDE    CSM5,R
LIST            INCLUDE    CSM6,R
LIST            INCLUDE    CSM7,R
LIST            INCLUDE    CSM12,R
LIST            INCLUDE    CSM13,R
LIST       PHASE      RUN,SORT,NOAUTO
LIST            INCLUDE    CSM10,R
LIST            INCLUDE    CSM11,R
LIST            INCLUDE    CSM8,R
LIST            INCLUDE    CSM8A,R
LIST            INCLUDE    SUB1,L
LIST            INCLUDE    SUB10001,L
LIST            INCLUDE    SUB3,R
LIST            INCLUDE    SUB4,R
LIST            INCLUDE    SUB5,R


LINKAGE EDITOR HIGHEST SEVERITY WAS  0
//SYS001   ACCESS   SDSRDR
//SYS002   ACCESS   SDSPCH
//SYS005   ACCESS   SDSOPT
//          EXEC          CONTINUOUS SYSTEM MODELING PROGRAM
```

CONFIGURATION SPECIFICATION

OUTPUT NAME	BLOCK	TYPE	INPUT 1	INPUT 2	INPUT 3
NOISE DRIVE	1	J	0	0	0
NOISE DRIVE GAIN	2	G	1	0	0
CORRECT DRIVE ME	3	+	48	-66	0
ZOH	4	Z	20	40	0
ZOH	5	Z	22	50	0
FILTER SUMMER	6	+	2	9	0
FILTER INT 1	7	I	0	6	8
FILTER SUGN INV	8	-	7	0	0
FILTER INT 2	9	I	0	8	0
SYSTEM	10	I	0	32	0
MODEL	11	I	0	34	0
SUMMER	12	+	10	23	-11
NOISE	13	J	0	0	0
SQUARER	14	X	12	12	0
Y	15	I	0	14	0
GAIN	16	G	17	0	0
NOISE	17	J	0	0	0
MULT	18	X	4	16	0
GAIN	19	G	13	0	0
SUMMER	20	+	-10	3	0
TAO	21	K	0	0	0
SUMMER	22	+	-11	3	0
SUM NOISE OFF	23	O	19	0	0
SYSTEM SQUARE	31	X	4	4	0
SYSTEM CUBE	32	X	31	4	0
MODEL SQUARE	33	X	5	5	0
MODEL CUBE	34	X	33	5	0
TX GEN	40	2	1	67	0
FILTER OUTPUT GA	48	G	9	0	0
DIVIDE CHECK MEA	49	/	67	64	0
SPECIAL	50	1	15	21	11
DRIVE MEAN OFFSE	64	O	76	0	0
MEAN OFFSET	66	K	0	0	0
CHECK DRIVE MEAN	67	I	3	0	0

```
                        INITIAL CUNDITIONS AND PARAMETERS

IC/PAR NAME       BLOCK      IC/PAR1          PAR2           PAR3
NOISE DRIVE GAIN    2        20.0000          0.0            0.0
HH, 2(ZW)           7         0.0             0.6300         0.8820
FILTER INT 2        9         0.0             1.0000         0.0
SYSTEM GAIN        10         0.0             0.0250         1.0000
MODEL GAIN         11         0.0             0.0025         0.0
Y GAIN             15         0.0           500.0000         0.0
SYS NOI GA(1.0)    16         0.0250          0.0            0.0
SUM NOISE GAIN     19        10.0000          0.0            0.0
IMPROPER PARAMETER SPECIFICATION FOR ELEMENT
N                  21         0.0             1.0000         0.0
SUM NOISE OFF      23         0.0             0.0            0.0
TX GEN RANDOM SE   40         0.0          7243.0000         0.2500
FILTER OUT GAIN    48        22.4000          0.0            0.0
TA                 50         0.0             0.0200         1.0000
DRIVE MEAN OFFSE   64         0.0001          0.0            0.0
MEAN COR(CORR)     66        10.8394          0.0            0.0
CHECK DRIVE MEAN   67         0.0             0.0            0.0


(        ¤ INTEGRATION INTERVAL
  0.01000


(        ¤ TOTAL TIME
 20.00000


*         ¤ PRINT INTERVAL
      1.


( ¤ BLOCK FOR Y-AXIS   *          ¤ MINIMUM VALUE     *           ¤ MAXIMUM VALUE

   TIME      OUTPUT(   )  OUTPUT(   )  OUTPUT(   )  OUTPUT 15       -20.0000                                   20.00
   0.000       0.0000       0.0000       0.0000        0.0000    I-----------------------+                       I
   0.500     -10.4422      -8.0462      -1.1985    16225.5742    I-----------------------------------------------+
   1.000      -9.9954      -8.1454      -1.8509    40247.8672    I-----------------------------------------------+
   1.500      -9.3959      -8.1505      -2.1552    56877.6484    I-----------------------------------------------+
   2.000      -8.1868      -7.9920      -2.2126    72203.5625    I-----------------------------------------------+
   2.500      -5.7907      -1.2149      -1.8785    84791.3125    I-----------------------------------------------+
   3.000      -3.3175       4.2648      -0.4866    93609.6875    I-----------------------------------------------+
   3.500      -1.1425       8.0584       1.4120   107672.6875    I-----------------------------------------------+
   4.000       0.0005       9.3713       1.7769   129631.3125    I-----------------------------------------------+
   4.005       0.0091       9.3709       1.7786   130064.2500    I-----------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START


SWITCHES SET ON WERE    0

   TIME      OUTPUT 49    OUTPUT 10    OUTPUT 11    OUTPUT 15      -20.0000                                 20.0000
   0.000       0.0000       0.0000       0.0000        0.0000    I-----------------------+                       I
   0.500     -10.4422      -8.0462      -1.1921    16237.8867    I-----------------------------------------------+
   1.000      -9.9954      -8.1454      -1.8425    40291.1523    I-----------------------------------------------+
   1.500      -9.3959      -8.1505      -2.1452    56948.0430    I-----------------------------------------------+
   2.000      -8.1868      -7.9920      -2.2013    72303.5000    I-----------------------------------------------+
   2.500      -5.7907      -1.2149      -1.8604    84910.3125    I-----------------------------------------------+
   3.000      -3.3175       4.2648      -0.4613    93698.6875    I-----------------------------------------------+
   3.500      -1.1425       8.0584       1.4133   107725.2500    I-----------------------------------------------+
   4.000       0.0005       9.3713       1.7717   129697.0625    I-----------------------------------------------+
   4.005       0.0091       9.3709       1.7735   130130.3125    I-----------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START


SWITCHES SET ON WERE    0

   TIME      OUTPUT 49    OUTPUT 10    OUTPUT 11    OUTPUT 15      -20.0000                                 20.0000
   0.000       0.0000       0.0000       0.0000        0.0000    I-----------------------+                       I
   0.500     -10.4422      -8.0462      -1.2039    16215.4062    I-----------------------------------------------+
   1.000      -9.9954      -8.1454      -1.8585    40209.9648    I-----------------------------------------------+
   1.500      -9.3959      -8.1505      -2.1647    56813.9766    I-----------------------------------------------+
   2.000      -8.1868      -7.9920      -2.2237    72111.4375    I-----------------------------------------------+
   2.500      -5.7907      -1.2149      -1.8961    84680.9375    I-----------------------------------------------+
   3.000      -3.3175       4.2648      -0.5115    93527.1875    I-----------------------------------------------+
   3.500      -1.1425       8.0584       1.4095   107626.6250    I-----------------------------------------------+
   4.000       0.0005       9.3713       1.7813   129574.6875    I-----------------------------------------------+
   4.005       0.0091       9.3709       1.7830   130007.3125    I-----------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START


SWITCHES SET ON WERE    0

   TIME      OUTPUT 49    OUTPUT 10    OUTPUT 11    OUTPUT 15      -20.0000                                 20.0000
   0.000       0.0000       0.0000       0.0000        0.0000    I-----------------------+                       I
   0.500     -10.4422      -8.0462      -1.1581    16302.6289    I-----------------------------------------------+
   1.000      -9.9954      -8.1454      -1.7957    40529.0039    I-----------------------------------------------+
   1.500      -9.3959      -8.1505      -2.0959    57327.5703    I-----------------------------------------------+
   2.000      -8.1868      -7.9920      -2.1534    72815.3750    I-----------------------------------------------+
   2.500      -5.7907      -1.2149      -1.8394    85494.6875    I-----------------------------------------------+
   3.000      -3.3175       4.2648      -0.5070    94305.5000    I-----------------------------------------------+
   3.500      -1.1425       8.0584       1.3391   108504.8750    I-----------------------------------------------+
   4.000       0.0005       9.3713       1.7011   130747.8750    I-----------------------------------------------+
   4.005       0.0091       9.3709       1.7028   131185.8125    I-----------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START


SWITCHES SET ON WERE    0

   TIME      OUTPUT 49´   OUTPUT 10    OUTPUT 11    OUTPUT 15      -20.0000                                 20.0000
   0.000       0.0000       0.0000       0.0000        0.0000    I-----------------------+                       I
   0.500     -10.4422      -8.0462      -1.2384    16149.5820    I-----------------------------------------------+
   1.000      -9.9954      -8.1454      -1.9051    39972.4062    I-----------------------------------------------+
   1.500      -9.3959      -8.1505      -2.2131    56438.7773    I-----------------------------------------------+
   2.000      -8.1868      -7.9920      -2.2704    71608.8750    I-----------------------------------------------+
   2.500      -5.7907      -1.2149      -1.9157    84109.1250    I-----------------------------------------------+
   3.000      -3.3175       4.2648      -0.4646    92932.2500    I-----------------------------------------------+
   3.500      -1.1425       8.0584       1.4844   106858.9375    I-----------------------------------------------+
   4.000       0.0005       9.3713       1.8517   128539.0000    I-----------------------------------------------+
1            2.0000      130130.3125     130007.3125            0.0615          0.0615         0.0203         0.0815
1            2.0000      131185.8125     128967.0000            1.1094          0.0100         0.0025         0.0125
   4.005       0.0091       9.3709       1.8534   128967.0000    I-----------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START
```

