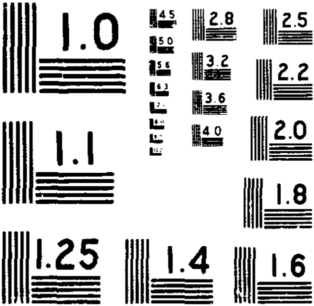


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# FUNCTION SPACE TOPOLOGIES—THE GRAPH TOPOLOGY

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# FUNCTION SPACE TOPOLOGIES—THE GRAPH TOPOLOGY

by
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## ABSTRACT

Title of Thesis: Function Space Topologies—The Graph Topology

Peter Bracken, Master of Arts, 1969

Thesis directed by: Dr. Richard A Holzsager
Assistant Professor of Mathematics

The graph topology $\Gamma$ on the set F of functions from a topological space X to a topological space Y is given by the basis $\{F_U \mid U$ is any open set in the product space $X \times Y\}$ where $F_U = \{f \in F \mid$ the graph of $f \subset U\}$.

A study was conducted to find those properties needed on the spaces X and Y to ensure that F under the graph topology possess certain topological properties. The properties of $\Gamma$ investigated included the separation properties of $T_0$, $T_1$, $T_2$ and regularity. Also included were comparisons between the graph topology and the pointwise convergence, the compact open, the uniform convergence and the sup metric topologies. Finally, continuity of the evaluation map with respect to the graph topology was investigated.

The conclusions reached concerning $\Gamma$ and the separation axioms included:

(1) X $T_1$ and Y $T_0$ implies $(F, \Gamma)$ is $T_0$.

(2) $(F, \Gamma)$ $T_0$ implies that X is $T_0$ and Y is $T_0$.

(3) X $T_1$, Y $T_1$ if and only if $(F, \Gamma)$ is $T_1$.

(4) $X\ T_1$, $Y\ T_2$ if and only if $(F, \Gamma)$ is $T_2$.

(5) $(F, \Gamma)$ regular implies $Y$ regular.

(6) $X$ regular and compact, $Y$ regular implies that $(\mathfrak{F}, \Gamma)$ is regular where $\mathfrak{F}$ is a set of continuous functions from $X$ to $Y$.

The conclusions reached concerning comparisons of $\Gamma$ with the usual function space topologies included:

(1) $X\ T_1$ implies that $\Gamma$ contains the topology of pointwise convergence on $F$.

(2) $X\ T_2$ implies that $\Gamma$ contains the compact open topology on $F$.

(3) $X\ T_2$, compact implies that $\Gamma$ is equivalent to the compact open topology on $\mathfrak{F}$, a space of continuous functions from $X$ to $Y$.

(4) $X$ and $Y$ uniform spaces and $X$ compact implies that $\Gamma$ is equivalent to the topology of uniform convergence on $\mathfrak{F}$, a space of continuous functions.

(5) $X$ and $Y$ metric spaces and $X$ compact implies that $\Gamma$ is equivalent to the sup metric topology on $\mathfrak{F}$, a space of continuous functions.

The main result with respect to the evaluation map is that if $X$ is regular then the evaluation map $e : (\mathfrak{F}, \Gamma) \times X \to Y$ is continuous with respect to the graph topology on $\mathfrak{F}$, a set of continuous functions.

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## INTRODUCTION

During a study of almost continuous functions, S. A. Naimpally [4] developed a new function space topology which he called the graph topology. In a later paper, Naimpally [5] investigated some further properties of the graph topology. In particular, Naimpally stated conditions under which the graph topology would be a $T_1$ or a Hausdorff topology. He also stated conditions under which the graph ($\Gamma$) topology would be comparable to the pointwise convergence (p.c.) topology, to the compact open (k) topology, to the uniform convergence (u.c.) topology and to the sup metric topology.

The purpose of this thesis is to review and extend Naimpally's work on the properties of the graph topology as given in Reference [5].

In Chapter I, relevant definitions are presented along with some general remarks and lemmas concerning the graph topology. In Chapter II, theorems and examples are stated concerning conditions under which the graph topology is $T_0$ and regular. Naimpally's conditions for $T_1$ and Hausdorff are also presented in Chapter II. In Chapter III, Naimpally's conditions for comparability of the graph topology with the pointwise convergence and the compact open topologies are reviewed. A theorem proven by Naimpally in Reference [5] concerning conditions for the equivalence of the graph topology and the compact open topology is shown to be false by counterexample and a correct set of conditions is given for the equivalence on a space of continuous functions. In Chapter III, theorems stronger than those proved by Naimpally are given concerning the comparability of the graph topology and the uniform convergence topology. Conditions for the equivalence of the graph topology and the sup metric topology are also given.

In Chapter IV, the evaluation map and its continuity with respect to several function space topologies is discussed. In particular, classical results relating continuity of the evaluation map with the pointwise convergence and the compact open topologies are reviewed. Conditions are presented for continuity of the evaluation map with respect to the graph topology.

In Appendix A, two general lemmas are presented. In Appendix B, a lemma proving that the graph of f is homeomorphic to X for continuous functions f is given. In Appendix C, theorems concerning the continuity of the evaluation map with respect to the uniform convergence topology and the sup metric topology are presented.

# CHAPTER I

## THE GRAPH TOPOLOGY

### A. Definitions

Let X and Y be topological spaces and let $F = Y^X$ be the set of all functions on X to Y. For $f \in F$, the graph of f is the set $G(f) = \{(x, f(x)) \mid x \in X\}$. $G(f)$ is a subset of the space $X \times Y$. It is understood that $X \times Y$ is assigned the usual product topology.

As standard notation throughout this thesis, a set of the form $F_U$ will be taken to mean the set $F_U = \{f \in F \mid G(f) \subset U\}$ where U is any subset of $X \times Y$.

Naimpally [5] defines the graph topology $\Gamma$ for F as that topology generated by the basis $\{F_U \mid U \text{ open in } X \times Y\}$ where U ranges over all of the open sets of $X \times Y$ and $X \times Y$ is assigned the usual product topology. The proof that $\{F_U\}$ is a basis for a topology on F is given in Lemma II of Appendix A.

If $\mathfrak{F}$ is any subset of F then for $U \subset X \times Y$, $\mathfrak{F}_U = \{f \in \mathfrak{F} \mid G(f) \subset U\} = \mathfrak{F} \cap F_U$.

### B. General Remarks and Lemmas

If U is an open set in $X \times Y$ then U is of the form $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ for some index set J where $U_\alpha$, $V_\alpha$ are open sets in X and Y respectively. This follows since the collection $\{U \times V \mid U \text{ open in } X, V \text{ open in } Y\}$ is a basis for the product topology on $X \times Y$.

If U is an open set in $X \times Y$ and if $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ then $F_U$ is the empty set if $\{U_\alpha \mid \alpha \in J\}$ does not cover X. That is, if $\{U_\alpha \mid \alpha \in J\}$ does not cover X then there is a point $x \in X \setminus \bigcup_{\alpha \in J} U_\alpha$ and $(x, f(x))$ cannot belong to U for any $f \in F$. Thus $F_U = \phi$ in this case. Henceforth in this paper, it will be assumed when dealing with sets of the form $F_U$, where $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$, that $\{U_\alpha \mid \alpha \in J\}$ covers X.

Most problems treated in this paper (and in Naimpally [5] also) will be of the type which require those conditions needed on the spaces X and Y to ensure that the space F possesses a certain property.

By examining Chapter 7 in Kelley [2], it can be seen that most of the desired properties for function spaces are obtained from conditions imposed only on the range space Y. In fact the domain space X seems to play a small role in determining properties of function spaces under the usual function space topologies. The fact that $F_U$ is empty (given U in $X \times Y$) if $p_X(U)$, the projection of U into the coordinate space X, does not cover X gives an indication that the properties of the graph topology may depend on properties of the domain space X. This is in fact true and in later chapters it will be seen that properties of F under $\Gamma$ do rely on properties of both topological spaces X and Y.

The following two lemmas concerning the structure of certain sets in $(F, \Gamma)$ will prove useful later on in the text.

LEMMA I.B.1. Let $U, V \subset X \times Y$. If $F_U \neq \phi$, then $U \subset V$ if and only if $F_U \subset F_V$.

Proof. Note that if $F_U \neq \phi$, there is a function $f \in F_U$. Then $G(f) \subset U$ and hence $p_X(U) \supset X$, that is $p_X(U)$ covers X.

Suppose $F_U \neq \phi$ and $F_U \subset F_V$. Let $(x, y)$ be any point of U and let $f \in F_U \subset F_V$. Then $f \in F_U \subset F_V$ implies that $G(f) \subset U$ and $G(f) \subset V$. Define $g \in F$ by

$$g(z) = \begin{cases} f(z) & z \neq x, \\ y & z = x. \end{cases}$$

Then for $z \neq x$, $\big(z, g(z)\big) = \big(z, f(z)\big) \in G(f)$. Also $\big(x, g(x)\big) = (x, y) \in U$. Thus $G(g) \subset U$ or $g \in F_U$ since $G(f) \subset U \cap V$. Therefore $g \in F_V$ since $F_U \subset F_V$. But $g \in F_V$ implies that $G(g) \subset V$ and hence $\big(x, g(x)\big) = (x, y) \in V$.

Thus $U \subset V$ since $(x, y)$ was an arbitrary point of U.

Suppose $U \subset V$ and $F_U \neq \phi$ and let $f \in F_U$. Then $G(f) \subset U$ which implies that $G(f) \subset V$ since $U \subset V$. Therefore $f \in F_V$ and $F_U \subset F_V$ since f was an arbitrary point of $F_U$.

LEMMA I.B.2. Let X be a $T_1$ space and F be the set of all functions from X to a topological space Y. If V is closed in $X \times Y$ then $F_V$ is closed in $(F, \Gamma)$, that is $F_V = \overline{F}_V$.

Proof. Suppose that X is $T_1$ and V is closed in $X \times Y$. Then $F_V \subset \overline{F}_V$ is immediate.

Let g be any point of $\overline{F}_V$ and suppose that $g \notin F_V$ then $G(g) \not\subset V$. This implies that there is a point $x \in X$ with $\big(x, g(x)\big) \notin V$. Thus since V is closed in $X \times Y$, there is a set $O_1 \times O_2 \subset X \times Y$ with $O_1$ open in X, $O_2$ open in Y, $\big(x, g(x)\big) \in O_1 \times O_2$ and $O_1 \times O_2 \cap V = \phi$.

Since X is $T_1$, the set $\{x\}$ is closed in X which implies that $X \setminus \{x\}$ is open in X. Let $P = \left[ (X \setminus \{x\}) \times Y \right] \cup (X \times O_2)$ then P is open in $X \times Y$. Since $g(x) \in O_2$, $G(g) \subset P$ or $g \in F_P$ an open set in $(F, \Gamma)$.

Suppose h is a point of $F_P$ then $G(h) \subset P$ and hence $h(x) \in O_2$. But this implies that $(x, h(x)) \in O_1 \times O_2$. Therefore $(x, h(x)) \notin V$ since $O_1 \times O_2 \cap V = \phi$. Thus $G(h) \not\subset V$ which implies that $F_P \cap F_V = \phi$ since h was an arbitrary point of $F_P$.

However $F_P \cap F_V = \phi$ and $g \in F_P$ an open set in $(F, \Gamma)$ contradicts the assumption that $g \in \overline{F}_V$. Therefore for each $g \in \overline{F}_V$, $G(g) \subset V$ or $g \in F_V$. Thus $\overline{F}_V \subset F_V$ and $F_V$ is closed in $(F, \Gamma)$.

CHAPTER II

# SEPARATION PROPERTIES OF THE GRAPH TOPOLOGY

## A. $T_0$

The following example shows that the implication $X\ T_0$ and $Y\ T_0 \Rightarrow \Gamma\ T_0$ is not true in general.

EXAMPLE II.A.1. Let X and Y be the topological spaces $X = \{a, b\}$, $Y = \{p, q\}$ with topologies $\Theta_X = \{\phi, X, \{a\}\}$ and $\Theta_Y = \{\phi, Y, \{p\}\}$ respectively where a and b and p and q are distinct points of X and Y. Define the functions f and g on X to Y by

$$
f: \begin{array}{l} a \to p \\ b \to q \end{array} \qquad g: \begin{array}{l} a \searrow \\ b \to q . \end{array}
$$

Then f and g are distinct points in $F = Y^X$. Note that $G(f) = \{(a, p), (b, q)\}$ and $G(g) = \{(a, q), (b, q)\}$.

