NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1361

Fuel-Optimal Retrothrust Control Programs for Propulsive Soft-Landing Maneuvers in the Presence of Drag

A. K. Bejczy

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Preface

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Abstract

A fuel-optimal retrothrust control program is demonstrated to be significantly superior to an acceleration-type control law for the terminal phase of a propulsive soft-landing mission to an atmospheric planet, using either a vertical trajectory or a gravity-turn ballistic trajectory. These trajectories are considered under conditions in which (1) the drag depends quadratically on the velocity and inverse-exponentially on the altitude, (2) it is desirable to reach zero velocity at zero altitude, and (3) the terminal value of the ballistic path angle is unspecified. The equivalent terminal-value optimal control problem (maximizing the final mass of the space vehicle), which could be termed an "Inverse" Goddard Problem, is treated by applying Pontryagin's Maximum Principle. It is proved that for both types of trajectories, the fuel-optimal retrothrust control program is always of the bang-bang type, composed of only two sections in the optimal control sequence. The fuel-optimal retrothrust control, when compared with the results of the acceleration-type control law, gives a decrease of 35% and 55% in fuel consumption for the vertical and ballistic trajectories, respectively, and a decrease of 45% and 75% in required controlled landing time for the two types of trajectories, respectively.

Fuel-Optimal Retrothrust Control Programs for Propulsive Soft-Landing Maneuvers in the Presence of Drag

I. Introduction

In this study, the terminal phase of a soft-landing mission of a space vehicle to an atmospheric planet (e.g., to Mars or back to earth) is considered. In the context of the present study, "terminal phase" means:

(1) The space vehicle, having been permanently captured by the gravitational field, is descending in the planetary atmosphere toward the planet's surface.

(2) During its atmospheric flight, the space vehicle has decelerated by some means (e.g., aeroshell, parachute, lifting maneuver) from its high-entry velocity to some low, subsonic velocity.

(3) The space vehicle has descended sufficiently close to the planet's surface, and is approaching the planet with such a velocity that to achieve a soft landing, it is necessary to initiate the final deceleration maneuver, which would be accomplished by propulsive means.

The following assumptions are made:

(1) During the final deceleration maneuver, the space vehicle would follow a vertical or gravity-turn ballistic trajectory.

(2) The space vehicle's trajectory and attitude controls are dynamically decoupled during the propulsive soft-landing maneuver.

(3) The main fuel consumption takes place in controlling the trajectory of the space vehicle during the propulsive soft-landing maneuver; i.e., in decelerating the space vehicle, from a given velocity at a given altitude to zero, or near-zero, velocity at zero altitude.

In the present study, these assumptions imply in more precise terms that only one- or two-dimensional motions of the center of gravity of the space vehicle will be considered.

The basic question which arises in connection with a soft-landing mission is the following: during the final deceleration maneuver, according to what control law should the space vehicle be guided from some given initial conditions to the prescribed terminal (landing) conditions? Selecting an appropriate control law for the final propulsive deceleration maneuver is certainly a matter of multilateral considerations. In this study, the question of an appropriate terminal control law is investigated from the point of view of fuel consumption, or more precisely, from the point of view of minimum fuel consumption.

Minimum fuel consumption of a propulsive soft lander is an important performance criterion, for several reasons. First of all, fuel consumption is a very natural cost function when searching for an optimal control law for the soft-landing problem. The performance of the soft-landing maneuver with a minimum of mass depletion is certainly desired. At the same time, minimum fuel consumption as a performance criterion does measure the necessary control effort in terms of physical forces which must be invoked to achieve a soft landing. This criterion also serves as an additional quantitative measure for judging the overall performance of a soft-landing mission. Maximum useful payload delivered to the planet's surface is a part of the overall performance of a soft-landing mission.

The question of a minimum fuel-consumption control program for a soft-landing maneuver on an atmospheric planet is essentially different from the problem of optimal thrust program for soft landing on a planet having no atmosphere; e.g., on the Moon. This question has been considered by several investigators (Ref. 1). In the case of an atmospheric planet, the presence of the gas-dynamic drag force implies a gratis, additional decelerating force; the drag, during the soft-landing maneuver, acts in the same dynamic sense as the retrothrust. The gas-dynamic drag force, however, is a strongly state-dependent, velocity- and altitude-dependent force. During a propulsive soft-landing maneuver on an atmospheric planet, the typical dynamic situation is the following: decreasing altitude implies increased drag; decreasing velocity implies decreased drag; and the retrothrust decreases the velocity and, at the same time, retards the descent to low altitudes. The retrothrust, therefore, reduces the decelerating effect of the drag force.

Reducing the decelerating effect of the drag force, however, implies increased integrated retrothrust effort (i.e., increased fuel consumption) in order to achieve a soft landing. Therefore, determining the minimum fuel-consumption burning program for a soft-landing maneuver on an atmospheric planet will necessarily imply the maximization of the decelerating effect of the drag force during the soft-landing maneuver. The time history of the drag, through its dependency on the velocity and on the altitude of the space vehicle, is implicitly controlled by the burning program of the retro-rockets.

It is noted that a fuel-optimal thrust program for propulsive soft landing on an atmospheric planet can be considered an Inverse Goddard Problem. The context of the "Original" Goddard Problem implies that the drag force always acts against the optimization of the terminal values of some of the state variables. When posing the question of minimizing the fuel consumption (or equivalently, maximizing the terminal mass) during a propulsive soft-landing maneuver in the presence of drag, the context of the Goddard Problem is actually reversed, since the drag acts for the performance criterion. The Original Goddard Problem and its different aspects are extensively treated in the scientific literature (Ref. 2 and 3). The Inverse Goddard Problem, as it is formulated in the present study however, has not received any significant attention. This problem is suggested by future space missions directed toward planetary surface exploration.

II. Pontryagin's Maximum Principle

The determination of a fuel-optimal thrust program belongs to the Mayer-type problem of the calculus of variations. Minimizing the fuel consumption is clearly equivalent to maximizing the terminal mass. In this study, this variational problem will be treated by applying Pontryagin's Maximum Principle (Refs. 3–5) as it is formulated for the Mayer-type problem. For the sake of quick reference, the fundamental equations of optimal terminal control, in the sense of Pontryagin's Maximum Principle, are briefly summarized below.

Consider the ordinary vector differential equation

$$\frac{dx}{dt} = f\,[x(t), u(t)] \tag{1}$$

describing the dynamic behavior of an autonomous system which has to be controlled, where

t = time

$x = \{x_1, \cdots, x_n\}$, n-dimensional state vector

$u = \{u_1, \cdots, u_m\}$, m-dimensional control vector; $m \leq n$

The control can be subject to the constraints

$$|u_k| \leq \beta_k$$

and there are given initial and/or terminal conditions on the state variables

$$x_i(0) = a_i \qquad x_j(\tau) = b_j \tag{2}$$

where

τ = terminal time (free or fixed)

a_i, b_j = constants

The performance criterion G, which has to be maximized, or minimized, with respect to the control vector $u(t)$ is given by the functional (which is assumed to be differentiable):

$$G = [x(\tau), \tau]; \tau = \text{terminal time} \tag{3}$$

The functional $G[x(\tau), \tau]$ describes the state variables of the control process at the end of the trajectory.

Pontryagin's Maximum (or Minimum) Principle states the following necessary condition for optimality: if the control vector is optimum, i.e., $u = u^*$ (the asterisk refers to the optimum control vector), then the Hamiltonian H obtains a constant extremum value with respect to the control vector u over the control interval. The Hamiltonian is defined by

$$H[x(t), u(t), \lambda(t)] \triangleq \langle \lambda(t), f[x(t), u(t)] \rangle \tag{4}$$

where

$\langle , \rangle$ = Euclidean inner product

$\lambda(t)$ = adjoint n-dimensional vector related to H by

$$\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x_i} \qquad i = 1, \cdots, n \tag{5}$$

In addition, there is

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial \lambda_i} = f_i[x(t), u(t)] \qquad i = 1, \cdots, n \tag{6}$$

The system of equations given by Eqs. (5) and (6) form the $2n$ Hamilton canonical equations. The additional necessary boundary conditions are determined by the transversality condition:

$$(G_t + H)\,dt + \langle G_x - \lambda, dx \rangle |_{t=\tau} = 0 \tag{7}$$

where

$$G_x \triangleq \left\{\frac{\partial G}{\partial x_1}, \cdots, \frac{\partial G}{\partial x_n}\right\}; \qquad G_t \triangleq \frac{\partial G}{\partial t}$$

It is necessary that Eq. (7) be evaluated on the terminal manifold.

The optimum design procedure is as follows: maximize H with respect to u and find $u^* = u^*(x, \lambda)$; substitute $u^*(x, \lambda)$ into the Hamilton canonical equations and solve the resulting boundary value problem for the optimum trajectory x^* and for the optimum adjoint vector λ^* subject to the boundary conditions given by Eqs. (2) and (7). By knowing $x^*(t)$ and $\lambda^*(t)$, the optimum control function $u^*(t)$ can be determined.

Hence, the optimal terminal control problem is transformed into a split boundary-value problem since, in integrating the canonical equations, some of the $2n$ boundary conditions are given on the initial manifold and some of the $2n$ boundary conditions are given on the terminal manifold.

In view of Eq. (7), it is noted that

$$H^* = 0 \tag{8}$$

when the terminal time is free and the performance-index functional G does not depend on the terminal time.

III. Dynamic Equations

In deriving the equations of motion of the space vehicle, consideration is only given to the following:

(1) The motion of the center of gravity of the space vehicle.

(2) Two-degrees-of-freedom (planar) motions.

As a convenient reference system, a trajectory-fixed coordinate system is used with its origin located at the center of gravity of the space vehicle and with unit vectors e_D and e_L parallel and perpendicular to the vehicle's motion. The path angle α is defined as the angle between the velocity vector and the local (instantaneous) horizontal (Fig. 1). Furthermore, it is assumed that the planet is nonrotating and has a quiet (nonmoving) atmosphere and

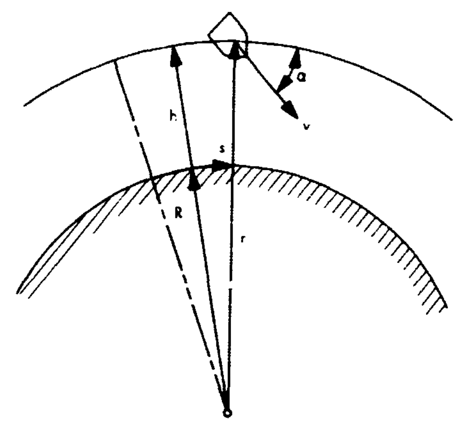

Fig. 1. State variables for the planar motion of a ballistic vehicle referred to a trajectory-fixed coordinate system

the retrothrust acts antiparallel to the velocity vector. The acting forces are shown in Fig. 2.

According to Newton's second law, the momentum balances in the tangential e_D and normal e_L directions of motion result in the following differential equations describing the time histories of velocity v and path angle α of the space vehicle:

$$\frac{dv}{dt} = \frac{\Gamma}{r^2}\sin\alpha - \frac{1}{m}(D + T) \tag{9}$$

$$\frac{d\alpha}{dt} = \left(\frac{\Gamma}{r^2} - \frac{v^2}{r}\right)\cos\alpha - \frac{L}{m} \tag{10}$$

where

m = mass of the space vehicle

D, L = gas dynamic drag and lift forces, respectively

$\frac{\Gamma}{r^2} = g$, acceleration of gravity

$\Gamma = \gamma M$; γ is the gravitational constant, M is the mass of the planet

r = distance of the space vehicle from the center of the planet

T = rocket thrust, $0 \leqq T \leqq T_{\max}$, defined by

$$T = -\frac{dm}{dt}v_e, \tag{11}$$

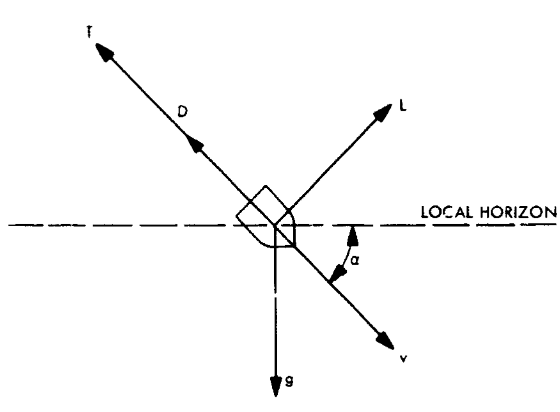

Fig. 2. Force diagram of a vehicle's retrothrust atmospheric flight

where

v_e = velocity of the exhaust gas, relative to the space vehicle; constant, $v_e > 0$

$\frac{dm}{dt}$ = mass flow rate, $\frac{dm}{dt} < 0$

The term $(v^2/r)\cos\alpha$ in Eq. (10) is a fictitious centrifugal acceleration which compensates for the curvature of the spherical planet.

The altitude h of the space vehicle above the planet's surface and the ground range s of the vehicle, measured from some reference vertical, are governed by the following kinematic relationships:

$$\frac{dh}{dt} = -v\sin\alpha \tag{12}$$

$$\frac{ds}{dt} = \frac{R}{r}v\cos\alpha \tag{13}$$

where

R = the radius of the planet.

In these calculations the following reasonable simplifying assumptions can be made:

$$g = \frac{\Gamma}{r^2} = \text{constant}; \tag{14a}$$

$$R >> h, \text{ hence } r \sim R = \text{constant} \tag{14b}$$

since consideration is given to a flight sufficiently close to the planet's surface. The fact that a pure gravity-turn ballistic flight is considered implies that

$$L = 0 \tag{14c}$$

In view of the simplifications introduced by Eqs. (14a–14c), the kinematic relationships (Eqs. 12 and 13) and the dynamic relationships (Eqs. 9 and 10) can be written

$$\dot{h} = -v\sin\alpha \tag{15}$$

$$\dot{s} = v\cos\alpha \tag{16}$$

$$\dot{\alpha} = \left(\frac{g}{v} - \frac{v}{R}\right)\cos\alpha \tag{17}$$

$$\dot{v} = g\sin\alpha - \frac{1}{m}[D(h, v) + uv_e] \tag{18}$$

where the dot means time derivative, and where u is introduced as the control force by defining

$$T \triangleq u v_e$$

In view of Eq. (11), therefore,

$$\dot{m} = -u \tag{19}$$

This means that the mass flow rate is regarded as the bounded "control force"

$$0 \leq u \leq u_{\max} \tag{20}$$

Equations (15–19) are five coupled, first-order, ordinary nonlinear differential equations describing a planar, descending, gravity-turn ballistic flight of a space vehicle in planetary atmosphere with reasonable accuracy, provided that the underlying assumptions are valid.

In the case of a vertical descent trajectory ($\alpha = 90°$), Eqs. (15–18) are reduced to:

$$\dot{h} = -v \tag{21}$$

$$\dot{v} = g - \frac{1}{m}[D(h, v) + uv_e] \tag{22}$$

Equation (19) remains unchanged.

In order to complete the dynamic description of the problem, the form of the drag force $D(h, v)$ is specified by the following functional model:

$$D(h, v) = Kv^2 \exp(-bh) \tag{23}$$

where

$K = (m_o/2\Delta)(\rho_o) =$ constant

$\Delta =$ ballistic coefficient; $\Delta = m_o/C_D A$

$m_o =$ reference mass of the space vehicle

$\rho_o =$ ground level atmospheric density on the planet

$b =$ inverse atmospheric scale factor constant (reflecting the shape of the atmospheric density distribution)

The exponential atmospheric-density distribution is valid for an isothermal atmosphere in hydrostatic equilibrium; it is a reasonable description of the atmospheric-density distribution in low altitudes. The quadratic dependence of the drag on the velocity is the usual aerodynamic description of the drag force in the subsonic region.

IV. Fuel-Optimal Thrust Program for a Vertical Soft-Landing Trajectory

A. Formulation of the Optimal Control Problem

The following notations for the dependent variables are introduced:

$$x_1 \triangleq h, \text{altitude}$$

$$x_2 \triangleq v, \text{velocity}$$

$$x_3 \triangleq m, \text{mass}$$

The differential equations, Eqs. (19), (21), and (22), describing the system to be controlled can then be rewritten

$$\dot{x}_1 = -x_2 \tag{24}$$

$$\dot{x}_2 = g - \frac{1}{x_3}[D(x_1, x_2) + v_e u] \tag{25}$$

$$\dot{x}_3 = -u \tag{26}$$

where $0 \leq u \leq u_{\max}$ is the bounded control.

The prescribed end conditions are as follows:

At $t = 0$:

$$\left.\begin{aligned} x_1 &= a_1 \quad (27a)\\ x_2 &= a_2 \quad (27b)\\ x_3 &= a_3 \quad (27c)\end{aligned}\right\} \text{given values} > 0$$

At $t = \tau$, where the terminal time τ is free:

$$x_1 = 0 \tag{27d}$$

$$x_2 = 0 \tag{27e}$$

$$x_3 = \text{free}; > 0 \tag{27f}$$

The optimal control problem is the following: Among all admissible controls u, find the one that maximizes the terminal value of x_3; i.e., that maximizes the performance criterion functional

$$G = x_3(\tau) \tag{28}$$

subject to the prescribed end conditions given by Eqs. (27a–27f). It is assumed that there is sufficient control capability to meet the given terminal conditions.