```
SWITCHES SET ON WERE     0

    TIME       OUTPUT 49    OUTPUT 10    OUTPUT 11    OUTPUT 15       -20.0000                                        20.0000
    0.000        0.0000       0.0000       0.0000        0.0000       I------------------------+                        I
    0.500      -10.4422      -8.0462      -3.9103    11407.3398       I-------------------------------------------------+
    1.000       -9.9954      -8.1454      -4.8328    25490.6055       I-------------------------------------------------+
    1.500       -9.3959      -8.1505      -5.0844    35691.9570       I-------------------------------------------------+
    2.000       -8.1868      -7.9920      -5.0606    45264.8359       I-------------------------------------------------+
    2.500       -5.7907      -1.2149      -2.1612    54875.5000       I-------------------------------------------------+
    3.000       -3.3175       4.2648       2.3092    61643.7070       I-------------------------------------------------+
    3.500       -1.1425       8.0584       5.6486    69799.3750       I-------------------------------------------------+
    4.000        0.0005       9.3713       5.8234    80528.0000       I-------------------------------------------------+
    4.005        0.0091       9.3709       5.8235    80735.6250       I-------------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE     0

    TIME       OUTPUT 49    OUTPUT 10    OUTPUT 11    OUTPUT 15       -20.0000                                        20.0000
    0.000        0.0000       0.0000       0.0000        0.0000       I------------------------+                        I
    0.500      -10.4422      -8.0462      -3.8704    11480.5742       I-------------------------------------------------+
    1.000       -9.9954      -8.1454      -4.8075    25648.0078       I-------------------------------------------------+
    1.500       -9.3959      -8.1505      -5.0621    35883.7148       I-------------------------------------------------+
    2.000       -8.1868      -7.9920      -5.0288    45486.8359       I-------------------------------------------------+
    2.500       -5.7907      -1.2149      -2.1230    55115.9883       I-------------------------------------------------+
    3.000       -3.3175       4.2648       2.3196    61882.0430       I-------------------------------------------------+
    3.500       -1.1425       8.0584       5.6224    70039.0000       I-------------------------------------------------+
    4.000        0.0005       9.3713       5.7922    80819.4375       I-------------------------------------------------+
    4.005        0.0091       9.3709       5.7923    81028.4375       I-------------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE     0

    TIME       OUTPUT 49    OUTPUT 10    OUTPUT 11    OUTPUT 15       -20.0000                                        20.0000
    0.000        0.0000       0.0000       0.0000        0.0000       I------------------------+                        I
    0.500      -10.4422      -8.0462      -3.9495    11356.6953       I-------------------------------------------------+
    1.000       -9.9954      -8.1454      -4.8655    25342.0586       I-------------------------------------------------+
    1.500       -9.3959      -8.1505      -5.1138    35501.4258       I-------------------------------------------------+
    2.000       -8.1868      -7.9920      -5.0887    45036.5391       I-------------------------------------------------+
    2.500       -5.7907      -1.2149      -2.1902    54620.2656       I-------------------------------------------------+
    3.000       -3.3175       4.2648       2.2964    61381.1289       I-------------------------------------------------+
    3.500       -1.1425       8.0584       5.6841    69535.6875       I-------------------------------------------------+
    4.000        0.0005       9.3713       5.8594    80205.4375       I-------------------------------------------------+
    4.005        0.0091       9.3709       5.8595    80411.3750       I-------------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE     0

    TIME       OUTPUT 49    OUTPUT 10    OUTPUT 11    OUTPUT 15       -20.0000                                        20.0000
    0.000        0.0000       0.0000       0.0000        0.0000       I------------------------+                        I
    0.500      -10.4422      -8.0462      -3.8949    11431.5273       I-------------------------------------------------+
    1.000       -9.9954      -8.1454      -4.8185    25555.2383       I-------------------------------------------------+
    1.500       -9.3959      -8.1505      -5.0713    35775.9922       I-------------------------------------------------+
    2.000       -8.1868      -7.9920      -5.0486    45365.0508       I-------------------------------------------------+
    2.500       -5.7907      -1.2149      -2.1716    54984.2383       I-------------------------------------------------+
    3.000       -3.3175       4.2648       2.2898    61766.5547       I-------------------------------------------------+
    3.500       -1.1425       8.0584       5.6295    69933.8125       I-------------------------------------------------+
    4.000        0.0005       9.3713       5.8057    80693.0000       I-------------------------------------------------+
    4.005        0.0091       9.3709       5.8058    80901.4375       I-------------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE     0

    TIME       OUTPUT 49    OUTPUT 10    OUTPUT 11    OUTPUT 15       -20.0000                                        20.0000
    0.000        0.0000       0.0000       0.0000        0.0000       I------------------------+                        I
    0.500      -10.4422      -8.0462      -3.9256    11383.3711       I-------------------------------------------------+
    1.000       -9.9954      -8.1454      -4.8469    25426.6094       I-------------------------------------------------+
    1.500       -9.3959      -8.1505      -5.0973    35608.8203       I-------------------------------------------------+
    2.000       -8.1868      -7.9920      -5.0724    45165.7695       I-------------------------------------------------+
    2.500       -5.7907      -1.2149      -2.1508    54768.0195       I-------------------------------------------------+
    3.000       -3.3175       4.2648       2.3285    61522.3047       I-------------------------------------------------+
    3.500       -1.1425       8.0584       5.6676    69666.2500       I-------------------------------------------------+
    4.000        0.0005       9.3713       5.8409    80364.8750       I-------------------------------------------------+
1          3.0000         81028.4375        80411.3750              0.1731           0.1000           0.0815           0.1815
1          3.0000         80901.4375        80571.6875              0.0925           0.0100           0.0125           0.0225
    4.005        0.0091       9.3709       5.8410    80571.6875       I-------------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE     0

    TIME       OUTPUT 49    OUTPUT 10    OUTPUT 11    OUTPUT 15       -20.0000                                        20.0000
    0.000        0.0000       0.0000       0.0000        0.0000       I------------------------+                        I
    0.500      -10.4422      -8.0462      -6.1054     8600.8867       I-------------------------------------------------+
    1.000       -9.9954      -8.1454      -6.5614    18358.3164       I-------------------------------------------------+
    1.500       -9.3959      -8.1505      -6.6563    26896.8125       I-------------------------------------------------+
    2.000       -8.1868      -7.9920      -6.5015    35114.9453       I-------------------------------------------------+
    2.500       -5.7907      -1.2149      -1.1240    43921.4414       I-------------------------------------------------+
    3.000       -3.3175       4.2648       3.8587    49912.2344       I-------------------------------------------------+
    3.500       -1.1425       8.0584       7.4942    57822.3203       I-------------------------------------------------+
    4.000        0.0005       9.3713       7.6017    66343.7500       I-------------------------------------------------+
    4.005        0.0091       9.3709       7.6017    66478.2500       I-------------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE     0

    TIME       OUTPUT 49    OUTPUT 10    OUTPUT 11    OUTPUT 15       -20.0000                                        20.0000
    0.000        0.0000       0.0000       0.0000        0.0000       I------------------------+                        I
    0.500      -10.4422      -8.0462      -5.9483     8800.4531       I-------------------------------------------------+
    1.000       -9.9954      -8.1454      -6.4686    18781.4336       I-------------------------------------------------+
    1.500       -9.3959      -8.1505      -6.5658    27377.2344       I-------------------------------------------------+
    2.000       -8.1868      -7.9920      -6.3423    35639.9766       I-------------------------------------------------+
    2.500       -5.7907      -1.2149      -1.0529    44464.2773       I-------------------------------------------------+
    3.000       -3.3175       4.2648       3.7249    50506.8945       I-------------------------------------------------+
    3.500       -1.1425       8.0584       7.4148    58443.3281       I-------------------------------------------------+
    4.000        0.0005       9.3713       7.4956    67056.9375       I-------------------------------------------------+
    4.005        0.0091       9.3709       7.4956    67195.3125       I-------------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START
```

```
SWITCHES SET ON WERE    0

    TIME      OUTPUT 49   OUTPUT 10   OUTPUT 11   OUTPUT 15       -20.0000                                   20.0000
    0.000       0.0000      0.0000      0.0000      0.0000     I------------------------+                         I
    0.500     -10.4422     -8.0462     -6.2394   8437.1484     I--------------------------------------------------+
    1.000      -9.9954     -8.1454     -6.6488  18009.0820     I--------------------------------------------------+
    1.500      -9.3959     -8.1505     -6.7425  26500.5234     I--------------------------------------------------+
    2.000      -8.1868     -7.9920     -6.5250  34675.1367     I--------------------------------------------------+
    2.500      -5.7907     -1.2149     -1.0521  43391.5820     I--------------------------------------------------+
    3.000      -3.3175      4.2648      3.7280  49410.3750     I--------------------------------------------------+
    3.500      -1.1425      8.0584      7.5245  57323.8633     I--------------------------------------------------+
    4.000       0.0005      9.3713      7.6354  65819.7500     I--------------------------------------------------+
    4.005       0.0091      9.3709      7.6354  65953.0000     I--------------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START
```

```
SWITCHES SET ON WERE    0

    TIME      OUTPUT 49   OUTPUT 10   OUTPUT 11   OUTPUT 15       -20.0000                                   20.0000
    0.000       0.0000      0.0000      0.0000      0.0000     I------------------------+                         I
    0.500     -10.4422     -8.0462     -6.0925   8610.7070     I--------------------------------------------------+
    1.000      -9.9954     -8.1454     -6.5514  18387.5508     I--------------------------------------------------+
    1.500      -9.3959     -8.1505     -6.6472  26931.6406     I--------------------------------------------------+
    2.000      -8.1868     -7.9920     -6.4942  35154.1211     I--------------------------------------------------+
    2.500      -5.7907     -1.2149     -1.1342  43963.1719     I--------------------------------------------------+
    3.000      -3.3175      4.2648      3.8495  49954.3750     I--------------------------------------------------+
    3.500      -1.1425      8.0584      7.4856  57861.9531     I--------------------------------------------------+
    4.000       0.0005      9.3713      7.5938  66389.5625     I--------------------------------------------------+
    4.005       0.0091      9.3709      7.5937  66524.3125     I--------------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START
```

```
SWITCHES SET ON WERE    0

    TIME      OUTPUT 49   OUTPUT 10   OUTPUT 11   OUTPUT 15       -20.0000                                   20.0000
    0.000       0.0000      0.0000      0.0000      0.0000     I------------------------+                         I
    0.500     -10.4422     -8.0462     -6.1183   8591.1680     I--------------------------------------------------+
    1.000      -9.9954     -8.1454     -6.5713  18329.2617     I--------------------------------------------------+
    1.500      -9.3959     -8.1505     -6.6653  26862.1992     I--------------------------------------------------+
    2.000      -8.1868     -7.9920     -6.5088  35076.0000     I--------------------------------------------------+
    2.500      -5.7907     -1.2149     -1.1138  43880.0078     I--------------------------------------------------+
    3.000      -3.3175      4.2648      3.8679  49870.4687     I--------------------------------------------------+
    3.500      -1.1425      8.0584      7.5026  57783.0703     I--------------------------------------------------+
    4.000       0.0005      9.3713      7.6097  66298.3125     I--------------------------------------------------+
1         4.0000       67195.3125      65953.0000             0.2485         0.1000         0.1815         0.2815
1         4.0000       66524.3125      66432.5000             0.0184         0.0184         0.0225         0.0409
    4.005       0.0091      9.3709      7.6096  66432.5000     I--------------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT
```

APPENDIX V

SPECIAL DIGITAL PROGRAMS FOR HUMAN OPERATOR MODELING

This appendix presents the special programs written for the human operator modeling studies of Chapter 5.

The first listing is for the special program CASS which was used to translate and punch the S.T.I. data into format 20A4.

The second listing is for special program NEAL by which the above data is read and stored for use during the human operator modeling studies. This special program replaced the standard CSMP subroutine CSMM.

The third listing illustrates the most complicated modeling situation considered. It is for model 5 of Table 5.3, and contains three special subroutines. The first subroutine performs the Kiefer-Wolfowitz stochastic approximation iterative calculations for the transport lag T, the gain K, and the time constant B of the sampled-data model. The second subroutine brings the stored data $i(kT_q)$ and $m(kT_q)$ into blocks 1 and 2 via linear interpolation. The third subroutine generates a transport lag of T seconds. However, the control of the transport lag is performed in the first subroutine. Several iterations of the Kiefer-Wolfowitz algorithm are included.