Let $F_U$ be any basic open set in the graph topology $\Gamma$ on F with $f \in F_U$ where $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ and $U_\alpha$, $V_\alpha$ are open in X and Y respectively for each $\alpha \in J$.

Then $f \in F_U$ implies that $G(f) \subset U$ by definition of $F_U$. But $G(f) \subset U$ implies that $(b, q) \in U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$. Then $(b, q) \in U_\beta \times V_\beta$ for some $\beta \in J$, and thus $b \in U_\beta$ an open set in X and $q \in V_\beta$ an open set in Y. By definition of $\Theta_X$, the only open set in X which contains b (is X) also contains a. That is $b \in U_\beta$ open in X implies that $a \in U_\beta$. Thus $(a, q) \in U_\beta \times V_\beta$ and $(b, q) \in U_\beta \times V_\beta$. However this implies that $G(g) \subset U_\beta \times V_\beta \subset U$. Therefore $G(g) \subset U$ or $g \in F_U$ and we have shown that if $f \in F_U$ then $g \in F_U$.

Similarly, suppose $F_V$ is any basic open set in $(F, \Gamma)$ and $g \in F_V$ where $V = \bigcup_{\alpha \in K} W_\alpha \times Z_\alpha$ where $W_\alpha$, $Z_\alpha$ are open respectively in X and Y for each $\alpha \in K$.

Then the following implications hold:

$$g \in F_V \Rightarrow G(g) \subset V \Rightarrow (b, q) \in V = \cup W_\alpha \times Z_\alpha$$

$\Rightarrow$ there is a $\beta \in K$ such that $(b, q) \in W_\beta \times Z_\beta$

$\Rightarrow b \in W_\beta$ and $q \in Z_\beta$

$\Rightarrow a \in W_\beta$ and $p \in Z_\beta$ by definition of the topologies $\Theta_X$ and $\Theta_Y$ respectively

$\Rightarrow \{(b, q), (a, p)\} \in W_\beta \times Z_\beta \subset V$

$\Rightarrow G(f) \subset V \Rightarrow f \in F_V$.

And thus if $F_V$ is any basic open set in $(F, \Gamma)$ containing g then $F_V$ also contains f.

In summary, we have shown in this example that the points f and g of the space $(F, \Gamma)$ cannot be separated by open sets in this space—that is we have shown that any open set in $(F, \Gamma)$ containing one of f or g also contains the other. Therefore $(F, \Gamma)$ of this example is not a $T_0$ space. However X and Y are $T_0$ spaces by construction of $\Theta_X$ and $\Theta_Y$. Thus X $T_0$ and Y $T_0$ are not sufficient conditions to ensure that $(F, \Gamma)$ be $T_0$.

The following theorem yields sufficient conditions for the space $(F, \Gamma)$ to be a $T_0$ space.

THEOREM II.A.2. If $X$ is a $T_1$ topological space and if $Y$ is a $T_0$ topological space then $(F, \Gamma)$ is $T_0$.

Proof. Let $X$ be a $T_1$ space and $Y$ a $T_0$ space and let $f, g \in F$ with $f$ and $g$ distinct points of $F$. Since $f$ and $g$ are distinct there is a point $x \in X$ such that $f(x) \neq g(x)$.

Since $Y$ is $T_0$, $f(x) \in Y$, $g(x) \in Y$ and $f(x) \neq g(x)$, one of the following two cases must hold:

Case i. There is an open set $U$ in $Y$ with $f(x) \in U$ and $g(x) \notin U$.

Since $X$ is $T_1$, the set $\{x\}$ is closed in $X$. Thus $X\backslash\{x\}$ is open in $X$ and the set $V = (X \times U) \cup \left[\left(X\backslash\{x\}\right) \times Y\right]$ is open in $X \times Y$.

If $y \in X$ and $y \neq x$ then $\left(y, f(y)\right) \in \left(X\backslash\{x\}\right) \times Y$. By assumption, $f(x) \in U$ so that $\left(x, f(x)\right) \in X \times U$. Thus $G(f) \subset V$. However, by assumption $g(x) \notin U$ so that $\left(x, g(x)\right) \notin X \times U$ and thus $G(g) \not\subset V$.

Therefore $F_V$ is an open set in $(F, \Gamma)$ which contains $f$ but not $g$.

Case ii. There is an open set $U$ in $Y$ with $g(x) \in U$ and $f(x) \notin U$.

In a manner entirely similar to case i above, an open set $F_V$ in $(F, \Gamma)$ can be constructed such that $g \in F_V$ and $f \notin F_V$.

Thus if $f \neq g$ there is an open set in $(F, \Gamma)$ containing one of $f$ or $g$ but not the other, which implies that $(F, \Gamma)$ is $T_0$.

Although Example II.A.1 indicates that $X$ $T_0$ and $Y$ $T_0$ are not sufficient conditions for $(F, \Gamma)$ to be $T_0$, they are necessary conditions as the following theorem shows.

THEOREM II.A.3. Suppose X and Y are topological spaces, Y contains at least two distinct points and $F = Y^X$, then $(F, \Gamma)$ $T_0$ implies that X is $T_0$ and Y is $T_0$.

Proof. Assume that $(F, \Gamma)$ is a $T_0$ space and let p, q be distinct points of Y. Define $f, g \in F$ as $f(x) = p$ and $g(x) = q$ for each $x \in X$. Then $p \neq q$ implies that $f \neq g$.

Since $(F, \Gamma)$ is $T_0$ one of the following cases must hold.

Case i. There is a basic open set $F_U$ in $(F, \Gamma)$ such that $f \in F_U$ and $g \notin F_U$ where $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ and $U_\alpha, V_\alpha$ are open in X and Y respectively for each $\alpha \in J$. $f \in F_U$ and $g \notin F_U$ implies that $G(f) \subset U$ and $G(g) \not\subset U$. But $G(g) \not\subset U$ implies that there is a point $x \in X$ such that $(x, g(x)) = (x, q) \notin U = \cup U_\alpha \times V_\alpha$. But $G(f) \subset U$ implies that $(x, f(x)) = (x, p) \in U$. Thus there is an index $\beta \in J$ such that $(x, p) \in U_\beta \times V_\beta$. Then $(x, q) \notin U$ implies that $(x, q) \notin U_\beta \times V_\beta$. Thus we must have $p \in V_\beta$ and $q \notin V_\beta$ where $V_\beta$ is open in Y.

Case ii. There is a basic open set $F_U$ in $(F, \Gamma)$ such that $g \in F_U$ and $f \notin F_U$. By a proof entirely similar to that of case i above, an open set $V_\beta$ in Y can be found such that $q \in V_\beta$ and $p \notin V_\beta$.

Therefore there is an open set $V_\beta$ in Y containing one of p or q but not the other which implies that Y is $T_0$.

Assume that $(F, \Gamma)$ is $T_0$ and X is not $T_0$. Then there are distinct points a, b in X such that every open set containing one of a or b also contains the other.

Define functions f and g belonging to F as follows

$$f(x) = \begin{cases} p & x \in X,\ x \neq b \\ q & x = b \end{cases}$$

$$g(x) = \begin{cases} p & x \in X \quad x \neq a \\ q & x = a \end{cases}$$

Let $F_U$ be any basic open set in $(F, \Gamma)$ with $f \in F_U$ where $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ and $U_\alpha$, $V_\alpha$ are open in X and Y respectively for each $\alpha \in J$. Then $G(f) \subset U$ so that there is a $\beta \in J$ with $(a, f(a)) = (a, p) \in U_\beta \times V_\beta$. Thus $a \in U_\beta$ and $p \in V_\beta$. By assumption, since $a \in U_\beta$ an open set in X, $b \in U_\beta$ and this implies that $(b, p) = (b, g(b)) \in U_\beta \times V_\beta$.

Similarly, there is a $\gamma \in J$ such that $(b, f(b)) = (b, q) \in U_\gamma \times V_\gamma$ which implies that $b \in U_\gamma$ and $q \in V_\gamma$. By assumption, since $b \in U_\gamma$, an open set in X, $a \in U_\gamma$ and this implies that $(a, q) = (a, g(a)) \in U_\gamma \times V_\gamma$. Thus we have

(i) if $x \in X$ and $x \neq a$, $x \neq b$ then $(x, g(x)) = (x, p) = (x, f(x)) \in U$ since $G(f) \subset U$

(ii) $(a, g(a)) \in U_\beta \times V_\beta \subset U$

(iii) $(b, g(b)) \in U_\gamma \times V_\gamma \subset U$

or $G(g) \subset U$ which implies that $g \in F_U$. Hence if $F_U$ is any basic open set in $(F, \Gamma)$ which contains f then $F_U$ also contains g.

Similarly, it can be shown that any basic open set in $(F, \Gamma)$ which contains g also contains f. The last two statements contradict the hypothesis that $(F, \Gamma)$ is $T_0$ and hence the assumption that X is not $T_0$ is false. Thus $(F, \Gamma)$ $T_0$ implies that X is $T_0$, and it has been shown that X and Y $T_0$ are necessary conditions for $(F, \Gamma)$ to be $T_0$.

Theorem II.A.2 indicates that X $T_1$ and Y $T_0$ are sufficient conditions for $(F, \Gamma)$ to be $T_0$. The following example shows that in general X $T_1$ is not a necessary condition for $(F, \Gamma)$ to be $T_0$.

EXAMPLE II.A.4. Let $X = \{a, b\}$ and $Y = \{p, q\}$ be topological spaces with the topologies $\Theta_X = \{\phi, \{a\}, X\}$ and $\Theta_Y = \{\phi, \{p\}, \{q\}, Y\}$ respectively where a and b and p and q are distinct points.

There are only four functions mapping X into Y, with

$$G(f_1) = \{(a, p), (b, p)\}$$

$$G(f_2) = \{(a, p), (b, q)\}$$

$$G(f_3) = \{(a, q), (b, p)\}$$

$$G(f_4) = \{(a, q), (b, q)\}.$$

The set $U = X \times \{q\} = \{(a, q), (b, q)\}$ is open in $X \times Y$ and $G(f_4) \subset U$ but $G(f_i) \not\subset U$ for $i = 1, 2$ or $3$. Thus $F_U = \{f_4\}$ is an open set in $(F, \Gamma)$.

The set $V = X \times \{p\} = \{(a, p), (b, p)\}$ is open in $X \times Y$. Also $G(f_1) \subset V$ but $G(f_i) \not\subset V$ for $i = 2, 3$ or $4$. Thus $F_V = \{f_1\}$ is an open set in $(F, \Gamma)$.

To prove $(F, \Gamma)$ is $T_0$, it is sufficient to show that there is an open set containing $f_3$ but not $f_2$.

Let $W = \{a\} \times \{q\} \cup V$ then W is open in $X \times Y$ and $W = \{(a, q), (a, p), (b, p)\}$. Also $W \supset G(f_3)$ but $W \not\supset G(f_2)$, that is $f_3 \in F_W$ and $f_2 \notin F_W$. Therefore $(F, \Gamma)$ is $T_0$. Although X is $T_0$, X is not $T_1$ since every open set containing b also contains a. Thus $(F, \Gamma)$ $T_0$ implies X is $T_0$ holds but $(F, \Gamma)$ $T_0$ does not imply that X is $T_1$ in general.

## B. $T_1$

The following example of Naimpally [5] further illustrates the premise that space $(F, \Gamma)$ does not inherit its separation properties from the space Y. The example shows that even though Y is taken as a discrete space, this in itself is not sufficient to guarantee that $(F, \Gamma)$ will be a $T_1$ space.

EXAMPLE II.B.1. Let $X = \{a, b\}$ and $Y = \{p, q\}$ be topological spaces with the topclogies $\Theta_X = \{\phi, \{a\}, X\}$ and $\Theta_Y = \{\phi, Y, \{p\}, \{q\}\}$ respectively where $a \neq b, p \neq q$.

Define $f, g \in F$ as

$$f: \begin{array}{l} a \to p \\ b \to q \end{array} \qquad g: \begin{array}{l} a \searrow \\ b \to q \end{array} .$$

Then $G(f) = \{(a, p), (b, q)\}$, $G(g) = \{(a, q), (b, q)\}$ and f and g are distinct points of F.

Let $F_U$ be any basic open set in $(F, \Gamma)$ such that $f \in F_U$ then $G(f) \subset U$. Suppose $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ where $U_\alpha$, $V_\alpha$ are open in X and Y respectively for each $\alpha \in J$. Then $G(f) \subset U$ implies that there is an index $\beta \in J$ such that $(b, q) \in U_\beta \times V_\beta$.