The Hamiltonian, defined by Eq. (4), for this problem is

$$H = -\lambda_1 x_2 + \lambda_2 \left[g - \frac{1}{x_3} D(x_1, x_2) \right] - \left[\lambda_2 \frac{v_e}{x_3} + \lambda_3 \right] u \tag{29}$$

The adjoint variables (λ_i) are nontrivial solutions of the system of adjoint equations, defined by Eq. (5), which now become:

$$\dot{\lambda}_1 = \frac{\lambda_2}{x_3} \frac{\partial D}{\partial x_1} \tag{30}$$

$$\dot{\lambda}_2 = \lambda_1 + \frac{\lambda_2}{x_3} \frac{\partial D}{\partial x_2} \tag{31}$$

$$\dot{\lambda}_3 = -\frac{\lambda_2}{x_3^2} \left[D(x_1, x_2) + v_e u \right] \tag{32}$$

The transversality condition, Eq. (7), gives

$$\lambda_3(\tau) = 1 \tag{33a}$$

and

$$H^* = 0 \tag{33b}$$

over the control interval.

Let us define

$$\sigma \triangleq \lambda_2 \left(\frac{v_e}{x_3} \right) + \lambda_3 \tag{34}$$

which is termed the switching function. Then, by formally performing

$$\frac{\partial H}{\partial u} = 0$$

it can be seen that the Hamiltonian is maximized by

$$u = \begin{cases} u_{\max}, \text{ whenever } \sigma < 0 \\ 0, \text{ whenever } \sigma > 0 \end{cases} \tag{35}$$

which is a bang-bang type of control. When

$$\sigma \equiv 0 \tag{36}$$

on a finite, closed-time interval, then the optimal control is indeterminate. This means that variable-thrust subarcs can be parts of the optimal control program when Eq. (36) is realized.

The condition expressed by Eq. (36) is called the singularity condition. To obtain a complete answer to the optimization problem, one has to investigate whether the singularity condition can hold on any finite, closed-time interval (Ref. 6). This is generally the case when the control appears linearly in the Hamiltonian.

B. The Singularity Condition and the General Behavior of the Switching Function

When the singularity condition $\sigma \equiv 0$ holds on a finite, closed-time interval, then, in view of the Hamiltonian in Eqs. (29) and (33b), there also is

$$\eta \triangleq -\lambda_1 x_2 + \lambda_2 \left[g - \frac{1}{x_3} D(x_1, x_2) \right] = 0 \tag{37}$$

The function σ is the collected coefficients of the control u as it appears in the Hamiltonian, while the function η is constituted by the remaining terms of the Hamiltonian $H^* = 0$.

Differentiating σ with respect to the time and combining the result with the relevant canonical equations (26), (31), and (32), we obtain

$$\lambda_1 v_e + \lambda_2 \frac{1}{x_3} \left[v_e \frac{\partial D}{\partial x_2} - D(x_1, x_2) \right] = 0 \tag{38}$$

Equations (37) and (38) form a homogeneous, linear, algebraic system for the two unknowns λ_1 and λ_2 along the singular optimal subarcs, if any.

A nontrivial solution of Eqs. (37) and (38) exists if, and only if, the coefficient determinant of these two equations vanishes:

$$\begin{vmatrix} -x_2 & \left(g - \dfrac{D}{x_3} \right) \\ v_e & \dfrac{1}{x_3} \left(v_e \dfrac{\partial D}{\partial x_2} - D \right) \end{vmatrix} = 0 \tag{39}$$

Solving the determinant, we obtain

$$x_3 g = D \left(1 + \frac{x_2}{v_e} \right) - x_2 \frac{\partial D}{\partial x_2} \tag{40}$$

A fundamental relation is established by Eq. (40) in terms of the state variables which must hold along the singular optimal subarcs, if any. To state the possibility of existence of singular optimal subarcs, it is necessary to investigate whether or not Eq. (40) can hold physically.

Since the mass x_3 and the acceleration of gravity g are positive quantities, we have $x_3 g > 0$; it must be true, therefore, that

$$\left[D\left(1+\frac{x_2}{v_e}\right)-x_2\frac{\partial D}{\partial x_2}\right]>0 \qquad (41)$$

Assuming the following functional form for the drag force D,

$$D(x_1,x_2)=K\,x_2^n f(x_1) \qquad (42)$$

where $K>0$, $x_2>0$, $f(x_1)>0$ (that is, $D>0$), we obtain

$$x_2\frac{\partial D}{\partial x_2}=nKx_2^n\,f(x_1)=nD \qquad (43)$$

Equations (42) and (43), substituted into Inequality (41), yield

$$\left[\left(1+\frac{x_2}{v_e}\right)-n\right]>0 \qquad (44)$$

Since $x_2 < v_e$ (as a matter of fact, $x_2 << v_e$), and since $n = 2$, Eq. (23) is the adopted model for the drag force, inequality (41) or (44) can never be satisfied. This fact rules out the possibility of singular optimal control. The fuel-optimal thrust program for a vertical soft-landing maneuver, therefore, is of the bang-bang type given by Eq. (35).

The possibility that variable-thrust subarcs can be part of the fuel-optimal thrust program have been ruled out; the remaining problem is the general behaviour of the switching function σ. More specifically, how many switchings of $u=0$ and $u=u_{max}$ constitute the fuel-optimal thrust program? For physical reasons, it is obvious that the last two sections of the fuel-optimal control program have to be in the following sequence:

$$u^*=\{\cdots,0,u_{max}\} \qquad (45)$$

In order to achieve the prescribed terminal conditions, the fuel-optimal thrust program must end with u_{max}. The question is whether it is possible to have another $u=u_{max}$ section in the optimal control sequence prior to the next to the last section, which is $u=0$. To have a section with $u=u_{max}$ preceding a section with $u=0$, it is necessary that the switching function σ attains at least one maximum point at a time t', $\dot{\sigma}(t')=0$, when $\sigma>0$ (which implies $u=0$ at that time) since σ, being made up of absolutely continuous functions, is itself an absolutely continuous function of time, as shown in Fig. 3.

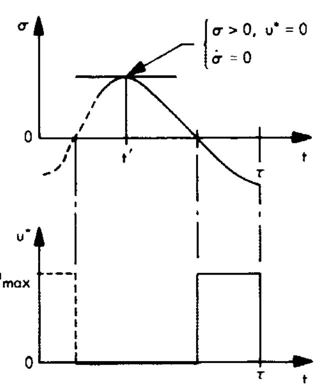

Fig. 3. A nonsingular switching function and corresponding fuel-optimal control

Now, when

$$\sigma=\lambda_2\left(\frac{v_e}{x_3}\right)+\lambda_3>0 \qquad (46)$$

then $H^*=0$ implies (since $u^*=0$)

$$-\lambda_1 x_2+\lambda_2\left(g-\frac{1}{x_3}D\right)=0 \qquad (47)$$

If $\dot{\sigma}=0$, then it implies (since $\dot{x}_3=0$)

$$\lambda_1 v_e+\frac{\lambda_2}{x_3}\left(v_e\frac{\partial D}{\partial x_2}-D\right)=0 \qquad (48)$$

The conditions for maximum $\sigma(\dot{\sigma}=0)$ when $\sigma>0$, as expressed by Eqs. (47) and (48), could be satisfied at an isolated time point t' in four different ways, as follows:

(1) Case 1. Eqs. (47) and (48) could be satisfied by

$$\begin{vmatrix} -x_2 & \left(g-\dfrac{D}{x_3}\right) \\ v_e & \dfrac{1}{x_3}\left(v_e\dfrac{\partial D}{\partial x_2}-D\right)\end{vmatrix}=0 \qquad (49)$$

yielding

$$x_3 g = D\left(1 + \frac{x_2}{v_e}\right) - x_2 \frac{\partial D}{\partial x_2} \tag{50}$$

However, Eq. (50) cannot be satisfied for the same physical reasons outlined in the analysis of the singularity condition. Consequently, Case 1 must be ruled out.

(2) Case 2. Eqs. (47) and (48) could be satisfied by

$$\lambda_1(t') = 0 \text{ with } \lambda_2(t') \neq 0 \tag{51}$$

with the result that Eqs. (52a) and (52b) must be true:

$$g - \frac{D}{x_3} = 0 \tag{52a}$$

$$v_e \frac{\partial D}{\partial x_2} - D = 0 \tag{52b}$$

simultaneously. However, Eq. (52b) can never be satisfied, since, according to the adopted model for the drag force D, we have

$$v_e \frac{\partial D}{\partial x_2} - D = \left(\frac{2v_e}{x_2} - 1\right) D > 0 \tag{53}$$

This inequality is always true, since $v_e > x_2$. Consequently, Case 2 also must be ruled out.

(3) Case 3. Eqs. (47) and (48) could be satisfied by

$$\lambda_2(t') = 0 \text{ with } \lambda_1(t') \neq 0 \tag{54}$$

This case also must be ruled out since x_2 and v_e are always different from zero.

(4) Case 4. Eqs. (47) and (48) could be satisfied by

$$\lambda_1(t') = \lambda_2(t') = 0 \tag{55}$$

The condition given by Eq. (55) does not imply a trivial solution since, in view of Eq. (46), $\lambda_3(t') \neq 0$. This case is also ruled out by examining the second derivative of σ, since at maximum σ we must have $\ddot{\sigma}(t') < 0$. By recalling that $\dot{x}_3 = 0$ when $u = 0$, we have

$$\ddot{\sigma} = \dot{\lambda}_1 v_e + \frac{\dot{\lambda}_2}{x_3}\left(v_e \frac{\partial D}{\partial x_2} - D\right) + \frac{\lambda_2}{x_3}\frac{d}{dt}\left(v_e \frac{\partial D}{\partial x_2} - D\right) \tag{56}$$

Equation (55), in view of the adjoint Eqs. (30) and (31), would imply that $\dot{\lambda}_1(t') = \dot{\lambda}_2(t') = 0$. Consequently, $\ddot{\sigma} = (t') = 0$ if $\lambda_1(t') = \lambda_2(t') = 0$, which contradicts the necessary condition; that is, $\ddot{\sigma}(t') < 0$, if σ has a maximum.

Having ruled out the possibility of the existence of maximum points of the switching function, when $\sigma > 0$ it means that there is no way to realize a $u = u_{max}$ section prior to the next to the last section (which is $u = 0$) in the optimal control sequence. This proves that the fuel-optimal thrust program for a vertical soft-landing trajectory, under the specified gas-dynamic drag conditions, is a bang-bang type of control composed of only two sections in the optimal control sequence

$$u^* = \{0, u_{max}\} \tag{57}$$

It is interesting to note that the result provided by Eq. (57) is in sharp contrast to the fuel-optimal thrust program for the equivalent Original Goddard Problem where the thrust is an accelerating force and the drag is a position- and velocity-dependent decelerating force; i.e., sending a rocket upward, vertically, from zero altitude and zero velocity to a prescribed altitude and velocity with a minimum amount of fuel consumption, under the influence of drag. Examining the equivalent singularity condition in that case yields the following relation, which must be compared with Eq. (40):

$$x_3 g = D\left(\frac{x_2}{v_e} - 1\right) + x_2 \frac{\partial D}{\partial x_2} \tag{58}$$

This relation can be physically realized for the adopted model drag force since, in view of Eq. (43) with $n = 2$, it yields for Eq. (58)

$$x_3 g = D\left(\frac{x_2}{v_e} + 1\right) \tag{59}$$

where it is always true that

$$D\left(\frac{x_2}{v_e} + 1\right) > 0 \tag{60}$$

This means that variable-thrust subarcs can be part of the fuel-optimal thrust program for the equivalent Original Goddard Problem (Ref. 7). This certainly contrasts the character of the fuel-optimal thrust program, Eq. (57), for the Inverse Goddard Problem where the thrust, as well as the drag, is a decelerating force.

C. Numerical Results

Having determined the character of the fuel-optimal thrust program for vertical soft-landing trajectories under the influence of drag, the optimization problem is reduced to solving the six canonical equations, Eqs. (30–32) and (34–36) with the prescribed end conditions which are specified partly at the beginning and partly at the end of the trajectory. More specifically, three conditions are given at $t = 0$: $x_1(0)$, $x_2(0)$, $x_3(0)$; and three conditions are given at $t = \tau$: $x_1(\tau)$, $x_2(\tau)$, $\lambda_3(\tau)$. The terminal time τ is free. Since, in this case, there is no analytical solution to the problem, we must rely on numerical integration by trial and error. That is, one must guess the missing initial conditions, integrate the canonical equations, and iterate the guessing and integrating procedure so long as the prescribed terminal conditions are not satisfied. This is usually a difficult way to solve the problem. Fortunately, in the present case, we can ease the guessing–iterating procedure by observing that at the (free) terminal time τ there may be not just three but five fixed conditions for the six canonical equations. We have

$$x_1(\tau) = 0, \qquad x_2(\tau) = 0, \qquad \lambda_3(\tau) = 1 \tag{61}$$

and, since $D(\tau) = 0$ and $u^*(\tau) = u_{\max}$, the Hamiltonian $H^* = 0$, evaluated at τ, yields

$$\lambda_2(\tau) = \frac{x_3(\tau)\, u_{\max}}{x_3(\tau)\, g - v_e u_{\max}} \tag{62}$$

Thus, if a proper value is selected for $x_3(\tau)$ = terminal mass, there will be two more fixed-end conditions $[x_3(\tau)$ and $\lambda_2(\tau)]$ in addition to the three end conditions given by Eq. (61). Therefore, when integrating the six canonical equations in backward time we had to guess only one missing condition. The guessing–iterating procedure for the missing value of $\lambda_1(\tau)$, however, actually becomes a tracking of some initial manifolds which are consistent with the given terminal conditions in the sense of the fuel-optimal retrothrust control program specified by Eq. (57). It is also noted that manipulating with only one open-end condition, in solving the canonical equations in backward time, is completely consistent with the posed optimization problem; in searching for the fuel-optimal thrust program, only one condition is left open: that is, the final mass.

When integrating the canonical equations in forward time, we can also find simplifications for guessing the missing initial conditions. If it is assumed, for example, that the given initial conditions are located in that control region where $u^* = 0$, the Hamiltonian, $H^* = 0$, evaluated at $t = 0$, yields:

$$\lambda_1(0) = \frac{\lambda_2(0)}{x_2(0)\, x_3(0)} \left[g x_3(0) - D(0) \right] \tag{63}$$

Since $u^* = 0$ implies $\sigma > 0$, there also is a bound on $\lambda_3(0)$:

$$\lambda_3(0) > -\lambda_2(0) \frac{v_e}{x_3(0)} \tag{64}$$

As an example, the fuel-optimal thrust program for a propulsive vertical soft-landing trajectory to Mars has been computed with the following end conditions:

At $t = 0$: altitude = 20000 ft
velocity = 900 ft/s
mass = 1000 lbm = 31.08 slugs

At $t = \tau$: altitude = 0
velocity = 0

In the calculations, the following Martian parameters were applied:

$$g = 12.3 \text{ ft/s}^2$$
$$\rho_0 = 1.32\ 10^{-5} \text{ slugs/ft}^3$$
$$b = 2.15\ 10^{-5} \text{ ft}^{-1}$$

(The values of ρ_o and b correspond to the VM-7 atmosphere.[1])

For the characteristics of the retrorocket, it has been assumed that

$$I_{sp} = 300 \text{ lbf-s/lbm}$$
$$T_{\max} = 1500 \text{ lbf}$$

These are equivalent to

$$v_e = 9652 \text{ ft/s}$$
$$u_{\max} = \frac{T}{I_{sp}} = 0.1554 \text{ slugs/s}$$

For the ballistic coefficient, the following was used:

$$\Delta = \frac{m_o}{C_D A} = 0.3 \text{ slug/ft}$$

[1]Internal communication, *1973 Voyager Capsule System Constraints and Requirements Document*, Jet Propulsion Laboratory, January 1, 1967.

The specified end conditions and parameter values were taken from JPL sources[2,3] where they were applied as typical values for a *Voyager*-type Martian propulsive soft-landing problem. In these documents the following feedback control law (acceleration-type thrust program) was applied for the soft-landing problem on Mars:

$$T = m\left(g + \frac{v^2}{2h}\right) \tag{65}$$

This corresponds to a mass flow rate

$$u = \frac{m}{v_e}\left(g + \frac{v^2}{2h}\right) \tag{66}$$

The Acceleration Type Control Law (ATCL) is obtained in the following manner:[2] In the case of a vertical descent in vacuum the equations of motion for a constant thrust and constant acceleration of gravity are

$$\dot{y} = -(a - g)\,t + v \tag{67a}$$

$$y = -\frac{1}{2}(a - g)\,t^2 + vt \tag{67b}$$

For a soft landing, the boundary conditions on these equations are $\dot{y} = 0$ when $y = h$. For these conditions Eqs. (67a) and (67b) result in

$$a = g + \frac{2v}{t} - \frac{2h}{t^2} \tag{67c}$$

$$t = \frac{v}{a - g} \tag{67d}$$

Solving these equations for the acceleration a yields

$$a = g + \frac{v^2}{2h} \tag{67e}$$

Through the command for acceleration a as given in Eq. (67e), a soft landing is achieved.