```
//CASS      JOB ,111899
IA55I  150842
//MAIN44       EXEC  FORTRAN

FORTRAN IV    MODEL 44  PS       VERSION 3,  LEVEL 1   DATE  68353          USC/SSL        PAGE  0001

        DIMENSION I(2010), M(2010)
        READ (5,1) I
        READ (5,1) M
        INTEGER*2 NED(600,2)
      1 FORMAT (15I5)
        DO 2 J = 1, 600
        NED(J,1) = I(J)
      2 NED(J,2) = M(J)
        WRITE (6,3) (NED(J,1), NED(J,2), J = 1, 600)
      3 FORMAT ('1', (2I20))
        WRITE (7,4) NED
      4 FORMAT (20A4)
        STOP
        END
```

```
//          JOB      ,290002

//NFAL   EXEC  FORTRAN(BCU)
 FORTRAN IV    MODEL 44  PS        VERSION 3,  LEVEL 1   DATE  68353            USC/SSL       PAGE  0001

 0001              REAL*4  DUMMY(10)
 0002              REAL    REALS(395)
 0003              INTEGER INTS(587), TEST1, TEST3, TEST4,TEST7
 0004              INTEGER*2  NED(600,2)
 0005              DIMENSION C(76)
 0006              COMMON  REALS, INTS
 0007              COMMON NED, DUMMY
 0008              EQUIVALENCE  ( INTS(380),   KEY1     ) , ( REALS(  2),     C(1)  )CSAA0140
 0009              EQUIVALENCE  ( INTS(381),   KEY2     )                             CSAA0150
 0010              EQUIVALENCE  ( INTS(382),   KEY3     )                             CSAA0160
 0011              EQUIVALENCE  ( INTS(383),   KEY4     )                             CSAA0170
 0012              EQUIVALENCE  ( INTS(391),   KEY12    )                             CSAA0180
 0013              EQUIVALENCE  ( INTS(392),   KEY13    )                             CSAA0190
 0014              EQUIVALENCE  ( INTS(393),   KEY14    )                             CSAA0200
 0015              EQUIVALENCE  ( INTS(394),   KEY15    )                             CSAA0210
 0016              EQUIVALENCE  ( INTS(395),   KEY16    )                             CSAA0220
 0017              EQUIVALENCE  ( INTS(525),  TEST1     )                             CSAA0230
 0018              EQUIVALENCE  ( INTS(527),  TEST3     )                             CSAA0240
 0019              EQUIVALENCE  ( INTS(528),  TEST4     )                             CSAA0250
 0020              EQUIVALENCE  ( INTS(531),  TEST7     )                             CSAA0260
 0021              EQUIVALENCE  ( INTS(587),  IARG      )
 0022              READ (1,527) NED
 0023          527 FORMAT (20A4)
             C          INITIALIZATION SUBROUTINE
 0024              CALL LOAD ('SORT')
 0025              CALL CSM0                                                          CSAA0330
             C        CONFIGURATION SECTION                                           CSAA0350
             C        PROGRAM WILL NOT BRANCH BEYOND THE CONFIGURATION SECTION        CSAA0360
             C        UNTIL SUCCESSFUL SORT TEST IS ACHIEVED AT WHICH TIME THE        CSAA0370
             C        SWITCH TEST1   IS SET TO 2                                      CSAA0380
             C                                                                        CSAA0390
 0026           10 CONTINUE
 0027              GO TO ( 12, 11), TEST1
 0028           11 GO TO ( 12,100), KEY1                                              CSAA0410
             C          GET CONFIG. SPECS                                             CSAA0420
 0029           12 CALL CSM1                                                          CSAA0430
             C          PREPARE FOR SORT                                              CSAA0440
 0030              CALL CSM2                                                          CSAA0450
             C              TEST1 # 1 IF PRE-SORT SCAN INDICATES ERROR                CSAA0460
             C              TEST1 # 2 IF PRE-SORT SCAN IS SUCCESSFUL                  CSAA0470
 0031              GO TO (12,13), TEST1                                               CSAA0480
             C          SORT                                                          CSAA0490
 0032           13 CALL CSM3                                                          CSAA0500
             C        TEST FOR SUCCESSFUL SORT                                        CSAA0510
             C              TEST1 # 1 IF SORT PROCEDURE IS UNSUCCESSFUL               CSAA0520
             C              TEST1 # 2 IF SORT PROCEDURE IS SUCCESSFUL                 CSAA0530
 0033              GO TO ( 12,100) , TEST1                                            CSAA0540
 0034          100 CONTINUE                                                           CSAA0550
             C                                                                        CSAA0560
             C          SET-UP SECTION                                                CSAA0570
             C                                                                        CSAA0580
             C          PARAMETERS AND INITIAL CONDITIONS                             CSAA0590
 0035              GO TO (110,109), TEST3                                             CSAA0600
 0036          109 GO TO (110,115), KEY2                                              CSAA0610
 0037          110 CALL CSM4                                                          CSAA0620

 0038          115 CONTINUE                                                           CSAA0630
             C          FUNCTION GENERATORS     ******                                CSAA0640
 0039              GO TO (121,118), TEST4                                             CSAA0650
 0040          118 GO TO (120,119), TEST3                                             CSAA0660
 0041          119 GO TO (120,121),KEY3                                               CSAA0670
 0042          120 CALL CSM5                                                          CSAA0680
 0043          121 CONTINUE                                                           CSAA0690
             C        SET TEST3#2 TO INDICATE COMPLETION OF INITIAL SPECIFICATION     CSAA0700
             C        OF CONFIGURATION, PARAMETERS, AND FUNCTION GENERATOR INTERCEPTSCSAA0710
 0044                   TEST3 = 2                                                     CSAA0720
             C          %INTERRUPT POINT¤                                             CSAA0730
 0045              CALL DATSW (0,KEY16)                                               CSAA0740
 0046              GO TO (225,125), KEY16                                             CSAA0750
 0047          125 CONTINUE                                                           CSAA0760
             C          PUNCH CARDS   ****************                                CSAA0770
 0048              GO TO (127,126), KEY12                                             CSAA0780
 0049          126 CALL DATSW(12,KEY12    )                                           CSAA0790
 0050              GO TO (127,128), KEY12                                             CSAA0800
 0051          127 CALL CSM6                                                          CSAA0810
             C          %INTERRUPT POINT¤                                             CSAA0820
 0052              CALL DATSW (0,KEY16)                                               CSAA0830
 0053              GO TO (225,128), KEY16                                             CSAA0840
 0054          128 CONTINUE                                                           CSAA0850
             C          TIMING   *********************                                CSAA0860
 0055              GO TO (130,129), TEST7                                             CSAA0870
             C        TEST7 # 1 UNTIL FIRST TIME THROUGH CSM7                         CSAA0880
             C        TEST7 # 2 AFTER FIRST TIME THROUGH CSM7                         CSAA0890
 0056          129 GO TO (130,135), KEY4                                              CSAA0900
 0057          130 CALL CSM7                                                          CSAA0910
             C          %INTERRUPT POINT¤                                             CSAA0920
 0058              CALL DATSW (0,KEY16)                                               CSAA0930
 0059              GO TO (225,135), KEY16                                             CSAA0940
 0060          135 CONTINUE                                                           CSAA0950
             C          PLOT SPECS    ****************                                CSAA0960
 0061                 IARG = 2                                                        CSAA0970
 0062          150 CALL CSM9                                                          CSAA0980
             C          %INTERRUPT POINT¤                                             CSAA0990
 0063              CALL DATSW (0,KEY16)                                               CSAA1000
 0064              GO TO (225,155), KEY16                                             CSAA1010
 0065          155 CONTINUE                                                           CSAA1020
             C          NEW PLOT FRAME      ***********                               CSAA1030
 0066                 IARG = 1                                                        CSAA1040
 0067          160 CALL CSM9                                                          CSAA1050
 0068          165 CONTINUE                                                           CSAA1060
             C          OUTPUT SPECS  ****************                                CSAA1070
 0069                 IARG = 1                                                        CSAA1080
 0070          170 CONTINUE
 0071              CALL LOAD ('RUN')
 0072              CALL CSM8
             C                                                                        CSAA1100
             C          %INTERRUPT POINT¤                                             CSAA1110
 0073              CALL DATSW (0,KEY16)                                               CSAA1120
 0074              GO TO (225,200), KEY16                                             CSAA1130
 0075          200 CONTINUE                                                           CSAA1140
             C                                                                        CSAA1150
```

```
FORTRAN IV    MODEL 44  PS       VERSION 3,  LEVEL 1   DATE  68353          USC/SSL       PAGE  0003

0076            GO TO (210,220) , KEY15                                          CSAA1160
0077        210 CONTINUE                                                         CSAA1170
0078            READ (1,1) (C(N), N = 1, 76)
0079          1 FORMAT (4E20.8)
          C           COMPUTE SECTION    ***********                             CSAA1190
0080        220 CALL CSM10                                          ,            CSAA1200
          C                                                                      CSAA1210
          C           CALLS INTERRUPT SUBROUTINE FOR NEW SENSE SWITCH SETTINGS   CSAA1220
0081        225 CONTINUE
0082            CALL LOAD ('SORT')
0083            CALL CSM12
0084            GO TO (230,240) ,KEY13                                           CSAA1240
0085        230 CALL CSM13                                                       CSAA1250
0086            GO TO 225                                                        CSAA1260
0087        240 CONTINUE                                                         CSAA1270
0088            GO TO (250,10),KEY14                                             CSAA1280
          C                                                                      CSAA1290
          C           SAVE STATUS                                                CSAA1300
0089        250 CONTINUE                                                         CSAA1310
0090            WRITE (2,1) (C(N), N = 1, 76)
0091            GO TO 10                                                         CSAA1330
0092            END                                                              CSAA1340
```