However since $U_\beta$ is open in X and since $b \in U_\beta$, $U_\beta$ must also contain a (i.e., $U_\beta = X$ by definition of $\Theta_X$). Thus $(a, q) \in U_\beta \times V_\beta$ which implies that $G(g) = \{(b, q), (a, q)\} \subset U_\beta \times V_\beta \subset U$.

Thus $g \in F_U$ and we have shown that any basic open set in $(F, \Gamma)$ which contains f also contains g. This implies that $(F, \Gamma)$ is not a $T_1$ space.

By definition of $\Theta_X$, X is a $T_0$ space but X is not $T_1$. Therefore this example indicates that in general X $T_0$ and Y discrete are not sufficient conditions for $(F, \Gamma)$ to be $T_1$.

In Reference [5] Naimpally presented the following necessary and sufficient conditions for the space $(F, \Gamma)$ to be $T_1$.

THEOREM II.B.2. If X and Y are topological spaces and if Y contains at least two distinct points then $(F, \Gamma)$ is $T_1$ if and only if X is $T_1$ and Y is $T_1$.

Proof. Assume X and Y are $T_1$ spaces and suppose $f \neq g$ with $f, g \in F$. Since $f \neq g$, there is a point $x \in X$ such that $f(x) \neq g(x)$.

Since X is $T_1$, the set $\{x\}$ is closed which implies that the set $X \setminus \{x\}$ is open in X. Also since Y is $T_1$ and $f(x) \neq g(x)$ there are open sets V and W in Y such that

$$f(x) \in V, \qquad g(x) \notin V$$

$$g(x) \in W, \qquad f(x) \notin W.$$

Thus the set $Z = \left[(X \setminus \{x\}) \times Y\right] \cup X \times V$ contains G(f) but does not contain G(g) and the set $P = \left[(X \setminus \{x\}) \times Y\right] \cup X \times W$ contains G(g) but does not contain G(f). Then the open sets $F_Z$ and $F_P$ in $(F, \Gamma)$ separate the points f and g in the following manner

$$f \in F_Z, \qquad g \notin F_Z$$

$$g \in F_P, \qquad f \notin F_P$$

which implies that $(F, \Gamma)$ is a $T_1$ space.

Next suppose $(F, \Gamma)$ is a $T_1$ space and let p and q be two distinct points of Y. Define $f, g \in F$ by $f(x) = p$, $g(x) = q$ for each $x \in X$, then $p \neq q$ implies $f \neq g$.

Since $(F, \Gamma)$ is $T_1$, there are open sets $F_U$ and $F_V$ in $(F, \Gamma)$ such that

$$f \in F_U, \qquad g \notin F_U$$

$$g \in F_V, \qquad f \notin F_V .$$

Suppose $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ where $U_\alpha$, $V_\alpha$ are open in X and Y respectively for each $\alpha \in J$. Then $g \notin F_U$ implies that $G(g) \not\subset U$ and hence that there is an $x \in X$ such that $\big(x, g(x)\big) = (x, q) \notin U$. Thus $(x, q) \notin U_\alpha \times V_\alpha$ for any $\alpha \in J$. However $f \in F_U$ implies that $G(f) \subset U$ and hence that there is an index $\beta \in J$ such that $\big(x, f(x)\big) = (x, p) \in U_\beta \times V_\beta$.

Thus we have $(x, q) \notin U_\beta \times V_\beta$ and $(x, p) \in U_\beta \times V_\beta$, which implies that $q \notin V_\beta$, $p \in V_\beta$ where $V_\beta$ is open in Y. A similar argument applied to the set $F_V$ will yield an open set $V_\gamma$ in Y such that $q \in V_\gamma$ and $p \notin V_\gamma$. Thus Y is a $T_1$ space.

Again assume $(F, \Gamma)$ is $T_1$. Let a and b be distinct points of X and p and q be distinct points of Y. Define $f, g \in F$ by

$$f(x) = p, \text{ for each } x \in X$$

$$g(x) = \begin{cases} p & \text{for} \quad x \in X \setminus \{a\} \\ q & \text{for} \quad x = a . \end{cases}$$

Since $f \neq g$ and $(F, \Gamma)$ is $T_1$, there is a basic open set $F_U$ in $(F, \Gamma)$ such that $g \in F_U$ and $f \notin F_U$. But $g \in F_U$, $f \notin F_U$ implies that $\big(a, f(a)\big) = (a, p) \notin U$ since f and g agree everywhere except at $a \in X$.

Suppose $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ where $U_\alpha, V_\alpha$ are open in X and Y respectively for each $\alpha \in J$. Then $G(g) \subset U$ implies that there is an index $\beta \in J$ such that $(b, g(b)) = (b, p) \in U_\beta \times V_\beta$. The point $a \in X$ cannot belong to $U_\beta$ otherwise $(a, p) \in U_\beta \times V_\beta \subset U$ since $p \in V_\beta$ and this contradicts the fact that $(a, p) \notin U$. Thus $U_\beta$ is an open set in X with $a \notin U_\beta$, $b \in U_\beta$.

By a similar argument, if a function $h \in F$ is defined as

$$h(x) = \begin{cases} p, & x \in X \setminus \{b\} \\ q, & x = b \end{cases}$$

then since $f \neq h$ and $(F, \Gamma)$ is $T_1$ there is a basic open set $F_V$ in $(F, \Gamma)$ such that $h \in F_V$, $f \notin F_V$. Then $h \in F_V$, $f \notin F_V$ implies that $(b, f(b)) = (b, p) \notin V$. Since $G(h) \subset V$ suppose $(a, h(a)) = (a, p) \in U_\gamma \times V_\gamma \subset V$ where $U_\gamma$ and $V_\gamma$ are open in X and Y respectively. Then $b \notin U_\gamma$ for otherwise $(b, p) \in U_\gamma \times V_\gamma \subset V$ contradicting the fact that $(b; p) \notin V$.

So $U_\gamma$ is an open set in X with $a \in U_\gamma$, $b \notin U_\gamma$. Thus a and b can be separated by open sets in X which implies that X is $T_1$.

## C. $T_2$

In Reference [5] Naimpally stated the following theorem without proof.

THEOREM II.C.1. If Y has at least two points then $(F, \Gamma)$ is $T_2$ if and only if X is $T_1$ and Y is $T_2$.

Proof. Assume $(F, \Gamma)$ is a $T_2$ space and let p, q be distinct points of Y. Define $f, g \in F$ by

$$f(x) = q, \text{ for each } x \in X \text{ and}$$

$$g(x) = \begin{cases} q, & \text{for } x \in X \setminus \{a\} \\ p, & \text{for } x = a . \end{cases}$$

Since $p \neq q$ implies that $f \neq g$ and since $(F, \Gamma)$ is $T_2$, there are basic open sets $F_U$ and $F_V$ in $(F, \Gamma)$ such that $f \in F_U$, $g \in F_V$ and $F_U \cap F_V = \phi$ where $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$, $V = \bigcup_{\beta \in K} U_\beta' \times V_\beta'$ and the sets $U_\alpha$, $U_\beta'$ and $V_\alpha$, $V_\beta'$ are open in X and Y respectively.

Since $G(f) \subset U$, there is an index $\beta \in J$ such that $(a, f(a)) = (a, p) \in U_\beta \times V_\beta$. Similarly, since $G(g) \subset V$ there is an index $\gamma \in K$ such that $(a, g(a)) = (a, q) \in U_\gamma' \times V_\gamma'$.

Thus $p \in V_\beta$ and $q \in V_\gamma'$ where $V_\beta$ and $V_\gamma'$ are open sets in Y.

Suppose $V_\beta \cap V_\gamma' \neq \phi$ then there is a point $z \in Y$ with $z \in V_\beta \cap V_\gamma'$. Define $h \in F$ by

$$h(x) = \begin{cases} q, & \text{for } x \in X \setminus \{a\} \\ z, & x = a . \end{cases}$$

Then since $(a, h(a)) = (a, z) \in (U_\beta \times V_\beta) \cap (U_\gamma' \times V_\gamma')$ and since $h = g = f$ on $X \setminus \{a\}$, $G(h) \subset U \cap V$. Thus $h \in F_U \cap F_V$ contradicting the assumption that $F_U \cap F_V = \phi$.

Thus $V_\beta \cap V_\gamma' = \phi$ which implies that Y is $T_2$.

Since $(F, \Gamma)$ $T_2$ implies that $(F, \Gamma)$ is $T_1$, then $(F, \Gamma)$ $T_2$ implies that X is $T_1$ by Theorem II.B.2.

Assume $X$ is $T_1$, $Y$ is $T_2$ and let $f, g \in F$ with $f \neq g$. Since $f \neq g$, there is a point $a \in X$ such that $f(a) \neq g(a)$. Since $Y$ is a $T_2$ space, there are open sets $V$ and $V'$ in $Y$ such that $f(a) \in V$, $g(a) \in V'$ and $V \cap V' = \phi$. Also since $X$ is a $T_1$ space, the set $\{a\}$ is closed in $X$ and hence the set $X\setminus\{a\}$ is open in $X$.

Define open sets $\mathcal{U}$ and $\mathcal{V}$ in $X \times Y$ as

$$\mathcal{U} \;=\; [X \times V] \cup \big[X\setminus\{a\} \times Y\big]$$

and

$$\mathcal{V} \;=\; [X \times V'] \cup \big[X\setminus\{a\} \times Y\big].$$

By construction, $G(f) \subset \mathcal{U}$ and $G(g) \subset \mathcal{V}$. Thus $f \in F_{\mathcal{U}}$ and $g \in F_{\mathcal{V}}$ where $F_{\mathcal{U}}$ and $F_{\mathcal{V}}$ are basic open sets in $(F, \Gamma)$.

Suppose $F_{\mathcal{U}} \cap F_{\mathcal{V}} \neq \phi$ then there is a function $h \in F$ such that $h \in F_{\mathcal{U}} \cap F_{\mathcal{V}}$. By Lemma I of Appendix A, $F_{\mathcal{U}} \cap F_{\mathcal{V}} = F_{\mathcal{U} \cap \mathcal{V}}$. Thus $h \in F_{\mathcal{U} \cap \mathcal{V}}$ which implies that $G(h) \subset \mathcal{U} \cap \mathcal{V}$. But this implies that $\big(a, h(a)\big) \in \mathcal{U} \cap \mathcal{V}$ and hence that $h(a) \in V \cap V'$ by construction of $\mathcal{U}$ and $\mathcal{V}$.

But $h(a) \in V \cap V'$ contradicts the assumption that $V \cap V' = \phi$.

Thus $F_{\mathcal{U}} \cap F_{\mathcal{V}} = \phi$ and $(F, \Gamma)$ is $T_2$.

## D. Regularity

In general, proofs and counterexamples involving the graph topology are harder to construct for the regularity separation axiom. In particular counterexamples tend to be either very complicated or trivial. In large measure, these difficulties are due to the fact that one can no longer restrict attention to finding appropriate basic open sets in $\Gamma$ that provide the required separations. For example to separate points from closed sets in $(F, \Gamma)$, arbitrary unions of basic, open sets in $\Gamma$ enter the picture. Arbitrary closed sets and arbitrary unions of open sets in $\Gamma$ are quite unwieldy and have few restrictions as to their structures.

The following example shows that in general X regular and Y regular are not sufficient conditions for $(F, \Gamma)$ to be regular.