The numerical results obtained for the fuel-optimal thrust program are displayed in Fig. 4. For the sake of comparison, the results obtained by applying the ATCL, given in Eq. (66), are also depicted on the same figure. The adjoint variables λ_1, λ_2, λ_3 and the switching function σ are depicted on Fig. 5. The basic results are presented in Table 1.

Table 1. Comparison of results for the fuel-optimal thrust program and the ATCL for a vertical descent

Program	Fuel consumption, Δm, slugs	Difference, %	Required controlled time for soft landing, s	Difference, %
Fuel-optimal	3.12	−35	32.5	−45
ATCL	4.23		56.5	

It must be emphasized that the fuel-optimal control program requires a much briefer controlled time for a soft landing than the ATCL.

This is interesting since no minimum time requirement in the optimal control problem was posed; the terminal time was merely left open.

It must also be observed that from the point of view of implementation, the fuel-optimal thrust program has the advantage of being a one-level constant-thrust program, while the ATCL provides a variable-thrust program.

The difference in fuel consumption for the two thrust programs can be best explained by examining the time histories of the drag force during the period of the two types of control (Fig. 3d). In the case of a fuel-optimal thrust program, the drag increases in the first 12.5 s while the retroengine is still *off*. The reason for increased drag is obvious if one examines Figs. 4a and 4b; during that period of time the velocity remains practically constant, but the space vehicle flies to more and more dense atmospheric regions.

In the case of the ATCL, however, the drag decreases during the entire period of control. When the retroengine is being turned *on* in the fuel-optimal control program, the drag is about 3 times greater than the corresponding value of drag in the ATCL. This amply illustrates that minimizing the fuel consumption implies the maximization of the deceleration effect of the drag force during the soft-landing maneuver; the time history of the drag force, through its dependency on the velocity and altitude, is implicately controlled by the burning program of the retrorockets.

In the numerical calculations, the pluming effects, that is, the change in aerodynamics due to the retroengine's

[2]Internal communication, *Propulsive Lander Control System Interim Report*, April 18, 1967.

[3]Internal communication, *Voyager Standardized Soft Lander Report Outline*, Jet Propulsion Laboratory, August 15, 1966.

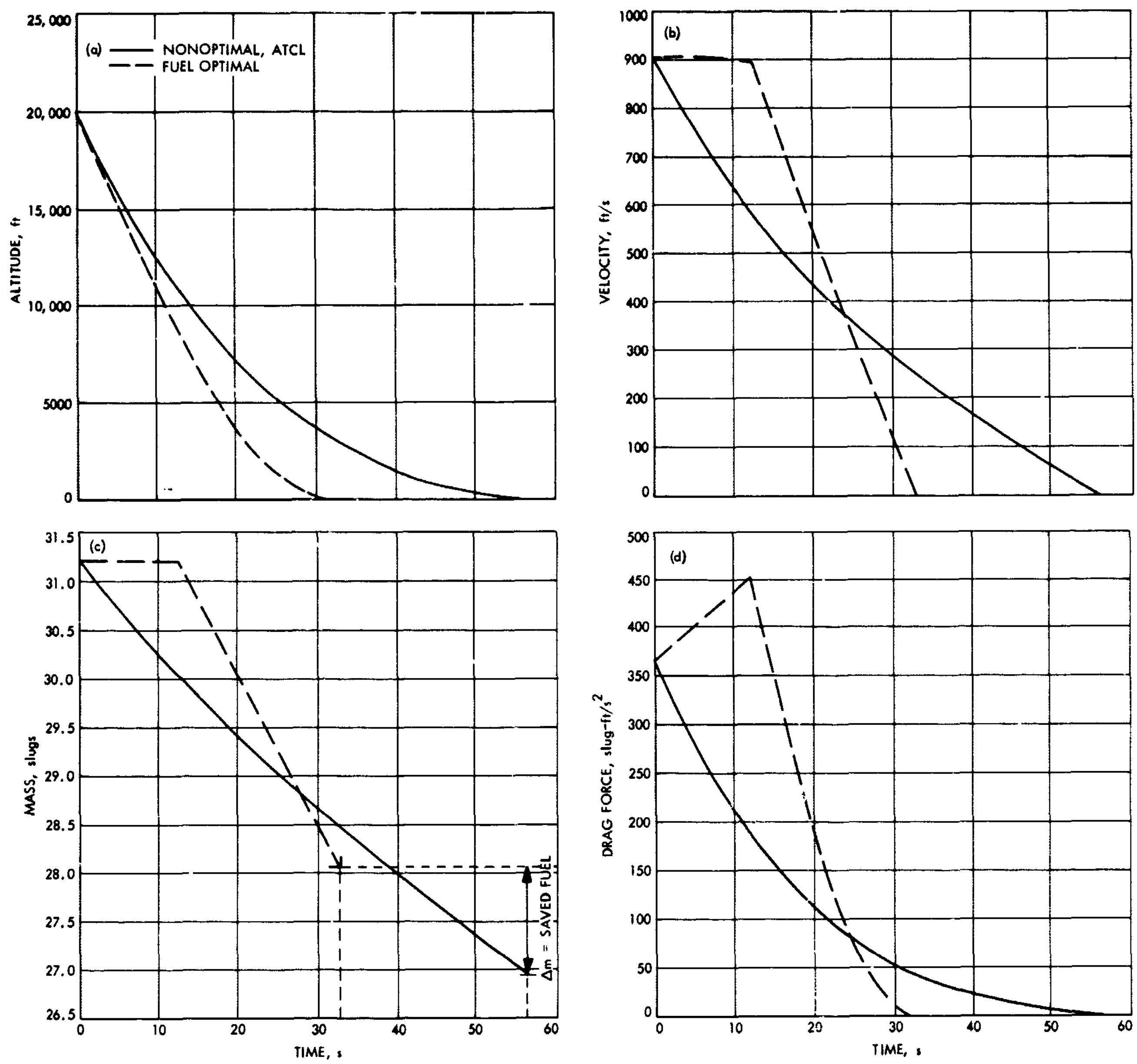

Fig. 4. Fuel-optimal and nonoptimal (ATCL) time histories of the state variables and of the atmosp[illegible] drag of a vehicle on a *vertical* soft-landing trajectory

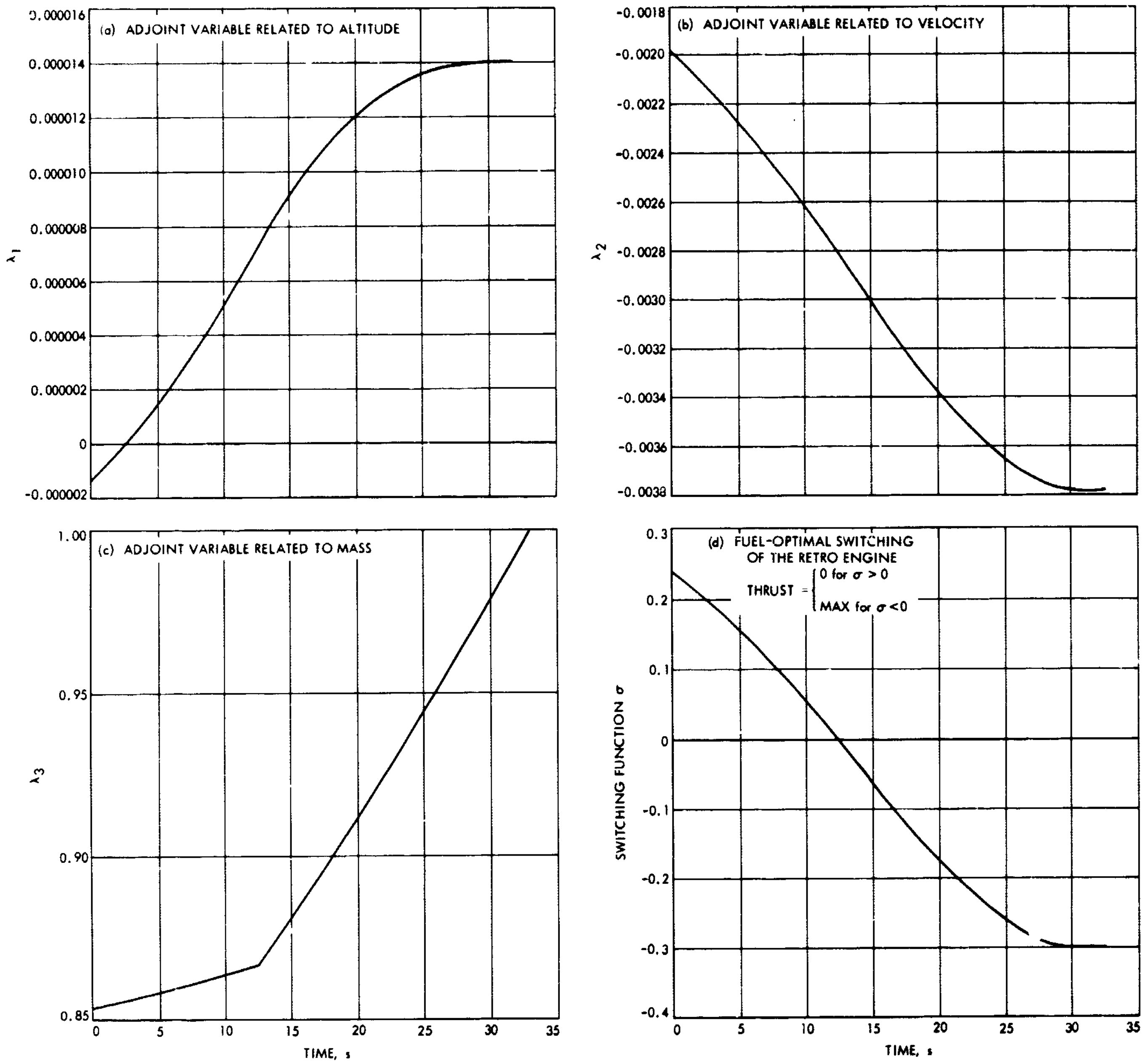

Fig. 5. Time histories of the system of adjoint variables and of the switching function for a fuel-optimal *vertical* soft-landing trajectory

exhaust plumes, were not considered since the quantitative description of these effects is not well known.

V. Fuel-Optimal Thrust Program for a Ballistic Soft-Landing Trajectory

A. Formulation of the Optimal Control Problem

The following notations for the dependent variables are introduced:

$x_1 \triangleq h$, altitude

$x_2 \triangleq s$, ground range

$x_3 \triangleq \alpha$, path angle

$x_4 \triangleq v$, velocity

$x_5 \triangleq m$, mass

The differential equations, Eqs. (15–19), describing the system to be controlled can then be rewritten

$$\dot{x}_1 = -x_4 \sin x_3 \tag{68}$$

$$\dot{x}_2 = x_4 \cos x_3 \tag{69}$$

$$\dot{x}_3 = \left(\frac{g}{x_4} - \frac{x_4}{R}\right) \cos x_3 \tag{70}$$

$$\dot{x}_4 = g \sin x_3 - \frac{1}{x_5}\left[D(x_1, x_4) + v_e u\right] \tag{71}$$

$$\dot{x}_5 = -u \tag{72}$$

where u is the bounded control: $0 \leq u \leq u_{max}$.

The prescribed end conditions are as follows:

At $t = 0$:

$$\left.\begin{array}{ll} x_1 = a_1 & (73a) \\ x_2 = a_2 & (73b) \\ x_3 = a_3 & (73c) \\ x_4 = a_4 & (73d) \\ x_5 = a_5 & (73e) \end{array}\right\} \text{given values} > 0$$

At $t = \tau$ (terminal time, τ free):

$$x_1 = 0 \tag{73f}$$

$$x_2 = \text{free}, > 0 \tag{73g}$$

$$x_3 = \text{free}, 0 < x_3(\tau) \leq \pi/2 \tag{73h}$$

$$x_4 = 0 \tag{73i}$$

$$x_5 = \text{free}, > 0 \tag{73j}$$

The optimal control problem is now the following: among all admissible controls u, find the one that maximizes the terminal value of x_5, that is, that maximizes the performance criterion functional

$$G = x_5(\tau) \tag{74}$$

subject to the prescribed end conditions, Eqs. (73a–73j). It is assumed that there is sufficient control capability to meet the prescribed terminal conditions.

As defined by Eq. (4), the Hamiltonian for this problem becomes

$$H = -\lambda_1 x_4 \sin x_3 + \lambda_2 x_4 \cos x_3 + \lambda_3 \left(\frac{g}{x_4} - \frac{x_4}{R}\right) \cos x_3 + \lambda_4 \left[g \sin x_3 - \frac{1}{x_5} D(x_1, x_4)\right] - \lambda_4 \left(\frac{v_e}{x_5} + \lambda_5\right) u \tag{75}$$

The adjoint variables λ_i are nontrivial solutions of the system of adjoint equations, defined by Eq. (5), which now become

$$\dot{\lambda}_1 = \frac{\lambda_4}{x_5} \frac{\partial D}{\partial x_1} \tag{76}$$

$$\dot{\lambda}_2 = 0 \tag{77}$$

$$\dot{\lambda}_3 = \lambda_1 x_4 \cos x_3 + \lambda_2 x_4 \sin x_3 + \lambda_3 \left(\frac{g}{x_4} - \frac{x_4}{R}\right) \sin x_3 - \lambda_4 g \cos x_3 \tag{78}$$

$$\dot{\lambda}_4 = \lambda_1 \sin x_3 - \lambda_2 \cos x_3 + \lambda_3 \left(\frac{g}{x_4^2} + \frac{1}{R}\right) \cos x_3 + \frac{\lambda_4}{x_5} \frac{\partial D}{\partial x_4} \tag{79}$$

$$\dot{\lambda}_5 = -\frac{\lambda_4}{x_5^2}\left[D(x_1, x_4) + v_e u\right] \tag{80}$$

The transversality condition, Eq. (7), gives

$$\lambda_2(\tau) = 0 \tag{81a}$$

$$\lambda_3(\tau) = 0 \tag{81b}$$

$$\lambda_5(\tau) = 1 \tag{81c}$$

and

$$H^* = 0 \tag{81d}$$

over the control interval.

In view of the terminal condition $\lambda_2(\tau) = 0$, Eq. (77) yields

$$\lambda_2 = 0 \tag{82}$$

over the entire control interval.

Let us define

$$\sigma \triangleq \lambda_4\left(\frac{v_e}{x_5}\right) + \lambda_5 \tag{83}$$

which again is termed a switching function. Then, by formally performing

$$\frac{\partial H}{\partial u} = 0$$

we maximize the Hamiltonian by

$$u = \begin{cases} u_{max}, \text{ whenever } \sigma < 0 \\ 0, \text{ whenever } \sigma > 0 \end{cases} \tag{84}$$

which is a bang-bang type of control.

When

$$\sigma \equiv 0 \tag{85}$$

on a finite, closed-time interval, then the optimal control is indeterminate; this implies that variable-thrust subarcs can be part of the optimal control program.

To obtain a complete answer to the optimization problem, it is necessary to determine whether the singularity condition, Eq. (85), can hold on any finite, closed-time interval.

B. The Singularity Condition and the General Behavior of the Switching Function

When the singularity condition, Identity (85), holds on a finite, closed-time interval, then in view of the Hamiltonian, Eqs. (75) and (81d), and taking into account Eq. (82), we also have

$$\eta = -\lambda_1 x_4 \sin x_3 + \lambda_3\left(\frac{g}{x_4} - \frac{x_4}{R}\right)\cos x_3$$
$$+ \lambda_4\left[g \sin x_3 - \frac{1}{x_5}D(x_1, x_4)\right] = 0 \tag{86}$$

The switching function σ is the collected coefficient of the control u as it appears in the Hamiltonian, while the function η is identical with the remaining terms of the Hamiltonian $H^* = 0$ with $\lambda_2 = 0$.

