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ANALYTICAL DETERMINATION OF RADIATION INTERCHANGE FACTORS

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FINAL REPORT

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FOREWORD

On 7 June 1968 the NASA Manned Spacecraft Center and Hughes Aircraft Company entered into Contract NAS 9-7980 for a study of analytical and empirical determination of radiation interchange factors. The study consisted of two concurrent phases: a 12 month analytical program (Phase I) with a generalized computer program as the end product, and a 6-month feasibility study (Phase II) to determine possible experimental techniques for measuring radiation interchange factors.

This document is submitted in partial fulfillment of the Phase I requirements. It contains the analytical derivation of the algorithm for computing thermal and solar radiation interchange factors, together with a very brief description of the computer program which implements the algorithm. A complementary volume, the program Users" Manual (Reference 15), describes the computer program in detail.

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ABSTRACT

A matrix formulation is presented for computing thermal and solar script-F radiation interchange factors in an enclosure containing surfaces with arbitrary emission and reflection characteristics. An abstract enclosure is postulated having a total of M nodal areas; N (≤ M) areas have directional emittance with bidirectional reflectance. The remaining (M-N) nodes emit diffusely and have diffuse-plus-specular components of reflectance. An additional node, A_{M+1}, is used to depict black space. The derivation starts with a generalized form of Bevans and Edwards "third approximation" for the emergent directional flux density from a nondiffuse node A_k to an arbitrary node A_w. The emergent directional flux, termed pseudoradiosity, is analogous to simple radiosity for surfaces that emit diffusely and have a finite diffuse component of reflectance. A total of $[N \times (M+1) + M+1-N]$ simultaneous algebraic equations are cast in matrix form to account for all the radiant exchange in the enclosure and to space. The solutions to the set of flux equations are expressed as sums of M products of excitation components and elements in an inverted transfer matrix. The concept of net heat flux is used with a thermodynamic argument to define expressions for script-F factors in terms of surface properties, geometry factors, and elements of the inverted transfer matrix. This formulation is exact within the framework of nodal analysis common to digital computer thermal analyzer programs.

An approximate algorithm is developed to a second order of approximation for use in minimizing computer time for inverting very large matrices. Groups of nodal areas are collected into subsets, and mean values of pseudoradiosity or radiosity are found for each subset by inverting a relatively small transfer matrix. The mean values are used iteratively to obtain second order approximate solutions for individual nodes. The iterated solutions are introduced in the net heat flux expressions to obtain node-to-node script-F factors.

A brief description is given of a computer program that implements the script-F algorithms. The computer program uses CONFAC as a subroutine to compute shape factors between two real surfaces and between a real surface and a virtual image "behind" a specular reflector. Input requirements include surface properties (bidirectional reflectance, diffuse and specular components of reflectance) and geometrical data to locate real surfaces; shape factors and specular exchange factors may be input directly in lieu of computing them. The program node capacity varies according to

the surface properties of the nodes and the choice of exact or mean-to-local algorithms. The exact algorithm can accommodate a maximum of 1500 nodal areas if all nodes are diffuse-plus-specular reflectors, but only 38 nodes if all surfaces reflect bidirectionally. In combinations, the number of nodes is $M \leq (1500 + N)/(N + 1)$. The program printout includes input data, shape factors and exchange factors, the transfer matrix and its inverse, the area of each node, and the product of interchange factors and areas for half the nodes (reciprocity may be used to obtain the other half).

CONTENTS

APPENDICES

INTRODUCTION

Problems of radiative transfer in nonabsorbing media are described by a Fredholm integral equation of the second kind. The dependent variable of the field equation is a vector quantity, surface spectral intensity, but analytical simplification sometimes permits the dependent variable to take the form of a vector-like quantity, radiosity, or hemispherical emergent flux. In the most general case, the kernel of the integral is the product of the field geometry (the shape factor kernel) and the local reflection function (bidirectional reflectance), an empirical or semiempirical entity. Both the analytical formulation and the solution of such an integral equation for a complex physical structure is beyond the state or the art of contemporary thermophysical analysis and technology.

Typically, thermal design engineers are not interested either in intensity and radiosity or integral equations which lead to exact solutions. In this sense, an inability to solve or even formulate an integral equation does not represent an impediment to most thermal design engineers. Local surface intensity and radiosity are of secondary interest compared to temperature, net heat flux, and other considerations which include product reliability, costs, and schedules. The temperatures and heat fluxes of interest to thermal designers are mean values for reasonably small surface areas. During the analytical design process, the engineer will predict temperatures and fluxes with the aid of a digital computer using coefficients to account for conduction, convection, contact resistance, and radiation. The thermal designer's usual concern for radiative transfer analysis is confined to obtaining numerical values of radiation coefficients and understanding the range of application and accuracy of those coefficients.

This document presents a detailed derivation that starts with Bevans' and Edwards (Reference 1) "third approximation" for the intensity of a nodal area in an arbitrary enclosure and leads to an explicit formulation for a gray radiant interchange factor, Hottel's script-F, $\mathcal{F}_{ki}$ (Reference 2). The derivation considers thermal and solar excitation separately so that the results are of special use to spacecraft designers and engineers working with applications of solar energy. The formulation is termed "explicit" insofar as surface properties and geometrical factors are treated as inputs to a transfer matrix which, in theory, may be inverted to obtain an algebraic expression identifying the influence of each surface on any other surface.

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NOMENCLATURE

The notation and jargon of radiative transfer is irksome at best; at worst, it may be sufficiently arcane as to discourage both students and workers in other fields of energy transport from learning or using even the simplest analytical concepts. Unfortunately, this document does not present any breakthrough in the "nomenclature barrier." The need to account for locations and directions, real surfaces and images, thermal and solar wavebands, and to distinguish between primary and matrix functions has led to a cumbersome notation which uses, at times, four subscripts, superscripts, asterisks, carets, and tildas. Similarly, the jargon is propagated, and several new terms are introduced for the sake of verbal convenience.

LIST OF SYMBOLS

A_k	=	k^{th} nodal surface area
A_γ	=	composite area of γ^{th} subset
$\hat{b}_{\substack{kw\\j}}$	=	row matrix in the $(kw)^{th}$ row of the inverted transfer matrix (Equation 40)
$C_{s,k}$	=	direct solar incidence factor (Equation 63a)
$C_{s(p),k}$ and $C_{s(p,b),k}$	=	specular solar incidence factors accounting for reflections from specular surfaces A_p, A_b, etc. (Equations 63b and 63c)
$\tilde{C}_{s,k}$	=	complete solar incidence factor accounting for direct and specular incidence (Equation 65)
$D_{\substack{kw\\jp}}$, $D_{\substack{k'\\j'}}$, etc.	=	elements of transfer matrix occupying $(kw)^{th}$ row $(jp)^{th}$ column and $(k')^{th}$ row, (j') column, respectively
E_k	=	blackbody thermal excitation of $(k)^{th}$ node, σT_k^4

E_γ	=	mean value of blackbody thermal excitation of A_γ (Equation 88)
F_{kj}, $F_{\gamma\zeta}$	=	shape factors between nodes A_k and A_j and subsets A_γ and A_ζ, respectively
$\mathcal{J}_{ki}$	=	thermal radiation interchange factor accounting for radiant energy emitted by node A_i and absorbed at A_k
$\mathcal{J}_{ki}^*$	=	solar radiation interchange factor accounting for solar energy (direct plus specular incidence) reflected from A_i and absorbed at A_k
G_k	=	irradiation (i. e., incident flux density) at A_k
H_{jkw}, H_{jk}	=	second-order approximations for $\hat{\delta}_{jk_w}$ and $\hat{\delta}_{j_w}$ arising from the mean-to-local concept
$\bar{I}_{kw}$	=	total intensity of A_k directed toward A_w, units of power per unit area per steradian
J_k	=	radiosity of A_k (i. e., diffuse emergent flux density)
J_γ	=	radiosity of subset A_γ (Equation 82)
$\mathcal{J}_{kw}$	=	pseudoradiosity of node A_k, $\pi\bar{I}_{kw}$ (i. e., directional emergent flux density of A_k toward A_w)
$\mathcal{J}_{\gamma\zeta}$	=	pseudoradiosity of subset A_γ directed toward subset A_ζ
k, j, i, p, b, w	=	indices for summation and subscripts for identifying nodal areas
M	=	total number of nodal areas in an enclosure
m	=	total number of subsets in an enclosure
N	=	total number of directionally emitting, bidirectionally reflecting nodal areas in an enclosure
n	=	number of directional emitting, bidirectional reflecting subsets in an enclosure
$q_{k,\,net}$	=	net heat flux at A_k