EXAMPLE II.D.1. Let $X = \{a, b\}$ and $Y = \{p, q\}$ be topological spaces with the topologies $\Theta_X = \{\phi, X\}$ and $\Theta_Y = \{\phi, Y, \{p\}, \{q\}\}$ respectively where $a \neq b$ and $p \neq q$. As in Example II.A.4, $F = Y^X$ consists of the functions $f_1$, $f_2$, $f_3$, and $f_4$ where

$$G(f_1) = \{(a, p), (b, p)\}$$

$$G(f_2) = \{(a, p), (b, q)\}$$

$$G(f_3) = \{(a, q), (b, p)\}$$

$$G(f_4) = \{(a, q), (b, q)\} .$$

The following table lists all the open sets in $X \times Y$ and the open and closed sets in $(F, \Gamma)$:

| open in $X \times Y$ | open in $(F, \Gamma)$ | closed in $(F, \Gamma)$ |
|---|---|---|
| $\phi$ | $\phi$ | $F$ |
| $U_1 = X \times Y$ | $F = F_{U_1}$ | $\phi$ |
| $U_2 = X \times \{p\} = \{(a, p), (b, p)\}$ | $F_{U_2} = \{f_1\}$ | $F \setminus F_{U_2} = \{f_2, f_3, f_4\}$ |
| $U_3 = X \times \{q\} = \{(a, q), (b, q)\}$ | $F_{U_3} = \{f_4\}$ | $F \setminus F_{U_3} = \{f_1, f_2, f_3\}$ |
| | $F_{U_2} \cup F_{U_3} = \{f_1, f_4\}$ | $F \setminus (F_{U_2} \cup F_{U_3}) = \{f_2, f_3\}$ |

Consider the set $F \setminus F_{U_2}$. $F \setminus F_{U_2}$ is closed in $(F, \Gamma)$ and $f_1 \notin F \setminus F_{U_2}$. Let $\bigcup_{\alpha \in J} F_{U_\alpha}$ be any open set in $(F, \Gamma)$ which contains $F \setminus F_{U_2} = \{f_2, f_3, f_4\}$, where $F_{U_\alpha}$ is a basic open set in $(F, \Gamma)$ for each $\alpha \in J$ (i.e., $U_\alpha$ is open in $X \times Y$ for each $\alpha \in J$). Then there is an index $\beta \in J$ such that $f_2 \in F_{U_\beta}$. So $G(f_2) \subset U_\beta$ an open set in $X \times Y$. From the above table, the only open set in $X \times Y$ which contains the graph of $f_2$ is $X \times Y$. Thus $U_\beta = X \times Y$ and $F_{U_\beta} = F_{X \times Y} = F$. Therefore the only open set in $(F, \Gamma)$ which contains $F \setminus F_{U_2}$ is $F$ and hence contains $f_1$. This implies that $(F, \Gamma)$ is not regular since the point $f_1$ and the closed set $F \setminus F_{U_2}$ cannot be separated by disjoint open sets.

However since $X$ is indiscrete and $Y$ is discrete, $X$ and $Y$ are regular spaces. Thus $X$ regular and $Y$ regular does not imply that $(F, \Gamma)$ is regular.

The next theorem shows that $Y$ regular is a necessary condition for the regularity of $(F, \Gamma)$.

THEOREM II.D.2. If $X$ and $Y$ are topological spaces and $F$ is the space of all functions from $X$ to $Y$ then $(F, \Gamma)$ regular implies that $Y$ is regular.

Proof. Assume that $(F, \Gamma)$ is regular. Let $p \in Y$ and let $V$ be any open neighborhood of $p$ in $Y$. To prove that $Y$ is regular, it is sufficient to find an open neighborhood $Q$ of $p$ in $Y$ such that $p \in Q \subset \overline{Q} \subset V$.

Define $f \in F$ by $f(x) = p$ for each $x \in X$, then $f \in F_{X \times V}$ since by construction $G(f) \subset X \times V$. Also $F_{X \times V}$ is a basic open set in $(F, \Gamma)$ because $V$ is open in $Y$.

Since $(F, \Gamma)$ is regular, there is an open neighborhood $\bigcup_{\alpha \in J} F_{U_\alpha}$ of $f$ with $f \in \bigcup_{\alpha \in J} F_{U_\alpha} \subset \overline{\bigcup_{\alpha \in J} F_{U_\alpha}} \subset F_{X \times V}$, where $U_\alpha$ is open in $X \times Y$ for each $\alpha \in J$. In particular, $f \in F_{U_\alpha}$ for some $\alpha \in J$ and the relation $f \in F_{U_\alpha} \subset \overline{F_{U_\alpha}} \subset F_{X \times Y}$ holds. By Lemma I.B.2, $F_{U_\alpha} \subset F_{X \times V}$ implies that $U_\alpha \subset X \times V$.

Let $a \in X$, then the point $(a, p) = \bigl(a, f(a)\bigr) \in U_\alpha$ since $G(f) \subset U_\alpha$. Since $U_\alpha$ is open in $X \times Y$, let $(a, p) \in W \times Q \subset U_\alpha$ where $W$ is open in $X$ and $Q$ is open in $Y$. Then $Q \subset V$ because $W \times Q \subset U_\alpha \subset X \times V$ and $p \in Q$.

Let $q$ be any point of $\overline{Q}$ and define $g \in F$ by

$$g(x) = \begin{cases} f(x) = p & \text{for } x \neq a,\ x \in X \\ q & \text{for } x = a\ . \end{cases}$$

Then $g \in \overline{F_{U_\alpha}}$ for suppose $F_A$ is any basic open set in $(F, \Gamma)$ with $g \in F_A$ and $A = \bigcup_{\beta \in K} A_\beta \times B_\beta$ where $A_\beta$, $B_\beta$ are open in $X$ and $Y$ respectively. Since $G(g) \subset A$, $\bigl(a, g(a)\bigr) = (a, q) \in A_\gamma \times B_\gamma$ for same $\gamma \in K$. So $q \in B_\gamma$, an open set in $Y$. However $q \in \overline{Q}$ which implies that the open set $B_\gamma$ and the set $Q$ have a point $r \neq q$ in common, that is $r \in B_\gamma \cap Q$.

Define a function $h \in F$ by

$$h(x) = \begin{cases} f(x) = g(x) = p & \text{for } x \neq a,\ x \in X \\ r & \text{for } x = a\ . \end{cases}$$

Then $\bigl(a, h(a)\bigr) = (a, q) \in \bigl(A_\gamma \times B_\gamma\bigr) \cap (W \times Q)$. This implies that $G(h) \subset A \cap U_\alpha$ or that $h \in F_{A \cap U_\alpha}$. By Lemma I of Appendix A, $F_{A \cap U_\alpha} = F_A \cap F_{U_\alpha}$. Therefore every open set $F_A$ in $(F, \Gamma)$ which contains $g$ also contains a point $h \neq g$ with $h \in F_A \cap F_{U_\alpha}$. That is $g \in \overline{F}_{U_\alpha}$.

Thus $g \in \overline{F_{U_\alpha}} \subset F_{X \times V}$ or $G(g) \subset X \times V$. Since $G(g) \subset X \times V$, $\bigl(a, g(a)\bigr) = (a, q) \in X \times V$. This implies that $q \in V$. But $q$ was an arbitrary point of $\overline{Q}$. Therefore $\overline{Q} \subset V$ and it has been shown that $p \in Q \subset \overline{Q} \subset V$ with $Q$ an open set in $Y$. This completes the proof of Theorem II.D.2.

Suppose $\mathfrak{F}$ is a space of continuous functions from a space X to a space Y, then the following lemma provides information about the structure of certain closed sets in $(\mathfrak{F}, \Gamma)$. The lemma is useful in proving that certain conditions on X and Y are sufficient to ensure the regularity of $(\mathfrak{F}, \Gamma)$.

LEMMA II.D.3. If $\mathfrak{F}$ is a space of continuous functions from the regular space X to the topological space Y and if V is a closed set in $X \times Y$ then $\mathfrak{F}_V$ is closed in $(\mathfrak{F}, \Gamma)$.

Proof. Assume that $\mathfrak{F}$ is a space of continuous functions from X to Y, that is regular and that V is closed in $X \times Y$.

Let g be any point of $\overline{\mathfrak{F}}_V$, the closure of $\mathfrak{F}_V$. It is sufficient to prove that $g \in \mathfrak{F}_V$ or that $G(g) \subset V$. By way of contradiction, suppose that $G(g) \not\subset V$. Then there is a point $(x, g(x)) \in G(g)$ with $(x, g(x)) \notin V$. Since V is a closed set in $X \times Y$ and $(x, g(x)) \notin V$ there is a basic open set $O \times B$ in $X \times Y$ (i.e., O open in X and B open in Y) such that $(x, g(x)) \in O \times B$ and $(O \times B) \cap Y = \phi$. Since $g \in \mathfrak{F}$, g is continuous and there is an open set U in X such that $x \in U$ and $g(U) \subset B$. Let $W = O \cap U$ then $x \in W$ and $g(W) \subset g(U) \subset B$.

Since X is regular, there is an open neighborhood $O_1$ of x in X with $x \in O_1 \subset \overline{O}_1 \subset W$. Thus $g(O_1) \subset g(\overline{O}_1) \subset g(W) \subset B$.

$X \backslash \overline{O}_1$ is open in X. Let $P = \left[(X \backslash \overline{O}_1) \times Y\right] \cup (X \times B)$, then P is open in $X \times Y$. Since $g(\overline{O}_1) \subset B$, $G(g) \subset P$ or $g \in \mathfrak{F}_P$. Let h be any point of $\mathfrak{F}_P$ then $h(\overline{O}_1) \subset B$. Therefore $(x, h(x)) \in \overline{O}_1 \times B$ since $x \in \overline{O}_1$. Thus $(x, h(x)) \notin V$ since $(\overline{O}_1 \times B) \cap V \subset (O \times B) \cap V = \phi$. However $(x, h(x)) \notin V$ implies that $G(h) \not\subset V$ or that $h \notin \mathfrak{F}_V$. Since h was an arbitrary point of $\mathfrak{F}_P$ and $h \notin \mathfrak{F}_V$, $\mathfrak{F}_P \cap \mathfrak{F}_V = \phi$. But $\mathfrak{F}_P$ is an open neighborhood of g in $(\mathfrak{F}, \Gamma)$. Thus $\mathfrak{F}_P \cap \mathfrak{F}_V = \phi$ contradicts the assumption that $g \in \overline{\mathfrak{F}}_V$. Therefore $G(g)$ must be contained in V. Hence $g \in \mathfrak{F}_V$ for each $g \in \overline{\mathfrak{F}}_V$ and the lemma is proved.

As an aside, Lemma I.B.2 provides a similar structural theorem for closed sets in a space of non-continuous functions when X is $T_1$.

The following theorem provides sufficient conditions for regularity of the space $(\mathfrak{F}, \Gamma)$.

THEOREM II.D.4. Let $\mathfrak{F}$ be a set of continuous functions from a compact, regular space X to a regular space Y then $(\mathfrak{F}, \Gamma)$ is regular.

Proof. Assume that X is a compact, regular space, that Y is regular and that $\mathfrak{F}$ is a set of continuous functions. Let $f \in \mathfrak{F}$ and let $\mathfrak{F}_U$ be any basic open set in $(\mathfrak{F}, \Gamma)$ containing f where $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ and $U_\alpha$, $V_\alpha$ in X and Y respectively for each $\alpha \in J$.

Since $f \in \mathfrak{F}_U$, $G(f) \subset U$. This implies that for each $x \in X$, there is an index $\alpha_x \in J$ with $(x, f(x)) \in U_{\alpha_x} \times V_{\alpha_x}$. Since Y is regular, for each $x \in X$ there is an open neighborhood $V'_{\alpha_x}$ of $f(x)$ with $f(x) \in V'_{\alpha_x} \subset \overline{V}'_{\alpha_x} \subset V_{\alpha_x}$. Since $f \in \mathfrak{F}$, f is continuous so that for each $x \in X$, there is an open neighborhood $W_x$ of x with $f(W_x) \subset V'_{\alpha_x}$. Since $U_{\alpha_x} \cap W_x$ is an open neighborhood of x and since X is regular, there is an open neighborhood $U'_{\alpha_x}$ of x with $x \in U'_{\alpha_x} \subset \overline{U}_{\alpha_x'} \subset U_{\alpha_x} \cap W_x$. Then for each $x \in X$, $f(x) \in f(U'_{\alpha_x}) \subset f(\overline{U}'_{\alpha_x}) \subset f(U_{\alpha_x} \cap W_x) \subset f(W_x) \subset V'_{\alpha_x}$ by choice of $W_x$.

Since X is compact and $\{U'_{\alpha_x} \mid x \in X\}$ is an open cover of X, there is a finite sub-cover $\{U'_{\alpha_{x_i}} \mid i = 1, \cdots, n\}$ of X. Let $V = \bigcup_{i=1}^{n} \overline{U}'_{\alpha_{x_i}} \times \overline{V}'_{\alpha_{x_i}}$ and $V' = \bigcup_{i=1}^{n} U'_{\alpha_{x_i}} \times V'_{\alpha_{x_i}}$. Then V is a finite union of closed sets in $X \times Y$ and $V'$ is a finite union of open sets in $X \times Y$ making $V$ and $V'$ closed and open respectively in $X \times Y$. Thus $\mathfrak{F}_{V'}$ is open in $(\mathfrak{F}, \Gamma)$ and by Lemma II.D.3, $\mathfrak{F}_V$ is closed in $(\mathfrak{F}, \Gamma)$.

