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A NOTE ON THE PLASMA MOMENT EQUATIONS IN A DIPOLE FIELD

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ABSTRACT

A coordinate system is discussed which is defined with respect to a magnetic dipole field by unit vectors in the direction of the field, in the azimuthal direction and in a direction along the dipole equipotential lines. The velocity terms of the continuity and momentum equations pertinent to plasma motion are expressed in terms of variations in these directions.

A NOTE ON THE PLASMA MOMENT EQUATIONS IN A DIPOLE FIELD

1. Introduction

The magnetic dipole field plays an important role in determining the properties of the plasma distribution and the nature of the waves propagating within the plasmasphere since the magnetic field in the region is approximately dipolar. Therefore coordinates should be used which reflect the geometry of the field keeping in mind that an investigation of the plasma moment equations is simplified by expressing vector quantities in terms of components along the directions in which the principal processes take place.

Gothard (1967) has summarized the different descriptions of the dipole field and has investigated the mathematical form of the differential distances along and across the field lines. This paper will develop the mathematical forms of the continuity and momentum equations in a curvilinear coordinate system defined at every point by unit vectors in the direction of the field ($\hat{i}_{\parallel}$), in the direction perpendicular to the field direction but lying in the plane of the field line ($\hat{i}_{\perp}$), and in the azimuthal direction about the dipole axis ($\hat{i}_{\varphi}$). The orientations of these vectors are depicted in Figure 1.

The distance along a field line is the coordinate with respect to which the thermal plasma moment equations are frequently expressed in studies of ionosphere-protonosphere coupling (see for example Angerami and Thomas, 1964, and Tamao, 1966). The azimuthal direction will correspond to the direction of the plasma velocity for corotation about the earth's dipole axis. The direction $\hat{i}_\perp$, which is identical to the radial direction in the equatorial plane, will be useful in studies of diffusion across field lines. Although many of the terms to be investigated have previously been used in ionospheric and magnetospheric computations, their complete expression in terms of these coordinates has not been previously developed.

2. The Coordinate System

In terms of spherical coordinates (r, θ, φ) where r is the geocentric distance, θ is the colatitude and φ is the azimuthal angle, the dipole magnetic field is:

$$\vec{B} = \frac{2a \cos\theta}{r^3}\,\hat{r} + \frac{a \sin\theta}{r^3}\,\hat{\theta}$$

or

$$\vec{B} = \mathrm{grad}\left(\frac{a}{r^2}\cos\theta\right)$$

where a is the dipole moment.

The unit vectors along and perpendicular to the field direction (Gothard, 1967) are

$$\hat{i}_{\parallel} = \frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{\sqrt{1 + 3\cos^2\theta}}$$

$$\hat{i}_{\perp} = \frac{-2\cos\theta\,\hat{\theta} + \sin\theta\,\hat{r}}{(1 + 3\cos^2\theta)^{1/2}}$$

$$\hat{i}_{\varphi} = \hat{\varphi}.$$

These unit vectors form an orthogonal system and are related to one another by

$$\hat{i}_{\perp} = \hat{i}_{\parallel} \times \hat{i}_{\varphi}.$$

The relative orientation between the direction $\hat{i}_{\perp}$ and the direction of the dipole gradient $\hat{i}_{\nabla}$ is given as a function of colatitude by

$$\hat{i}_{\nabla} \cdot \hat{i}_{\perp} = \frac{-\sin\theta\,(1 + \cos\theta)}{[(1 + 3\cos^2\theta)^2 + \cos^2\theta\,\sin^2\theta]^{1/2}\,[4 - 3\sin^2\theta]^{1/2}}.$$