$R_{\zeta\gamma\chi}$, $R'_{\zeta\gamma\chi}$, etc.	=	composite bidirectional reflection factors accounting for radiant energy originating at subset A_ζ and being reflected from subset A_γ toward subset A_χ (Equation 89)
$R_{\zeta\gamma}$, $R'_{\zeta\gamma}$, etc.	=	composite diffuse reflection factors for radiant energy originating at subset A_ζ and being reflected from subset A_γ (Equation 89)
$\overline{r}_{jkw}$, $\overline{r}^*_{jkw}$	=	nondimensional bidirectional reflectances, thermal and solar, respectively, of A_k for radiation arriving from A_j and directed toward A_w
S	=	solar constant, power/unit area
T	=	absolute temperature
$Z_{s\gamma}$	=	composite complete solar incidence factor of subset A_γ (Equation 100)
α_k, α^*_k	=	hemispherical thermal and solar absorptance, respectively, of diffusely emitting node A_k
$\beta_{\substack{kw \\ jp}}$	=	element of $\hat{b}_{\substack{kw \\ j}}$ occupying the pth position (Equation 40 and 39)
γ, ζ, ν, ξ, χ	=	indices for summation and subscripts for identifying subset areas
$\Gamma_{j\gamma\chi}$, $\Gamma_{j\gamma_k\chi_i}$	=	first-order approximation to H_{jki} (Equation 92a)
$\Gamma_{j\gamma}$, $\Gamma_{j\gamma_k}$	=	first-order approximation to H_{jk} (Equation 92b)
δ^i_k	=	Kronecker delta
$\hat{\delta}_{\substack{kw \\ j}}$, $\hat{\delta}_{\substack{k' \\ j}}$	=	a row matrix of the modified inverse transfer matrix (Equation 47), occupying (kw)th and (k')th rows, respectively. Modified resolvants of the linear set of radiosity and pseudoradiosity equations (Equation 48)
$\hat{\delta}_{\substack{\gamma\chi \\ \zeta}}$, $\hat{\delta}_{\substack{\gamma \\ \zeta}}$	=	modified resolvants for mean-to-local first approximation

ϵ_k, $\bar{\epsilon}_k$	=	hemispherical emittance of diffusely emitting and directionally emitting surfaces, respectively
$\bar{\epsilon}_{kw}$	=	directional emittance of A_k toward A_w
ρ_k, ρ_γ	=	semigray reflectances of k^{th} node and γ^{th} subset, respectively
ρ_k^d, ρ_k^m	=	diffuse and specular components of reflectance, respectively
$\bar{\rho}_{kj}$	=	hemispherical-directional semigray reflectance of A_k in the direction of A_j
σ	=	Stefan-Boltzmann constant
φ_{kj}	=	complete exchange factor. Fraction of diffuse (or pseudodiffuse) energy that leaves A_k and arrives at A_j both directly and by all possible intervening specular reflections
$\varphi_{kj(p)}$	=	specular component of φ_{kj} accounting for energy leaving A_k and directed toward all possible images of A_j appearing "behind" A_p
$\varphi_{k,j(p,b)}$	=	specular component of φ_{kj} accounting for pseudodiffuse radiation leaving A_k and directed toward a secondary image of bidirectional surface A_j which appears "behind" both a real specular surface A_p and a specular image of A_b (in A_p)

SUBSCRIPTS AND SUPERSCRIPTS

$(\ )_{\gamma_k}$	=	node A_k contained in subset A_γ
$(\overline{\ })$	=	a mean value, areal or directional
$(\widehat{\ })$	=	a matrix
$(\ )^*$	=	pertaining to the solar waveband
$(\ )^d$	=	diffuse component
$(\ )^m$	=	specular (mirror-like) component

INTERCHANGE FACTOR FORMULATION PROBLEM STATEMENT

Consider an enclosure containing M nodal surfaces. The surfaces $k = 1, 2, \ldots, N$ are assumed to absorb and emit with a directional character and to have bidirectional reflectances. The surfaces $k = N + 1, N + 2, \ldots, M$ absorb and emit diffusely and have diffuse-plus-specular components $(\rho_k = \rho_k^d + \rho_k^m)$ of reflectance. Black space at absolute zero temperature is designated $k = M + 1$. The problem at hand is to derive coefficients $\mathcal{J}_{ki}$ and $\mathcal{J}_{ki}^*$ which account for the direct and reflected radiative energy exchange in thermal and solar wavebands, respectively. The coefficients will be used to express net heat flux at a surface as

$$\text{Thermal flux: } q_{k,net} = \sum_{i=1}^{M+1} \sigma \mathcal{J}_{k,i} \left(T_k^4 - T_i^4\right), \text{ Btu/hr-ft}^2$$

$$\text{Solar flux: } q_{k,net}^* = -S \left[\alpha_{ks}^* C_{sk} + \sum_{i=1}^{M} C_{s,i} \mathcal{J}_{ki}^* \right], \text{ Btu/hr-ft}^2$$

The node-to-node radiation interchange factors $\mathcal{J}_{ki}$ and $\mathcal{J}_{ki}^*$ will be formulated in terms of matrix elements obtained by inverting a matrix of size $(N^2 + M + 1 - N) \times (N^2 + M + 1 - N)$. An "exact" formulation will be developed first without regard to the magnitudes of N and M. Subsequently, an approximate formulation will be developed for a circumstance in which N and/or M are too large to accommodate matrix inversion in a reasonable amount of machine time. The approximate formulation is termed the "mean-to-local" method.

INTENSITY AND RADIOSITY EQUATIONS: THERMAL INTERCHANGE

The approach to the problem used here is due to Bevans and Edwards (Reference 1) and is generalized to permit both real (directional/bidirectional) and ideal (diffuse/specular) surfaces in an enclosure. The enclosure is shown schematically in Figure 1 and is assumed to have the following types and numbers of surfaces:

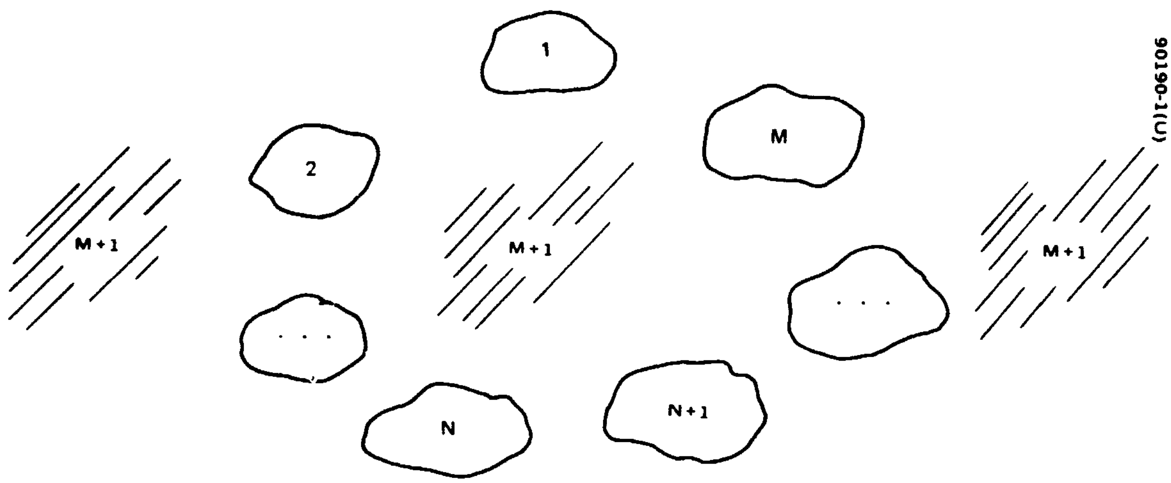

Figure 1. General Enclosure

Directional emitting, bidirectional reflecting	k = 1 (1) N
Diffuse emitting, diffuse-plus-specular reflecting	k = N+1 (1) M
Black space	k = M+1

Bevans and Edwards (Reference 1) derived a set of N^2 equations for mean directional intensity of a surface A_k in a directional/bidirectional enclosure:

$$\bar{I}_{kw} = \bar{\epsilon}_{kw} \frac{E_k}{\pi} + \sum_{j=1}^{n} \bar{r}_{jkw} \bar{I}_{jk} F_{kj} \quad (1)$$

where

$\bar{I}_{kw}$ = mean intensity of A_k in the direction of A_w, power/unit area-solid angle

E_k = Planckian emissive power, power/unit area

F_{kj} = conventional geometric shape factor

$\bar{\epsilon}_{kw}$ = mean directional emittance of A_k in the direction of A_w, dimensionless

$\bar{r}_{jkw}$ = mean bidirectional reflectance of A_k for energy incident from A_j and redirected to A_w, dimensionless.