Also $G(f) \subset V' \subset V$ since $\left\{U'_{\alpha_{x_i}} \mid i = 1, \cdots, n\right\}$ covers X, since $f\left(U'_{\alpha_{x_i}}\right) \subset f\left(\overline{U}'_{\alpha_{x_i}}\right) \subset V'_{\alpha_{x_i}} \subset \overline{V}'_{\alpha_{x_i}}$ by construction of $U'_{\alpha_{x_i}}$ and since $U'_{\alpha_{x_i}} \times V'_{\alpha_{x_i}} \subset \overline{U}'_{\alpha_{x_i}} \times \overline{V}'_{\alpha_{x_i}}$ for $i = 1, \cdots, n$. By Lemma I.B.1 $V' \subset V$ implies that $\mathfrak{F}_{V'} \subset \mathfrak{F}_V$. Therefore $f \in \mathfrak{F}_{V'} \subset \mathfrak{F}_V$ and $\mathfrak{F}_V$ is a closed neighborhood of f.

Finally it remains to show that $\mathfrak{F}_V \subset \mathfrak{F}_U$. By construction, $\overline{U}'_{\alpha_x} \subset U_{\alpha_x}$ and $\overline{V}'_{\alpha_x} \subset V_{\alpha_x}$ for each $x \in X$. Therefore $V = \bigcup_{i=1}^{n} \overline{U}'_{\alpha_{x_i}} \times \overline{V}'_{\alpha_{x_i}} \subset \bigcup_{x \in X} U_{\alpha_x} \times V_{\alpha_x} = U$.

Again by Lemma I.B.1, $V \subset U$ implies that $\mathfrak{F}_V \subset \mathfrak{F}_U$ and the theorem is proved.

As a final note on regularity, it is shown in Kelley [2], p. 141, that if X is $T_2$ and compact then X is a regular compact space also. Thus

COROLLARY II.D.5. If $\mathfrak{F}$ is a set of continuous functions from the $T_2$, compact space X to the regular space Y then $(\mathfrak{F}, \Gamma)$ is regular.

Proof. The proof follows immediately from Theorem II.D.4 and the above remark as Kelley's proof.

# CHAPTER III

## THE GRAPH TOPOLOGY AND OTHER FUNCTION SPACE TOPOLOGIES

In this chapter comparison of the graph topology with the pointwise convergence topology, with the compact open topology, with the uniform convergence topology and with the sup metric topology will be made.

### A. Comparison With the Topology of Pointwise Convergence

Naimpally's Example II.B.1 will serve in this section to shed some light on a comparison of the p.c. and the graph topologies.

EXAMPLE III.A.1. Let the spaces $X = \{a, b\}$, $\Theta_X = \{\phi, \{a\}, X\}$ and $Y = \{p, q\}$, $\Theta_Y = \{\phi, Y, \{p\}, \{q\}\}$ be given as in Example II.B.1. Then as was remarked previously, X is a $T_0$ space and Y is discrete. The four functions in $F = Y^X$ are:

$$f: \begin{tikzcd} a \arrow[r] & p \\ b \arrow[r] & q \end{tikzcd} \qquad g: \begin{tikzcd} a \arrow[dr] & \\ b \arrow[r] & q \end{tikzcd}$$

$$h: \begin{tikzcd} a \arrow[r] & p \\ b \arrow[ur] & \end{tikzcd} \qquad i: \begin{tikzcd} a \arrow[dr] & p \\ b \arrow[ur] & q \end{tikzcd} .$$

By definition, a subbasic open set for the p.c. topology is a set of the form $W(x, U) = \{f \in F \mid f(x) \in U\}$ where x is a fixed point in X and U is a fixed open set in Y.

The following is a listing of some of the subbasic open sets for the p.c. topology, $\mathcal{P}$, on F in this example:

$$W(a, \{p\}) = \{f, h\}$$

$$W(a, \{q\}) = \{g, i\}$$

$$W(b, \{p\}) = \{h, i\}$$

$$W(b, \{q\}) = \{f, g\} .$$

Note that the following relations hold:

$$\{f\} = W(b, \{q\}) \cap W(a, \{p\})$$

$$\{g\} = W(a, \{q\}) \cap W(b, \{q\})$$

$$\{h\} = W(a, \{p\}) \cap W(b, \{p\})$$

$$\{i\} = W(a, \{q\}) \cap W(b, \{p\}) .$$

Therefore every point of F is a finite intersection of open sets in $(F, \mathcal{P})$. However this implies that every point of F is open in $(F, \mathcal{P})$ or that F is a discrete space under the p.c. topology.

As was shown in Example II.B.1, $(F, \Gamma)$ is not a $T_1$ space. Therefore the p.c. topology and the graph topology on F of this example are not comparable.

To get meaningful results on comparisons between the p.c. and the graph topologies, it is necessary to consider X a $T_1$ space as the following theorem due to Naimpally [5] indicates.

THEOREM III.A.2. If X is a $T_1$ space and if F is the set of all functions from X to a space Y then graph topology on F contains the p.c. topology on F.

Proof. Suppose X is $T_1$ and $F = Y^Z$. Let $W(x, U) = \{f \in F \mid f(x) \in U\}$ (where x is a fixed point in X and U is open in Y) be any subbasic open set for the p.c. topology on F. Since X is $T_1$, the set $\{x\}$ is closed in X. Let $V = \left[(X\setminus\{x\}) \times Y\right] \cup (X \times U)$, then V is open in $X \times Y$.

Suppose $f \in W(x, U)$ then $f(x) \in U$ so that $G(f) \subset V$ or $f \in F_V$. Therefore $W(x, U) \subset F_V$. Similarly, if $f \in F_V$ then $G(f) \subset V$ so that $f(x) \in U$ or $f \in W(x, U)$. Therefore $W(x, U) \subset F_V \subset W(x, U)$ or $W(x, U) = F_V$ which implies that $W(x, U)$ is open in $(F, \Gamma)$.

## B. Comparison With the Compact Open Topology

Since the compact open topology contains the pointwise convergence topology, Example III.A.1 also shows that F of that example is discrete under the k-topology. Therefore since F under the graph topology was not even $T_1$, this example shows that the k topology and the graph topology may not be comparable when X is only a $T_0$ space. In fact Naimpally showed in Reference [5] that the stronger condition of X being $T_2$ is needed for a meaningful comparison of the k-topology with the graph topology. Naimpally's result is:

THEOREM III.B.1. If X is a $T_2$ space and if F is the set of all functions from X to Y then the graph topology contains the compact open topology on F.

Proof. Suppose X is $T_2$ and $F = Y^X$. Let $W(K, U) = \{f \mid f(K) \subset U\}$ (where K is a fixed compact subset of X and U is a fixed open set in Y) be any subbasic open set for the k-topology on F.

Since X is $T_2$ and K is compact, K is closed in X. Therefore the set $V = (X \times U) \cup [(X \setminus K) \times Y]$ is open in $X \times Y$.

Suppose $f \in W(K, U)$ then $f(K) \subset U$ so that $G(f) \subset V$ or $f \in F_V$. Therefore $W(K, U) \subset F_V$. Similarly, if $f \in F_V$ then $G(f) \subset V$ so that $f(K) \subset U$ or $f \in W(K, U)$. Therefore $W(K, U) \subset F_V \subset W(K, U)$ or $W(K, U) = F_V$ which implies that $W(K, U)$ is open in $(F, \Gamma)$.

Naimpally stated in his Theorem 42 of Reference [5] that if X is a $T_2$, compact space then the graph topology coincides with the k-topology on F. In fact, this statement is false as the following counterexample shows.

EXAMPLE III.B.2. Let $X = Y = [0, 1]$ with the usual topology. Define $f \in F = Y^X$ by

$$f(x) = \begin{cases} 0 & x \in (0, 1] \\ 1 & x = 0 \end{cases}.$$

Let $U = \{(x, y) \in X \times Y \mid y < x \text{ or } x + 3/4 < y\}$. By construction, U is open in $X \times Y$ and $G(f) \subset U$. Thus $f \in F_U$ and $F_U$ is an open neighborhood of f in $(F, \Gamma)$. It suffices to show that $F_U$ is not a neighborhood of f in $(F, k)$ to show that the graph topology is not contained in the k topology.

Let $\{K_i \mid i = 1, \cdots, n\}$ be a collection of compact sets in X and $\{U_i \mid i = 1, \cdots, n\}$ be a collection of open sets in Y. Then the set $\bigcap_{i=1}^{n} W(K_i, U_i)$

is an arbitrary basic open neighborhood of f in the k topology on F where $W(K_i, U_i) = \{g \in F \mid g(K_i) \subset U_i\}$ for $i = 1, \cdots, n$, when $f(K_i) \subset U_i$.

Let J be the set of indices j from 1 to n such that $K_j \cap (0, 1] \neq \phi$. Then there are two cases to consider:

Case i: If $J = \phi$ then $K_i \cap (0, 1] = \phi$ and $K_i = \{0\}$ for each $i = 1, \cdots, n$. Define $h \in F$ by $h(x) = 1$ for each $x \in [0, 1]$. Then since $f(\{0\}) = f(K_i) \subset U_i$ and since $f(0) = 1$, $1 \in U_i$ for each $i = 1, \cdots, n$. But $1 \in U_i$ implies that $h(K_i) \subset U_i$ for each $i = 1, \cdots, n$ or that $h \in \bigcap_{i=1}^{n} W(K_i, U_i)$. However $G(h) \not\subset U$ since the point $(1, 1) = (1, h(1)) \notin U$ by construction of U. Thus if $J = \phi$, $\bigcap_{i=1}^{n} W(K_i, U_i) \not\subset F_U$.

Case ii: Suppose that $J \neq \phi$ and define $W = \bigcap_{j \in J} U_j$. By definition, $f \equiv 0$ on $(0, 1]$. Thus for each $j \in J$, since $K_j \cap (0, 1] \neq \phi$ and since $f \in W(K_j, U_j)$, $0 \in U_j$. Therefore $0 \in \bigcap_{j \in J} U_j = W$ and W is an open neighborhood of 0 in Y.

Since W is an open neighborhood of 0, there is a point $y_0 \in W \setminus \{0\}$. Define a function $h \in F$ by

$$h(x) = \begin{cases} y_0 & x \in (0, 1] \\ 1 & x = 0 \ . \end{cases}$$

Then by definition, $h \in \bigcap_{i=1}^{n} W(K_i, U_i)$.

Also $h(y_0) = y_0$ since $y_0 \neq 0$. However the point $(y_0, h(y_0)) = (y_0, y_0) \notin U$ by construction. This implies that $G(h) \not\subset U$ or that $h \notin F_U$.

Thus $\bigcap_{i=1}^{n} W(K_i, U_i) \not\subset F_U$. Since $\bigcap_{i=1}^{n} W(K_i, U_i)$ was an arbitrary open neighborhood of f in $(F, k)$, this shows that $F_U$ cannot be a k neighborhood of f. Therefore $F_U$ is not open in the k topology and $\Gamma \not\subset k$.

The next theorem gives a correct set of conditions insuring the equivalence of the k topology and the graph topology on a space of continuous functions.

THEOREM III.B.3. If $\mathfrak{F}$ is a set of continuous functions from a compact, $T_2$ space X to a topological space Y then the graph topology on $\mathfrak{F}$ is equivalent to the compact open topology on $\mathfrak{F}$.

Proof. Suppose X is a compact, $T_2$ space and $\mathfrak{F}$ is a set of continuous functions in $Y^X$. Let $\mathfrak{F}_U$ be any basic open set in $(\mathfrak{F}, \Gamma)$ where $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ and $U_\alpha$, $V_\alpha$ are open in X and Y respectively for each $\alpha \in J$.

Let f be any point of $\mathfrak{F}_U$. To show that $\mathfrak{F}_U$ is open in the k-topology on $\mathfrak{F}$, it suffices to show that $\mathfrak{F}_U$ is a k-neighborhood of f.

Since $f \in \mathfrak{F}_U$, $G(f) \subset U$. Therefore for each $x \in X$, there is an index $\alpha_x \in J$ such that $(x, f(x)) \in U_{\alpha_x} \times V_{\alpha_x}$. Since f is continuous there is an open set $O_x$ in X with $x \in O_x$ and $f(O_x) \subset V_{\alpha_x}$ for each $x \in X$.

Since X is compact and $T_0$, X is regular. Therefore there is an open set $U'_{\alpha_x}$ such that $x \in U'_{\alpha_x} \subset \bar{U}'_{\alpha_x} \subset O_x \cap U_{\alpha_x}$ for each $x \in X$. Thus $f(x) \in f(U'_{\alpha_x}) \subset f(\bar{U}'_{\alpha_x}) \subset f(O_x \cap U_{\alpha_x}) \subset f(O_x) \subset V_{\alpha_x}$.