The length coordinates corresponding to these unit vectors are easily determined. Measured from the center of the dipole axis, the distances along these coordinate directions are in terms of the dipole latitude λ and the geocentric distance r:

$$S_{\parallel} = \frac{r}{2\sqrt{3}\cos^2\lambda}\left[\ell n\left(\frac{\sqrt{3} + 2}{\sqrt{3}\sin\lambda + \sqrt{1 + 3\sin^2\lambda}}\right) - \sqrt{3}\sin\lambda\,\sqrt{1 + 3\sin^2\lambda} + 2\sqrt{3}\right]$$

$$S_{\perp} = \frac{r}{2\sin^{1/2}\lambda}\,\lim_{\epsilon\to 0}\int_{\epsilon}^{\lambda}\left(\frac{1 + 3\sin^2\lambda}{\sin\lambda}\right)^{1/2} d\lambda$$

and along the azimuthal direction

$$S_{\varphi} = r\,\varphi\,\sin\theta.$$

The distance $S_{||}$ is the distance along a field line (i.e., along the curve $r = \text{constant} \circ \cos^2 \lambda$) whereas $S_\perp$ is the distance along an equipotential line of the dipole ($r^2 = \text{constant} \circ \sin \lambda$). The reference level for S_φ is taken as $\varphi = 0$.

The expression for $S_\perp$ contains an integral which has not been evaluated analytically. However, near the equator where $\sin \lambda \sim \lambda$ the integral can be evaluated approximately to yield

$$S_\perp \approx r \left[1 + \frac{3}{10} \lambda^2 \right] .$$

It should be noted that near the poles ($\lambda = \pm \pi / 2$) the differential distance $dS_\perp$ approximates to $r\,d\varphi$ just as near the equator $dS_{||} \sim r\,d\varphi$. However, these coordinates should be applied with caution near the poles since $\hat{i}_\perp$ is a zero vector at the pole and the divergence of this unit vector has a singularity at the pole. In a dipole meridian plane, the contours of $\hat{i}_\perp$ as defined are mirror reflections about the dipole axis and $\hat{i}_\perp$ is directed in the polar regions towards the pole. Hence the vectors in the perpendicular direction converge at the pole forming the singularity.

3. Continuity Equation

The continuity equation has the general form

$$\frac{\partial \rho}{\partial t} + \text{div}\, \vec{J} = S$$

where ρ is the density, $\vec{J}$ is the flux density and S represents source and sink terms. Only the divergence term is to be considered since the other terms do not explicitly involve particular spatial directions.

Expressed in terms of the three previously discussed coordinates, the divergence term is

$$\operatorname{div} \vec{J} = \operatorname{div} (J_{\parallel} \hat{i}_{\parallel} + J_{\perp} \hat{i}_{\perp} + J_{\varphi} \hat{i}_{\varphi})$$

or using the relationship $\hat{i}_s \cdot \operatorname{grad} f = \frac{\partial f}{\partial s}$:

$$\operatorname{div} \vec{J} = \frac{\partial J_{\parallel}}{\partial S_{\parallel}} + \frac{\partial J_{\perp}}{\partial S_{\perp}} + J_{\parallel} \operatorname{div} \hat{i}_{\parallel} + J_{\perp} \operatorname{div} \hat{i}_{\perp} + \frac{\partial J_{\varphi}}{\partial S_{\varphi}}$$

where use has been made of $\operatorname{div} \hat{i}_{\varphi} = 0$. In terms of spherical coordinates the differentiation operators are

$$\frac{\partial}{\partial S_{\parallel}} = \frac{2 \cos \theta}{\sqrt{1 + 3 \cos^2 \theta}} \frac{\partial}{\partial r} + \frac{\sin \theta}{r \sqrt{1 + 3 \cos^2 \theta}} \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial S_{\perp}} = \frac{\sin \theta}{\sqrt{1 + 3 \cos^2 \theta}} \frac{\partial}{\partial r} - \frac{2 \cos \theta}{r \sqrt{1 + 3 \cos^2 \theta}} \frac{\partial}{\partial \theta}$$

and

$$\frac{\partial}{\partial S_{\varphi}} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} .$$

The divergence quantities in terms of spherical coordinates are:

$$\operatorname{div} \hat{i}_{\parallel} = \frac{3 \cos \theta}{r} \frac{(3 + 5 \cos^2 \theta)}{(1 + 3 \cos^2 \theta)^{3/2}}$$

$$\operatorname{div} \hat{i}_{\perp} = \frac{2 \sin \theta}{r \sqrt{1 + 3 \cos^2 \theta}} \left\{ 1 + \frac{\sin^2 \theta - \cos^2 \theta - 3 \cos^4 \theta}{(1 + 3 \cos^2 \theta) \sin^2 \theta} \right\} .$$