Equation 1 describes an enclosure in which each surface "sees" itself (I_{kk}), and it is assumed that the node representing black space may be treated as a bidirectional surface. Despite its generality, Equation 1 is not suitable for an enclosure containing specularly (or diffuse-plus-specular) reflecting surfaces, except within the framework of Bevans' and Edwards' "one bounce approximation" (Reference 1). The image method (References 3 through 5) will be used below to develop a more general form of Equation 1. Additional insight to the "one-bounce" and image formulations is given in Appendix A.

If diffuse emitters and diffuse-plus-specular reflectors are introduced in an enclosure, it is necessary to write only a single intensity equation for each surface based on diffuse emission and diffuse reflection. Indeed, the concept of radiosity and specular exchange factors may be used in place of intensity

$$J_k = \epsilon_k E_k + \rho_k^d \left[\sum_{j=1}^{N} \left(\pi \bar{I}_{jk} F_{kj} + \sum_{p=N+1}^{M} \pi \bar{I}_{jp} \varphi_{kj(p)} \right) + \sum_{j=N+1}^{M} J_j \varphi_{kj} \right] \tag{2}$$

$$J_k = \epsilon_k E_k + \rho_k^d \left[\underbrace{G_k'}_{j=1(1)N} + \sum_{j=N+1}^{M} J_j \varphi_{kj} \right] \tag{3}$$

The partial irradiation G_k' for j=1(1)N will be discussed below. The exchange factors φ_{kj} and $\varphi_{kj(p)}$ take into account the direct exchange (F_{kj}) plus all specular reflections (e.g., $\rho_w^m F_{kj(w)}$)(Reference 5).

In the fully mixed enclosure, it is necessary to modify Equation 1 to account for the intensity directed at a specular reflector which arrives at another surface. The mean directional intensity is replaced by a directional flux which shall be called "pseudoradiosity" and is defined as

$$\mathcal{J}_{kw} = \pi \bar{I}_{kw} \tag{4}$$

The script-J and double subscript are used to distinguish this directional flux from the diffuse flux of true radiosity. A general expression for pseudoradiosity is

$$\mathcal{J}_{kw} = \bar{\epsilon}_{kw} E_k + \sum_{j=1}^{N} \left(\bar{r}_{jkw} F_{kj} \mathcal{J}_{jk} + \sum_{p=N+1}^{M} \mathcal{J}_{jp} \sum_{b=N+1}^{M} \bar{r}_{j(b)kw} \varphi_{kj(p,b)} \right)$$

$$+ \sum_{j=N+1}^{M} \left(\bar{r}_{jkw} F_{kj} + \sum_{p=N+1}^{M} \bar{r}_{j(p)kw} \varphi_{kj(p)} \right) J_j \quad (5)$$

The corresponding radiosity expression for a diffuse-plus-specular surface is

$$J_k = \epsilon_k E_k + \rho_k^d \left[\sum_{j=1}^{N} \left(\mathcal{J}_{jk} F_{kj} + \sum_{p=N+1}^{M} \mathcal{J}_{jp} \varphi_{kj(p)} \right) + \sum_{j=N+1}^{M} J_j \varphi_{kj} \right] \quad (6)$$

The use of "wrong way" shape factors and exchange factors is based on the reciprocity relation. Equations 5 and 6 represent a linear set of $[N^2 + M - N]$ algebraic equations which must be solved for the radiosities and pseudo-radiosities. The shape factors, exchange factors, emittances, reflectances, and emissive powers are assumed known; the product ϵE represents the excitation, or forcing function at each surface. The term $\mathcal{J}_{jp} \varphi_{k,j(p)}$ represents the radiant flux leaving a directional surface A_j which is specularly reflected from A_p. The notation on the exchange factor $\varphi_{k,j(p)}$ denotes that the energy appears to reach A_k from the images of A_j lying behind A_p.

The solution of Equations 5 and 6 by matrix algebra follows the discussion of geometrical factors and surface properties below.

Example: Exchange Factors for a Mixed Enclosure

The construction of Equations 5 and 6 may be explained by considering a specific enclosure. Figure 2 shows an enclosure of five surfaces; three of these (k = 1,2,3) are directional/bidirectional; the other two (k = 4,5) are diffuse-plus-specular. The image technique is used for this illustration, but alternative procedures may be used to construct the exchange factors.

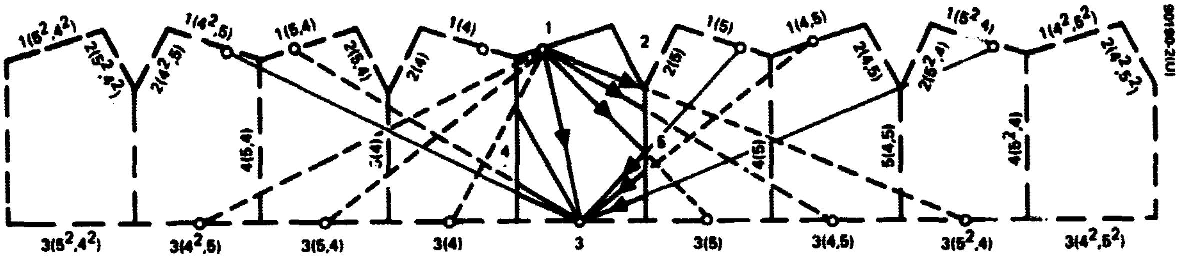

Figure 2. Mixed Enclosure

The pseudoradiosity from A_3 to A_2 may be expressed as

$$\mathcal{J}_{32} = \bar{\epsilon}_{32} E_3 + \sum_{j=1}^{3} \left(\bar{r}_{j32} F_{3j} \mathcal{J}_{j3} + \sum_{p=4}^{5} \mathcal{J}_{jp} \sum_{b=4}^{5} \bar{r}_{j(b)32} \varphi_{3j(p,b)} \right)$$

$$+ \sum_{j=4}^{5} \left(\bar{r}_{j32} F_{3j} + \sum_{p=4}^{5} \bar{r}_{j(p)32} \varphi_{3j(p)} \right) J_j \qquad (7)$$

The first summation will be expanded for $j = 1$:

$$\bar{r}_{132} F_{31} \mathcal{J}_{13} + \mathcal{J}_{14} \left[\bar{r}_{1(4)32} \varphi_{3,1(4,4)} + \bar{r}_{1(5)32} \varphi_{3,1(4,5)} \right]$$

$$+ \mathcal{J}_{15} \left[\bar{r}_{1(4)32} \varphi_{3,1(5,4)} + \bar{r}_{1(5)32} \varphi_{3,1(5,5)} \right] \qquad (8)$$

The first term, $\bar{r}_{132} F_{31} \mathcal{J}_{13}$, accounts for energy that originates at A_1, goes directly to A_3, and is redirected to A_2. The pseudoradiosity $\mathcal{J}_{13}$ is shown in Figure 2 as a vector from A_1 to A_3. The second term in Equation 8 accounts for the pseudoradiosity of A_1 which is initially directed toward A_4:

$$\mathcal{J}_{14} \left[\bar{r}_{1(4)32} \varphi_{3,1(4,4)} + \bar{r}_{1(5)32} \varphi_{3,1(4,5)} \right]$$