Since X is compact and since $\{U'_{\alpha_x} \mid x \in X\}$ is an open cover of X, there is a finite subcover $\{U'_{\alpha_{x_i}} \mid i = 1, \cdots, n\}$. Then $f(\bar{U}'_{\alpha_{x_i}}) \subset V_{\alpha_{x_i}}$. But X compact and $\bar{U}'_{\alpha_{x_i}}$ closed in X implies that $\bar{U}'_{\alpha_{x_i}}$ is compact. Therefore $f \in \bigcap_{i=1}^{n} W(\bar{U}'_{\alpha_{x_i}}, V_{\alpha_{x_i}})$, a basic open set in the k topology on $\mathfrak{F}$.

Suppose $g \in \bigcap_{i=1}^{n} W(\bar{U}'_{\alpha_{x_i}}, V_{\alpha_{x_i}})$ then $g(U'_{\alpha_{x_i}}) \subset V_{\alpha_{x_i}}$ since $U'_{\alpha_{x_i}} \subset \bar{U}'_{\alpha_{x_i}}$ for each $i = 1, \cdots, n$. But $\{U'_{\alpha_{x_i}} \mid i = 1, \cdots, n\}$ is an open cover of X. Therefore

$G(g) \subset \cup U'_{\alpha_{x_i}} \times V_{\alpha_{x_i}} \subset U$ or $g \in \mathfrak{F}_U$. Since g was an arbitrary point of $\cap W\left(\overline{U}'_{\alpha_{x_i}}, V_{\alpha_{x_i}}\right)$, this implies that $\cap W\left(\overline{U}'_{\alpha_{x_i}}, V_{\alpha_{x_i}}\right) \subset \mathfrak{F}_U$.

Therefore $f \in \bigcap_{i=1}^{n} W\left(\overline{U}'_{\alpha_{x_i}}, V_{\alpha_{x_i}}\right) \subset \mathfrak{F}_U$ which implies that $\mathfrak{F}_U$ is a k-neighborhood of each of its points. That is $\mathfrak{F}_U$ is k-open. Hence $\Gamma \subset k$. By Theorem III.B.1, $k \subset \Gamma$ on $\mathfrak{F}$ since $\mathfrak{F} \subset F$ and $k \subset \Gamma$ on F.

Therefore k and $\Gamma$ are equivalent topologies on $\mathfrak{F}$.

## C. Comparison With the Topology of Uniform Convergence

The two theorems of this section are stronger than those in Naimpally [5] since uniform continuity is not required.

THEOREM III.C.1. Let $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ be uniform spaces. If $\mathfrak{F}$ is a set of functions from X to Y which are continuous with respect to the uniform topologies on X and Y then the graph topology on $\mathfrak{F}$ contains the uniform convergence topology on $\mathfrak{F}$.

Proof. A basis for the u.c. uniformity on $\mathfrak{F}$ consists of sets of the form $W(V) = \left\{(f, g) \in \mathfrak{F} \times \mathfrak{F} \mid \big(f(x), g(x)\big) \in V \text{ for each } x \in X\right\}$ where V is an element of the uniformity $\mathfrak{V}$ on Y.

Let O be any open set in the uniform topology on $\mathfrak{F}$ and let f be any point of O. Then O is a neighborhood of f in the u.c. topology and thus contains a set of the form $W(V)[f] = \left\{g \in \mathfrak{F} \mid \big(f(x), g(x)\big) \in V \text{ for each } x \in X\right\}$ where V is an element of $\mathfrak{V}$.

From page 179 of Kelley [2], the family of open symmetric members of a uniformity form a basis for the uniformity. Let $V_1$ be an open symmetric member of $\mathfrak{V}$ such that $V_1 \circ V_1 \subset V$.

Let $x \in X$ then $V_1\,[f(x)]$ is a neighborhood of $f(x)$ in Y. Since f is continuous with respect to the uniform topologies on X and Y, there is an open symmetric member $U_x$ of $\mathfrak{U}$ such that $f(U_x\,[x]) \subset V_1\,[f(x)]$ for each $x \in X$. Since $U_x$ and $V_1$ are open members of $\mathfrak{U}$ and $\mathfrak{V}$ respectively, the set $P = \bigcup_{x \in X} U_x\,[x] \times V_1\,[f(x)]$ is open in $X \times Y$. Also $G(f) \subset P$ since $U_x$ and $V_1$ contain the diagonals in $X \times X$ and $Y \times Y$ respectively. Thus $f \in \mathfrak{F}_P$.

Suppose $g \in \mathfrak{F}_P$ then $G(g) \subset P$. This implies that for any $x \in X$, there is a $y \in X$ such that $(x, g(x)) \in U_y\,[y] \times V_1\,[f(y)]$. Thus $x \in U_y\,[y]$ and $(f(y), g(x)) \in V_1$. By definition of $U_y$, $f(U_y\,[y]) \subset V_1\,[f(y)]$. Thus $x \in U_y\,[y]$ implies that $f(x) \in V_1\,[f(y)]$ or that $(f(y), f(x)) \in V_1$. Since $V_1$ is symmetric, $(f(x), f(y)) \in V_1$. Therefore $(f(x), g(x)) \in V_1 \circ V_1 \subset V$ for each $x \in X$.

This implies that $(f, g) \in W(V)$ or that $g \in W(V)\,[f]$. Thus $f \in \mathfrak{F}_P \subset W(V)\,[f] \subset O$ and O is a $\Gamma$ neighborhood of each of its points. Therefore the topology of uniform convergence is contained in the graph topology.

THEOREM III.C.2. Let $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ be uniform spaces and let $\mathfrak{F}$ be a set of functions which are continuous with respect to the uniform topologies on X and Y. If X is compact then the graph topology on $\mathfrak{F}$ is equivalent to the uniform convergence topology on $\mathfrak{F}$.

Proof. By Theorem III.C.1, the u.c. topology is contained in the graph topology.

Let $\mathfrak{F}_U$ be a basic open set in $(\mathfrak{F}, \Gamma)$ and let $f \in \mathfrak{F}_U$ where $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ and $U_\alpha$, $V_\alpha$ are open in the uniform topologies on X and Y respectively. Then $G(f) \subset U$.

By Lemma I of Appendix B, $G(f)$ is homeomorphic to $X$ since $f$ is continuous. Thus $G(f)$ is compact since $X$ is compact.

From page 199 in Kelley [2], the open cover $\{U_\alpha \times V_\alpha \mid \alpha \in J\}$ of the compact set $G(f)$ in the uniform space $X \times Y$ is a uniform cover. Therefore there exists open symmetric sets $U_1 \in \mathfrak{U}$ and $V_1 \in \mathfrak{V}$ such that

$$G(f) \subset \bigcup_{x \in X} U_1[x] \times V_1[f(x)] \subset \bigcup_{\alpha \in J} U_\alpha \times V_\alpha \subset U .$$

Let $g \in \mathfrak{F} \cap W(V_1)[f]$ then $g(x) \in V_1[f(x)]$ for each $x \in X$. Therefore $(x, g(x)) \in U_1[x] \times V_1[f(x)]$ for each $x \in X$. This implies that $G(g) \subset \bigcup_{x \in X} U_1[x] \times V_1[f(x)] \subset U$ or that $g \in \mathfrak{F}_U$. Therefore $f \in \mathfrak{F} \cap W(V_1)[f] \subset \mathfrak{F}_U$ which implies that $\mathfrak{F}_U$ is a u.c. neighborhood of each of its points. Therefore the graph topology is contained in the u.c. topology on $\mathfrak{F}$.

## D. Comparison With the Sup Metric Topology

Let $F$ be a set of functions from a set $X$ to a metric space $(Y, d)$. If $f, g \in F$ then $\rho(f, g) = \sup_{x \in X} d(f(x), g(x))$ is a metric on $F$ called the sup metric.

THEOREM III.D.1. If $(X, d')$ and $(Y, d)$ are metric spaces with $d$ and $d'$ bounded metrics and if $\mathfrak{F}$ is a set of continuous functions from $X$ to $Y$ then the sup metric topology on $\mathfrak{F}$ is contained in the graph topology on $\mathfrak{F}$.

Proof. Let $O$ be any open set in the sup metric topology on $\mathfrak{F}$ and let $f$ be any point of $O$. It suffices to show that $O$ is a $\Gamma$ neighborhood of $f$. Since $O$ is open in the sup metric topology, there exists an open $\rho$ ball $N_\epsilon(f)$ of radius $\epsilon > 0$ about $f$ with $f \in N_\epsilon(f) \subset O$.

Since $f$ is continuous, given $x \in X$ and given $\epsilon > 0$ there is a $\delta_x > 0$ ($\delta_x$ depends on $x$) such that $f\left(N_{\delta_x}(x)\right) \subset N_{\epsilon/3}\left(f(x)\right)$ where $N_{\delta_x}(x)$ is a $d'$ ball of radius $\delta_x$ about $x$ and $N_{\epsilon/3}\left(f(x)\right)$ is a $d$ ball of radius $\epsilon/3$ about $f(x)$.

Consider $U = \bigcup_{x \in X} N_{\delta_x}(x) \times N_{\epsilon/3}\left(f(x)\right)$. By construction of $U$, $G(f) \subset U$ and $U$ is open in $X \times Y$. That is $f \in \mathfrak{F}_U$ a basic open set in $(\mathfrak{F}, \Gamma)$.

Let $g \in \mathfrak{F}_U$ then $G(g) \subset U$. Therefore for each point $z \in X$, $\left(z, g(z)\right) \in U$. This implies that for each $z \in X$, there is an $x' \in X$ such that $\left(z, g(z)\right) \in N_{\delta_{x'}}(x') \times N_{\epsilon/3}\left(f(x')\right)$. Thus $d'(z, x') < \delta_{x'}$ and $d\left(g(z), f(x')\right) < \epsilon/3$. However $d'(z, x') < \delta_{x'}$ implies that $d\left(f(z), f(x')\right) < \epsilon/3$ by definition of $\delta x'$. Thus

$$d\left(f(z), g(z)\right) \leq d\left(f(z), f(x')\right) + d\left(f(x'), g(z)\right)$$

$$< \epsilon/3 + \epsilon/3 .$$

But $d\left(f(z), g(z)\right) < 2\epsilon/3$ for each $z \in X$ implies that $\rho(f, g) = \operatorname{Sup}_{z \in X} d\left(f(z), g(z)\right) < \epsilon$. Therefore $g \in N_\epsilon(f)$ and $f \in \mathfrak{F}_U \subset N_\epsilon(f) \subset O$ or $O$ is a $\Gamma$ neighborhood of $f$. This implies that $O$ is a $\Gamma$ open set and that the sup metric topology on $\mathfrak{F}$ is contained in the graph topology on $\mathfrak{F}$.

If $(X, d')$ and $(Y, d)$ are metric spaces with bounded metrics then the product topology on $X \times Y$ is induced by the metric

$$D\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) = d'\left(x_1, x_2\right) + d\left(y_1, y_2\right)$$

where $x_1$, $x_2 \in X$ and $y_1$, $y_2 \in Y$.

THEOREM III.D.2. If $(X, d')$ is a compact metric space and $(Y, d)$ is a metric space ($d'$, $d$ bounded metrics) and if $\mathfrak{F}$ is a set of continuous functions from $X$ to $Y$ then the graph topology is equivalent to the sup metric topology on $\mathfrak{F}$.

Proof. Let $\mathfrak{Z}_U$ be an open set in $(\mathfrak{Z}, \Gamma)$ where $U = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha$ and $U_\alpha$, $V_\alpha$ are open in X and Y respectively for each $\alpha \in J$. If f is any point of $\mathfrak{Z}_U$, then it suffices to show that $\mathfrak{Z}_U$ is a $\rho$-neighborhood of f since the sup metric topology $\subset \Gamma$ by Theorem III.D.1.

By Lemma I of Appendix B, G( f ) is homeomorphic to the compact set X since f is continuous. Thus G( f ) is compact. The collection $\{U_\alpha \times V_\alpha \mid \alpha \in J\}$ is an open cover of the compact set G( f ) in the metric space $(X \times Y, D)$. Let $\epsilon > 0$ be the Lebesgue number of this open cover. Then by definition of the Lebesgue number, given $x \in X$ there is an index $\alpha_x \in J$ such that $N_\epsilon^D\left(x, f(x)\right) \subset U_{\alpha x} \times V_{\alpha x}$. Therefore $\bigcup_{x \in X} N_\epsilon^D\left(x, f(x)\right) \subset \bigcup_{\alpha_x \in J} U_{\alpha_x} \times V_{\alpha_x} \subset U$.