Differentiating σ with respect to the time and combining the result with the relevant canonical equations, Eqs. (72), (79), and (80), we obtain for the singular control domain, if any (where, in view of Eq. (85), all derivatives of σ must vanish),

$$\dot{\sigma} = \lambda_1 \sin x_3 + \lambda_3\left(\frac{g}{x_4^2} + \frac{1}{R}\right)\cos x_3$$
$$+ \lambda_4 \frac{1}{x_5}\left[\frac{\partial D}{\partial x_4} - \frac{1}{v_e}D(x_1, x_4)\right] = 0 \tag{87}$$

Equations (86) and (87) are two homogeneous, linear, algebraic equations for the three unknowns λ_1, λ_3, and λ_4. A third, independent relation can be obtained by forming the second derivative of σ, which also must vanish, and combining the result with the relevant canonical equations. It is noted that $\ddot{\sigma} = 0$ necessarily yields a homogeneous, linear algebraic relation for λ_1, λ_3, and λ_4, since λ_5 does not appear in the canonical equations and $\lambda_2 = 0$; furthermore, $\ddot{\sigma}$ is linear in the derivatives of the adjoint variables λ_i. After some algebra, we obtain, for $\ddot{\sigma} = 0$,

$$
\lambda_1 \left[2g \frac{\cos^2 x_3}{x_4} + \frac{1}{x_5} \frac{\partial D}{\partial x_4} \sin x_3 - \frac{1}{x_5 v_e} D \sin x_3 \right] + \lambda_3 \cos x_3 \left[-2g^2 \frac{\sin x_3}{x_4^3} + 2g \frac{D + v_e u}{x_4^3 x_5} + \frac{g}{x_4^2 x_5} \frac{\partial D}{\partial x_4} \right.
$$

$$
\left. - \frac{gD}{v_e x_4^2 x_5} + \frac{1}{R x_5} \frac{\partial D}{\partial x_4} - \frac{D}{R v_e x_5} \right] + \lambda_4 \left[\frac{\sin x_3}{x_5} \frac{\partial D}{\partial x_1} - \frac{g^2 \cos^2 x_3}{x_4^2} - \frac{g \cos^2 x_3}{R} - g \sin x_3 \left(\frac{1}{v_e x_5} \frac{\partial D}{\partial x_4} - \frac{1}{x_5} \frac{\partial^2 D}{\partial x_4^2} \right) \right.
$$

$$
\left. + \frac{1}{x_5^2} \left(\frac{\partial D}{\partial x_4} \right)^2 + \frac{D + v_e u}{x_5} \left(\frac{1}{v_e x_5} \frac{\partial D}{\partial x_4} - \frac{1}{x_5} \frac{\partial^2 D}{\partial x_4^2} \right) - \frac{D}{v_e x_5^2} \frac{\partial D}{\partial x_4} + \frac{u}{x_5^2} \left(\frac{\partial D}{\partial x_4} - \frac{D}{v_e} \right) - \frac{x_4 \sin x_3}{x_5} \left(\frac{\partial^2 D}{\partial x_1 \partial x_4} - \frac{1}{v_e} \frac{\partial D}{\partial x_1} \right) \right] = 0 \tag{88}
$$

A homogeneous, linear algebraic system is formed by Eqs. (86–88) in the three unknowns λ_1, λ_3, and λ_4. To have a nontrivial solution for the three unknowns, the system's coefficient determinant must vanish. Note that the coefficients of Eqs. (86–88) are in terms of the state variables and of the control u. Consequently, the condition of the vanishing of the determinant will necessarily be expressed in a synthetic form; i.e., in terms of the state variables, and eventually in terms of the control. If the resulting expression will explicitly contain the control u, then the condition of the vanishing determinant immediately yields an expression for the singular control u_{sing} in terms of the state variables. (Provided that the resulting expression will give such values that are consistent with the imposed constraint on the control.)

Solving the coefficient determinant of the homogeneous, linear algebraic system for λ_1, λ_3, and λ_4, given by Eqs. (86–88), and after extensive algebraic manipulations, we obtained the following expression:

$$
u_{sing} = \frac{1}{v_e} \left(\frac{Q_1 + Q_2 + Q_3}{Q_4} - D \right) \tag{89}
$$

where

$$
Q_1 = D g x_5 \sin x_3 \left[\frac{x_4}{v_e} (ctg^2 x_3 - 1) - \frac{1}{\sin^2 x_3} \right] + (g x_5 \sin x_3)^2 \tag{90}
$$

$$
Q_2 = g x_5 \sin x_3 \left[x_4^2 \frac{\partial^2 D}{\partial x_4^2} - x_4 \frac{\partial D}{\partial x_4} \left(ctg^2 x_3 - 1 + \frac{x_4}{v_e} \right) + \frac{x_4^2}{g} \frac{\partial D}{\partial x_1} \left(1 + \frac{x_4}{v_e} \right) - \frac{x_4^3}{g} \frac{\partial^2 D}{\partial x_1 \partial x_4} \right] \tag{91}
$$

$$
Q_3 = g x_5 \sin x_3 ctg^2 x_3 \frac{x_4^2}{gR} \left[x_4 \frac{\partial D}{\partial x_4} - D \left(1 + \frac{x_4}{v_e} \right) \right] \tag{92}
$$

$$
Q_4 = x_4^2 \frac{\partial^2 D}{\partial x_4^2} + x_4 \frac{\partial D}{\partial x_4} \left(1 - \frac{2x_4}{v_e} \right) + D \left(\frac{x_4^2}{v_e^2} - \frac{x_4}{v_e} - 1 \right) + g x_5 \sin x_3 \tag{93}
$$

Note that $Q_3 = 0$ if $R = \infty$; i.e., if the curvature of the spherical planet is neglected and the landing ground is regarded as a plane. This can be done, without making serious error in computing the vehicle's altitude, if the horizontal distance traveled during the ballistic soft-landing maneuver is relatively short. (In that case, the assumption $R = \infty$ should have been introduced into the dynamic equations and then Q_3 would not have appeared in Eq. (89), since the expression for Q_3 originates from the fictitious centrifugal acceleration due to the spherical curvature of the planet).

If $\sigma = 0$ in a finite, closed-time interval and

$$
0 < u_{sing} < u_{max} \tag{94}
$$

where u_{sing} is given by Eq. (89) with Eqs. (90–93), then singular control, that is, variable-thrust subarc, can be part of the fuel-optimal control program. If so, then the singular optimal control law is given by Eq. (89), with Eqs. (90–93) forming a synthetic expression.

To determine the possible existence of singular optimal subarcs, it is necessary to investigate whether Eq. (89), with Eqs. (90–93), can physically hold. Assuming that the functional form of the model drag force is given by

Eq. (42), then the following partials of the drag are obtained:

$$\frac{\partial D}{\partial x_1} = -bD \tag{95a}$$

$$\frac{\partial D}{\partial x_4} = \frac{n}{x_4} D \tag{95b}$$

$$\frac{\partial^2 D}{\partial x_1 \partial x_4} = -\frac{bn}{x_4} D \tag{95c}$$

$$\frac{\partial^2 D}{\partial x_4^2} = \frac{n(n-1)}{x_4^2} D \tag{95d}$$

Assuming the quadratic drag law ($n = 2$) and substituting the resulting expressions from Eqs. (95a–95d) into Eqs. (90–93), we obtain for the singular control, if any,

$$u_{sing} = \frac{1}{v_e} \cdot \frac{Dgx_5 \sin x_3}{gx_5 \sin x_3 + D\left[3 - \frac{x_4}{v_e}\left(5 - \frac{x_4}{v_e}\right)\right]} \left\{ \frac{gx_5 \sin x_3}{D} + 3\left(1 - \frac{x_4}{v_e}\right) + \frac{D}{gx_5 \sin x_3}\left[3 - \frac{x_4}{v_e}\left(5 - \frac{x_4}{v_e}\right)\right] + \frac{x_4^2}{g}\left(b + \frac{1}{R} ctg^2 x_3\right)\left(1 - \frac{x_4}{v_e}\right) - \frac{1}{\sin^2 x_3} - ctg^2 x_3 \left(2 - \frac{x_4}{v_e}\right) \right\} \tag{96}$$

If we assume that $x_4 << v_e$, i.e., the traveling velocity of the space vehicle is much less than the exhaust velocity of the burning gas, and $R \simeq \infty$, i.e., disregarding the planet's curvature, then Eq. (96) simplifies to

$$u_{sing} = \frac{1}{v_e} \cdot \frac{Dgx_5 \sin x_3}{gx_5 \sin x_3 + 3D}\left[\frac{gx_5 \sin x_3}{D} + \frac{bx_4^2}{g} + 3\left(1 + \frac{D}{gx_5 \sin x_3}\right) - \frac{1 + 2\cos^2 x_3}{\sin^2 x_3}\right] \tag{97}$$

It is interesting to note that in the case of vertical trajectory, an expression has been obtained for the singular control hypersurface, Eq. (40), and by examining its location in the state space it could be generally concluded that there is no way for realizing $\sigma = 0$ in a finite, closed-time interval; this would imply that the trajectory has to lie in a physically excluded region of the state space. In the case of a ballistic trajectory, however, the analysis of the singular condition did not result in an expression for the singular control hypersurface. Instead of that, a synthetic expression for the singular control itself was obtained, Eqs. (96) or (97). From the expression for u_{sing}, general conclusions cannot be made regarding the existence of a singular control hypersurface. Equation (96) or (97) merely states that if $\sigma = 0$ in a finite, closed-time interval, then u_{sing}, as given in Eq. (96) or (97), enters into the candidate extremal curves, provided that u_{sing} is in the interior of the admissible set U.

Regarding the general behavior of the switching function σ in the case of ballistic soft-landing trajectories, with the specified end conditions, the following two statements can be proved:

(1) At the terminal time τ,

$$\sigma(\tau) \neq 0 \tag{98}$$

(2) For $\sigma(t') > 0$ with $\dot{\sigma}(t') = 0$ and $\ddot{\sigma}(t') < 0$, there is a unique condition which must be satisfied:

$$\lambda_1(t') = 0, \qquad \lambda_3(t') \neq 0, \qquad \lambda_4(t') \neq 0 \tag{99a}$$

such that

$$x_5(t')\, g \sin x_3(t') = 3D(t') \tag{99b}$$

and

$$\frac{\lambda_3(t')}{\lambda_4(t')} > \frac{2 + \frac{b}{g} x_4^2(t') - 3\, ctg^2 x_3(t')}{2\, ctg x_3(t')} \tag{99c}$$

simultaneously. If these conditions cannot be satisfied, then $\sigma > 0$ is a continuously decreasing function of time.

Statement 1 can be easily proved. If we assume that $\sigma(\tau) = 0$, then to satisfy $H^* = 0$ at the terminal time τ, we must have $\lambda_4(\tau) = 0$, since $x_4(\tau) = \lambda_3(\tau) = D(\tau) = 0$. The defining equation for σ, Eq. (83), however, yields a fixed value, different from zero, for $\lambda_4(\tau)$ if $\sigma(\tau) = 0$, since $\lambda_5(\tau) = 1$. Therefore, the singularity condition at the terminal time is not compatible with the given terminal conditions. Consequently, at the terminal time τ, we must have $\sigma(\tau) \neq 0$ in order to satisfy $H^* = 0$ and the terminal conditions.

Regarding Statement 1, it is noted that Eq. (96) or (97) yields the following value for u_{sing} at the terminal time τ when $x_4(\tau) = D(\tau) = 0$:

$$u_{sing}(\tau) = \frac{1}{v_e} x_5(\tau)\, g \sin x_3(\tau) \tag{100}$$

This value of u_{sing} is always in the interior of the admissible control set U since there must always be

$$T_{max} = v_e u_{max} > x_5 g \tag{101}$$

to be able to stop the space vehicle. The maximum available retrothrust must always be greater than the weight of the space vehicle in order to achieve a soft landing. Therefore, Eq. (100) and Inequality (101) seemingly indicate the possibility of having a singular optimal subarc (variable-thrust subarc) as the terminating section of the fuel-optimal burning program. This possibility, however, is ruled out by Statement 1. The proof of Statement 1 illustrates, furthermore, that the manner in which singular subarcs enter into the candidate extremal curves of the canonical equations, is essentially determined by the split-boundary conditions for the canonical equations.

It follows immediately from Statement 1 that the terminating section of the fuel-optimal retrothrust program is u_{max}; that is,

$$u^* = \{\, \cdots, u_{max}\} \tag{102}$$

since, in view of Eq. (77), we must have $u(\tau) \neq 0$ to satisfy the prescribed terminal condition $x_1(\tau) = 0$. In view of Eq. (84), Eq. (102) implies that

$$\sigma(\tau) < 0 \tag{103}$$

Statement 2 can be proved, step by step, as follows. When $\dot{\sigma} = 0$ then, by taking into account the relevant canonical equations, we must have

$$\lambda_1 \sin x_3 + \lambda_3 \cos x_3 \frac{g}{x_4^2} + \lambda_4 \frac{1}{x_5}\left(\frac{\partial D}{\partial x_4} - \frac{D}{v_e}\right) = 0 \tag{104}$$

When $\sigma > 0$ then, since $u^* = 0$, $H^* = 0$ yields

$$-\lambda_1 \sin x_3 + \lambda_3 \frac{g}{x_4^2} + \frac{\lambda_4}{x_4 x_5}(x_5 g \sin x_3 - D) = 0 \tag{105}$$

The maximum point requirement implies, furthermore, that $\ddot{\sigma}(t') < 0$. That is, the expression given by Eq. (88) must be less than zero. In deriving Eqs. (104) and (105) it was assumed that $R \simeq \infty$. There are seven different ways in which Eqs. (104) and (105) can be satisfied:

(1) Case 1.

$$\lambda_1(t') = \lambda_3(t') = \lambda_4(t') = 0, \qquad \lambda_5(t') > 0$$

However, in view of Eq (88), this implies that $\ddot{\sigma}(t') = 0$, which contradicts the necessary condition for having maximum point. Therefore, Case 1 must be ruled out.

(2) Case 2.

$$\lambda_1(t') = \lambda_4(t') = 0, \qquad \lambda_3(t') \neq 0, \qquad \lambda_5(t') > 0$$

(3) Case 3.

$$\lambda_3(t') = \lambda_4(t') = 0, \qquad \lambda_1(t') \neq 0, \qquad \lambda_5(t') > 0$$

These two cases must be ruled out, too, since the coefficients of λ_1 and λ_3 are always different from zero.

(4) Case 4.

$$\lambda_1(t') = \lambda_3(t') = 0, \qquad \lambda_4(t') \neq 0$$

In this case we must simultaneously have

$$x_5 g \sin x_3 - D = 0 \tag{106a}$$

$$v_e \frac{\partial D}{\partial x_4} - D = 0 \tag{106b}$$

But, in view of the model drag force, Eq. (23), we have

$$v_e \frac{\partial D}{\partial x_4} = \frac{2v_e}{x_4} D$$

which is always greater than D since $v_e > x_4$. Therefore, Case 4 must also be ruled out.

(5) Case 5.

$$\lambda_4(t') = 0, \qquad \lambda_1(t') \neq 0, \qquad \lambda_3(t') \neq 0, \qquad \lambda_5(t') > 0$$

Then we must have

$$\begin{vmatrix} -1 & \cos x_3 \\ 1 & 1 \end{vmatrix} = 0 \tag{107}$$

which, in view of Eq. (73h), can not be satisfied. Therefore, this case must be excluded.

(6) Case 6.

$$\lambda_3(t') = 0, \qquad \lambda_1(t') \neq 0, \qquad \lambda_4(t') \neq 0$$

Then we must have

$$\begin{vmatrix} -1 & \frac{1}{x_4}(x_5 g \sin x_3 - D) \\ 1 & \left(\frac{\partial D}{\partial x_4} - \frac{D}{v_e}\right) \end{vmatrix} = 0 \tag{108a}$$

yielding

$$x_5 g \sin x_3 = D\left(1 + \frac{x_4}{v_e}\right) - x_4 \frac{\partial D}{\partial x_4} \tag{108b}$$

which is physically meaningless since the left-hand side of this equation is always positive while the right-hand side, in view of the model drag force Eq. (23), is always negative. Consequently, this case also must be eliminated.

(7) Case 7.

$$\lambda_1(t') = 0, \qquad \lambda_3(t') \neq 0, \qquad \lambda_4(t') \neq 0$$

Then we must have

$$\begin{vmatrix} 1 & \frac{1}{x_4}(x_5 g \sin x_3 - D) \\ 1 & \left(\frac{\partial D}{\partial x_4} - \frac{D}{v_e}\right) \end{vmatrix} = 0 \tag{109a}$$

yielding

$$x_5 g \sin x_3 = D\left(1 - \frac{x_4}{v_e}\right) + x_4 \frac{\partial D}{\partial x_4} \tag{109b}$$

In view of the model drag force, Eq. (23), and by assuming $x_4 << v_e$, Eq. (109b) results in

$$x_5 g \sin x_3 = 3D \tag{109c}$$

which is a realizable equality. Substituting Eq. (109c) into Eq. (88), where now $\lambda_1 = 0$, we find, after some algebra, that in order to satisfy $\ddot{\sigma}(t') < 0$ we must have

$$\frac{\lambda_3}{\lambda_4} > \frac{2 + \frac{b}{g} x_4^2 - 3\, ctg^2 x_3}{2\, ctg x_3} \tag{109d}$$

This completes the proof of Statement 2.