Part of this energy arrives at A3 after one or more specular reflections from the direction of A4, $\mathscr{J}_{14} \overline{r}_{1(4)32} \varphi_{3,1(4,4)}$, and the remainder from the direction of A5, $\mathscr{J}_{14} r_{1(5)32} \varphi_{3,1(4,5)}$. In the general case, the bidirectional reflectances $\overline{r}_{1(4)32} (\approx r_{432})$ and $\overline{r}_{1(5)32} (\approx \overline{r}_{532})$ are different so that they cannot be combined. This distribution of the pseudoradiosity $\mathscr{J}_{14}$ is shown in Figure 2 as three vectors from A_1 to A_4; two of these vectors account for the first and third images of A_1 lying behind A_4 and comprise the first two terms of $\varphi_{3,1(4,4)}$

$$\varphi_{3,1(4,4)} = \rho_4^m \left\{ F_{3,1(4)} + \rho_4^m \rho_5^m \left[F_{3,1(4^2,5)} + \cdots \right] \right\}$$

The third vector of $\mathscr{J}_{14}$ "arrives" at A_3 from the second image of A_1 behind A_5 and is accounted in the first term of $\varphi_{3,1(4,5)}$:

$$\varphi_{3,1(4,5)} = \rho_4^m \rho_5^m \left\{ F_{3,1(4,5)} + \rho_4^m \rho_5^m \left[F_{3,1(4^2,5^2)} + \cdots \right] \right\}$$

Similarly, the second term in Equation 8 accounts for the pseudoradiosity $\mathscr{J}_{15}$ which arrives at A_3 from images behind both A_4 and A_5.

The first summation in Equation 7 for j = 2 is similar to that of j = 1. The case for j = 3 is also similar except that (r_{332} F_{33} $\mathscr{J}_{33}$) may appear if A_3 is concave; however, if A_3 is plane or convex, then $r_{332} = 0$, $F_{33} = 0$, and $\mathscr{J}_{33} = 0$.

The second summation in Equation 7 contains the diffuse contribution of the diffuse-plus-specular surfaces. The expansion for j = 4 is

$$\left(r_{432} F_{34} + r_{4(4)32} \varphi_{34(4)} + r_{4(5)32} \varphi_{34(5)} \right) J_4 \quad (9)$$

The first term r_{432} F_{34} J_4 accounts for diffuse energy, leaving A_4 which is reflected at A_3 toward A_2. The second term in Equation 9 accounts for diffuse energy which appears to originate at images of A_4 lying behind A_4. For example

$$\varphi_{34(4)} = \rho_4^m \rho_5^m \left[F_{34(5,4)} + \rho_4^m \cdot \rho_5^m \left(F_{3,4(5^2,4^2)} + \cdots \right) \right] \quad (10)$$

The last term accounts for images of A_4 lying behind A_5:

$$\varphi_{34(5)} = \rho_5^m \left[F_{34(5)} + \rho_5^m \rho_4^m \left(F_{3,4(5^2,4)} + \cdots \right) \right] \quad (11)$$

The exchange factors $\varphi_{34(4)}$ and $\varphi_{34(5)}$ are once removed from conventional exchange factors insofar as they do not include the direct shape factor F_{34}. The conventional shape factor, φ_{34}, would occur in Equation 9 only if $r_{432} = r_{4(4)32} = r_{4(5)35}$ and would be

$$\varphi_{34} = F_{34} + \varphi_{34(4)} + \varphi_{34(5)} \tag{12}$$

The radiosity of surface A_4 follows from Equation 6 as

$$J_4 = \bar{\epsilon}_4 E_4 + \rho_4^d \left[\sum_{j=1}^{3} \mathcal{J}_{j4} F_{4j} + \sum_{p=4}^{5} \mathcal{J}_{jp} \varphi_{4j(p)} + \sum_{j=4}^{5} J_j \varphi_{4j} \right] \tag{13}$$

The expansion of the first summation for j = 1 is

$$\mathcal{J}_{14} F_{41} + \mathcal{J}_{14} \varphi_{41(4)} + \mathcal{J}_{15} \varphi_{41(5)} \tag{14}$$

The shape factor $\varphi_{41(4)}$ will be non-zero if A_4 is concave and/or if A_4 can "see" the image $A_{1(4,5)}$. In the case at hand, A_4 is plane and sees $A_{1(4,5)}$, then

$$\varphi_{41(4)} = \rho_4^m \rho_5^m \left[F_{41(4,5)} + \rho_4^m \rho_5^m \left(F_{41(4^2,5^2)} + \ldots \right) \right] \tag{15}$$

The factor $\varphi_{41(5)}$ will be

$$\varphi_{41(5)} = \rho_5^m \left[F_{41(5)} + \rho_5^m \rho_4^m \left(F_{41(5^2,4)} + \ldots \right) \right] \tag{16}$$

The second summation for j = 4, 5 contains conventional exchange factors:

$$\varphi_{44} = \rho_5^m \left[F_{44(5)} + \rho_5^m \rho_4^m \left(F_{44(5^2,4)} + \ldots \right) \right] + \tag{17}$$

and

$$\varphi_{45} = F_{45} + \rho_4^m \rho_5^m \left[F_{45(4,5)} + \rho_5^m \rho_4^m \left(F_{44(4^2,5^2)} + \ldots \right) \right] \tag{18}$$

All other exchange factors for the enclosure of Figure 2 may be generated in a manner comparable to those described above. It should be remarked that many of the shape factors from real surfaces to images are partial view factors insofar as nonspecular surfaces occlude a complete view.

Surface Properties: Bidirectional, Directional, Specular, and Diffuse

Radiative surface properties are measured parameters whose proper use in thermal analysis requires some discussion. The tutorial paper of Edwards (Reference 6) provides excellent insight to the measurement and interpretation of radiative transfer data. Love (Reference 7) devotes a chapter to the description of radiative characteristics of surfaces and shows the relationships between bidirectional reflectance and other surface properties. The graphical clarity of Siegel and Howell (Reference 8) provides considerable insight to the significance of surface properties and their measurement. The discussion which follows is intended to complement the information in References 6 through 8 by showing how surface property data should be applied to the computation of radiation interchange factors.

The use of mean values of bidirectional and directional properties for nodal surface analysis is postulated in Reference 1. The bidirectional reflectance $\bar{r}_{jkw}$ denotes a property of nodal surface A_k which accounts for the redistribution of radiant energy originating at nodal area A_j and subsequently directed to nodal area A_w. The measured bidirectional reflectance is designated without the bar simply as r_{jkw} and denotes a property of dA_k which accounts for the redistribution of radiant energy originating at dA_j and subsequently redirected to dA_w. The mean value $\bar{r}_{jkw}$ must be computed from knowledge of the geometries and orientation of A_j, A_k, and A_w and a catalog of measured data of r_{jkw}. The fundamental relationship defining area mean bidirectional reflectance is given as a multiple integral in Reference 1.

$$\pi A_w F_{wk} F_{kj} \bar{r}_{jkw} = \int_{A_k} \int_{A_j} \frac{1}{A_{k'}} \int_{A_{k'}} \int_{A_w} r_{jk'w} K_{wk} K_{kj} \, dA_w \, dA_{k'} \, dA_j \, dA_k \tag{19}$$

where

$$K_{ab} = \frac{\cos \theta_a \cos \theta_b}{r_{ab}^2}, \text{ shape factor kernal}$$

k' = dummy variable for k

Equation 19 may be cast in a form suitable for digital computation by recognizing that

$$\frac{K_{wk}}{\pi} dA_w = F_{dA_k - dA_w}$$

and by subdividing each nodal area (e.g., A_j) into a number of elemental areas ($\Delta A_j'$):

$$A_j = \sum_{j'=1}^{M_j} \Delta A_{j'} \quad , \qquad A_{k'} = \sum_{k'=1}^{M_k} \Delta A_{k'}, \text{ etc.}$$