```
//NEAL      JOB ,111899
IA55I  142027
//SUB1  EXEC  FORTRAN(BCU)
 FORTRAN IV    MODEL 44  PS       VERSION 3,  LEVEL 1   DATE  68354           USC/SSL       PAGE  0001

                  SUBROUTINE    SUB1
        C         THIS SUBROUTINE GENERATES THE APPROXIMATE GRADIENT SEARCH (KIEFER-
        C         WOLFOWITZ) FOR OPTIMAL VALUES OF  T, K, AND B.  THE CONTROL OF THE
        C         TRANSPORT LAG (T) SEARCH IS EXERCISED IN THIS BLOCK RATHER THAN IN
        C         SPEC SUBROUTINE SUB3 BECAUSE THE GRADIENT CALCULATIONS REQUIRED ARE
        C         THE SAME AS THOSE FOR THE SAMPLING INTERVAL  FOR WHICH THIS SUBROUTINE
        C         WAS ORIGINALLY DESIGNED.
                  REAL  REALS(395)
                  REAL*4  DD(10)
                  INTEGER INTS(587)
                  INTEGER  TEST5
                  INTEGER*2 NED(600,2)
                  DIMENSION C(76),MTRX2(75),MTRX3(75),MTRX4(75)
                  DIMENSION PAR1(75), PAR2(75), PAR3(75)
                  COMMON REALS, INTS
                  COMMON NED
                  COMMON DD
                  EQUIVALENCE (INTS(76), MTRX2(1)), (REALS(2), C(1))
                  EQUIVALENCE (INTS(376),   I     ), (REALS(79), DTS2 )
                  EQUIVALENCE                        (REALS(81), PAR1(1))
                  EQUIVALENCE (INTS(151),  MTRX3(1)),(INTS(226),MTRX4(1))
                  EQUIVALENCE (REALS(156), PAR2(1)) ,(REALS(231),PAR3(1))
                  EQUIVALENCE (INTS(529) , TEST5 )
                  EQUIVALENCE(DD(1),YMT),(DD(2),YPT),(DD(3),YMK),(DD(4),YPK)
                  EQUIVALENCE(DD(5),YMB),(DD(6),YPB)
                  IF(C(76))1,1,2
                1 PAR1(I)=DTS2/2.0
                  C(I)=1.0
                4 RETURN
                2 PAR1(I)=PAR1(I)+DTS2
                  IF(PAR1(I)-PAR2(I))3,1,1
                3 C(I)=0.0
                  IF(C(76)-29.4)4,5,5
                5 TEST5=6
        C            J=MTRX3(I)=BL(10)
                  J=MTRX3(I)
                  CN1=0.1*(PAR2(J)**(-.166))
                  CN2=3.0*CN1
                  CN3=CN2
                  N=PAR3(I)
                  GO TO (20,21, 22, 23, 24, 25, 26),N
        C         PAR2(I)= T
               20 PAR2(I)=PAR2(I)-CN1
                  PAR3(I)=2.0
                  RETURN
               21 YMT=C(9)
                  PAR2(I)=PAR2(I)+2.0*CN1
                  PAR3(I)=3.0
                  RETURN
               22 YPT=C(9)
                  PAR2(I)=PAR2(I)-CN1
        C            MTRX4(I)=BL(4)
                  J=MTRX4(I)
        C         PAR2(J)=PAR2(4)= K
                  PAR2(J)=PAR2(J)-CN2


                  PAR3(I)=4.0
                  RETURN
               23 YMK=C(9)
                  J=MTRX4(I)
                  PAR2(J)=PAR2(J)+2.0*CN2
                  PAR3(I)=5.0
                  RETURN
               24 YPK=C(9)
                  J=MTRX4(I)
                  PAR2(J)=PAR2(J)-CN2
        C            PAR3(J)=PAR3(4)=B
                  PAR3(J)=PAR3(J)-CN3
                  PAR3(I)=6.0
                  RETURN
               25 YMB=C(9)
                  J=MTRX4(I)
                  PAR3(J)=PAR3(J)+2.0*CN3
                  PAR3(I)=7.0
                  RETURN
               26 YPB=C(9)
                  J=MTRX4(I)
                  PAR3(J)=PAR3(J)-CN3
                  P1=PAR2(I)
                  P2=PAR2(J)
                  P3=PAR3(J)
        C            JJ=MTRX3(I)=BL(10), PAR2(10)=N (AN INDEX)
                  JJ=MTRX3(I)
                  AN=0.000001/PAR2(JJ)
                  AL= (AN/CN1)
                  D1=AL*(YMT-YPT)
                  D2=10.0*AL*(YMK-YPK)
                  D3=10.0*AL*(YMB-YPB)
                  P4=D1
                  P5=D2
                  P6=D3
                  IF(ABS(D1).LE.0.1)GO TO 9
                  D1=0.1*D1/ABS(D1)
                9 CONTINUE
                  IF((PAR2(I)+D1).LE.0.018)GO TO 10
                  PAR2(I)=PAR2(I)+D1
               10 CONTINUE
                  IF(ABS(D2).LE.2.0)GO TO 11
                  D2=2.0*D2/ABS(D2)
               11 CONTINUE
                  IF((PAR2(J)+D2).LE.0.002)GO TO  12
                  PAR2(J)=PAR2(J)+D2
               12 CONTINUE
                  IF(ABS(D3).GE.5.0)D3=5.0*D3/ABS(D3)
                  PAR3(J)=PAR3(J)+D3
                  IF(PAR3(J).LE.0.0)PAR3(J)=0.0
                  PAR3(I)=1.0
                  PAR2(JJ)=PAR2(JJ)+1.0
                  WRITE(3,30) PAR2(JJ), YMT, YPT, P4, D1, P1, PAR2(I)
                  WRITE(3,30) PAR2(JJ), YMK, YPK, P5, D2, P2, PAR2(J)
                  WRITE(3,30) PAR2(JJ), YMB, YPB, P6, D3, P3, PAR3(J)
                  RETURN
               30 FORMAT(1H1,7F17.4)
                  END
```

```
      SUBROUTINE SUB2
C     PROGRAM TO BRING DATA  I  AND [redacted]  INTO BLOCKS 1  AND  2  VIA LINEAR
C     INTERPOLATION
      REAL  REALS(395)
      REAL*4  DD(10)
      INTEGER INTS(587)
      INTEGER*2  NED(600,2)
      DIMENSION C(76),MTRX2(75),MTRX3(75),MTRX4(75)
      DIMENSION PAR1(75), PAR2(75), PAR3(75)
      COMMON REALS, INTS
      COMMON NED, DD
      EQUIVALENCE (INTS(76), MTRX2(1)), (REALS(2), C(1))
      EQUIVALENCE (INTS(376),    I    ), (REALS(79), DTS2 )
      EQUIVALENCE                         (REALS(81), PAR1(1))
      EQUIVALENCE (INTS(151),  MTRX3(1)),(INTS(226),MTRX4(1))
      EQUIVALENCE (REALS(156), PAR2(1)) ,(REALS(231),PAR3(1))
      EQUIVALENCE (INTS(529) , TEST5 )
      INTEGER  TEST5
C PROGRAM TO BRING DATA I AND M INTO BLOCKS 1 AND 2 VIA INTERPOLATION
      P=PAR2(I)
      IF(C(76))1,1,2
    1 PAR1(I)=DTS2/2.0
      PAR2(I)=PAR2(I)+1.0
      K=MTRX3(I)
      L=MTRX4(I)
      TK=C(76)
    4 RETURN
    2 PAR1(I)=PAR1(I)+DTS2
   10 FORMAT(1H ,2I6, F17.4,I6,F17.4,I6,F17.4)
      TT=(C(76)-TK)/.05
      C(K)=NED(P,1)+TT*(NED(P+1,1)-NED(P,1))
      C(L)=NED(P,2)+TT*(NED(P+1,2)-NED(P,2))
      IF(PAR1(I)-.05)3,1,1
    3 CONTINUE
      IF(C(76)-29.4)4,5,5
    5 TEST5=6
      PAR2(I)=0.0
      RETURN
      END
```

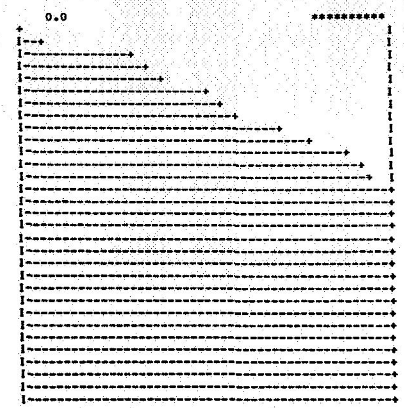
```
      SUBROUTINE SUB3
C     PROGRAM TO GENERATE A TRANSPORT LAG OF  T SECONDS.  THE
C     CONTROL OF THE TRANSPORT LAG VIA GRADIENT CALCULATIONS IS PERFORMED
C     IN SUB1 IN THE SAME MANNER AS WAS DONE PREVIOUSLY FOR THE SAMPLING
C     INTERVAL. THIS SPECIAL ONLY PROVIDES A TIME DELAY OF T  SECONDS IN
C     THE ERROR SIGNAL.
      REAL  REALS(395)
      REAL*4 DD(10),E(10)
      INTEGER INTS(587)
      INTEGER*2 NED(600,2)
      DIMENSION C(76),MTRX2(75),MTRX3(75),MTRX4(75)
      DIMENSION PAR1(75), PAR2(75), PAR3(75)
      COMMON REALS, INTS
      COMMON NED, DD
      COMMON  E
      EQUIVALENCE (INTS(76), MTRX2(1)), (REALS(2), C(1))
      EQUIVALENCE (INTS(376),    I    ), (REALS(79), DTS2 )
      EQUIVALENCE                         (REALS(81), PAR1(1))
      EQUIVALENCE (INTS(151),  MTRX3(1)),(INTS(226),MTRX4(1))
      EQUIVALENCE (REALS(156), PAR2(1)) ,(REALS(231),PAR3(1))
      EQUIVALENCE (INTS(529) , TEST5 )
      INTEGER  TEST5
      J=MTRX2(I)
      TD=PAR2(J)
      IF(C(76))7,7,2
    7 E(1)=0.0
      E(2)=0.0
      E(3)=0.0
      E(4)=0.0
      E(5)=0.0
      E(6)=0.0
      E(7)=0.0
      E(8)=0.0
      E(9)=0.0
      E(10)=0.0
    1 PAR1(I)=DTS2/2.0
      E(1)=E(2)
      E(2)=E(3)
      E(3)=E(4)
      E(4)=E(5)
      E(5)=E(6)
      E(6)=E(7)
      E(7)=E(8)
      E(8)=E(9)
      E(9)=E(10)
      E(10)=C(2)
C        C(2) IS THE ERROR SUMMER
      C(12)=E(1)
   10 FORMAT(1H ,11F9.4)
    4 RETURN
    2 PAR1(I)=PAR1(I)+DTS2
      IF(PAR1(I)-TD/10.0)3,1,1
    3 CONTINUE
      IF(C(76)-29.4)4,5,5
    5 TEST5=6
      RETURN
      END
```