It follows from Statement 2 that if the very unique conditions $\lambda_1(t') = 0$ and Eqs. (109c) and (109d) are not or cannot be satisfied simultaneously at an isolated time point t' when $\sigma > 0$, then the fuel-optimal retrothrust program can have, at most, three sections in the following sequence:

$$u^* = \{0, u_{sing}, u_{max}\} \tag{110}$$

provided that $\sigma = 0$ in a finite, closed-time interval. If σ never vanishes in a finite, closed-time interval, then there is

$$u^* = \{0, u_{max}\} \tag{111}$$

The conditions involved in Statement 2 and the vanishing of the switching function σ can be tested only by actual computation of given cases.

C. Numerical Results

Having derived general information on the character of the fuel-optimal retrothrust program for ballistic soft-landing trajectories under the influence of drag, we reduced the optimization problem to solving the eight canonical equations, Eqs. (68), (70–72), (76), and (78–80), with the prescribed end conditions, which are specified partly at the beginning and partly at the end of the trajectory. More specifically, four conditions are given at $t = 0$, $[x_1(0), x_3(0), x_4(0), x_5(0)]$, and four conditions are given at $t = \tau$, $[x_1(\tau), x_4(\tau), \lambda_3(\tau), \lambda_5(\tau)]$, and the terminal time τ is free. It should be noted that two canonical equations, Eq. (69) and Eq. (77), are decoupled from the optimum searching procedure, since Eq. (77) yields zero over the entire control interval and Eq. (69) simply means integration along the optimal trajectory determined by the remaining canonical equations. Since, in this case, the canonical equations cannot be integrated analytically, numerical integration by trial-and-error-technique must be applied. The guessing–iterating procedure is eased, however, by integrating the eight canonical equations in backward time, since it is observed that, in view of $u^*(\tau) = u_{\max}$ and the given terminal conditions, the Hamiltonian $H^* = 0$ evaluated at the (free) terminal time τ yields

$$\lambda_4(\tau) = \frac{x_5(\tau)\, u_{\max}}{g x_5(\tau) \sin x_3(\tau) - v_e u_{\max}} \tag{112}$$

By selecting a proper $x_5(\tau) =$ terminal mass, and a proper $x_3(\tau) =$ landing angle, there are three more fixed terminal conditions in addition to the four given terminal conditions, $x_1(\tau) = 0, x_4(\tau) = 0, \lambda_3(\tau) = 0, \lambda_5(\tau) = 1$. The eighth necessary terminal condition remains open. When integrating the eight canonical equations in backward time, therefore, it is necessary to guess only one missing condition; that is, $\lambda_1(\tau)$. The guessing–iterating procedure for the missing value of $\lambda_1(\tau)$, however, actually becomes a tracking of some initial manifolds which are consistent with the given and selected terminal conditions in the sense of the fuel-optimal retrothrust control program specified by Eq. (84), or eventually by Eqs. (110) and (111).

As an example, the fuel-optimal retrothrust program for a propulsive ballistic soft-landing trajectory to Mars has been computed. For the characteristics of the retrorocket (v_e and $u_{\max}$) and for the Martian parameters (g, b, ρ_0) the same values were applied as in the case of a vertical soft-landing trajectory. The radius of Mars was assumed to be

$$R = 11.2\,10^6\,\text{ft}$$

The computations were carried out for the following end conditions:

At $t = 0$:

Altitude = 14900 ft

Path angle = 24°25′

Velocity = 940 ft/s

Mass = 31.24 slugs

At $t = \tau$:

Altitude = 0

Velocity = 0

These end conditions correspond to those typical values applied in two JPL internal documents[4,5] for calculating trajectories for a *Voyager*-type Martian soft-landing problem. In these same documents, the following feed-back control law (acceleration-type thrust program) was applied for the soft-landing problem on Mars:

$$T = m\left(g + \frac{v^2}{2S}\right) \tag{113}$$

where S is the slant range, related to the altitude h by $S = h/\sin\alpha$. In terms of the mass flow rate u and altitude h, the control law given by Eq. (113) corresponds to

$$u = \frac{m}{v_e}\left(g + \frac{v^2 \sin\alpha}{2h}\right) \tag{114}$$

The justification for applying this control law is analogous to that given for the vertical trajectory.

The numerical results obtained for the fuel-optimal thrust program are displayed in Figs. 6 and 7. For the sake of comparison, the results obtained by applying the ATCL, given by Eq. (114), are also depicted in the same figures. The adjoint variables are depicted in Fig. 8, while Fig. 7c displays the behavior of the switching function σ.

[4]Internal communication, *Propulsive Lander Control System Interim Report*, April 18, 1967.

[5]Internal communication, *Voyager Standardized Soft Lander Report Outline*, Jet Propulsion Laboratory, August 15, 1966.

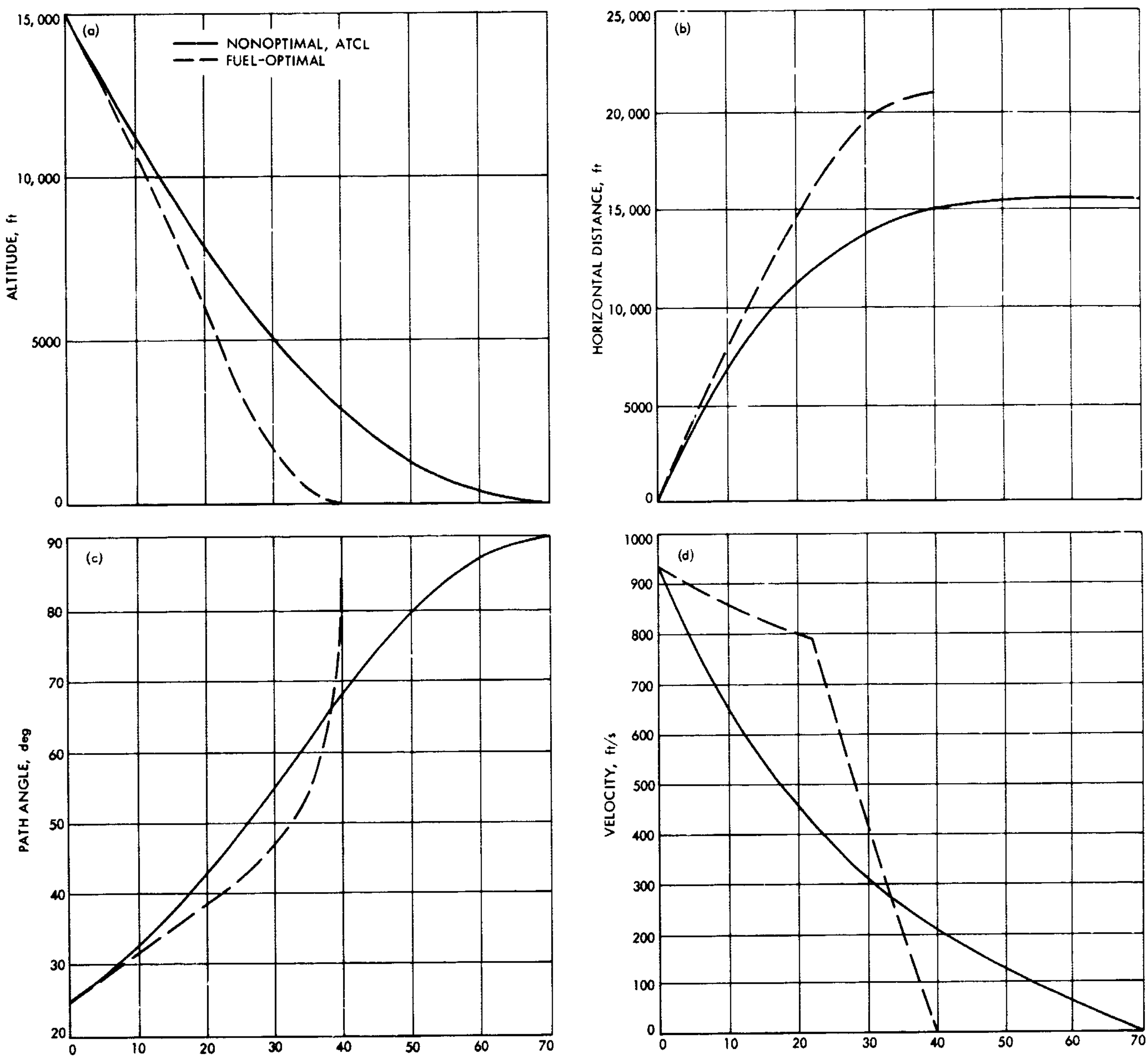

Fig. 6. Fuel-optimal and nonoptimal (ATCL) time histories of state variables of a vehicle on a *ballistic* soft-landing trajectory

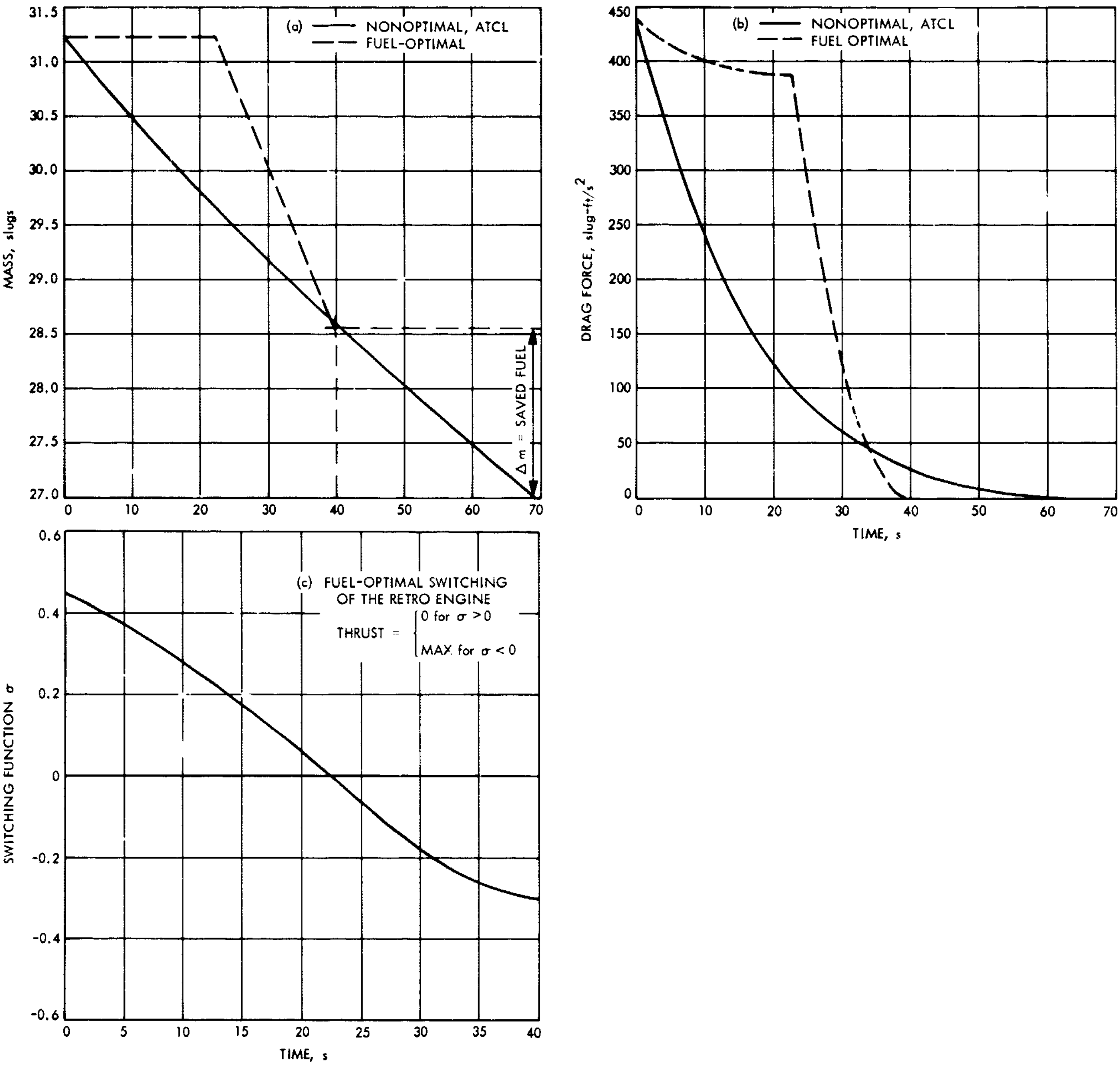

Fig. 7. Time histories of mass depletion, atmospheric drag, and fuel-optimal switching function for a *ballistic* soft-landing trajectory

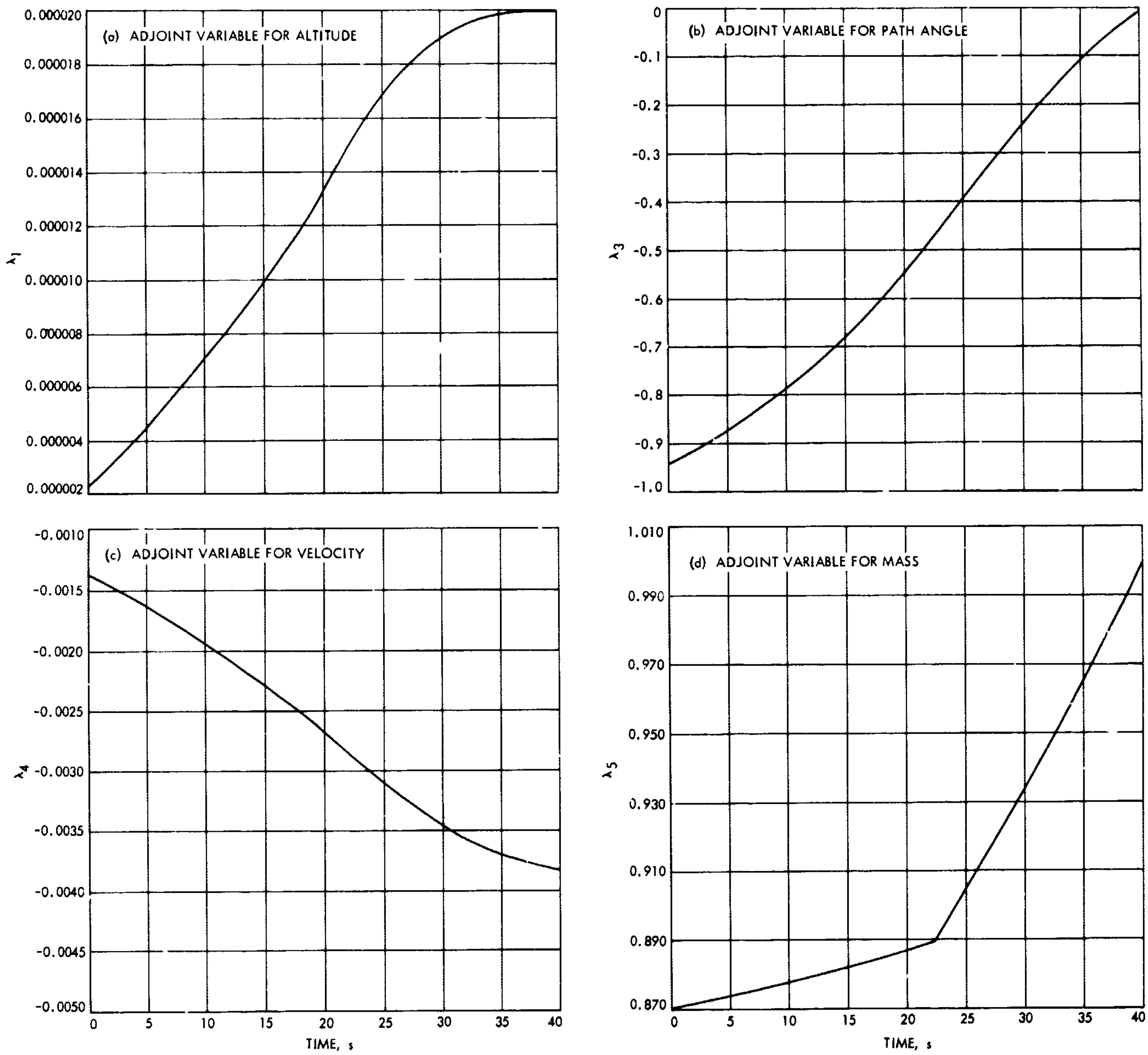

Fig. 8. Time histories of the system of adjoint variables for a fuel-optimal *ballistic* soft-landing trajectory

The basic results are as follows:

(1) The switching function σ is a monotonic, strictly decreasing function of time over the entire control period. This implies that the fuel-optimal retrothrust control program for the ballistic soft-landing trajectory, with the specified end conditions and under the specified gas-dynamic drag conditions, is a bang-bang type of control composed of only two sections in the optimal control sequence:

$$u^* = \{0, u_{\max}\} \tag{115}$$

(2) The fuel consumption Δm: (a) in the case of the fuel-optimal thrust program is 2.74 slugs, and (b) in the case of the ATCL, 4.23 slugs. The difference in the consumed fuel is about 55% in favor of the fuel-optimal thrust program (Table 2).