The quadruple integral of Equation 58 may be approximated by a quadruple summation as

$$\bar{r}_{jkw} F_{kj} = \frac{\sum_{w'=1}^{M_w} \left\{ \sum_{k'=1}^{M_k} \left[\sum_{k''=1}^{M_k} \left(\sum_{j'=1}^{M_j} r_{j'k''w'} F_{k''j'} \right) A_{k''} \right] F_{w'k'} \right\} A_{w'}}{A_k \left[A_w F_{wk} \right]} \tag{20}$$

The element-to-element shape factors, $F_{w'k'}$, and the node-to-node factors, F_{wk}, may be found from closed form expressions, computed by CONFAC, or measured. The local or triple element bidirectional reflectance $r_{j'k''w'}$ must be stored as input data. If the source surface A_j appears as an image in a surface A_p, then the bidirectional reflectance may be computed as $\left(\bar{r}_{j(p)kw} F_{kj(p)} \right)$ by subdividing the image

$$A_{j(p)} = \sum_{j'(p)=1}^{M_{j(p)}} \Delta A_{j'(p)}$$

However, an approximation which would be suitable for many hardware design situations would be to let

$$\bar{r}_{j(p)kw} = \bar{r}_{pkw}$$

The algorithm for mean directional emittance is

$$\bar{\epsilon}_{kw} = \frac{\sum_{k'=1}^{M_k} \sum_{w'=1}^{M_w} \epsilon_{k'w'} F_{k'w'} \Delta A_{k'}}{A_k F_{kw}} \tag{21}$$

where

$$A_k = \sum_{k=1}^{M_k} \Delta A_k \quad , \qquad A_w = \sum_{w=1}^{M_w} \Delta A_w$$

and $\epsilon_{k'w'}$ represents local directional emittance from ΔA_k to ΔA_w. Ordinarily, the physical property measured in a laboratory is a directional reflectance. For example, an integrating sphere measures directional-hemispherical reflectance $\rho_{j,k}$ of a specimen ΔA_k and directional absorptance is found as

$$\alpha_{j,k} = 1 - \rho_{j,k} \text{ (properties of } \Delta A_k)$$

Alternately, a heated hohlraum is used to measure hemispherical-directional reflectance, $\rho_{k,w}$, of a specimen, and directional emittance is found as

$$\epsilon_{k,w} = 1 - \rho_{k,w} \text{ (properties of } \Delta A_k)$$

Bevans and Edwards (Reference 1) observe that when the measurements are monochromatic, Kirchoff's Law yields

whereby
$$\left.\begin{aligned} \alpha_{j,k}(\lambda) &= \epsilon_{j,k}(\lambda) \\ \rho_{j,k}(\lambda) &= \rho_{k,j}(\lambda) \end{aligned}\right\} \text{ (properties of } \Delta A_k)$$

That is, directional-hemispherical reflectance is identical to hemispherical-directional reflectance when the directions are identical. Additional justification of this equivalence is provided in Reference 6.

In the present study, it is assumed that a complementary relationship exists between mean directional emittance and mean directional reflectance:

$$\bar{\epsilon}_{k,w}(k) = 1 - \bar{\rho}_{k,w}(k) = 1 - \bar{\rho}_{w,k}(k) \tag{22}$$

The nomenclature $\bar{X}_{k,w}(k)$ is intended to denote that X is a mean directional property of the surface A_k for direction from A_k toward A_w; $\bar{X}_{w,k}(k)$ is a mean directional property of A_k from A_w to A_k.

Insofar as local reflectance represents a measured quantity, it may be used to obtain mean reflectance which, in turn, may be used to find emittance. The algorithm for $\bar{\rho}_{kw}(k)$ follows from Equations 21 and 22:

$$\bar{\rho}_{k,w}(k) = \frac{\sum_{k'=1}^{M_k} \sum_{w'=1}^{M_w} \rho_{k'w'} F_{k'w'} \Delta A_{k'}}{A_k F_{kw}}, \quad \text{(property of } A_k\text{)} \tag{23}$$

The local directional reflectance in the summation, $\rho_{k'w'}$, may be obtained from stored data for A_k. It may be provided as a direct input from measurements or it may be computed from bidirectional reflectance data. In the latter case, the bidirectional data should be preprocessed so that directional reflectance is available directly (in storage) rather than computed each time it is needed.

The relationship between a directional reflectance and the bidirectional reflectance will be illustrated in terms of the directional-hemispherical reflectance of a nodal surface A_k in an enclosure of M+1 nodal surfaces. Formally, the local reflectances are related by integrating as follows:

$$\rho_{j',k'}(k') = \frac{1}{\pi} \int_{(2\pi)_{emergent}} r_{j'k'w'} \cos\theta \, d\omega$$

$$= \sum_{w=1}^{M+1} \int_{A_w} r_{j'k'w'} \, dF_{k',w'}$$

The mean reflectances may be found by integrating over the several areas to obtain

$$\bar{\rho}_{j,\,k}(k) = \sum_{w=1}^{M+1} \bar{r}_{jkw}\, F_{kw} \tag{24a}$$

Similarly, the hemispherical-directional reflectance is related to bidirectional reflectance as

$$\bar{\rho}_{k,\,j}(k) = \sum_{j=1}^{M+1} \bar{r}_{jkw}\, F_{kj} \tag{24b}$$

The hemispherical reflectance may be found by integrating the directional-hemispherical reflectance over all incoming directions to obtain

$$\bar{\rho}_{k} = \sum_{j=1}^{M+1} \bar{\rho}_{jk}(k)\, F_{kj} \tag{25}$$

Equation 24 may be substituted in Equation 25 to find

$$\bar{\rho}_{k} = \sum_{j=1}^{M+1} \sum_{w+1}^{M+1} \bar{r}_{jkw}\, F_{kw}\, F_{kj} \tag{26}$$

The last two equations are presented in the interest of completeness rather than in the belief that a hemispherical reflectance will be computed in preference to being measured.

The utility of Equations 24 and 25 resides in being able to use them to obtain values of $\rho_{k\to(space)}$ and $\bar{r}_{j,\,k,\,(space)}$ when the space node $k = M+1$ is arbitrarily distributed in an enclosure (see Figure 1). That is, if all $\bar{\rho}_{k,\,w}$ and $F_{k,\,w}\,(w \neq M+1)$ are known and $\bar{\rho}_k$ is known, then

$$\bar{\rho}_{k,\,M+1} = \frac{\bar{\rho}_k - \sum_{w=1}^{M} \bar{\rho}_{k,\,w}\, F_{k,\,w}}{1 - \sum_{w=1}^{M} F_{k,\,w}} = \bar{\rho}_{M+1,\,k} \tag{27}$$

and

$$\bar{r}_{j,\,k,\,M+1} = \frac{\bar{\rho}_{j,\,k}(k) - \sum_{w=1}^{M} \bar{r}_{j,\,k,\,w}\, F_{k,\,w}}{1 - \sum_{w=1}^{M} F_{k,\,w}} \tag{28}$$

The bar indicating an area mean value is used in the pseudoradiosity expression (Equation 5). It is omitted from the radiosity in Equation 6 because the diffuse and specular properties are assumed to be uniform over a nodal surface and independent of directions of incidence and emergence.

MATRIX SOLUTION OF NODAL EQUATIONS

The design equations for net heat flux appearing in the Problem Statement are derived from a more fundamental relationship:

$$q_{k,\,net} = \frac{Q_{k,\,net}}{A_k} = \begin{pmatrix} \text{radiant flux} \\ \text{emerging at } A_k \end{pmatrix} - \begin{pmatrix} \text{radiant flux} \\ \text{incident at } A_k \end{pmatrix} \tag{29}$$

or, equivalently,

$$q_{k,\,net} = \begin{pmatrix} \text{hemispherical radiant} \\ \text{flux emitted at } A_k \end{pmatrix} - \begin{pmatrix} \text{radiant flux} \\ \text{absorbed at } A_k \end{pmatrix} \tag{30}$$

From Reference 1, it follows that when A_k is a directional/bidirectional surface, Equation 29 becomes

$$q_{k,\,net} = \sum_{j=1}^{M+1} \mathcal{J}_{kj}\, F_{kj} - G_k, \qquad 1 \leq k \leq N \tag{31}$$

For the special case of a fully diffuse surface (i.e., $\rho_k = \rho_k^d$, $\rho_k^m = 0$) Equation 31 reduces to

$$q_{k,net} = J_k - G_k \quad \text{(fully diffuse } A_k\text{, only)} \tag{32}$$

The most general expression for net heat flux follows from Equation 30

$$q_{k,net} = \bar{\epsilon}_k \sigma T_k^4 - \overline{\alpha_k G_k} \tag{33}$$

where the hemispherical emittance is indicated and the product $\overline{\alpha_k G_k}$ is separable only when k represents a diffuse-plus-specular surface. The use of the symbol α does not denote absorptance of solar energy; the solar absorptance will be indicated as α^*. In the case of surfaces which are both gray and which emit diffusely, it is possible to let $\epsilon = \alpha$, but in the present analysis of gray surfaces which emit and absorb with a directional dependence, the hemispherical values of emittance and absorptance may be different. The two symbols ϵ and α will be retained in this discussion although they refer to the same spectral interval.