```
//SYS002 ACCESS SDSLIB
//              EXEC RLNKEDT(MAP,SYS002)

LIST        PHASE   ROOT,ROOT,NOAUTO
LIST             INCLUDE    NEAL,R
LIST             INCLUDE    CSM9,R
LIST             INCLUDE    DATSW,R
LIST             INCLUDE    LOAD,R
LIST             INCLUDE     IBCOM#,R
LIST             INCLUDE FIOCS#,R
LIST             INCLUDE    USEROPT,R
LIST             INCLUDE    UNITAB#,R
LIST             INCLUDE    SQRT,R
LIST             INCLUDE    FRXPR#,R
LIST             INCLUDE    ALOG,R
LIST             INCLUDE    EXP,R
LIST        PHASE   SORT,*,NOAUTO
LIST             INCLUDE    CSM0,R
LIST             INCLUDE    CSM1,R
LIST             INCLUDE    CSM2,R
LIST             INCLUDE    CSM3,R
LIST             INCLUDE    CSM4,R
LIST             INCLUDE    CSM5,R
LIST             INCLUDE    CSM6,R
LIST             INCLUDE    CSM7,R
LIST             INCLUDE    CSM12,R
LIST             INCLUDE    CSM13,R
LIST        PHASE   RUN,SORT,NOAUTO
LIST             INCLUDE    CSM10,R
LIST             INCLUDE    CSM11,R
LIST             INCLUDE    CSM8,R
LIST             INCLUDE    CSM8A,R
LIST             INCLUDE     SUB1,L
LIST             INCLUDE SUB10001,L
LIST             INCLUDE    SUB10002,L
LIST             INCLUDE    SUB4,R
LIST             INCLUDE    SUB5,R

LINKAGE EDITOR HIGHEST SEVERITY WAS  0
//SYS001   ACCESS   SDSRUR
//SYS002   ACCESS    SDSPCH
//SYS005   ACCESS   SDSOPT
//           EXEC          CONTINUOUS SYSTEM MODELING PROGRAM

                           CONFIGURATION SPECIFICATION

 OUTPUT NAME       BLOCK        TYPE    INPUT 1   INPUT 2   INPUT 3
 INPUT(II)             1          K          0         0         0
 SUMMER                2          +          1        -4         0
 MODEL INTEG           4          I          0        12        14
 TD GRAD. CONT.(S      5          1          9        10         4
 SYSTEM OUTPUT         6          K          0         0         0
 ERROR SUM             7          +          6        -4         0
 SQUARE                8          X          7         7         0
 INTEG ER SQUARE       9          I          0         8         0
 N                    10          K          0         0         0
 P(SPEC)  COUNT       11          2          1         1         6
 TRANS LAG OUTPUT     12          K          0         0         0
 TRANS LAG(SPECIA     13          3          5         0         0
 SIGN REVERSER        14          +         -4         0         0

                         INITIAL CONDITIONS AND PARAMETERS

 IC/PAR NAME      BLOCK      IC/PAR1            PAR2            PAR3
 INPUT (II)           1          0.0            0.0             0.0
 MODEL I.C., K, B     4          0.0            0.1000          0.0
 TD TIME DELAY        5          0.0            0.2750          1.0000
 OUTPUT(MM)           6          0.0            0.0             0.0
 INTER. SQ. ER.       9          0.0            5.0000          0.0
 IMPROPER PARAMETER SPECIFICATION FOR ELEMENT
 N                   10          0.0            1.0000          1.0000
 P COUNTER(SPEC)     11          0.0            0.0             0.0
 TRANS. LAG (SPEC    13          0.0            0.0             0.0

(        ¤ INTEGRATION INTERVAL
   0.01000

(        ¤ TOTAL TIME
  30.00000

%         ¤ PRINT INTERVAL
       1.