Table 2. Comparison of results for the fuel-optimal thrust program and the ATCL for a ballistic descent

Program	Fuel consumption, Δm, slugs	Difference, %	Required controlled time for soft landing, s	Difference, %
Fuel-optimal	2.74	−55	40	−75
ATCL	4.23		69	

(3) The required controlled time for soft landing is 40 s in the case of the fuel-optimal thrust program, and 69 s in the case of the ATCL. Thus, the difference in the necessary (controlled) time for the two types of soft-landing maneuvers is approximately 75% in favor of the fuel-optimal thrust program. This fact is interesting since a minimum time requirement in the optimal control problem had not been posed. The terminal time was merely left open.

(4) From the point of view of implementation, the fuel-optimal thrust program has the advantage of being a one-level constant-thrust program, while the ATCL yields a variable thrust program.

The difference in fuel consumption for the two thrust programs can best be explained by examining the time histories of the drag force and path angle during the period of the two types of control, Figs. 6c and 7b. In the case of the fuel-optimal thrust program, the drag decreases only slightly in the first 22.5 s when the retroengine is still *off*. The reason for this is obvious if one examines Fig. 6; during the first 22.5 s the velocity decreases only slightly while the space vehicle flies to atmospheric regions of greater and greater density. In the case of the ATCL, however, the drag decreases drastically during the first 22.5 s. The drag in the acceleration-type burning program is four times less than the value of the drag in the fuel-optimal thrust program at $t = 22.5$ s when the retroengine is being turned *on*.

This again illustrates that minimizing the fuel consumption does imply the maximization of the decelerating effect of the drag force during the soft-landing maneuver; the time history of the drag force, through its dependency on the velocity and on the altitude of the space vehicle, is implicitly controlled by the burning program of the retrorockets. It is observed, furthermore, that the fuel-optimal retrothrust program considerably lengthens the atmospheric flight path of the space vehicle. The reason for this becomes obvious by examining Fig. 5. During the first 22.5 s, the path angle increases less and the velocity decreases less in the fuel-optimal thrust program than in the acceleration-type control program, while the space vehicle reaches roughly identical altitudes at $t = 22$ s in both cases. This necessarily results in lengthening the horizontal, and hence the total, distance traveled by the space vehicle when controlled by the fuel-optimal thrust program. Lengthening the total traveled path implies, however, that under the specified circumstances the slowing down effect of the drag force increases.

It is noted that the landing angle is approximately 85 deg in the case of the fuel-optimal thrust program, while in the case of the ATCL, the landing angle is 90 deg. This insignificant difference in the landing angles for the two types of thrust programs, however, has to be expected since no constraint was imposed on the terminal value of the path angle in formulating the fuel-optimal control problem for a ballistic trajectory.

In the numerical calculations presented, the pluming effects (that is, the change in aerodynamics due to the retroengine's exhaust plumes) were not considered because of a lack of adequate quantitative description of these effects.

VI. Conclusion

The main results of this study can be summarized as follows:

(1) The fuel-optimal retrothrust control program for a propulsive soft-landing maneuver under the influence of drag, which depends quadratically on the velocity and inverse-exponentially on the altitude, is

(a) in the case of vertical trajectories, always a bang-bang type of control

$$T^* = \{0, T_{max}\} \tag{116}$$

and, (b) in the case of gravity-turn ballistic trajectories, with the specified end conditions as well as atmospheric parameters, also a bang-bang type of control

$$T^* = \{0, T_{max}\} \tag{117}$$

(2) Having specified the character of the fuel-optimal retrothrust control programs, Eqs. (116–117), we reduced the synthesis of these programs to determining a switching hypersurface:

$$F_1 (h, v, m) = 0, \text{for vertical trajectories} \tag{118a}$$

$$F_2 (h, \alpha, v, m) = 0, \text{for ballistic trajectories} \tag{118b}$$

for given atmospheric parameters and retrorocket characteristics. The development of the switching hypersurface, Eqs. (118a) and (118b) can be eased by integrating the canonical equations (or, eventually, only the system equations) in backward time. The switching hypersurface can then be obtained as tabulated numbers with the time, or, with the mass, as parameters.

(3) The numerical results clearly demonstrate the superiority of the fuel-optimal retrothrust control programs over other types of control law proposed and applied elsewhere. This applies to (a) fuel consumption, (b) necessary controlled time, and (c) implementation. The reduction of the necessary controlled time and the simplicity of implementing the fuel-optimal control program are consequences rather than conditions of the formulated optimal control problem. They can be regarded as fringe benefits of fuel optimality.

It is needless to emphasize that the fuel-optimal retrothrust programs, given by Eqs. (116–118b), are open-loop control laws which are necessarily very sensitive to changes in the values of the involved parameters, as well as to the accuracy in determining the state variables at a single instant of time. The practical mechanization aspects of the fuel-optimal retrothrust control law are beyond the scope of this study. Two means are mentioned, however, by which the inherent difficulties of the open-loop optimal-control law can be circumvented in the present problem. For example, a closure section may be introduced into the retrothrust control program by terminating the fuel-optimal full-thrust phase, say, at $h = 200$ ft and $v = 100$ ft/s. From there on, a suitable feed-back control might be applied until $h = v = 0$ is reached. Introducing a closure section might be necessary for other reasons, too. The uncertainties in the involved atmospheric parameter values, as well as in the determination of state variables, can be also improved considerably by application of a proper nonlinear sequential-estimation technique in a few seconds prior to the initiation of the terminal phase of a soft-landing mission.[6]

It is believed, therefore, that the present fuel-optimization study can be used as a guide for developing a system which accomplishes a soft-landing mission with an efficient utilization of fuel.

[6]A proper nonlinear sequential-estimation technique related to the soft-landing problem under the influence of drag will be treated in a subsequent report.

Appendix A
State Constraints

The vertical or gravity-turn ballistic motions of a rocket-thrusted space vehicle are restricted to the following bounded region of state space during a propulsive soft-landing maneuver:

$0 \leq h < h_o$, altitude

$0 < \alpha_o < \alpha \leq \pi/2$, path angle

$0 \leq v < v_o$, velocity

$0 < m \leq m_o$, mass

where h_o, α_o, v_o, and m_o are given numbers (initial values). These state constraints seem to indicate that the application of the Restricted Maximum Principle, Ref. 3, would be an adequate procedure in searching for the fuel-optimal retrothrust control program for a soft-landing maneuver under the influence of drag. It can be easily shown, however, that this is not the case since all the described state constraints are natural constraints; that is, they do belong internally to the nature of the problem instead of being imposed externally on it.

Since there is no lifting force present, and the thrust direction is fixed such that it always acts antiparallel to the velocity vector, as does the drag force, there necessarily is $\alpha(t) > \alpha_o$. The other boundary $\alpha \leq \pi/2$ is also a natural constraint since there are no forces acting in a direction other than parallel or antiparallel to $\pi/2$ when $\pi/2$ has been reached. This fact also has the consequence that the trajectory remains vertical when it starts or when it becomes vertical.

The restriction $m(t) \leq m_o$ is a natural consequence of the constraint on the control u (= mass flow rate) which is $0 \leq u(t) \leq u_{max}$. This implies that m can only decrease or be constant since $\dot{m} = -u$. The lower boundary $m(t;\tau) > 0$ is a natural consequence of the assumption that there is sufficient control capability (= amount of fuel) on board to stop the space vehicle. That is, it is assumed that there exists at least one admissible thrust program which can stop the space vehicle at zero altitude. This is essentially a question of selecting the retrorocket's parameters (u_{max} and v_e) properly. It is also noted that leaving the terminal time τ free in the formulation of the optimal control problem is consistent with the assumption of the existence of at least one admissible thrust program which can accomplish the soft-landing maneuver with the prescribed end conditions.

Furthermore, to show that the phase restrictions on the altitude as well as on the velocity are natural consequences of the problem statement, the Restricted Maximum Principle (Ref. 3) is referenced. This principle states that for the bounded-state variable problem, the optimal trajectory consists of segments for which the trajectory is inside the constraint, and of segments for which the trajectory is on the constraint. In the first case, the optimal control satisfies the Unrestricted Maximum Principle; that is, the Hamiltonian, given by Eq. (4), remains unchanged for that part of the optimal trajectory. In the second case, the optimal control is determined by holding the trajectory on the constraint. This implies that the Hamiltonian, as given by Eq. (4), has to be augmented for this part of the optimal trajectory. It is obvious, however, that a trajectory segment with $v = 0$ and $h > 0$ cannot be a part of the optimal trajectory since it necessarily would require more integrated retrothrust effort to stop the space vehicle at $h > 0$, and, after maintaining it there, to stop it again at $h = 0$, than to stop it only once at $h = 0$. On the other hand, there is no physical possibility for realizing $h = 0$ with $v \neq 0$ in a finite time interval. The only way to realize $h = 0$ in a finite time interval is to keep $h = v = 0$ simultaneously. Achieving $h = v = 0$ simultaneously, however, is the termination of the control process itself, which is an instantaneous event. It is concluded, therefore, that the implications of the Restricted Maximum Principle and the dynamics of the system as well as the control problem itself exclude the possibility of having values other than $0 < h < h_o$, $0 < v < v_o$ on the optimal trajectory in a finite time interval.

This completes the demonstration of the statement that the described restrictions on the state space are natural constraints, which, being natural constraints, do not require the application of the Restricted Maximum Principle in the present problem. In other words, the maximum principle in its unrestricted form, as it is employed in this report, is indeed the adequate principle for the optimizaton problem in question.

Appendix B
The Physical Consequences of Fuel Optimality

It can be easily demonstrated that in the absence of drag the fuel-optimal and time-optimal retrothrust control programs for the vertical-trajectory soft-landing problem are equivalent. We put $D = 0$ in Eq. (22). Since

$$\frac{\dot{m}}{m} = \frac{d}{dt}\ln(m) \qquad \text{and} \qquad \dot{m} = -u$$

Equation (22) then reads

$$\frac{d}{dt}v = g + v_e \frac{d}{dt}\ln(m) \tag{B-1}$$

which, integrated in $[0, \tau]$ yields

$$v(\tau) = v(0) + g\tau + v_e \ln\left[\frac{m(\tau)}{m(0)}\right] \tag{B-2}$$

In order to achieve $v(\tau) = 0$, Eq. (B-2) gives for $m(\tau)$

$$m(\tau) = m(0)\exp\frac{1}{v_e}[-v(0) - g\tau] \tag{B-3}$$

The integral form of the minimum fuel consumption performance index J is

$$J = \int_0^\tau \dot{m}dt = m(0) - m(\tau) = \Delta m \tag{B-4}$$

Equation (B-4), combined with Eq. (B-3), results in

$$J = m(0)\left\{1 - \exp\frac{1}{v_e}[-v(0) - g\tau]\right\} \tag{B-5}$$

which clearly shows that J is a monotonic, strictly increasing function of the terminal time τ. Hence, minimizing τ is equivalent to minimizing J, since $m(0)$, $v(0)$, v_e, and g are fixed constants.

In the presence of drag, however, when $D \neq 0$ in Eq. (22), for $m(\tau)$ we obtain

$$m(\tau) = m(0)\exp\frac{1}{v_e}\left[\int_0^\tau D'(h,v)\,dt - v(0) - g\tau\right] \tag{B-6}$$

where

$$D'(h,v) = \frac{D(h,v)}{m}\text{, reduced drag}$$

Equation (B-4), combined with Eq. (B-6), results in

$$J = m(0)\left\{1 - \exp\frac{1}{v_e}\left[\int_0^\tau D'(h,v)\,dt - v(0) - g\tau\right]\right\} \tag{B-7}$$

This expression shows that in the presence of drag the fuel-optimal and time-optimal retrothrust control programs can not be claimed to be equivalent since the maximum value of the exponential in Eq. (B-7), which implies the minimum value of J, also clearly depends on

$$\int_0^\tau D'(h,v)\,dt;$$

that is, on the integrated time history of the drag force and not on the terminal time alone. In addition, Eq. (B-7) also shows that the fuel-optimal retrothrust program necessarily implies the relative maximization of the decelerating effect of the drag force during the soft-landing maneuver.

In the case of ballistic trajectories, Eqs. (B-5) and (B-7) must be modified to

$$J = m(0)\left\{1 - \exp\frac{1}{v_e}\left[-v(0) - g\int_0^\tau \sin\alpha(t)\,dt\right]\right\} \tag{B-8}$$

and

$$J = m(0)\left\{1 - \exp\frac{1}{v_e} \times\left[\int_0^\tau D'(h,v)\,dt - v(0) - g\int_0^\tau \sin\alpha(t)\,dt\right]\right\} \tag{B-9}$$

Equation (B-8) shows that in the absence of drag the minimization of J implies the minimization of the total path and not the minimization of the terminal time; Equation (B-9) shows that in the presence of drag the minimization of J still implies the relative maximization of the decelerating effect of the drag force during a ballistic soft-landing maneuver. Finally, it must be remembered that the time history of the drag is implicitly controlled by the retrothrust program since the drag is a position- and velocity-dependent force.

Nomenclature

D drag force

G performance index (maximum final mass)

H Hamiltonian

J performance index (minimum fuel consumption)

K constant, related to the drag

R radius of the planet

T thrust, $T = v_e u$

T_{max} maximum thrust

U admissible control set

b atmospheric scale factor

g acceleration of gravity

h altitude

m mass

n exponent of the velocity

s ground range

t time

u control (mass flow rate)

u_{max} maximum mass flow rate

u_{sing} singular control (variable mass flow rate)

v velocity

v_e exhaust velocity of the burning gas

x_i state variables (defined in different ways in Sections IV and V)

α path angle

λ_i adjoint variables

σ switching function

τ terminal time

* denotes optimal trajectories

· (dot) denotes time derivative

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CONFIGURATION INTERACTION MATRIX ELEMENTS FOR d^n CONFIGURATIONS. II. $d^n - l'^{4l'+1} d^{n+1}$.*

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ABSTRACT

The matrix elements of the configuration interactions $d^n - l'^{4l'+1} d^{n+1}$ for $n = 3, 4, 5$ and $l' = s, d$ are computed and tabulated for general usage.

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CONFIGURATION INTERACTION MATRIX ELEMENTS FOR d^n CONFIGURATIONS. II. $d^n - l'^{4l'+1} d^{n+1}$.

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Introduction:

In this second paper of a series[1] on configuration interaction (CI) in d^n configurations, the interaction matrix elements between the core excitation $l'^{4l'+1}d^{n+1}$ and the d^n configuration are calculated and tabulated. This weak interaction is one of the five basic types of CI discussed by Rajnak and Wybourne[2].

As noted in Paper I, the accumulative effect of these weak interactions should lead to further depression of the higher excited term-multiplets. However, it is generally accepted that only a complete CI treatment, including both strong and weakly interacting configurations, will erase this major remaining discrepancy between theoretically predicted and the observed energy level schemes. Since this problem is currently insurmountable, we must be content for the present to introduce all the available CI into our calculations in hopes that these additional refinements will further enhance the value of our model.

[1]J. A. Barnes, B. L. Carroll and L. M. Flores, J. Chem. Phys., (To be published), hereafter referred to as Paper I.

[2]K. Rajnak and B. G. Wybourne, Phys. Rev. 132, 280 (1963).

Configuration Interaction Matrix Elements

To obtain the second-order correction to the Coulomb matrix elements, we apply

$$H^{2}_{CI} = -\sum_{\psi} \frac{(l^n\gamma SL|H_c|\psi)(\psi|H_c|l^n\gamma' SL)}{\Delta E_{\psi}} \quad . \qquad (1)$$

As noted in Paper I, ΔE_{ψ} may be approximated by the average energy separation, ΔE, of the l^n configuration and the particular perturbing configuration to allow the summation over the perturbing states, ψ, to be simplified.

To apply Eq. (1), we must evaluate the matrix elements $(l^n\gamma SL|H_c|\psi)$, where H_c is the Coulomb operator, between the interacting configurations, d^n and $l'^{4l'+1}d^{n+1}$. These matrix elements are given by the general expression[2],

$$(l^n\gamma SL|H_c|l^{n+1}(\gamma''S''L'')[l'^{4l'+1}s];SL)$$

$$= (-1)^{S''-S-s} \left\{\frac{(n+1)(2S''+1)(2L''+1)}{(2S+1)(2L+1)}\right\}^{1/2}$$

$$\times \sum_{k}\sum_{\bar{\psi}} (l^{n+1}\psi''\{|l^n\bar{\psi})(l^n\bar{\psi}||U^{(k)}||l^n\psi)$$

$$\times \begin{Bmatrix} L & L & k \\ l' & l & L'' \end{Bmatrix} \Omega\,(k,lll l')$$

$$- \sum_{k} \frac{\delta(l,l')}{(2l+1)} (l^{n+1}\psi''\{|l^n\psi)(-1)^{L''+l+L}$$

$$\times \quad \Omega(k,llll') \quad , \qquad (2)$$

where

$$\Omega(k,llll') = (l||C^{(k)}||l)(l||C^{(k)}||l')R^k(llll') ,$$

and sum over k is given by ,

$$k = |l-l'| , \ldots, \min(2l,l+l') .$$

In Eq. (2), the reduced matrix elements of the tensor operators $U^{(k)}$ and $C^{(k)}$ are given by ,

$$(l^n\gamma SL||U^{(k)}||l^n\gamma' S' L')$$

$$= n\{(2L+1)(2L'+1)\}^{1/2} \sum_{\bar{\gamma}\bar{S}\bar{L}} (l^n\gamma SL\{|l^{n-1}\bar{\gamma}\bar{S}\bar{L})$$

$$\times (l^{n-1}\bar{\gamma}\bar{S}\bar{L}|\}l^n\gamma' S' L')(-1)^{\bar{L}+L+k}$$

$$\times \begin{Bmatrix} L & k & L' \\ 2 & \bar{L} & 2 \end{Bmatrix} , \qquad (3)$$

and

$$(l||C^k||l') = (-1)^l\{(2l+1)(2l'+1)\}^{1/2} \begin{pmatrix} l & k & l' \\ 0 & 0 & 0 \end{pmatrix} \qquad (4)$$

In Eqs. (1), (3) and (4),

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$$

are the Wigner 3-j and 6-j symbols. The coefficients $(l^n\gamma SL\{|l^{n-1}\gamma' S' L')$ are the coefficients of fractional parentage introduced by Racah[3]. The reduced matrix elements

[3] G. Racah, Phys. Rev. 62; 438 (1942); 63, 367 (1943); 76,

of $U^{(k)}$ and appropriate fractional parentage coefficients have been tabulated by Nielson and Koster[4]. The integral $R^k(1111')$, occurring in Eq. (2), is the Slater integral[5].