When A_k is in the bidirectional set, the absorbed flux must take account of directional absorptance:

$$\overline{\alpha_k G_k} = \sum_{j=1}^{N} \left[\mathcal{J}_{jk} \bar{\alpha}_{kj} F_{kj} + \sum_{p=N+1}^{M} \mathcal{J}_{jp} \left(\sum_{b=N+1}^{M} \bar{\alpha}_{kj(b)} \varphi_{kj(p,b)} \right) \right]$$

$$+ \sum_{j=N+1}^{M+1} J_j \left[\bar{\alpha}_{kj} F_{kj} + \sum_{p=N+1}^{M+1} \bar{\alpha}_{kj(p)} \varphi_{kj(p)} \right] \tag{34a}$$

When A_k is in the diffuse plus specular set,

$$\alpha_k G_k = \alpha_k \left\{ \sum_{j=1}^{N} \left[\mathcal{J}_{jk} F_{kj} + \sum_{p=N+1}^{M} \mathcal{J}_{jp} \varphi_{kj(p)} \right] \right.$$

$$\left. + \sum_{j=N+1}^{M} J_j \varphi_{kj} \right\} , \tag{34b}$$

The irradiation G_k of a surface may be obtained from Equation 34 by deleting absorptances from the expressions.

It follows from Equations 31 through 34 that the net heat flux at a nodal surface explicitly depends upon all enclosure surface radiosities and pseudoradiosities, geometrical factors, and surface properties. The factors and properties are known a priori; the radiosities and pseudo-radiosities must be found. The net heat flux equations are general within the framework of nodal analysis and are not constrained to further elucidation by any specific method of solving for the radiosities and pseudo-radiosities. In theory, the $\mathcal{J}$'s and J's may be found by: 1) solving a linear set of integral equations, 2) using Monte Carlo procedures for ray tracing, or 3) inverting a matrix implied by Equations 5 and 6. The matrix solution will be described in the remainder of this report.

<u>Explicit Solutions for $\mathcal{J}_{kj}$ and J_k</u>

Matrix solutions for diffuse and diffuse-plus-specular enclosures are well documented in the literature of engineering radiation analysis (References 9 through 11). The analysis starts by writing a matrix equation for radiosity in terms of a transfer matrix, $\hat{D}$, a response vector, $\hat{\mathcal{J}}$, and an excitation vector, $\widehat{\epsilon\sigma T^4}$:

$$[\hat{D}] \cdot [\hat{\mathcal{J}}] = \left[\widehat{\epsilon\sigma T^4}\right]$$

The elements of the transfer matrix are of the form

$$\delta_k^j - \rho_k^d \varphi_{kj}$$

where δ_k^j is a Kronecker delta, ρ_k^d is the diffuse component of reflectance for a diffuse-plus-specular surface, and φ_{kj} is the exchange factor discussion previously. In an enclosure of M surface, $\hat{D}$ contains M^2 elements while the $\hat{J}$ and $\epsilon\sigma T^4$ vectors are column matrices of order $1 \times M$

$$\hat{J} = \left[J_1, J_2 \ldots J_M\right]^T$$

$$\widehat{\epsilon\sigma T^4} = \left[\epsilon_1 \sigma T_1^{\ 4}, \quad \epsilon_2 \sigma T_2^{\ 4}, \ldots, \epsilon_m \sigma T_m^{\ 4}\right]^T$$

The radiosity solution is obtained by inverting $\hat{D}$ and may be expressed as

$$[\hat{J}] = [\hat{D}]^{-1} \left[\widehat{\epsilon\sigma T^4}\right]$$

The elements of $[\hat{D}]^{-1}$ are presumed known and designated as β_{kj}. The radiosity of a surface A_k is then

$$J_k = \sum_{j=1}^{M} \epsilon_j \sigma T_j^4 \beta_{kj}$$

The only real problem in this conceptually simple formulation is that of evaluating the β_{jk} elements in $[\hat{D}]^{-1}$. This last aspect of matrix algebra is not discussed in this report.

The matrix solution for radiosities and pseudoradiosities in a mixed enclosure containing both bidirectional and diffuse-plus-specular surfaces is also conceptually simple. However, the notation is much more cumbersome because up to four indices are required to identify an element of the transfer matrix and, correspondingly, the inverted transfer matrix. The analysis below applies to a mixed enclosure containing a total M nodal areas of which N emit directionally and reflect bidirectionally while M-N nodal areas emit diffusely and reflect in a diffuse-plus-specular manner. Black space is designated with an index M+1. Submatrices associated with the bidirectional set are identified with plain indices

$$\left.\begin{matrix} j \\ k \\ p \\ w \end{matrix}\right\} = 1, 2, \ldots, N$$

Primed indices b', k', etc., refer to nodes in the diffuse-plus-specular set and black space

$$\left.\begin{matrix} b' \\ j' \\ k' \\ p' \end{matrix}\right\} = N+1,\ N+2, \ldots,\ M,\ M+1$$

This "plain or primed" conventional is used in describing the matrix elements only and discontinued when there is no sacrifice in clarity. A matrix equation based on Equations 5 and 6 may be written as

$$\begin{vmatrix} \hat{D}_{kw\atop jp} & \hat{D}_{kw\atop j'} \\ \hat{D}_{k'\atop jp} & \hat{D}_{k'\atop j'} \end{vmatrix} \cdot \begin{vmatrix} \hat{J}_{kw} \\ \hat{J}_{k'} \end{vmatrix} = \begin{vmatrix} \widehat{\bar{\epsilon}_{kw} E_k} \\ \widehat{\epsilon_{k'} E_{k'}} \end{vmatrix} \tag{35}$$

Transfer matrix — Response vector — Excitation vector

where $E = \sigma T^4$ is introduced for notational convenience. The column matrices comprising the response and excitation vectors are shown below:

$$\hat{\mathcal{J}}_{kw} = \left[\mathcal{J}_{1,1}, \mathcal{J}_{1,2}, \ldots, \mathcal{J}_{1,M+1} \ldots \ldots \mathcal{J}_{N,1}, \ldots, \mathcal{J}_{N,M+1}\right]^T \quad \text{order } 1 \times N(M+1) \tag{36a}$$

$$\hat{J}_{k'} = \left[J_{N+1}, J_{N+2}, \ldots J_M\right]^T \quad \text{order } 1 \times (M+1-N) \tag{36b}$$

$$\widehat{\bar{\epsilon}_{kj} E_k} = \left[\bar{\epsilon}_{1,1} E_1, \bar{\epsilon}_{1,2} E_1, \ldots, \bar{\epsilon}_{1,M+1} E_1, \ldots\right.$$

$$\left.\ldots \bar{\epsilon}_{N,1} E_N, \ldots, \bar{\epsilon}_{N,M+1} E_N\right]^T \quad \text{order } 1 \times N(M+1) \tag{37a}$$

$$\widehat{\epsilon_{k'} E_{k'}} = \left[\epsilon_{N+1} E_{N+1}, \epsilon_{N+2} E_{N+2}, \ldots \epsilon_{M+1} E_{M+1}\right]^T \quad \text{order } 1 \times (M+1-N) \tag{37b}$$

The elements of the transfer matrix have been partially generalized and are given below. The notation requires some clarification:

$D_{\substack{kw\\jp}}$ ↙ Row: A_k toward A_w or $A_{w'}$
↖ Column: A_j toward A_p or $A_{p'}$

$D_{\substack{kw\\j'}}$ ↙ Row: A_k toward A_w or $A_{w'}$
↖ Column: $A_{j'}$

$D_{\substack{k'\\jp}}$ ↙ Row: $A_{k'}$
↖ Column: A_j toward A_p or $A_{p'}$

$D_{\substack{k'\\j'}}$ ↙ Row: $A_{k'}$
↖ Column: $A_{j'}$

Elements of $\hat{D}_{\substack{kw\\jp}}$ (order $N(M+1) \times N(M+1)$):

$$D_{\substack{kw\\jp}} = \delta_k^j \, \delta_w^p - \delta_k^p \, (\bar{r}_{jkw} \, F_{kj}) \tag{38a}$$

$$D_{\substack{kw\\jp'}} = \delta_k^j \, \delta_w^p - \sum_{b'=N+1}^{M} r_{j(b')kw} \, \varphi_{kj(p',b')} \tag{38b}$$

Elements of $\hat{D}_{\substack{kw\\j'}}$ (order $N(M+1) \times (M+1-N)$):

$$D_{\substack{kw\\j'}} = - \left[\bar{r}_{j'kw} \, F_{kj} + \sum_{p'=N+1}^{M} \bar{r}_{j'(p')kw} \, \varphi_{kj'(p')} \right] \tag{38c}$$

Elements of $\hat{D}_{\substack{k' \\ jp}}$ (order $(M+1-N) \times N(M+1)$):

$$D_{\substack{k' \\ jp}} = 0 \qquad (38d')$$

$$D_{\substack{k' \\ jp'}} = -\rho^{d}_{k'} \left[\delta^{p'}_{k'} \varphi_{k'j} + \left(1 - \delta^{p'}_{k'}\right) \sum_{p'=N+1}^{M} \varphi_{k'j(p')} \right] \qquad (38d)$$

Elements of $\hat{D}_{\substack{k' \\ j'}}$ (order $(M+1-N) \times (M+1-N)$):

$$D_{\substack{k' \\ j'}} = \delta^{j'}_{k'} - \rho^{d}_{k'} \varphi_{k'j'} \qquad (38e)$$

The solution of Equation 35 may be expressed as

$$\begin{vmatrix} \hat{\mathcal{J}}_{kw} \\ \\ \hat{J}_{k'} \end{vmatrix} = \underbrace{\begin{vmatrix} \hat{B}_{\substack{kw \\ jp}} & \hat{B}_{\substack{kw \\ j'}} \\ \\ \hat{B}_{\substack{k' \\ jp}} & \hat{B}_{\substack{k' \\ j'}} \end{vmatrix}}_{\text{Resolvant matrix}} \cdot \begin{vmatrix} \widehat{\bar{\epsilon}_{jp} E_j} \\ \\ \widehat{\epsilon_{j'} E_{j'}} \end{vmatrix} \qquad (38)$$

The resolvant is the inverse of the transfer matrix, and the terminology used here is intended to recall the integral equation origin for radiant transport. The matrices comprising the resolvant are expanded below where it is assumed that each element is known. The transpose in the excitation matrix corresponds to the transpose implied in the resolvant elements. The four $\hat{B}$ submatrices are expanded in terms of B-elements for tutorial purposes:

$$\hat{\beta}^{kw}_{jp} = \begin{vmatrix} \beta^{1,1}_{1,1} & \cdots & \beta^{1,1}_{1,M+1} & \beta^{1,1}_{2,1} & \cdots & \beta^{1,1}_{2,M+1} & \cdots & \beta^{1,1}_{N,M+1} \\ \beta^{1,2}_{1,1} & \cdots & \beta^{1,2}_{1,M+1} & \beta^{1,2}_{2,1} & \cdots & \beta^{1,2}_{2,M+1} & \cdots & \beta^{1,2}_{N,M+1} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \beta^{N,M+1}_{1,1} & \cdots & \beta^{N,M+1}_{1,M+1} & \beta^{N,M+1}_{2,1} & \cdots & \beta^{N,M+1}_{2,M+1} & \cdots & \beta^{N,M+1}_{N,M+1} \end{vmatrix}_{\substack{\text{order} \\ N(M+1) \\ \times N(M+1)}} \qquad (39a)$$

$$\hat{\beta}^{kw}_{j'} = \begin{vmatrix} \beta^{1,1}_{N+1} & \cdots & \beta^{1,1}_{N+2} & \cdots & \beta^{1,1}_{M+1} \\ \beta^{1,2}_{N+1} & \cdots & \beta^{1,2}_{N+2} & \cdots & \beta^{1,2}_{M+1} \\ \vdots & & \vdots & & \vdots \\ \beta^{N,M+1}_{N+1} & & \beta^{N,M+1}_{N+2} & \cdots & \beta^{N,M+1}_{M+1} \end{vmatrix}_{\substack{\text{order} \\ N(M+1) \times (M+1-N)}} \qquad (39b)$$

$$\hat{\beta}^{k'}_{jp} = \begin{vmatrix} \beta^{N+1}_{1,1} & \cdots & \beta^{N+1}_{1,M+1} & \beta^{N+1}_{2,1} & \cdots & \beta^{N+1}_{2,M+1} & \cdots & \beta^{N+1}_{N,M+1} \\ \beta^{N+2}_{1,1} & \cdots & \beta^{N+2}_{1,M+1} & \beta^{N+2}_{2,1} & \cdots & \beta^{N+2}_{2,M+1} & \cdots & \beta^{N+2}_{N,M+1} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \beta^{M+1}_{1,1} & \cdots & \beta^{M+1}_{1,M+1} & \beta^{M+1}_{2,1} & & \beta^{M+1}_{2,M+1} & \cdots & \beta^{M+1}_{N,M+1} \end{vmatrix}_{\substack{\text{order} \\ (M+1-N) \\ \times N(M+1)}} \qquad (39c)$$

$$\hat{\beta}_{\substack{k'\\j'}} = \left|\begin{matrix} \beta_{\substack{N+1\\N+1}} & \beta_{\substack{N+1\\M+2}} & \cdots & \beta_{\substack{N+1\\M+1}} \\ \beta_{\substack{N+2\\N+1}} & \beta_{\substack{N+2\\N+2}} & \cdots & \beta_{\substack{N+2\\M+1}} \\ \vdots & \vdots & & \vdots \\ \beta_{\substack{M+1\\N+1}} & \beta_{\substack{M+1\\N+2}} & \cdots & \beta_{\substack{M+1\\M+1}} \end{matrix}\right| \quad \text{order } (M+1-N) \times (M+1-N) \qquad (39d)$$

The matrices $\hat{\beta}_{\substack{kj\\pw}}$ and $\hat{\beta}_{\substack{k'\\jp}}$ may be cast in terms of submatrices: Let

$$\hat{\beta}_{\substack{kw\\jp}} = \left|\begin{matrix} \hat{b}_{\substack{1,1\\1}} & \hat{b}_{\substack{1,1\\2}} & \cdots & \hat{b}_{\substack{1,1\\N}} \\ \hat{b}_{\substack{1,2\\1}} & \hat{b}_{\substack{1,2\\2}} & \cdots & \hat{b}_{\substack{1,2\\N}} \\ \vdots & \vdots & & \vdots \\ \hat{b}_{\substack{N,M+1\\1}} & \hat{b}_{\substack{N,M+1\\2}} & \cdots & \hat{b}_{\substack{N,M+1\\N}} \end{matrix}\right| \quad \text{order } N(M+1) \times N \qquad (39)$$

where the $\left(\hat{b}_{\substack{k,w\\j}}\right)$'s are row matrices

$$\hat{b}_{\substack{k,w\\j}} = \begin{bmatrix} \beta_{\substack{k,w\\j,1}} & \beta_{\substack{k,w\\j,2}} & \cdots & \beta_{\substack{k,w\\j,M}} & \beta_{\substack{k,w\\j,M+1}} \end{bmatrix} \text{ order of } 1 \times (M+1) \qquad (40)$$