( ¤ BLOCK FOR Y-AXIS   %           ¤ MINIMUM VALUE     %          ¤ MAXIMUM VALUE

     TIME      OUTPUT(    )   OUTPUT(    )   OUTPUT(    )  OUTPUT  9          0.0                              ********
    0.000        0.0000         0.0000         0.0000         0.0000     +                                                  I
    1.000     -174.0000        -5.5247      -145.0000     20762.9336     I--+                                               I
    2.000      -76.0000       -20.5907       -71.0001     94031.6875     I--------------+                                   I
    3.000       -7.0001       -22.3442        33.0001    103845.1875     I----------------+                                 I
    4.000     -111.0001       -22.3946       -99.0000    119391.6250     I-------------------+                              I
    5.000     -153.9997       -32.7464      -143.0001    155144.1875     I-------------------------+                        I
    6.000     -157.0000       -41.8270       -65.0000    168111.4375     I---------------------------+                      I
    7.000     -137.0001       -50.3761      -100.0001    179303.6875     I-----------------------------+                    I
    8.000     -148.0001       -61.4614      -160.0000    218466.2500     I-----------------------------------+              I
    9.000       70.0001       -65.8497       -59.0000    237347.4375     I--------------------------------------+           I
   10.000      -40.9999       -57.3481        33.0001    272134.6250     I--------------------------------------------+     I
   11.000      -50.0000       -55.8409       -29.0001    280318.8750     I---------------------------------------------+    I
   12.000      116.0000       -52.3286        51.0001    288519.6875     I-----------------------------------------------+  I
   13.000      256.0000       -32.9429       183.0000    483362.5625     I-------------------------------------------------+
   14.000      138.9999        -8.5259       152.0003    713336.1875     I-------------------------------------------------+
   15.000       51.0000         1.3414        70.0000    758001.8125     I-------------------------------------------------+
   16.000       -5.0001         4.8665        52.0001    771992.5000     I-------------------------------------------------+
   17.000     -110.9995         1.2392       -54.0002    782290.3750     I-------------------------------------------------+
   18.000      -62.9993        -9.5273       -77.9995    809060.3125     I-------------------------------------------------+
   19.000      128.0023        -7.8476       114.0004    830436.4375     I-------------------------------------------------+
   20.000      128.9986         5.9312       169.9999    928983.3125     I-------------------------------------------------+
   21.000      -39.9998        13.3071        17.0000    982066.6250     I-------------------------------------------------+
   22.000       49.0002        10.1029        -1.9999    984392.8125     I-------------------------------------------------+
   23.000       21.9994        15.8490       106.9997   1015204.1875     I-------------------------------------------------+
   24.000       15.9999        20.1324        60.0003   1027280.5625     I-------------------------------------------------+
   25.000       42.9995        19.8030        13.0000   1032235.0625     I-------------------------------------------------+
   26.000      100.9999        22.7592        18.0000   1036194.3125     I-------------------------------------------------+
   27.000       -3.0005        28.1706        30.9997   1045705.1875     I-------------------------------------------------+
   28.000       17.9996        24.0136       -16.9996   1067649.0000     I-------------------------------------------------+
   29.000      156.9997        27.0920       112.0007   1073173.0000     I-------------------------------------------------+
   29.405      172.3003        32.5272       154.3995   1100442.0000     I-------------------------------------------------+
RUN TERMINATED BY QUIT ELEMENT
```

AFTER SELECTING DESIRED UPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
0.000	0.0000	0.0000	0.0000	0.0000
1.000	-174.0000	-6.9304	-145.0000	20291.1914
2.000	-76.0000	-21.4701	-71.0001	92027.0625
3.000	-7.0001	-21.4392	33.0001	101693.1250
4.000	-111.0001	-22.7534	-99.0000	117050.9375
5.000	-153.9997	-33.4642	-143.0001	152098.5625
6.000	-157.0000	-42.4310	-65.0000	164800.4375
7.000	-137.0001	-50.9917	-100.0001	175767.8125
8.000	-148.0001	-61.9558	-160.0000	214322.8125
9.000	70.0001	-65.0906	-59.0000	233142.8125
10.000	-40.9999	-56.4722	33.0001	267044.3750
11.000	-50.0000	-55.7884	-29.0001	275073.5000
12.000	116.0000	-50.9875	51.0001	282971.9375
13.000	256.0000	-30.7637	183.0000	473802.5000
14.000	138.9999	-7.2803	152.0003	699051.7500
15.000	51.0000	1.3185	70.0000	743066.1875
16.000	-5.0001	5.1154	52.0001	757029.6875
17.000	-110.9995	-0.1727	-54.0002	767196.3750
18.000	-62.9993	-10.2478	-77.9995	793114.6875
19.000	128.0023	-7.0775	114.0004	814111.5625
20.000	128.9986	7.2558	169.9999	911004.7500
21.000	-39.9998	13.1068	17.0000	963441.1875
22.000	49.0002	10.0333	-1.9999	965653.7500
23.000	21.9994	16.4161	106.9997	996141.1875
24.000	15.9999	20.1237	60.0003	1008163.8125
25.000	42.9995	19.6004	13.0000	1013177.0625
26.000	100.9999	23.0317	18.0000	1017105.6250
27.000	-3.0005	27.9789	30.9997	1026531.3125
28.000	17.9996	23.4250	-16.9996	1048065.9375
29.000	156.9997	28.2540	112.0007	1053420.0000
29.405	172.3003	33.4162	154.3995	1080216.0000



RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED UPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
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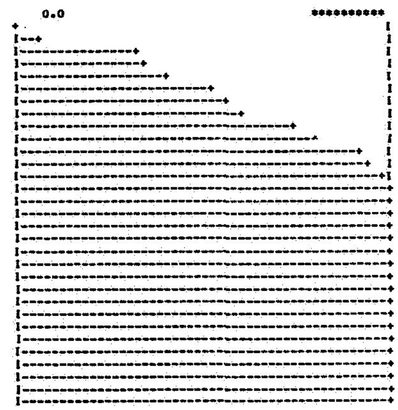
1.000	-174.0000	-4.1935	-145.0000	21194.1211
2.000	-76.0000	-19.4920	-71.0001	96070.6875
3.000	-7.0001	-22.9824	33.0001	106090.5625
4.000	-111.0001	-22.1841	-99.0000	121867.0625
5.000	-153.9997	-31.6476	-143.0001	158298.8125
6.000	-157.0000	-41.3509	-65.0000	171569.5000
7.000	-137.0001	-49.6659	-100.0001	183005.2500
8.000	-148.0001	-60.7063	-160.0000	222794.5625
9.000	70.0001	-66.3619	-59.0000	241805.7500
10.000	-40.9999	-58.2823	33.0001	277547.4375
11.000	-50.0000	-56.0104	-29.0001	285936.0625
12.000	116.0000	-53.7289	51.0001	294413.2500
13.000	256.0000	-35.1583	183.0000	493242.8750
14.000	138.9999	-10.0942	152.0003	728239.8750
15.000	51.0000	1.2610	70.0000	773695.5625
16.000	-5.0001	4.5566	52.0001	787698.2500
17.000	-110.9995	2.2631	-54.0002	798179.4375
18.000	-62.9993	-8.5551	-77.9995	825874.5000
19.000	128.0023	-8.7830	114.0004	847602.7500
20.000	128.9986	4.6794	169.9999	947801.9375
21.000	-39.9998	13.4916	17.0000	1001697.2500
22.000	49.0002	10.4015	-1.9999	1004152.9375
23.000	21.9994	15.3105	106.9997	1035249.0000
24.000	15.9999	20.1716	60.0003	1047420.1875
25.000	42.9995	20.1335	13.0000	1052293.0000
26.000	100.9999	22.8678	18.0000	1056233.0000
27.000	-3.0005	28.2746	30.9997	1065817.0000
28.000	17.9996	24.9294	-16.9996	1088195.0000
29.000	156.9997	26.0104	112.0007	1093866.0000
29.405	172.3003	31.5694	154.3995	1121640.0000



RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED UPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
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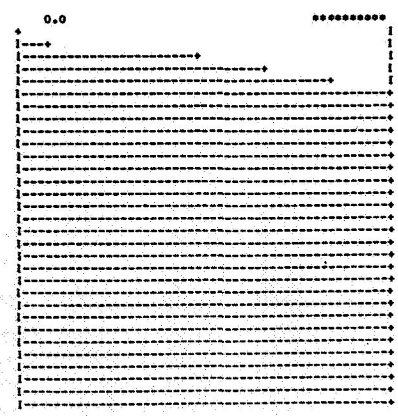
1.000	-174.0000	11.1552	-145.0000	24566.4141
2.000	-76.0000	46.9335	-71.0001	146603.9375
3.000	-7.0001	65.1500	33.0001	198104.1875
4.000	-111.0001	82.9694	-99.0000	255912.2500
5.000	-153.9997	126.7769	-143.0001	486107.3750
6.000	-157.0000	179.3927	-65.0000	754566.2500
7.000	-137.0001	243.9871	-100.0001	1216057.0000
8.000	-148.0001	328.8323	-160.0000	2134839.0000
9.000	70.0001	419.8401	-59.0000	3347409.0000
10.000	-40.9999	502.8679	33.0001	4339199.0000
11.000	-50.0000	616.4277	-29.0001	6054445.0000
12.000	116.0000	749.7339	51.0001	8538682.0000
13.000	256.0000	876.2056	183.0000	10749105.0000
14.000	138.9999	1013.5715	152.0003	13557814.0000
15.000	51.0000	1205.5273	70.0000	18766704.0000
16.000	-5.0001	1449.2410	52.0001	26844512.0000
17.000	-110.9995	1757.8867	-54.0002	39583712.0000
18.000	-62.9993	2147.5427	-77.9995	60144816.0000
19.000	128.0023	2594.5701	114.0004	87839392.0000
20.000	128.9986	3108.4060	169.9999	*************
21.000	-39.9998	3739.2134	17.0000	*************
22.000	49.0002	4523.2500	-1.9999	*************
23.000	21.9994	5452.5195	106.9997	*************
24.000	15.9999	6577.3867	60.0003	*************
25.000	42.9995	7945.8828	13.0000	*************
26.000	100.9999	9592.9766	18.0000	*************
27.000	-3.0005	11577.4141	30.9997	*************
28.000	17.9996	13995.5000	-16.9996	*************
29.000	156.9997	16903.9727	112.0007	*************
29.405	172.3003	18239.3594	154.3995	*************

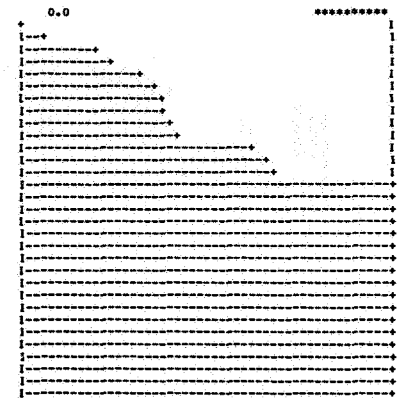


RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED UPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