The first quantity can also be written in the familiar form:

$$\operatorname{div} \hat{i}_{\parallel} = - \frac{1}{B} \frac{\partial B}{\partial S_{\parallel}}$$

so that

$$\operatorname{div} (J_{\parallel} \hat{i}_{\parallel}) = B \frac{\partial J_{\parallel}/B}{\partial S_{\parallel}} .$$

It is interesting to speculate as to what function F is required in order to have a similar expression for the divergence of the perpendicular vector. That is

$$\operatorname{div} \hat{i}_{\perp} = \frac{1}{F} \frac{\partial F}{\partial S_{\perp}} .$$

Such a functional form would be an aid in exploring plasma phenomena in which the perpendicular flux is important. In terms of spherical coordinates the differential equation for F can be written

$$\frac{\partial F}{\partial r} - \frac{2}{r} \frac{\cos \theta}{\sin \theta} \frac{\partial F}{\partial \theta} = - \frac{2}{r} \left\{ 1 + \frac{1}{1 + 3 \cos^2 \theta} - \cot^2 \theta \right\} F .$$

Solving this equation by separation of variables yields

$$F = K\, r^{m} \frac{(1 + 3\cos^2\theta)^{1/2}}{\sin\theta\,(\cos\theta)^{1/2\, m + 2}}$$

where K is an arbitrary constant and m is determined by the geometry.

Interpreting F as the inverse of the cross sectional area of a tube defined by equipotential lines (i.e., lines parallel to $\hat{i}_{\perp}$), in analogy to the concept of a magnetic flux tube, the constant m must be -4 in order to have a non-vanishing area at the equator. This value can also be obtained by considering the geometry of an equipotential tube of infinitesimal cross section at the equatorial plane – a procedure which illuminates some properties of this coordinate system.

In a meridian plane the equipotential lines are of the general form as shown in Figure 2. In the equatorial plane the equipotential lines are radial (see Figure 1b). Therefore the differential cross sectional area of a potential tube at the equator is

$$(r\, d\varphi)\,(r\, d\,|\lambda|)$$

where $|\lambda|$ is the angle between the equator and the wall of the flux tube in a meridian plane. But along an orthogonal line (i.e., an equipotential)

$$r = M \sin^{1/2} |\lambda|$$

where M is a constant. Therefore near the equator

$$|\lambda| \sim \frac{r^2}{M^2}.$$

The differential area is then

$$\frac{r^4}{M^2} d\varphi$$

and hence the value of m is -4. For this value the function F is

$$F = \frac{KB}{r \sin\theta}$$

and the divergence of $\vec{J}$ can be put into the particularly simple form

$$\text{div } \vec{J} = B \frac{\partial (J_{\parallel}/B)}{\partial S_{\parallel}} + \frac{B}{r \sin\theta} \frac{\partial \left(\frac{J_{\perp} r \sin\theta}{B} \right)}{\partial S_{\perp}} + \frac{\partial J_{\varphi}}{\partial S_{\varphi}}.$$

Near the equator in the approximation $\sin\lambda \approx \lambda$ and $\cos\lambda \approx 1$ the divergence becomes in spherical coordinates

$$\text{div } \vec{J} = 2\lambda \frac{\partial J_{\parallel}}{\partial r} + \frac{1}{r}\frac{\partial J_{\parallel}}{\partial \lambda} + \frac{\partial J_{\perp}}{\partial r} - \frac{2\lambda}{r}\frac{\partial J_{\perp}}{\partial \lambda} + \frac{4}{r} J_{\perp} + \frac{9\lambda}{r} J_{\parallel} + \frac{1}{r}\frac{\partial J_{\varphi}}{\partial \varphi}.$$

At the equator ($\lambda = 0$) this expression reduces to a form which is similar to that of the corresponding divergence in a spherically symmetric system. That is, at $\lambda = 0$,

$$\text{div}\,(J_\perp,\ J_{\parallel},\ J_\varphi) = \frac{\partial J_\perp}{\partial r} + \frac{4}{r} J_\perp + \frac{1}{r}\frac{\partial J_{\parallel}}{\partial \lambda} + \frac{1}{r}\frac{\partial J_\varphi}{\partial \varphi}$$

whereas in a spherical coordinate system

$$\text{div}\,(J_r,\ J_\theta,\ J_\varphi) = \frac{\partial J_r}{\partial r} + \frac{2}{r} J_r + \frac{1}{r}\frac{\partial J_\theta}{\partial \lambda} + \frac{1}{r}\frac{\partial J_\varphi}{\partial S_\varphi}.$$