Consider the sup metric open neighborhood $N_\epsilon^\rho(f)$ of f. Then $f \in N_\epsilon^\rho(f)$ and if $g \in N_\epsilon^\rho(f)$, $\rho(f, g) = \operatorname{Sup}_{x \in X} d\left(f(x), g(x)\right) < \epsilon$.

Therefore given $x \in X$,

$$D\Big(\big(x, g(x)\big), \big(x, f(x)\big)\Big) = d'(x, x) + d\big(g(x), f(x)\big)$$

$$< 0 + \epsilon = \epsilon .$$

Thus $\left(x, g(x)\right) \in N_\epsilon^D\left(x, f(x)\right)$ for each $x \in X$. Therefore $G(g) \subset \bigcup_{x \in X} N_\epsilon^D\left(x, f(x)\right) \subset U$ or $G(g) \subset U$. Thus if $g \in N_\epsilon^\rho(f)$, $g \in \mathfrak{Z}_U$ which implies that $f \in N_\epsilon^\rho(f) \subset \mathfrak{Z}_U$ or that $\mathfrak{Z}_U$ is a $\rho$-neighborhood of f.

Therefore $\mathfrak{Z}_U$ is $\rho$-open in $\mathfrak{Z}$ and the graph topology is contained in the sup metric topology on $\mathfrak{Z}$.

## CHAPTER IV

## CONTINUITY OF THE EVALUATION MAP

If $\mathfrak{F}$ is a set of functions defined on the space X with range in the space Y then the evaluation map is the map $e : \mathfrak{F} \times X \to Y$ where $e(f, x) = f(x)$ for each point $(f, x) \in \mathfrak{F} \times X$. Because of this definition, two types of continuity can be considered for the map e: separate continuity and joint continuity. The map e is separately continuous in f and x when e is continuous in each coordinate separately. That is when e is continuous as a function of f when x is held fixed and e is continuous as a function of x when f is held fixed. The map e is jointly continuous if e is continuous. That is if e is continuous when both f and x vary simultaneously. Note that separate continuity is a necessary condition for joint continuity.

Considering separate continuity of e, suppose $f \in \mathfrak{F}$ is fixed and $x \in X$ is allowed to vary. Then since $e(f, x) = f(x)$, the continuity of e in x is equivalent to the continuity of f. Thus to consider either separate or joint continuity of f, it is necessary that $\mathfrak{F}$ be a space of continuous functions. Hence throughout this chapter, attention will be centered on spaces $\mathfrak{F}$ of continuous functions from X to Y.

A good reference on the evaluation map is McCarty [3].

## A. Classic Results—The Pointwise Convergence and the Compact Open Topologies

The following theorem yields necessary and sufficient conditions for separate continuity of $e$.

THEOREM IV.A.1. If $(\mathfrak{F}, \Theta)$ a topological space of continuous functions on the space X to the space Y then the evaluation map e is separately continuous with respect to $\Theta$ if and only if $\Theta \supset \mathbb{P}$, the topology of pointwise convergence.

Proof. Assume e is separately continuous with respect to $(\mathfrak{F}, \Theta)$ and suppose O is a subbasic open set in $(\mathfrak{F}, \mathbb{P})$. Then by definition of the topology $\mathbb{P}$, O is of the form $O = \left\{f \in \mathfrak{F} \mid f\left(x_0\right) \in V, \text{ for some fixed } x_0 \in X \text{ and fixed } V \text{ open in } Y\right\}$.

Since e is separately continuous in f and x with respect to $\Theta$, the function $e\left(f, x_0\right)$, where $x_0$ is fixed and f is allowed to vary, is continuous with respect to $\Theta$. Denote the function $e\left(f, x_0\right)$ by $e_{x_0}(f)$, then $e_{x_0}$ maps $\mathfrak{F}$ into Y by $e_{x_0}(f) = e\left(f, x_0\right) = f\left(x_0\right)$ for each $f \in \mathfrak{F}$.

Since $e_{x_0}$ is continuous on $(\mathfrak{F}, \Theta)$ and since V is open in Y, $e_{x_0}^{-1}(V)$ is open in $(\mathfrak{F}, \Theta)$. However $e_{x_0}^{-1}(V) = \left\{f \in \mathfrak{F} \mid e_{x_0}(f) \in V\right\} = \left\{f \in \mathfrak{F} \mid f\left(x_0\right) \in V\right\} = O$. Thus O is open in $\Theta$ and $\mathbb{P} \subset \Theta$.

Next suppose that $\mathbb{P} \subset \Theta$ and consider e as a function of f only. That is let x be fixed at $x_0 \in X$, allow f to vary and consider the function $e\left(f, x_0\right) = e_{x_0}(f)$ for $f \in \mathfrak{F}$. Again $e_{x_0}$ maps $\mathfrak{F}$ into Y. Let V be any open set in Y, then $e_{x_0}^{-1}(V) = \left\{f \in \mathfrak{F} \mid e_{x_0}(f) = f\left(x_0\right) \in V\right\}$. By definition of the topology $\mathbb{P}$, $e_{x_0}^{-1}(V)$ is a subbasic open set in $\mathbb{P}$. Thus $e_{x_0}^{-1}(V)$ is open in $\mathbb{P} \subset \Theta$ and hence is open in $\Theta$. Thus the function $e_{x_0}(f) = e\left(f, x_0\right)$ for $f \in \mathfrak{F}$ is continuous in f.

The continuity of $e$ in $x$ for fixed $f$ is equivalent to the continuity of $f$ by a remark made in the introduction to Chapter IV. By assumption, $(\mathfrak{F}, \Theta)$ is a space of continuous functions from X to Y. Thus $e$ is continuous in $x$ for fixed $f$ by assumption.

So $e$ is separately continuous in $f$ and in $x$ with respect to $(\mathfrak{F}, \Theta)$ when $P \subset \Theta$ and the proof is complete.

COROLLARY IV.A.2. The pointwise convergence topology on $\mathfrak{F}$ is the smallest topology on $\mathfrak{F}$ for which $e$ is separately continuous.

Proof. The proof of this corollary follows immediately from the proof of Theorem IV.A.1.

Now turning to the continuity of $e$, that is the joint continuity of $e$, the following definition can be made.

Definition: If $\mathfrak{F}$ is a space of continuous functions from a topological space X to a topological space Y, then an admissible topology for $\mathfrak{F}$ is a topology on $\mathfrak{F}$ which makes the evaluation map $e : \mathfrak{F} \times X \to Y$ continuous (i.e. jointly continuous).

The following theorem was presented by Arens in Reference [1].

THEOREM IV.A.3. If $\mathfrak{F}$ is a space of continuous functions from the space X to the space Y and if $\Theta$ is an admissible topology for $\mathfrak{F}$ then the compact open topology is coarser than $\Theta$, that is $k \subset \Theta$.

Proof. Let $W(K, U)$ be any subbasic open set for the k-topology on $\mathfrak{F}$ where K is a compact set in X, U is open in Y and $W(K, U) = \{f \in \mathfrak{F} \mid f(K) \subset U\}$. It will be shown that $W(K, U)$ is a $\Theta$ neighborhood of each of its points.

Let f be any point of $W(K, U)$ then $f(K) \subset U$ so that $e(f, x) = f(x) \in U$ for each $x \in K$. This implies that $(f, x) \in e^{-1}(U)$ for each $x \in K$. Since $\Theta$ is admissible, $e^{-1}(U)$ is open in $\mathfrak{F} \times X$. Thus for each $x \in K$, there is a $\Theta$ open set $W_x$ and an X open set $V_x$ such that $(f, x) \in W_x \times V_x \subset e^{-1}(U)$.

The collection $\{V_x \mid x \in K\}$ is an open cover of the compact set K. Let $\{V_{x_i} \mid i = 1, \cdots, n\}$ be a finite subcover of K and let $O = \bigcap_{i=1}^{n} W_{x_i}$. Then O, a finite intersection of $\Theta$ open sets, is $\Theta$ open and $f \in O$ since $f \in W_x$ for each $x \in K$.

Let g be any element of O and $x \in K$ then $x \in V_{x_j}$ for some $j = 1, \cdots, n$ since $K \subset \bigcup_{i=1}^{n} V_{x_i}$. Since $g \in W_{x_j}$, $(g, x) \in W_{x_j} \times V_{x_j} \subset e^{-1}(U)$. It follows that $e(g, x) = g(x) \in U$ for each $x \in K$. Thus $g(K) \subset U$ or $g \in W(K, U)$.

Since g was an arbitrary element of O, we have shown that $f \in O \subset W(K, U)$ which implies that $W(K, U)$ is a $\Theta$ neighborhood of each of its points.

Thus $W(K, U)$ is a $\Theta$ open set and hence $k \subset \Theta$.

Theorem IV.A.3. does not indicate that the k topology is the smallest admissible topology for $\mathfrak{F}$. In fact in Reference [1], Arens shows that in general there is no smallest admissible topology for $\mathfrak{F}$. However, as the next theorem shows, Arens proved that if the space X is locally compact and $T_2$ then the compact open topology on $\mathfrak{F}$ is the smallest admissible topology on $\mathfrak{F}$.

THEOREM IV.A.4. If $\mathfrak{F}$ is a set of continuous functions from the locally compact, $T_2$ space X to the space Y then the k-topology is admissible.

Proof. It must be shown that the map $e: \mathfrak{F} \times X \to Y$ is continuous with respect to the k-topology on $\mathfrak{F}$.

Let $(f, x)$ be any point of $\mathfrak{F} \times X$ and $V$ be any open neighborhood of $e(f, x) = f(x)$ in $Y$. Since $f \in \mathfrak{F}$, $f$ is continuous by hypothesis and there exists an open neighborhood $O$ of $x$ in $X$ such that $f(O) \subset V$.

Since $X$ is a locally compact, $T_2$ space the family of all closed compact neighborhoods of any point in $X$ is a base for the neighborhood system of the point. (See Kelley, Reference [2], p 146.)

Let $K$ be a closed compact neighborhood of $x$ such that $K \subset O$, then $f(K) \subset f(O) \subset V$. Thus $f \in W(K, V)$, that is $W(K, V)$ is an open neighborhood of $f$ in the k-topology.

Since $K$ is a neighborhood of $x$, $x$ belongs to the interior of $K$, $x \in K^{\circ}$. Then $W(K, V) \times K^{\circ}$ is an open set in the space $(\mathfrak{F}, k) \times X$ containing $(f, x)$.

Let $(g, y)$ be any point in $W(K, V) \times K^{\circ}$ then $g \in W(K, V)$ and $y \in K^{\circ} \subset K$. However, $g \in W(K, V)$ implies that $g(K) \subset V$ so that $g(y) \in V$ since $y \in K$.

Therefore for each $(g, y) \in W(K, V) \times K^{\circ}$, $e(g, y) = g(y) \in V$ or $e\big(W(K, V) \times K^{\circ}\big) \subset V$. Thus $e$ maps the open neighborhood $W(K, V) \times K^{\circ}$ of $(f, x)$ into $V$ which implies that $e$ is continuous at $(f, x)$.

So $e$ is continuous on $\mathfrak{F} \times X$ with respect to the k topology and the theorem is proved.

## B. Continuity With Respect to the Graph Topology

In Chapter III it was shown that rather strong conditions on the space $X$ (i.e. $T_2$, compact) are required to force equivalence of the k-topology and the graph topology on a space of continuous functions. Because of this, it might be expected that weaker conditions on the space X could result in admissibility for the graph topology. This is in fact so as the following theorem shows.

THEOREM IV.B.1. If $\mathfrak{F}$ is a space of continuous functions from the regular space $X$ to the space $Y$ then the graph topology on $\mathfrak{F}$ is admissible.

Proof. Let $(f, x)$ be any point of $\mathfrak{F} \times X$ and $V$ be any open neighborhood of $f(x)$ in $Y$. Since $f \in \mathfrak{F}$, $f$ is continuous by hypothesis and there is an open neighborhood $O$ of $x$ in $X$ with $f(O) \subset V$.

Since $X$ is regular, there is an open neighborhood $O_1$ of $x$ in $X$ with $x \in O_1 \subset \bar{O}_1 \subset O$. It follows that $f(x) \in f(O_1) \subset f(\bar{O}_1) \subset f(O) \subset V$. Since $\bar{O}_1$ is closed in $X$, $X \backslash \bar{O}_1$ is open in $X$.