0.000	0.0000	0.0000	0.0000	0.0000
1.000	-174.0000	-21.8875	-145.0000	17366.8516
2.000	-76.0000	-71.8715	-71.0001	63301.0234
3.000	-7.0001	-58.8853	33.0001	76125.6875
4.000	-111.0001	-46.7541	-99.0000	97690.6250
5.000	-153.9997	-77.0421	-143.0001	111074.1250
6.000	-157.0000	-94.7957	-65.0000	116759.9375
7.000	-137.0001	-106.9582	-100.0001	118096.8750
8.000	-148.0001	-128.0737	-160.0000	122664.9375
9.000	70.0001	-119.6102	-59.0000	126697.8750
10.000	-40.9999	-68.4682	33.0001	186863.8125
11.000	-50.0000	-58.9490	-29.0001	197174.9375
12.000	116.0000	-43.5873	51.0001	205228.1250
13.000	256.0000	26.6037	183.0000	338474.3750
14.000	138.9999	95.6380	152.0003	423516.4375
15.000	51.0000	93.9562	70.0000	427801.5000
16.000	-5.0001	73.2788	52.0001	431748.2500
17.000	-110.9995	33.8580	-54.0002	450870.0000
18.000	-62.9993	-17.9013	-77.9995	485543.6250
19.000	128.0023	-7.3839	114.0004	508213.8125
20.000	128.9986	44.4962	169.9999	580438.6250
21.000	-39.9998	56.5897	17.0000	604530.0625
22.000	49.0002	28.7025	-1.9999	617622.2500
23.000	21.9994	43.7624	106.9997	635275.3750
24.000	15.9999	49.5452	60.0003	638661.8125
25.000	42.9995	37.4698	13.0000	641006.4375
26.000	100.9999	42.0739	18.0000	642500.5000
27.000	-3.0005	54.5768	30.9997	645553.5000
28.000	17.9996	29.1696	-16.9996	678265.5625
29.000	156.9997	39.7158	112.0007	682722.4375
29.405	172.3003	59.8642	154.3995	701147.3750

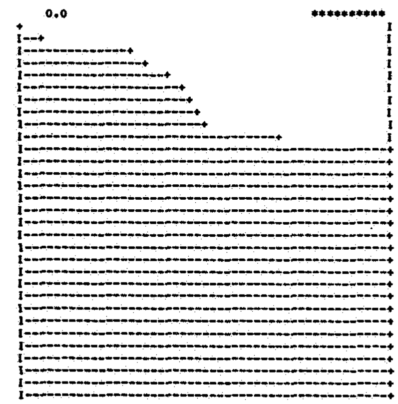


RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
0.000	0.0000	0.0000	0.0000	0.0000
1.000	-174.0000	-5.8966	-145.0000	20699.7812
2.000	-76.0000	-25.5459	-71.0001	92307.1875
3.000	-7.0001	-36.2193	33.0001	102400.1250
4.000	-111.0001	-46.7005	-99.0000	119020.4375
5.000	-153.9997	-71.7085	-143.0001	133319.2500
6.000	-157.0000	-102.4758	-65.0000	140040.5000
7.000	-137.0001	-140.7222	-100.0001	145914.5000
8.000	-148.0001	-191.3815	-160.0000	148833.6250
9.000	70.0001	-247.5364	-59.0000	210719.5625
10.000	-40.9999	-301.6790	33.0001	640131.6875
11.000	-50.0000	-375.1914	-29.0001	1124475.0000
12.000	116.0000	-463.6306	51.0001	1928804.0000
13.000	256.0000	-553.3020	183.0000	4114139.0000
14.000	138.9999	-654.7759	152.0003	7284060.0000
15.000	51.0000	-793.4495	70.000010551471.0000	
16.000	-5.0001	-970.3796	52.000114921296.0000	
17.000	-110.9995	-1195.2434	-54.000220795808.0000	
18.000	-62.9993	-1480.7031	-77.999528711472.0000	
19.000	128.0023	-1818.5718	114.000442619920.0000	
20.000	128.9986	-2218.9468	169.999965820608.0000	
21.000	-39.9998	-2715.8164	17.000098493056.0000	
22.000	49.0002	-3338.1858	-1.9999	*************
23.000	21.9994	-4092.9299	106.9997	*************
24.000	15.9999	-5020.5391	60.0003	*************
25.000	42.9995	-6164.8945	13.0000	*************
26.000	100.9999	-7566.9062	18.0000	*************
27.000	-3.0005	-9285.7305	30.9997	*************
28.000	17.9996	-11407.5859	-16.9996	*************
29.000	156.9997	-14006.7617	112.0007	*************
29.405	172.3003	-15216.5391	154.3995	*************

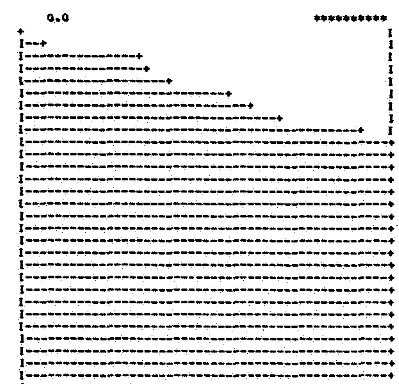


RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
0.000	0.0000	0.0000	0.0000	0.0000
1.000	-174.0000	-5.1864	-145.0000	20821.2148
2.000	-76.0000	-16.8405	-71.0001	95516.6875
3.000	-7.0001	-14.0848	33.0001	105814.0000
4.000	-111.0001	-11.5446	-99.0000	121776.3125
5.000	-153.9997	-18.6897	-143.0001	168590.5000
6.000	-157.0000	-22.9381	-65.0000	189280.7500
7.000	-137.0001	-26.0382	-100.0001	212569.3750
8.000	-148.0001	-31.1899	-160.0000	279308.6875
9.000	70.0001	-29.4025	-59.0000	321087.9375
10.000	-40.9999	-17.6783	33.0001	333533.3125
11.000	-50.0000	-15.3618	-29.0001	337805.5625
12.000	116.0000	-11.4857	51.0001	340094.2500
13.000	256.0000	5.0411	183.0000	466497.8750
14.000	138.9999	21.5164	152.0003	628757.7500
15.000	51.0000	21.8154	70.0000	653275.2500
16.000	-5.0001	17.6089	52.0001	659901.3750
17.000	-110.9995	8.6428	-54.0002	669836.0625
18.000	-62.9993	-3.4102	-77.9995	701502.1875
19.000	128.0023	-1.1667	114.0004	721342.0000
20.000	128.9986	10.6403	169.9999	811890.9375
21.000	-39.9998	13.4804	17.0000	862132.5625
22.000	49.0002	7.2479	-1.9999	864143.9375
23.000	21.9994	10.8019	106.9997	897915.5625
24.000	15.9999	12.1279	60.0003	912953.8750
25.000	42.9995	9.4020	13.0000	919990.3750
26.000	100.9999	10.4651	18.0000	927188.4375
27.000	-3.0005	13.3924	30.9997	942626.9375
28.000	17.9996	7.5269	-16.9996	956136.6875
29.000	156.9997	10.0559	112.0007	965750.4375



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1	2.0000	2857371904.0000	1001791.0000	285636.8750	5.0000	0.0	5.0000

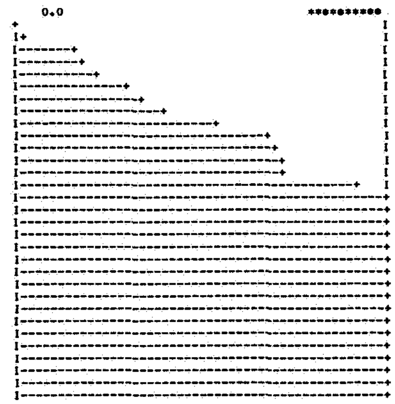
29.405	172.3003	14.6619	154.3995	1001791.0000

RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
0.000	0.0000	0.0000	0.0000	0.0000
1.000	-174.0000	-45.2623	-145.0000	10070.1367
2.000	-76.0000	-39.7702	-71.0001	48549.6094
3.000	-7.0001	9.7984	33.0001	56352.4492
4.000	-111.0001	-20.3564	-99.0000	66979.1875
5.000	-153.9997	-43.5258	-143.0001	93843.1875
6.000	-157.0000	-37.9466	-65.0000	105703.2500
7.000	-137.0001	-44.4886	-100.0001	120968.0000
8.000	-148.0001	-49.2124	-160.0000	166219.7500
9.000	70.0001	-12.4853	-59.0000	205705.6250
10.000	-40.9999	1.4997	33.0001	211997.1875
11.000	-50.0000	-21.0805	-29.0001	215348.2500
12.000	116.0000	21.0114	51.0001	217032.9375
13.000	256.0000	55.8797	183.0000	278746.6875
14.000	138.9999	44.8090	152.0003	363730.5000
15.000	51.0000	10.7908	70.0000	383544.6875
16.000	-5.0001	18.2055	52.0001	393742.8750
17.000	-110.9995	-28.2632	-54.0002	401080.1875
18.000	-62.9993	-24.0891	-77.9995	411825.8750
19.000	128.0023	26.5291	114.0004	421977.6875
20.000	128.9986	46.5701	169.9999	471598.1875
21.000	-39.9998	4.8997	17.0000	505645.8750
22.000	49.0002	4.6267	-1.9999	505915.2500
23.000	21.9994	29.0659	106.9997	529244.1875
24.000	15.9999	12.3220	60.0003	541458.3750
25.000	42.9995	6.2162	13.0000	550427.1250
26.000	100.9999	16.0872	18.0000	555521.8750
27.000	-3.0005	12.0316	30.9997	566252.2500
28.000	17.9996	-7.1710	-16.9996	573086.6250
29.000	156.9997	48.7063	112.0007	576947.7500
29.405	172.3003	44.6534	154.3995	596595.7500

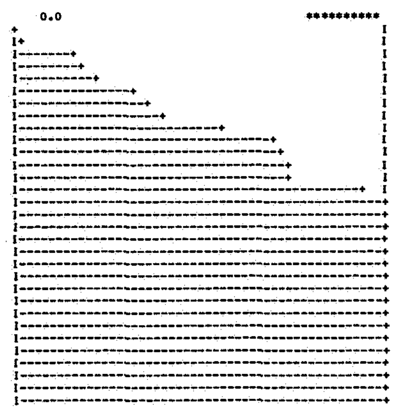


RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
0.000	0.0000	0.0000	0.0000	0.0000
1.000	-174.0000	-48.7552	-145.0000	9835.0508
2.000	-76.0000	-34.1190	-71.0001	48996.0234
3.000	-7.0001	12.7032	33.0001	58202.1875
4.000	-111.0001	-23.3258	-99.0000	69603.5000
5.000	-153.9997	-36.3724	-143.0001	96250.3750
6.000	-157.0000	-42.8667	-65.0000	108022.6875
7.000	-137.0001	-39.5017	-100.0001	123706.5625
8.000	-148.0001	-43.8015	-160.0000	169337.9375
9.000	70.0001	-2.2624	-59.0000	211080.0625
10.000	-40.9999	-0.2508	33.0001	218306.8125
11.000	-50.0000	-21.9471	-29.0001	222003.5625
12.000	116.0000	22.4733	51.0001	224590.6250
13.000	256.0000	59.7474	183.0000	284637.0625
14.000	138.9999	39.9629	152.0003	372142.5625
15.000	51.0000	11.2460	70.0000	393229.8750
16.000	-5.0001	12.0521	52.0001	403218.8750
17.000	-110.9995	-35.0306	-54.0002	411928.5625
18.000	-62.9993	-21.7213	-77.9995	423385.5625
19.000	128.0023	22.1376	114.0004	433857.8750
20.000	128.9986	48.7589	169.9999	482417.5625
21.000	-39.9998	0.8982	17.0000	519732.5625
22.000	49.0002	9.0508	-1.9999	520305.3125
23.000	21.9994	24.8760	106.9997	543297.1250
24.000	15.9999	10.1856	60.0003	556271.4375
25.000	42.9995	10.4634	13.0000	565518.1875
26.000	100.9999	25.6708	18.0000	570587.5625
27.000	-3.0005	9.2478	30.9997	582322.8125
28.000	17.9996	0.0928	-16.9996	589453.3750
29.000	156.9997	47.3995	112.0007	592640.2500
29.405	172.3003	43.3290	154.3995	613114.2500

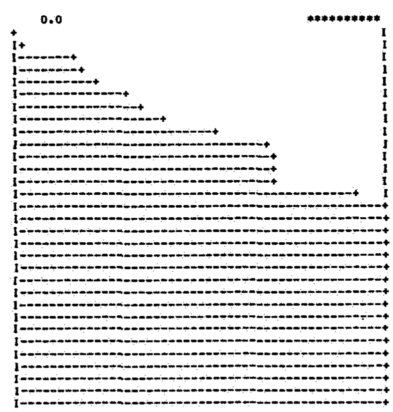


RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
0.000	0.0000	0.0000	0.0000	0.0000
1.000	-174.0000	-46.1095	-145.0000	10402.6406
2.000	-76.0000	-41.4292	-71.0001	48538.6367
3.000	-7.0001	5.7759	33.0001	55594.2930
4.000	-111.0001	-19.6623	-99.0000	65832.8750
5.000	-153.9997	-47.4184	-143.0001	93186.6875
6.000	-157.0000	-35.9555	-65.0000	104757.0000
7.000	-137.0001	-43.8013	-100.0001	119975.6875
8.000	-148.0001	-51.7337	-160.0000	165181.5000
9.000	70.0001	-15.7108	-59.0000	203563.7500
10.000	-40.9999	2.5381	33.0001	209316.8750
11.000	-50.0000	-19.1707	-29.0001	212482.3125
12.000	116.0000	22.1246	51.0001	213818.5625
13.000	256.0000	56.5473	183.0000	276324.7500
14.000	138.9999	46.8542	152.0003	360282.0000
15.000	51.0000	12.6227	70.0000	379474.3750
16.000	-5.0001	17.7874	52.0001	389785.9375
17.000	-110.9995	-23.6720	-54.0002	396289.2500
18.000	-62.9993	-26.6261	-77.9995	406863.6875
19.000	128.0023	27.8635	114.0004	417059.5000
20.000	128.9986	44.8588	169.9999	466785.0625
21.000	-39.9998	3.3139	17.0000	499585.2500
22.000	49.0002	0.7998	-1.9999	499787.9375
23.000	21.9994	30.0027	106.9997	523382.3750
24.000	15.9999	12.0244	60.0003	535014.3750
25.000	42.9995	5.1204	13.0000	543852.1875
26.000	100.9999	11.6912	18.0000	549001.0000
27.000	-3.0005	12.5182	30.9997	559221.1875
28.000	17.9996	-8.8055	-16.9996	565947.8750
29.000	156.9997	48.0151	112.0007	570347.8750
29.405	172.3003	45.9811	154.3995	589457.3750