4. Momentum Equation

The momentum equation for one component of a thermal plasma has the general form

$$\frac{d\vec{v}}{dt} = \frac{-\,\text{grad}\,P}{\rho} + \frac{\vec{F}}{\rho}$$

where P is the pressure, ρ the density and **F** denotes forces due to gravity, electric and magnetic fields and collisional interactions. The time derivative is the convective acceleration.

In terms of the length coordinates discussed in the previous sections the gradient operator is

$$\text{grad} = \hat{i}_{\parallel}\,\frac{\partial}{\partial S_{\parallel}} + \hat{i}_\perp\,\frac{\partial}{\partial S_\perp} + \hat{i}_\varphi\,\frac{\partial}{\partial S_\varphi}\ .$$

This form of the operator is needed to express the convective derivative

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot \text{grad}\,\vec{v}$$

in the desired form.

The local variation of the velocity with time can be written

$$\frac{\partial \vec{v}}{\partial t} = \hat{i}_{\parallel} \frac{\partial v_{\parallel}}{\partial t} + \hat{i}_{\perp} \frac{\partial v_{\perp}}{\partial t} + \hat{i}_{\varphi} \frac{\partial v_{\varphi}}{\partial t} .$$

The spatial derivative is evaluated by a straightforward expansion of the vector components along the $\hat{i}_{\parallel}$, $\hat{i}_{\perp}$ and $\hat{i}_{\varphi}$ directions. The three orthogonal components are

$$[(\vec{v} \cdot \text{grad})\,\vec{v}]_{\parallel} = v_{\parallel} \frac{\partial v_{\parallel}}{\partial S_{\parallel}} + v_{\perp} \frac{\partial v_{\parallel}}{\partial S_{\perp}} + v_{\varphi} \frac{\partial v_{\parallel}}{\partial S_{\varphi}} - v_{\varphi}^2 \frac{3 \cos \theta}{r \sqrt{1 + 3 \cos^2 \theta}}$$

$$+ \frac{3 v_{\perp} \left(v_{\parallel} - 2 v_{\perp} \dfrac{\cos \theta}{\sin \theta} \right) \sin \theta \, (1 + \cos^2 \theta)}{r \, (1 + 3 \cos^2 \theta)^{3/2}}$$

$$[(\vec{v} \cdot \text{grad})\,\vec{v}]_{\perp} = v_{\parallel} \frac{\partial v_{\perp}}{\partial S_{\parallel}} + v_{\perp} \frac{\partial v_{\perp}}{\partial S_{\perp}} + v_{\varphi} \frac{\partial v_{\perp}}{\partial S_{\varphi}} - v_{\varphi}^2 \frac{(1 - 3 \cos^2 \theta)}{r \sin \theta \, (1 + 3 \cos^2 \theta)}$$

$$+ 3 v_{\parallel} \left(v_{\parallel} - 2 v_{\perp} \frac{\cos \theta}{\sin \theta} \right) \frac{\sin \theta \, (1 + \cos^2 \theta)}{r \, (1 + 3 \cos^2 \theta)^{3/2}}$$

$$[(\vec{v} \cdot \text{grad})\,\vec{v}]_{\varphi} = v_{\parallel} \frac{\partial v_{\varphi}}{\partial S_{\parallel}} + v_{\perp} \frac{\partial v_{\varphi}}{\partial S_{\perp}} + v_{\varphi} \frac{\partial v_{\varphi}}{\partial S_{\varphi}}$$

$$+ \frac{v_{\varphi}}{r \sqrt{1 + 3 \cos^2 \theta}} \left[v_{\parallel} \; 3 \cos \theta + v_{\perp} \frac{(1 - 3 \cos^2 \theta)}{\sin \theta} \right] .$$

The terms quadratic in $v_{\parallel}$, $v_{\perp}$ or v_{φ} correspond to centripetal acceleration components whereas those terms which consist of the product of two different velocity components correspond to components of the coriolis acceleration.