Discussion of Tables:

Tables I - III present a tabulation of the matrix elements, $(d^n\ SL|H_c|d^{n+1}(\ ''S''L'')1,^{4l'+1};SL)$, for the d^3, d^4 and d^5 configurations, respectively. The calculations were performed in single precision on the Univac 1108 computer. An accuracy of seven-significant figures has been established.

The matrix elements for a given d^n-$d^{n+1}l,^{4l'+1}$ interaction are listed in columns below the interacting configurations. Each matrix element is formed from the sum of the appropriate Slater integrals, R^k, multiplied by the coefficients of the integrals presented in the tables.

Acknowledgement

We wish to express our gratitude to Mrs. D. A. Barton for her assistance in the preparation of this work.

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[4] C. W. Nielson and G. F. Koster; Spectroscopic Coefficients for p^n, d^n and f^n Configurations, (M.I.T. Press, Cambridge, Mass., 1964).

[5] J. C. Slater, Quantum Theory of Atomic Structure Vol. I, (McGraw-Hill Book Company, New York, 1960).

TABLE I. CONFIGURATION INTERACTION MATRIX ELEMENTS FOR d^3

$l^{N-1}l'l^{N+1}$ V,SL	(V,SL)SL	d^3-sd^4 $R^{(2)}$	d^3-dd^4 $R^{(0)}$	d^3-dd^4 $R^{(2)}$	d^3-dd^4 $R^{(4)}$
3,2H	(4,1I)2H		2.5725818	-.60376932	-.35876150
	(4,3H)2H	-.15649211	3.2249021	-.82267924	-.66911247
	(4,3G)2H		-1.8973660	.38721762	.44745146
	(4,1G)2H		-.4264013	-.13923308	.19241240
	(2,1G)2H		-1.4142132	.14430749	.13308359
	(4,3F)2H		1.4142132	-.057723023	-.46178394
	(2,3F)2H		1.4142132	.028861460	-.34152772
	(4,1F)2H		1.0954448	-.089424095	-.24591622
	(4,3D)2H				
	(4,1D)2H				
	(2,1D)2H				
	(4,3P)2H				
	(2,3P)2H				
3,2G	(4,1I)2G		-1.8618982	.34198134	.31242741
	(4,3H)2G		2.0976169	-.42808519	-.49467621
	(4,3G)2G	.23473819	2.0598190	-.31527849	-.54414731
	(4,1G)2G	-.19342943	2.0059429	-.28183977	-.38470864
	(2,1G)2G	-.11664235	-1.0235323	.36554731	-.048739613
	(4,3F)2G		2.1213198	-.41127635	-.51229157
	(2,3F)2G		-1.4142132	.33190721	.14119067
	(4,1F)2G		-.1825741	.005589089	.046161039
	(4,3D)2G		1.6903080	-.13798437	-.51360848
	(4,1D)2G		-.6900654	.16899563	.092321690
	(2,1D)2G		.9758998	.000000000	-.14715952
	(4,3P)2G				
	(2,3P)2G				
	(4,1S)2G				
	(0,1S)2G				
3,4F	(4,3H)4F		-2.5071320	.40932775	.34963413
	(4,3G)4F		-2.2677863	.25454748	.38053562
	(4,3F)4F	.22131328	.8944270	-.28293108	-.048676316
	(2,3F)4F	.000000000	-1.7888540	.12777532	.12777532
	(4,5D)4F		-3.1622770	.64536270	.87123965
	(4,3D)4F		-.9258199	.037768586	.19209190
	(4,3P)4F		-.9561826	.15611148	.13334522
	(2,3P)4F		.8944270	-.20078975	.012169057
3,2F	(4,1I)2F				
	(4,3H)2F		1.7728100	-.72359600	-.21707883
	(4,3G)2F		-2.4053504	.85905384	.34271165
	(4,1G)2F		1.9820618	-.38427737	-.32135475
	(2,1G)2F		1.1952283	-.085373480	-.13280316

$l^{N-1}l'l^{N+1}$ V,SL	(V,SL)SL	d^3-sd^4 $R^{(2)}$	d^3-dd^4 $R^{(0)}$	d^3-dd^4 $R^{(2)}$	d^3-dd^4 $R^{(4)}$
3,2F	(4,3F)2F	.23473819	-1.5811364	.42271258	.31730334
	(2,3F)2F	-.15649213	-.6324554	-.032268115	.16349187
	(4,1F)2F	.30304568	1.2247446	-.28704088	-.17079774
	(4,3D)2F		1.3093078	-.37408776	-.24939184
	(4,1D)2F		1.6035670	-.19635619	-.32342241
	(2,1D)2F		.7559287	.21597964	-.23397797
	(4,3P)2F		-1.6903080	.25527105	.44044913
	(2,3P)2F		-1.2649107	.12987255	.21942332
3,2D	(4,1I)2D				
	(4,3H)2D				
	(4,3G)2D		2.2677861	-.67107969	-.41918391
	(4,1G)2D		-1.1212235	.35467279	.10551199
	(2,1G)2D		.8451540	-.026147280	-.11402874
	(4,3F)2D		-1.5491930	.36035377	.37412487
	(2,3F)2D		-.7745965	.45643469	-.076405753
	(4,1F)2D		1.9999996	-.34693874	-.34693874
	(4,3D)2D	-.31290424	2.6186138	-.58705221	-.58705221
	(4,1D)2D	-.38332586	-1.0690447	-.043634462	.31271378
	(2,1D)2D	.13552616	-1.1338931	.080992390	.12570816
	(4,3P)2D		-1.2649187	.43084663	.19791123
	(2,3P)2D		1.1832156	-.15695721	-.18612981
	(4,1S)2D		-1.2649108	.14197981	.26244751
	(0,1S)2D		.0000000	-.17744539	.098580774
1,2D	(4,1I)2D				
	(4,3H)2D				
	(4,3G)2D		.0000000	-.31813172	.17673983
	(4,1G)2D		.0000000	-.15720053	.067302514
	(2,1G)2D		-2.3237094	.52166709	.42945321
	(4,3F)2D		.0000000	-.21732534	.12873629
	(2,3F)2D		-3.5496470	.94174317	.06722777
	(4,1F)2D		.0000000	-.28856508	.15586988
	(4,3D)2D	.000000000	.0000000	-.36734685	.20408159
	(4,1D)2D	.000000000	.0000000	-.14996871	.083315964
	(2,1D)2D	.20701963	-1.7320504	.26510978	.38042767
	(4,3P)2D		.0000000	-.17744539	.098580769
	(2,3P)2D		-2.3237094	.49795493	.62705436
	(4,1S)2D		.0000000	-.17744539	.098580774
	(0,1S)2D		-1.2649108	.090350000	.090350000
3,4P	(4,3H)4P				
	(4,3G)4P				
	(4,3F)4P		2.7325194	-.44612571	-.38106572

TABLE I. (continued)

	$l^N{-}l'l^{N+1}$	$d^3{-}sd^4$	$d^3{-}dd^4$		
V,SL	(V,SL)SL	$R^{(2)}$	$R^{(0)}$	$R^{(2)}$	$R^{(4)}$
3,4P	(2,3F)4P		1.3662597	.11153139	-.21376856
	(4,5D)4P		-3.1622767	1.12938470	.60233854
	(4,3D)4P		2.1602464	-.41882332	-.26452000
	(4,3P)4P	-.41403925	1.3662598	-.013941452	-.30671141
	(2,3P)4P	.000000000	1.4605932	-.10432811	-.10432811
3,2P	(4,1I)2P				
	(4,3H)2P				
	(4,3G)2P				
	(4,1G)2P				
	(2,1G)2P				
	(4,3F)2P		-2.5819881	.54801390	.59719463
	(2,3F)2P		1.0327953	-.42154913	-.003512934
	(4,1F)2P		-1.9999994	.16326530	.44897953
	(4,3D)2P		1.6329927	-.21662153	-.44990622
	(4,1D)2P		1.9999995	-.34693874	-.34693874
	(2,1D)2P		-1.4142132	.24532273	.076964010
	(4,3P)2P	-.15649211	2.5819881	-.43735727	-.65867054
	(2,3P)2P	.29276995	-.9660915	.54219428	-.078864601

TABLE II. CONFIGURATION INTERACTION MATRIX ELEMENTS FOR d^4

l^N-l·l^{N+1} V,SL	(V,SL)SL	d^4-sd^5 $R^{(2)}$	d^4-dd^5 $R^{(0)}$	d^4-dd^5 $R^{(2)}$	d^4-dd^5 $R^{(4)}$
4,1I	(5,2I)1I	.090350760	-5.3512941	.83727740	.83727748
	(3,2H)1I		-3.5496470	.50707248	.39440828
	(5,2G)1I		-2.7136014	.22151682	.53738748
	(3,2G)1I		-2.3237804	-.063232384	.47778849
	(5,2F)1I				
	(3,2F)1I				
	(5,2D)1I				
	(3,2D)1I				
	(1,2D)1I				
	(3,2P)1I				
	(5,2S)1I				
4,3H	(5,2I)3H		3.6588730	-.55126765	-.42876377
	(3,2H)3H	-.090350767	-2.7928472	.36098032	.18365663
	(5,4G)3H		-4.2426395	.63495293	.79208773
	(5,2G)3H		-.6396020	.043510352	.097656552
	(3,2G)3H		1.6431672	.067068026	-.26330423
	(3,4F)3H		-2.4494891	.16663194	.43079741
	(5,2F)3H		1.6431672	-.067068074	-.27572424
	(3,2F)3H		-1.2247445	.18329512	.066652787
	(5,4D)3H				
	(5,2D)3H				
	(3,2D)3H				
	(1,2D)3H				
	(3,4P)3H				
	(3,2P)3H				
4,3G	(5,2I)3G		-2.7928472	.49397305	.25754151
	(3,2H)3G		-1.9165897	.29650610	.085131221
	(5,4G)3G	.23389467	-3.0708971	.56398728	.51529764
	(5,2G)3G	.19342943	3.0089146	-.38703215	-.40255325
	(3,2G)3G	.13852616	-1.7838867	-.012135846	.25213967
	(3,4F)3G		2.4494892	-.24994791	-.39251076
	(5,2F)3G		-.2738612	.020493019	.040779033
	(3,2F)3G		-1.0371169	.33742966	.065264194
	(5,4D)3G		2.9276995	-.39832652	-.34872173
	(5,2D)3G		-1.0350981	.056331063	.16586608
	(3,2D)3G		-1.4639408	.059748989	.16818231
	(1,2D)3G		.0000000	.15888091	-.087822732
	(3,4P)3G				
	(3,2P)3G				
	(5,2S)3G				
4,1G	(5,2I)1G		-3.2613429	.75432429	.37469704

l^N-l·l^{N+1} V,SL	(V,SL)SL	d^4-sd^5 $R^{(2)}$	d^4-dd^5 $R^{(0)}$	d^4-dd^5 $R^{(2)}$	d^4-dd^5 $R^{(4)}$
4,1G	(3,2H)1G		.7071066	.26937292	-.28433017
	(5,2G)1G	-.40657846	-2.4325304	.30060006	.36040004
	(3,2G)1G	.19342945	3.0089144	-.42039706	-.33987230
	(5,2F)1G		-3.5178109	.45468510	.60358515
	(3,2F)1G		-2.6220214	.33890077	.31115451
	(5,2D)1G		2.6592150	-.28943043	-.40641732
	(3,2D)1G		-1.2535660	.28994847	.077696480
	(1,2D)1G		.0000000	-.13537242	.075206894
	(3,2P)1G				
	(5,2S)1G				
2,1G	(5,2I)1G		.0000000	.29433065	-.16351781
	(3,2H)1G		-4.6904144	.89341240	.76932734
	(5,2G)1G	.000000000	.0000000	.21953176	-.12196207
	(3,2G)1G	-.23328470	3.0705971	-.37599155	-.61960973
	(5,2F)1G		.0000000	.31747844	-.17637581
	(3,2F)1G		3.1622769	-.47326590	-.59038734
	(5,2D)1G		.0000000	.21817239	-.12128686
	(3,2D)1G		-1.6000220	.26708339	.36139455
	(1,2D)1G		-2.9999993	.16324630	.40136050
	(3,2P)1G				
	(5,2S)1G				
4,3F	(5,2I)3F				
	(3,2H)3F		-1.5362906	.041776800	.10799677
	(5,4G)3F		3.5884847	-.78610692	-.58747026
	(5,2G)3F		2.9730926	-.30427736	-.38881605
	(3,2G)3F		2.0030946	-.26824307	-.13090387
	(3,4F)3F	-.10070155	1.0954448	-.49920539	.030744017
	(5,2F)3F	-.30304578	1.0371168	-.18744093	-.24678211
	(3,2F)3F	.13552614	1.3693060	-.19907116	-.077625061
	(5,4D)3F		2.2677063	-.37828886	-.40624747
	(5,2D)3F		2.4853604	-.19630619	-.34987387
	(3,2D)3F		-1.1338931	-.015427099	.16455594
	(1,2D)3F		.0000000	-.12244895	.068027195
	(3,4P)3F		2.1908896	-.32780829	-.29311226
	(3,2P)3F		1.4638496	.043815007	-.22571836
2,3F	(5,2I)3F				
	(3,2H)3F		3.0705971	-.58487570	-.34117751
	(5,4G)3F		.0000000	-.29270898	.16261609
	(5,2G)3F		.0000000	-.24270146	.13483415
	(3,2G)3F		2.7774595	-.34809714	-.41357490
	(3,4F)3F	.000000000	4.3817792	-.83462472	-.83462472

TABLE II. (continued)

$1^{N}-1,1^{N+1}$ V,SL	(V,SL)SL	d^4-sd^5 $R^{(2)}$	d^4-dd^5 $R^{(0)}$	d^4-dd^5 $R^{(2)}$	d^4-dd^5 $R^{(4)}$
2,3F	(5,2F)3F	.000000000	.0000000	-.14996871	.083315957
	(3,2F)3F	.18070154	-1.0954448	.16394413	.14655612
	(5,4D)3F		.0000000	-.18512540	.10284745
	(5,2D)3F		.0000000	.26180687	-.14544825
	(3,2D)3F		1.1338931	-.030854252	-.22883560
	(1,2D)3F		-2.9999992	.24489795	.19727891
	(3,4P)3F		-2.1908896	.41731235	.41731235
	(3,2P)3F		1.1710798	-.055765722	-.22306284
4,1F	(5,2I)1F				
	(3,2H)1F		2.0598190	-.50444555	-.11209903
	(5,2G)1F		3.9686225	-.73264097	-.56380191
	(3,2G)1F		.3105294	-.023236901	-.046239083
	(5,2F)1F	.31622770	-4.1079179	.64273556	.64273556
	(3,2F)1F	-.30304570	1.8371168	-.48739838	-.079150189
	(5,2D)1F		-1.7928424	.24392418	.30083980
	(3,2D)1F		-2.5354619	.31046480	.31046480
	(1,2D)1F		.0000000	.27380417	-.15211342
	(3,2P)1F		-1.9639604	-.080161626	.41862199
4,5D	(5,4G)5D		4.0249214	-.49284760	-.38636819
	(3,4F)5D		-3.5496471	.14488360	.31391443
	(5,4D)5D	.23904569	2.9999994	-.22448979	-.36734691
	(3,4P)5D		-2.3237894	.33196997	.073771127
	(5,6S)5D		3.2863346	-.62596857	-.62596855
4,3D	(5,2I)3D				
	(3,2H)3D				
	(5,4G)3D		2.5354622	-.58643348	-.35837602
	(5,2G)3D		-1.6818353	.25170328	.18051448
	(3,2G)3D		-1.9639605	.080161682	.22564024
	(3,4F)3D		1.3416405	-.20078975	-.17949387
	(5,2F)3D		2.9999993	-.36734690	-.36734690
	(3,2F)3D		1.3416405	-.31031141	-.012169098
	(5,4D)3D	-.27105232	-3.4016794	.62479833	.57080340
	(5,2D)3D	.38332586	-1.6035670	.16362932	.21453620
	(3,2D)3D	-.18070152	-2.2677863	.20055255	.20055257
	(1,2D)3D	.000000000	.0000000	.24489789	-.13608439
	(3,4P)3D		-2.0493896	.51583280	.15800285
	(3,2P)3D		1.0954448	.081972050	-.19623616
	(5,2S)3D		-1.8973661	.1548?706	.27535474
4,1D	(5,2I)1D				
	(3,2H)1D				