Let $U = (X \backslash \bar{O}_1) \times Y \cup X \times V$ then $G(f) \subset U$ since $f(\bar{O}_1) \subset V$. Thus $f \in \mathfrak{F}_U$ and $\mathfrak{F}_U$ is an open neighborhood of $f$ in the graph topology on $\mathfrak{F}$.

The set $\mathfrak{F}_U \times O_1$ is an open neighborhood of $(f, x)$ in the space $(\mathfrak{F}, \Gamma) \times X$.

Let $(g, y)$ be any point of $\mathfrak{F}_U \times O_1$ then $G(g) \subset U$ and $y \in O_1$. Since $G(g) \subset U$ and $y \in O_1 \subset \bar{O}_1$, $(y, g(y)) \in X \times V$. Thus $e(g, y) = g(y) \in V$ and we have shown that for every point $(g, y) \in \mathfrak{F}_U \times O_1$, $e(g, y) = g(y) \in V$. This implies that $e(\mathfrak{F}_U \times O_1) \subset V$ or that $e$ is continuous at $(f, x)$ with respect to the $\Gamma$ topology on $\mathfrak{F}$ and the theorem is proved.

The following corollary adds additional information to the results obtained in Chapter III, Section B concerning comparison of the $\kappa$-topology and the graph topology.

COROLLARY IV.B.2. If $\mathfrak{F}$ is a set of continuous functions from the regular space X to the space Y, then the k-topology on $\mathfrak{F}$ is contained in the graph topology on $\mathfrak{F}$, that is $k \subset \Gamma$.

Proof. The proof follows immediately from Theorem IV.B.1 and Theorem IV.A.3.

Two additional results concerning the admissibility of the topology of uniform convergence and the sup metric topology on $\mathfrak{F}$ appear in Appendix C.

The following additional result holds regarding the separate continuity of e and the graph topology.

THEOREM IV.B.3. If $\mathfrak{F}$ is a set of continuous functions from the $T_1$ space X to the space Y then the evaluation map is separately continuous with respect to the graph topology on $\mathfrak{F}$.

Proof. If X is $T_1$ then the pointwise convergence topology is contained in the graph topology by Theorem III.A.2. Therefore e is separately continuous with respect to the graph topology by Theorem IV.A.1.

## APPENDIX A

## A BASIS FOR THE GRAPH TOPOLOGY

Let $F = Y^X$, the set of all functions from X to Y. Define $F_U = \{f \in F \mid G(f) \subset U\}$ for $U \subset X \times Y$.

LEMMA I. If $U, V \subset X \times Y$ then $F_{U \cap V} = F_U \cap F_V$.

Proof. Suppose $F_{U \cap V} = \phi$. If $f \in F_U \cap F_V$ then $G(f) \subset U$ and $G(f) \subset V$ which implies that $G(f) \subset U \cap V$ Thus $f \in F_{U \cap V}$ contradicting the assumption that $F_{U \cap V} = \phi$. Therefore if $F_{U \cap V} = \phi$ then $F_U \cap F_V = \phi$ and $F_{U \cap V} = F_U \cap F_V$.

Suppose $F_U \cap F_V = \phi$. If $f \in F_{U \cap V}$ then $G(f) \subset U \cap V$ which implies that $G(f) \subset U$ and $G(f) \subset V$. Thus $f \in F_U \cap F_V$ contradicting the assumption that $F_U \cap F_V = \phi$. Therefore if $F_U \cap F_V = \phi$ then $F_{U \cap V} = \phi$ and $F_{U \cap V} = F_U \cap F_V$.

Suppose neither $F_U \cap F_V$ nor $F_{U \cap V}$ is empty and let $f \in F_{U \cap V}$. Then $G(f) \subset U \cap V$ or $f \in F_U$ and $f \in F_V$. Thus $f \in F_U \cap F_V$ and $F_{U \cap V} \subset F_U \cap F_V$.

Let $g \in F_U \cap F_V$ then $G(g) \subset U$ and $G(g) \subset V$ or $G(g) \subset U \cap V$. Thus $g \in F_{U \cap V}$ and $F_U \cap F_V \subset F_{U \cap V}$.

Therefore in any case $F_{U \cap V} = F_U \cap F_V$.

LEMMA II. The collection $\mathcal{B} = \{F_U \mid U \text{ an open set in } X \times Y\}$ is a basis for a topology on F.

Proof. A sufficient condition for $\mathcal{B}$ to be a basis for a topology on F is: for every two members $F_U$ and $F_V$ of $\mathcal{B}$ and for each point $f \in F_U \cap F_V$ there is an $F_W \in \mathcal{B}$ such that $f \in F_W \subset F_U \cap F_V$.

Let $F_U$ and $F_V \in \mathcal{B}$ and suppose $f \in F_U \cap F_V$. Then by definition, $G(f) \subset U$ and $G(f) \subset V$ or $G(f) \subset U \cap V$. Since U and V are open in $X \times Y$, $U \cap V$ is open in $X \times Y$. Therefore $F_{U \cap V} \in \mathcal{B}$. But $G(f) \subset U \cap V$ implies that $f \in F_{U \cap V}$. By Lemma I above, $F_{U \cap V} = F_U \cap F_V$. Therefore $f \in F_{U \cap V} \subset F_U \cap F_V$ holds. Let $F_W = F_{U \cap V} \in \mathcal{B}$ and the condition assuring that $\mathcal{B}$ is a basis for a topology on F is met.

## APPENDIX B

## CONDITIONS FOR THE GRAPH OF A FUNCTION TO BE HOMEOMORPHIC TO THE DOMAIN SPACE

LEMMA I. Let f be a function from a topological space X to a topological space Y. If f is continuous, then G(f) is homeomorphic to X.

Proof. The map $p : G(f) \to X$ by $p(x, f(x)) = x$ is the required homeomorphism.

Suppose $p(x, f(x)) = p(y, f(y))$ then $x = y$ by definition of p. But then $f(x) = f(y)$ since f is a function. Therefore $(x, f(x)) = (y, f(y))$ and p is 1-1.

Suppose $x \in X$ then $(x, f(x)) \in G(f)$ and $p(x, f(x)) = x$ which implies that p is onto X.

Note that $G(f) \subset X \times Y$ and G(f) is given the subspace topology induced by the product topology on $X \times Y$. Let $p_X$ and $p_Y$ be the projections of $X \times Y$ onto X and Y respectively. Then $p_X$ and $p_Y$ are continuous since the product topology is the smallest topology on $X \times Y$ such that the projection maps are continuous.

However $p = p_X|_{G(f)}$. Therefore p is continuous since it is a restriction of a continuous map.

Consider $p^{-1}$. $p^{-1} : X \to G(f) \subset X \times Y$. Since p is onto X. Therefore by Theorem 3, p. 91 in Kelley [2], $p^{-1}$ is continuous if and only if $p_X \circ p^{-1}$ and $p_Y \circ p^{-1}$ are continuous.

Let $x \in X$ then $p_X \circ p^{-1}(x) = p_X\big(x, f(x)\big) = x$ so that $p_X \circ p^{-1}$ is the identity of the space X into itself. Thus $p_X \circ p^{-1}$ is continuous.

Also $p_Y \circ p^{-1}(x) = p_Y\big(x, f(x)\big) = f(x)$ so that $p_Y \circ p^{-1} = f$. Thus $p_Y \circ p^{-1}$ is continuous since f is continuous by hypothesis. Therefore $p^{-1}$ is continuous which proves that p is a homeomorphism of $G(f)$ onto X.

## APPENDIX C

## CONTINUITY OF THE EVALUATION MAP WITH RESPECT TO THE UNIFORM CONVERGENCE AND THE SUP METRIC TOPOLOGIES

Although the two theorems of this appendix do not involve the graph topology, the fact that they are of interest in connection with the evaluation map justifies their appearence here.

THEOREM I. If $\mathfrak{F}$ is a set of continuous functions from a topological space X to a metric space $(Y, d)$ then the sup metric topology on $\mathfrak{F}$ is admissible.

Proof. Let $(f, x) \in \mathfrak{F} \times X$ and let $V_\epsilon(f(x))$ be an open ball of radius $\epsilon > 0$ about $f(x)$ in Y. Since f is continuous, there is an open set O in X with $x \in O$ and $f(O) \subset V_{\epsilon/3}(f(x))$. Let $N_{\epsilon/3}(f)$ be a $\rho$-ball of radius $\epsilon/3$ about f, then $(f, x) \in N_{\epsilon/3}(f) \times O$ an open subset of $\mathfrak{F} \times X$.

Suppose $(g, y) \in N_{\epsilon/3}(f) \times O$ then $g \in N_{\epsilon/3}(f)$ and $y \in O$. However $g \in N_{\epsilon/3}(f)$ implies that $\rho(f, g) = \sup_{x \in X} d(f(x), g(x)) < \epsilon/3$. Thus $d(f(x), g(x)) < \epsilon/3$ for each $x \in X$ and in particular, $d(f(y), g(y)) < \epsilon/3$.

Since $y \in O$, $f(y) \in f(O) \subset V_{\epsilon/3}(f(x))$.

By the triangle inequality,

$$d\big(f(x),\, g(y)\big) \leq d\big(f(y),\, g(y)\big) + d\big(f(x),\, f(y)\big)$$

$$< \epsilon/3 + \epsilon/3 < \epsilon \ .$$

Then since $d\big(f(x),\, g(y)\big) < \epsilon$, $g(y) \in V_\epsilon\ \big(f(x)\big)$. Thus $e(g,\, y) = g(y) \in V_\epsilon\ \big(f(x)\big)$ which implies that $e\big[N_{\epsilon/3}\ (f) \times O\big] \subset V_\epsilon\ \big(f(x)\big)$ since $(g,\, y)$ was an arbitrary point of $N_{\epsilon/3}\ (f) \times O$. Therefore the open neighborhood $N_{\epsilon/3}\ (f) \times O$ of $(f,\, x)$ is mapped into $V_\epsilon\ \big(f(x)\big)$ by $e$. This implies that $e$ is continuous on $\mathfrak{F} \times X$ since $(f,\, x)$ was an arbitrary point of $\mathfrak{F} \times X$.

THEOREM II. If $\mathfrak{F}$ is a set of continuous functions from the topological space $X$ to the uniform space $(Y,\, \mathcal{U})$ then the topology of uniform convergence on $\mathfrak{F}$ is admissible.

Proof. Let $(f,\, x) \in \mathfrak{F} \times X$ and let $V\big[f(x)\big]$ be any open neighborhood of $f(x)$ in $Y$ where $V \in \mathcal{U}$. Choose $U \in \mathcal{U}$ such that $U$ is open, symmetric and $U \circ U \subset V$ then $U\big[f(x)\big]$ is also a neighborhood of $f(x)$ in $Y$. Since $f$ is continuous, $f^{-1}\Big(U\big[f(x)\big]\Big)$ is a neighborhood of $x$ in $X$.

Also $W(U)\ [f] = \Big\{g \in \mathfrak{F} \mid \big(f(x),\, g(x)\big) \in U \text{ for every } x \in X\Big\}$ is a neighborhood of $f$ in the u.c. topology on $\mathfrak{F}$.

Therefore $W(U)\ [f] \times f^{-1}\Big(U\big[f(x)\big]\Big)$ is a neighborhood of $(f,\, x)$ in $\mathfrak{F} \times X$. Claim that $e\Big(W(U)\ [f] \times f^{-1}\Big(U\big[f(x)\big]\Big)\Big) \subset V\big[f(x)\big]$. Let $(g,\, y)$ be any point of $W(U)\ [f] \times f^{-1}\Big(U\big[f(x)\big]\Big)$. Then $g \in W(U)\ [f]$ which implies that $\big(f(x),\, g(x)\big) \in U$ for each $x \in X$. Thus in particular, $\big(f(y),\, g(y)\big) \in U$. Also $y \in f^{-1}\Big(U\big[f(x)\big]\Big)$ implies that $f(y) \in U\big[f(x)\big]$ or that $\big(f(x),\, f(y)\big) \in U$.

Thus $(f(y), g(y))$ and $(f(x), f(y)) \in U$ which implies that $(f(x), g(y)) \in U \circ U \subset V$ or that $g(y) \in V[f(x)]$.

However $e(g, y) = g(y)$. Therefore $e\Big(W(U)\,[f] \times f^{-1}\big(U[f(x)]\big)\Big) \subset V[f(x)]$ and $e$ is continuous at $(f, x)$.

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