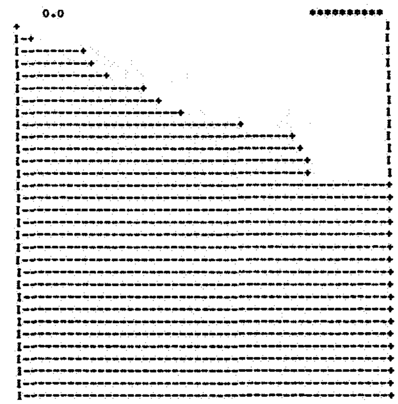


RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
0.000	0.0000	0.0000	0.0000	0.0000
1.000	-174.0000	-40.5701	-145.0000	11127.8906
2.000	-76.0000	-36.2134	-71.0001	53236.4687
3.000	-7.0001	8.5651	33.0001	61439.6719
4.000	-111.0001	-18.4122	-99.0000	72636.0625
5.000	-153.9997	-39.4776	-143.0001	102336.2500
6.000	-157.0000	-34.2487	-65.0000	115663.0000
7.000	-137.0001	-39.8119	-100.0001	132875.3125
8.000	-148.0001	-44.7283	-160.0000	182671.0000
9.000	70.0001	-11.7337	-59.0000	224614.0000
10.000	-40.9999	1.5344	33.0001	230854.8750
11.000	-50.0000	-18.8592	-29.0001	234480.2500
12.000	116.0000	18.4341	51.0001	236275.0625
13.000	256.0000	50.7523	183.0000	303240.5000
14.000	138.9999	41.1371	152.0003	396038.2500
15.000	51.0000	10.0913	70.0000	417305.5625
16.000	-5.0001	16.0853	52.0001	427912.6250
17.000	-110.9995	-25.4340	-54.0002	435402.2500
18.000	-62.9993	-22.0724	-77.9995	447551.9375
19.000	128.0023	23.7083	114.0004	458549.4375
20.000	128.9986	42.2120	169.9999	512090.0625
21.000	-39.9998	4.4905	17.0000	548174.3125
22.000	49.0002	3.6532	-1.9999	548463.6250
23.000	21.9994	26.3570	106.9997	573370.8125
24.000	15.9999	11.4212	60.0003	586199.0000
25.000	42.9995	5.5372	13.0000	595274.5625
26.000	100.9999	14.7146	18.0000	600862.3750
27.000	-3.0005	11.0975	30.9997	612428.5625
28.000	17.9996	-6.4414	-16.9996	619530.3750
29.000	156.9997	43.3814	112.0007	624120.8750
29.405	172.3003	40.6713	154.3995	645^37.3125

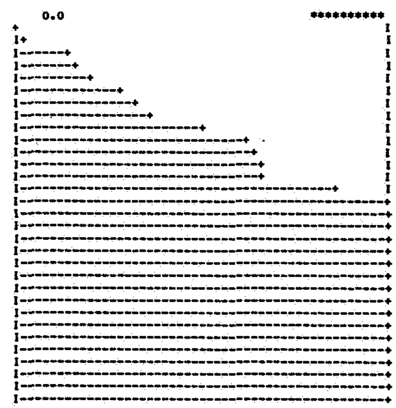


RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
0.000	0.0000	0.0000	0.0000	0.0000
1.000	-174.0000	-49.6579	-145.0000	9124.0859
2.000	-76.0000	-43.0469	-71.0001	44425.8516
3.000	-7.0001	11.0008	33.0001	51917.1133
4.000	-111.0001	-22.1326	-99.0000	62040.5234
5.000	-153.9997	-47.2632	-143.0001	86379.3750
6.000	-157.0000	-41.4434	-65.0000	97000.5000
7.000	-137.0001	-49.0015	-100.0001	110586.9375
8.000	-148.0001	-53.3376	-160.0000	151804.3125
9.000	70.0001	-13.0966	-59.0000	189136.1875
10.000	-40.9999	1.4381	33.0001	195522.3125
11.000	-50.0000	-23.2254	-29.0001	198643.5625
12.000	116.0000	23.5150	51.0001	200269.0000
13.000	256.0000	60.5561	183.0000	257236.8750
14.000	138.9999	48.1356	152.0003	335355.6875
15.000	51.0000	11.3844	70.0000	353903.3750
16.000	-5.0001	20.3060	52.0001	363738.8750
17.000	-110.9995	-30.9064	-54.0002	370998.6250
18.000	-62.9993	-25.9048	-77.9995	380527.2500
19.000	128.0023	29.1847	114.0004	389916.6875
20.000	128.9986	50.6140	169.9999	436038.7500
21.000	-39.9998	5.3377	17.0000	468313.6250
22.000	49.0002	5.6879	-1.9999	468570.8125
23.000	21.9994	31.5442	106.9997	490460.1875
24.000	15.9999	13.0753	60.0003	502166.5000
25.000	42.9995	6.8690	13.0000	511048.5625
26.000	100.9999	17.3193	18.0000	515704.8125
27.000	-3.0005	12.8882	30.9997	525727.9375
28.000	17.9996	-7.8838	-16.9996	532322.1250
29.000	156.9997	53.8011	112.0007	535541.0000
29.405	172.3003	48.2590	154.3995	553543.2500

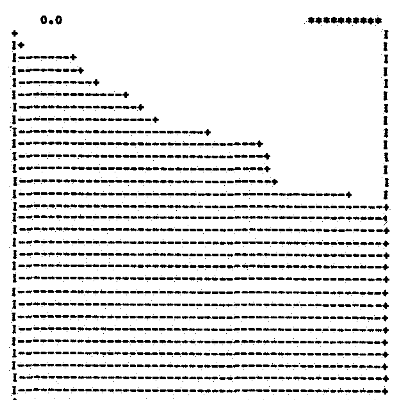


RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

SWITCHES SET ON WERE 0

TIME	OUTPUT 1	OUTPUT 4	OUTPUT 6	OUTPUT 9
0.000	0.0000	0.0000	0.0000	0.0000
1.000	-174.0000	-47.0113	-145.0000	9776.3008
2.000	-76.0000	-41.4153	-71.0001	46823.3945
3.000	-7.0001	9.8971	33.0001	54323.6758
4.000	-111.0001	-21.1196	-99.0000	64730.8164
5.000	-153.9997	-45.3883	-143.0001	90583.7500
6.000	-157.0000	-39.3142	-65.0000	101801.9375
7.000	-137.0001	-46.0304	-100.0001	116276.6875
8.000	-148.0001	-51.2681	-160.0000	159767.6875
9.000	70.0001	-13.1820	-59.0000	198017.9375
10.000	-40.9999	1.6836	33.0001	204227.4375
11.000	-50.0000	-21.7468	-29.0001	207464.1250
12.000	116.0000	21.7782	51.0001	209067.6875
13.000	256.0000	58.1677	183.0000	268946.0625
14.000	138.9999	46.7514	152.0003	350637.6250
15.000	51.0000	11.3853	70.0000	369693.5000
16.000	-5.0001	18.7493	52.0001	379705.1875
17.000	-110.9995	-29.1219	-54.0002	386893.2500
18.000	-62.9993	-25.2065	-77.9995	397080.1250
19.000	128.0023	27.5366	114.0004	406948.8750
20.000	128.9986	48.3108	169.9999	455064.1250
21.000	-39.9998	4.9761	17.0000	488073.3750
22.000	49.0002	4.4983	-1.9999	488333.0000
23.000	21.9994	30.2485	106.9997	511130.4375
24.000	15.9999	12.8204	60.0003	522986.7500
25.000	42.9995	6.4144	13.0000	531865.1875
26.000	100.9999	16.5610	18.0000	536787.4375
27.000	-3.0005	12.5620	30.9997	547128.3125
28.000	17.9996	-7.4544	-16.9996	553872.9375
29.000	156.9997	50.3918	112.0007	557555.5000
29.405	172.3003	46.5022	154.3995	576459.5000



RUN TERMINATED BY QUIT ELEMENT

AFTER SELECTING DESIRED OPTION PRESS START

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SWITCHES SET ON WERE     0

    TIME      OUTPUT  1    OUTPUT  4    OUTPUT  6    OUTPUT  9         0.0                                **********
    0.000        0.0000       0.0000       0.0000       0.0000     +                                                    I
    1.000     -174.0000     -43.6279    -145.0000   10353.3516     I+                                                   I
    2.000      -76.0000     -38.2468     -71.0001   50193.4023     I-------+                                            I
    3.000       -7.0001       9.7036      33.0001   58278.9805     I--------+                                           I
    4.000     -111.0001     -19.6505     -99.0000   69112.9375     I----------+                                         I
    5.000     -153.9997     -41.7961    -143.0001   96936.8750     I---------------+                                    I
    6.000     -157.0000     -36.6734     -65.0000  109411.4375     I-----------------+                                  I
    7.000     -137.0001     -43.0402    -100.0001  125428.6250     I--------------------+                               I
    8.000     -148.0001     -47.3075    -160.0000  172348.3125     I----------------------------+                       I
    9.000       70.0001     -11.8479     -59.0000  212990.8750     I-----------------------------------+                I
   10.000      -40.9999       1.3317      33.0001  219356.4375     I------------------------------------+               I
   11.000      -50.0000     -20.4558     -29.0001  222815.5000     I-------------------------------------+              I
   12.000      116.0000      20.2813      51.0001  224578.1250     I-------------------------------------+              I
   13.000      256.0000      53.7627     183.0000  288030.5625     I------------------------------------------------+ I
   14.000      138.9999      43.0193     152.0003  376122.4375     I--------------------------------------------------+
   15.000       51.0000      10.2450      70.0000  396649.0000     I--------------------------------------------------+
   16.000       -5.0001      17.6873      52.0001  407021.3750     I--------------------------------------------------+
   17.000     -110.9995     -27.4680     -54.0002  414498.9375     I--------------------------------------------------+
   18.000      -62.9993     -23.0569     -77.9995  425776.8125     I--------------------------------------------------+
   19.000      128.0023      25.5862     114.0004  436197.9375     I--------------------------------------------------+
   20.000      128.9986      44.9578     169.9999  487239.5000     I--------------------------------------------------+
   21.000      -39.9998       4.8280      17.0000  522259.1250     I--------------------------------------------------+
   22.000       49.0002       4.7317      -1.9999  522538.4375     I--------------------------------------------------+
   23.000       21.9994      27.9700     106.9997  546370.0625     I--------------------------------------------------+
   24.000       15.9999      11.8700      60.0003  558918.0625     I--------------------------------------------------+
   25.000       42.9995       6.0300      13.0000  567968.8125     I--------------------------------------------------+
   26.000      100.9999      15.6572      18.0000  573227.5625     I--------------------------------------------------+
   27.000       -3.0005      11.5429      30.9997  584325.8750     I--------------------------------------------------+
   28.000       17.9996      -6.9049     -16.9996  591245.5625     I--------------------------------------------------+
   29.000      156.9997      47.1215     112.0007  595280.2500     I--------------------------------------------------+
1          3.0000     613114.2500     589457.3750            0.1327         0.1000           0.1750           0.2750
1          3.0000     645637.3125     553543.2500            5.1662         2.0000           2.1000           4.1000
1          3.0000     576459.5000     615632.3750           -2.1975        -2.1975           5.0000           2.8025
   29.405      172.3003      42.9465     154.3995  615632.3750     I--------------------------------------------------+
 RUN TERMINATED BY QUIT ELEMENT
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