$1^{N}-1,1^{N+1}$ V,SL	(V,SL)SL	d^4-sd^5 $R^{(2)}$	d^4-dd^5 $R^{(0)}$	d^4-dd^5 $R^{(2)}$	d^4-dd^5 $R^{(4)}$
4,1D	(5,2G)1D		3.5677115	-.72810446	-.46382952
	(3,2G)1D		-1.3887298	.58572282	-.060081559
	(5,2F)1D		2.1213198	-.37519946	-.30785596
	(3,2F)1D		-2.8460491	.50338296	.26244791
	(5,2D)1D	-.090350763	3.4016794	-.53223561	-.53223561
	(3,2D)1D	.38332587	-1.6035670	-.16362926	.39634654
	(1,2D)1D	.000000000	.0000000	-.17316896	.096204982
	(3,2P)1D		-2.3237894	.33196997	.25819886
	(5,2S)1D		-2.6832809	.36507229	.45825580
2,1D	(5,2I)1D				
	(3,2H)1D				
	(5,2G)1D		.0000000	-.41187806	.22882114
	(3,2G)1D		-3.9279210	.85505772	.50488326
	(5,2F)1D		.0000000	-.24489791	.13608439
	(3,2F)1D		2.6832809	-.65713088	-.35890773
	(5,2D)1D	.000000000	.0000000	.065451720	-.036362064
	(3,2D)1D	.27105232	3.4016792	-.57831697	-.59651528
	(1,2D)1D	-.23904566	-2.9999992	.59163665	.16326536
	(3,2P)1D		-3.2863346	.46947642	.62596855
	(5,2S)1D		.0000000	.30977406	-.17209671
4,3P	(5,2I)3P				
	(3,2H)3P				
	(5,4G)3P				
	(5,2G)3P				
	(3,2G)3P				
	(3,4F)3P		-1.7888539	.63279191	.036507253
	(5,2F)3P		-2.9999992	.40816322	.34467117
	(3,2F)3P		2.2360674	-.20687438	-.19267704
	(5,4D)3P		-4.2426395	.83698342	.67904060
	(5,2D)3P		2.9999992	-.34693874	-.37868477
	(3,2D)3P		-1.4142132	.41368145	-.035275139
	(1,2D)3P		.0000000	-.15272067	.084044819
	(3,4P)3P	.33806162	1.6733196	-.079681913	-.31872758
	(3,2P)3P	-.090350766	-2.2360674	.033464905	.28901555
2,3P	(5,2I)3P				
	(3,2H)3P				
	(5,4G)3P				
	(5,2G)3P				
	(3,2G)3P				
	(3,4F)3P		-3.3466393	.63745517	.63745517
	(5,2F)3P		.0000000	.30544133	-.16968963

TABLE II. (continued)

1^{N}-1,1^{N+1}		d^4-sd^5	d^4-dd^5		
V,SL	(V,SL)SL	(2) R	(0) R	(2) R	(4) R
2,3P	(3,2F)3P		-3.3466392	.50085764	.44773638
	(5,4D)3P		.0000000	.43195929	-.23997739
	(5,2D)3P		.0000000	.15272067	-.084644816
	(3,2D)3P		-2.6457506	.34196781	.38396385
	(1,2D)3P		-2.9999993	.14285715	.25396824
	(3,4P)3P	.000000000	-3.5777079	.68146823	.68146823
	(3,2P)3P	-.33806162	-1.6733197	.079681915	.31872758
4,1S	(5,2I)1S				
	(5,2G)1S				
	(3,2G)1S				
	(5,2D)1S		-5.7999992	1.06122430	.87074820
	(3,2D)1S		-4.2426401	.69267593	.42330199
	(1,2D)1S		.0000000	-.45816202	.25453447
	(5,2S)1S	.72280619			
0,1S	(5,2I)1S				
	(5,2G)1S				
	(3,2G)1S				
	(5,2D)1S		.0000000	.000000000	.000000000
	(3,2D)1S		.0000000	.79355992	-.44086663
	(1,2D)1S		7.3484682	-1.399082	-1.399082
	(5,2S)1S	.000000000			

TABLE III. CONFIGURATION INTERACTION MATRIX ELEMENTS FOR d^5

$1^{N}-1'1^{N+1}$		d^5-rd^6	d^5-dd^6		
V,SL	(V,SL)SL	$R^{(2)}$	$R^{(0)}$	$R^{(2)}$	$R^{(4)}$
5,2I	(4,1I)2I	.063887641	5.0452483	-.48908026	-.48908026
	(4,3H)2I		-5.7965492	.62105890	.75907198
	(4,3G)2I		-3.7947322	.30977409	.55070946
	(4,1G)2I		2.5584079	-.10442484	-.32777791
	(2,1G)2I		.0000000	.17316896	-.096204975
	(4,3F)2I				
	(2,3F)2I				
	(4,1F)2I				
	(4,3D)2I				
	(4,1D)2I				
	(2,1D)2I				
	(4,3P)2I				
	(2,3P)2I				
	(4,1S)2I				
	(0,1S)2I				
3,2H	(4,1I)2H		3.6381803	-.38980508	-.47642842
	(4,3H)2H	-.11065664	4.5607004	-.53518429	-.75236051
	(4,3G)2H		-2.6832806	.24642377	.48067845
	(4,1G)2H		-.6030225	.30151127	-.052644815
	(2,1G)2H		3.9999989	-.28571425	-.36507931
	(4,3F)2H		1.9999994	-.38775501	-.24489792
	(2,3F)2H		-3.9999988	.28571425	.52380943
	(4,1F)2H		1.5491929	-.047424296	-.26873755
	(4,3D)2H				
	(4,1D)2H				
	(2,1D)2H				
	(4,3P)2H				
	(2,3P)2H				
5,4G	(4,3H)4G		-5.4160240	.55265559	.40221047
	(4,3G)4G	.20203046	3.5456200	-.27134851	-.31355827
	(4,3F)4G		3.6514827	-.20493019	-.36432032
	(2,3F)4G		.0000000	.22356017	-.12420005
	(4,5D)4G		4.4721348	-.54760840	-.63634124
	(4,3D)4G		-2.1821783	.089068525	.23627897
	(4,3P)4G				
	(2,3P)4G				
5,2G	(4,1I)2G		3.0748237	-.47063631	-.20219932
	(4,3H)2G		1.1547002	-.18852250	-.11520820
	(4,3G)2G	-.23690169	4.9135368	-.67686483	-.55987585
	(4,1G)2G	-.28749438	2.2934216	-.22232055	-.22232055
	(2,1G)2G	.000000000	.0000000	.15523238	-.086240210

$1^{N}-1'1^{N+1}$		d^5-sd^6	d^5-dd^6		
V,SL	(V,SL)SL	$R^{(2)}$	$R^{(0)}$	$R^{(2)}$	$R^{(4)}$
5,2G	(4,3F)2G		-4.2817429	.50244948	.53643187
	(2,3F)2G		.0000000	-.26214754	.14563752
	(4,1F)2G		3.3166239	-.25382330	-.35911893
	(4,3D)2G		-2.0470647	.20888416	.27387036
	(4,1D)2G		-2.5071320	.15349790	.29278301
	(2,1D)2G		.0000000	-.21707878	.12059932
	(4,3P)2G				
	(2,3P)2G				
	(4,1S)2G				
	(0,1S)2G				
3,2G	(4,1I)2G		-2.6331815	.61797784	.18822615
	(4,3H)2G		2.9664785	-.72648459	-.27915844
	(4,3G)2G	.16898497	2.9130241	-.63908187	-.31541251
	(4,1G)2G	-.13677526	2.8368317	-.31062844	-.36777873
	(2,1G)2G	.16495717	2.8949866	-.35448819	-.18216756
	(4,3F)2G		2.9999992	-.35204075	-.49489786
	(2,3F)2G		3.9999989	-.48979583	-.41043077
	(4,1F)2G		-.2881988	.040837571	.020493192
	(4,3D)2G		2.3905566	-.43906347	-.30626034
	(4,1D)2G		-.9758998	-.099581600	.24120881
	(2,1D)2G		-2.7602614	.14082970	.20322414
	(4,3P)2G				
	(2,3P)2G				
	(4,1S)2G				
	(0,1S)2G				
3,4F	(4,3H)4F		-3.5486201	.57887679	.28340849
	(4,3G)4F		-3.2071339	.44179912	.30180517
	(4,3F)4F	.15649211	1.2649107	.16134065	-.30947166
	(2,3F)4F	.000000000	5.0596429	-.36140312	-.36140312
	(4,5D)4F		-4.4721349	.52141259	.66169348
	(4,3D)4F		-1.3093070	.13360278	.14918976
	(4,3P)4F		-1.3522464	-.068992161	.26906946
	(2,3P)4F		-2.5298214	.18070165	.18070165
5,2F	(4,1I)2F				
	(4,3H)2F		3.3636705	-.61781709	-.28744781
	(4,3G)2F		.5070924	-.080203397	-.052031597
	(4,1G)2F		-3.7606961	.44130643	.32191920
	(2,1G)2F		.0000000	-.25454744	.14141525
	(4,3F)2F	.37115368	2.9999992	-.41326524	-.34183668
	(2,3F)2F	.000000000	.0000000	.18367342	-.10204079
	(4,1F)2F	.22360674	3.8729823	-.37544221	-.37544221

TABLE III. (continued)

$1^N\text{-}1'1^{N+1}$ V,SL	(V,SL)SL	$d^5\text{-}sd^6$ (2) R	$d^5\text{-}dd^6$ (0) R	(2) R	(4) R
5,2F	(4,3D)2F		-4.1403922	.50698686	.50698686
	(4,1D)2F		1.6903081	-.13798434	-.17822978
	(2,1D)2F		.0000000	.14635447	-.081308082
	(4,3P)2F		-3.2071339	.35998446	.41089135
	(2,3P)2F		.0000000	-.24489789	.13605439
3,2F	(4,1I)2F				
	(4,3H)2F		2.5071318	-.25582983	-.43491068
	(4,3G)2F		-3.4016792	.26033264	.63829701
	(4,1G)2F		2.8030587	-.32893039	-.35117690
	(2,1G)2F		-3.3806159	.34496087	.25105485
	(4,3F)2F	.16598496	-2.2360674	.23501527	.38408644
	(2,3F)2F	.22131328	1.7888539	-.18253613	-.20383202
	(4,1F)2F	.21428566	1.7320503	-.026810992	-.31518605
	(4,3D)2F		1.8516397	-.075577152	-.38418376
	(4,1D)2F		2.2677861	-.18512548	-.32911186
	(2,1D)2F		-2.1380892	.065451738	.24362585
	(4,3P)2F		-2.3904565	.34637231	.35775544
	(2,3P)2F		3.5777076	-.36507227	-.40766402
5,4D	(4,3H)4D				
	(4,3G)4D		-4.5355725	.50909492	.31111359
	(4,3F)4D		3.0983860	-.28454567	-.24766012
	(2,3F)4D		.0000000	.18969709	-.10538728
	(4,5D)4D	-.26726118	4.4721349	-.70732751	-.54740841
	(4,3D)4D	-.23473819	3.9279211	-.30060624	-.34736720
	(4,3P)4D		-3.7947322	.25169147	.35710070
	(2,3P)4D		.0000000	-.28976711	.16098171
5,2D	(4,1I)2D				
	(4,3H)2D				
	(4,3G)2D		-2.2677860	.39339152	.21340848
	(4,1G)2D		-3.3636705	.44620124	.25933065
	(2,1G)2D		.0000000	.20697650	-.11498694
	(4,3F)2D		-4.6475787	.71136416	.49005087
	(2,3F)2D		.0000000	.37939420	-.21077455
	(4,1F)2D		-1.9999995	.22448976	.17687072
	(4,3D)2D	-.46947636	-2.6186140	.36072747	.29837952
	(4,1D)2D	-.063887636	-3.2071339	.31089567	.31089569
	(2,1D)2D	.000000000	.0000000	.046281350	-.025711862
	(4,3P)2D		-3.7947321	.48402201	.45390508
	(2,3P)2D		.0000000	.14488354	-.080490859
	(4,1S)2D		2.5298215	-.20651600	-.26674990
	(0,1S)2D		.0000000	.000000000	.000000008

$1^N\text{-}1'1^{N+1}$ V,SL	(V,SL)SL	$d^5\text{-}sd^6$ (2) R	$d^5\text{-}dd^6$ (0) R	(2) R	(4) R
3,2D	(4,1I)2D				
	(4,3H)2D				
	(4,3G)2D		3.2071338	-.58936848	-.41089135
	(4,1G)2D		-1.5856494	.064730408	.26607273
	(2,1G)2D		-2.3904563	.25085961	.16939179
	(4,3F)2D		-2.1908896	.49183240	.23101220
	(2,3F)2D		2.1908896	-.42476434	-.13786212
	(4,1F)2D		2.8284261	-.34633796	-.34633796
	(4,3D)2D	-.22131328	3.7032793	-.54793425	-.54793426
	(4,1D)2D	-.27105231	-1.5118574	.43967286	.043710183
	(2,1D)2D	-.19166292	3.2071339	-.27816983	-.26544309
	(4,3P)2D		-1.7888538	-.009126788	.41679081
	(2,3P)2D		-8.3464390	.39271791	.39207697
	(4,1S)2D		-1.7888538	.10428252	.24946604
	(0,1S)2D		.0000000	-.25094566	.13941426
1,2D	(4,1I)2D				
	(4,3H)2D				
	(4,3G)2D		.0000000	-.25975344	.14430746
	(4,1G)2D		.0000000	-.12842554	.071347521
	(2,1G)2D		3.7947322	-.65826990	-.43239298
	(4,3F)2D		.0600000	-.17744539	.098580768
	(2,3F)2D		5.7965494	-.88722699	-.95623352
	(4,1F)2D		.0000000	-.22908100	.12726722
	(4,3D)2D	.000000000	.0000000	-.29993745	.16663191
	(4,1D)2D	.000000000	.0000000	-.12244895	.068027191
	(2,1D)2D	-.16903081	2.8284264	-.18759973	-.49064542
	(4,3P)2D		.0000000	-.14488354	.080490854
	(2,3P)2D		3.7947322	-.67763080	-.57222155
	(4,1S)2D		.0000000	-.14488354	.080490862
	(0,1S)2D		-3.0983859	.22131331	.22131331
3,4P	(4,3H)4P				
	(4,3G)4P				
	(4,3F)4P		3.8643660	-.39432313	-.44032747
	(2,3F)4P		-3.8643660	.27602620	.27602620
	(4,5D)4P		-4.4721349	.47918736	.85183527
	(4,3D)4P		3.0558497	-.077934973	-.47800101
	(4,3P)4P	-.29276995	1.9321831	-.34603271	-.13801309
	(2,3P)4P	.000000000	-4.1311812	.29508442	.29508442
3,2P	(4,1I)2P				
	(4,3H)2P				
	(4,3G)2P				

TABLE III. (continued)

	$1^{N}-1,1^{N+1}$	d^5-sd^6	d^5-dd^6		
V,SL	(V,SL)SL	(2) R	(0) R	(2) R	(4) R
3,2P	(4,16)2P				
	(2,16)2P				
	(4,3F)2P		-3.6514827	.86443271	.36016082
	(2,3F)2P		-2.9211862	.52164043	.20865618
	(4,1F)2P		-2.8284263	.69267589	.15392800
	(4,3D)2P		2.3094004	-.62448077	-.18459497
	(4,1D)2P		2.8284263	-.30304870	-.37038919
	(2,1D)2P		3.9999989	-.42857137	-.28571428
	(4,3P)2P	-.11065664	3.6514827	-.74147461	-.42849036
	(2,3P)2P	-.41403924	2.7328194	-.48794995	-.19518000
5,6S	(4,5D)6S		-8.9442705	.63887651	.63887651
5,2S	(4,1I)2S				
	(4,3G)2S				
	(4,1G)2S				
	(2,1G)2S				
	(4,3D)2S		-6.9282018	1.06043909	.77052473
	(4,1D)2S		5.6568532	-.63495293	-.40026596
	(2,1D)2S		.0000000	.48979580	-.27210879
	(4,1S)2S	.51110113			
	.0,1S)2S	.000000000			