Similarly

$$\hat{\beta}_{\substack{kw\\jp}} = \begin{vmatrix} \hat{b}_{\substack{N+1\\1}} & \hat{b}_{\substack{N+1\\2}} & \cdots & \hat{b}_{\substack{N+1\\M+1}} \\ \hat{b}_{\substack{N+2\\1}} & \hat{b}_{\substack{N+2\\2}} & \cdots & \hat{b}_{\substack{N+2\\M+1}} \\ \vdots & \vdots & & \vdots \\ \hat{b}_{\substack{M+1\\1}} & \hat{b}_{\substack{M+1\\2}} & \cdots & \hat{b}_{\substack{M+1\\M+1}} \end{vmatrix} \text{ order } (M+1-N) \times (M+1) \tag{41}$$

where $$\hat{b}_{\substack{k'\\j}} = \begin{vmatrix} \beta_{\substack{k'\\j,1}} & \beta_{\substack{k'\\j,2}} & \cdots & \beta_{\substack{k'\\j,M+1}} \end{vmatrix} \text{ order } 1 \times (M+1) \tag{42}$$

Next, form column matrices of the directional emissivities so that

$$\hat{\bar{\epsilon}}_j = \begin{vmatrix} \bar{\epsilon}_{j,1} \\ \bar{\epsilon}_{j,2} \\ \vdots \\ \bar{\epsilon}_{j,M+1} \end{vmatrix} \text{ order } (MH) \times 1 \tag{43}$$

and take the product of $\left(\hat{b}_{\substack{k,w\\j}}\right) \cdot \left(\hat{\bar{\epsilon}}_j\right)$:

$$\hat{\delta}_{\substack{k,w\\j}} = \left(\hat{b}_{\substack{k,w\\j}}\right) \cdot \left(\hat{\bar{\epsilon}}_j\right) = \sum_{p=1}^{M+1} \bar{\epsilon}_{jp}\, \beta_{\substack{k,w\\j,p}} \tag{44}$$

The product $\left(\hat{b}_{\substack{k'\\j}}\right) \cdot \left(\hat{\bar{\epsilon}}_j\right)$ is

$$\hat{\delta}_{\substack{k'\\j}} = \left(\hat{b}_{\substack{k'\\j}}\right) \cdot \left(\hat{\bar{\epsilon}}_j\right) = \sum_{p=1}^{M+1} \epsilon_{jp}\, \beta_{\substack{k'\\j,p}} \tag{45}$$

For the sake of uniformity define 1 × 1 matrices

$$\hat{\delta}_{\substack{kw\\ j'}} = \epsilon_{j'}\, \beta_{\substack{kw\\ j'}} \tag{46a}$$

$$\hat{\delta}_{\substack{k'\\ j'}} = \epsilon_{j'}\, \beta_{\substack{k'\\ j'}} \tag{46b}$$

$$\left|\begin{array}{c} \mathcal{J}_{1,1} \\ \mathcal{J}_{1,2} \\ \vdots \\ \mathcal{J}_{1,M+1} \\ \vdots \\ \mathcal{J}_{N,M+1} \\ \hline J_{N+1} \\ \vdots \\ J_{M+1} \end{array}\right| = \left|\begin{array}{cccc:ccc}
\hat{\delta}_{\substack{1,1\\1}} & \hat{\delta}_{\substack{1,1\\2}} & \cdots & \hat{\delta}_{\substack{1,1\\N}} & \hat{\delta}_{\substack{1,1\\N+1}} & \cdots & \hat{\delta}_{\substack{1,1\\M+1}} \\
\hat{\delta}_{\substack{1,2\\1}} & \hat{\delta}_{\substack{1,2\\2}} & \cdots & \hat{\delta}_{\substack{1,2\\N}} & \hat{\delta}_{\substack{1,2\\N+1}} & \cdots & \hat{\delta}_{\substack{1,2\\M+1}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\hat{\delta}_{\substack{1,M+1\\1}} & \hat{\delta}_{\substack{1,M+1\\2}} & \cdots & \hat{\delta}_{\substack{1,M+1\\N}} & \hat{\delta}_{\substack{1,M+1\\N+1}} & \cdots & \hat{\delta}_{\substack{1,M+1\\M+1}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\hat{\delta}_{\substack{N,M+1\\1}} & \hat{\delta}_{\substack{N,M+1\\2}} & \cdots & \hat{\delta}_{\substack{N,M+1\\N}} & \hat{\delta}_{\substack{N,M+1\\N+1}} & & \hat{\delta}_{\substack{N,M+1\\M+1}} \\
\hdashline
\hat{\delta}_{\substack{N+1\\1}} & \hat{\delta}_{\substack{N+1\\2}} & \cdots & \hat{\delta}_{\substack{N+1\\N}} & \hat{\delta}_{\substack{N+1\\N+1}} & \cdots & \hat{\delta}_{\substack{N+1\\M+1}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\hat{\delta}_{\substack{M+1\\1}} & \hat{\delta}_{\substack{M+1\\2}} & \cdots & \hat{\delta}_{\substack{M+1\\N}} & \hat{\delta}_{\substack{M+1\\N+1}} & \cdots & \hat{\delta}_{\substack{M+1\\M+1}}
\end{array}\right| \cdot \left|\begin{array}{c} E_1 \\ E_2 \\ \vdots \\ E_N \\ \hdashline E_{N+1} \\ \vdots \\ E_{M+1} \end{array}\right| \tag{47}$$

order $N(M+1) \times 1$ — order $N(M+1) + (M+1-N) \times (M+1)$ — order $(M+1) \times 1$

(47)

The matrix of $\hat{\delta}$'s is shown partitioned in the manner of Equation 35. For convenience, the matrix of $\hat{\delta}$'s will be referred to as the modified resolvant matrix by way of relating it to the solution of an integral equation.

From Equation 47 the pseudoradiosities and radiosities may be written as

$$\mathcal{J}_{kw} = \sum_{j=1}^{M+1} \hat{\delta}_{\substack{kw \\ j}} E_j \, , \quad \begin{cases} 1 \le k \le N \\ 1 \le j \le M+1 \end{cases} \tag{48a}$$

$$J_k = \sum_{j=1}^{M+1} \hat{\delta}_{\substack{k \\ j}} E_j \, , \qquad N+1 \le k \le M \tag{48b}$$

These emergent fluxes are shown in terms of "influence coefficients" (the $\hat{\delta}$'s) and black body emissive power (the E's).

Script-F, $\mathcal{J}$, and Hottel's Radiation Interchange Factor

The radiosity and pseudoradiosity solutions of Equations 48 may be used with any of the appropriate expressions for net heat flux to obtain a formulation for a radiation interchange factor. Equation 33 is the most convenient and general form of the net heat flux equations for use in a mixed enclosure. The derivation proceeds by combining Equations 48 to 34 to obtain the absorbed flux at a surface in terms of nodal temperatures, geometrical factors, $\hat{\delta}$ matrices:

A_k bidirectional: $1 \le k \le N$

$$\overline{\alpha_k G_k} = \sum_{i=1}^{M+1} \sigma T_i^4 \left\{ \sum_{j=1}^{N} \left[\bar{\alpha}_{kj} F_{kj} \hat{\delta}_{\substack{jk \\ i}} + \sum_{p=N+1}^{M} \hat{\delta}_{\substack{jp \\ i}} \left(\sum_{b=N+1}^{M} \bar{\alpha}_{kj(b)} \varphi_{kj(p,b)} \right) \right] + \sum_{j=N+1}^{M} \hat{\delta}_{\substack{j \\ i}} \left[\bar{\alpha}_{kj} F_{kj} + \sum_{p=N+1}^{M} \bar{\alpha}_{kj(p)} \varphi_{kj(p)} \right] \right\} \tag{49}$$

A_k diffuse-plus-specular: $N+1 < k \le M$

$$\alpha_k G_k = \alpha_k \sum_{i=1}^{M+1} \sigma T_i^4 \left\{ \sum_{j=1}^{N} \left[F_{kj} \hat{\delta}_{jk,i} + \sum_{p=N+1}^{M} \varphi_{kj(p)} \hat{\delta}_{jp,i} \right] + \sum_{j=N+1}^{M} \varphi_{kj} \hat{\delta}_{j,i} \right\} \quad (50)$$

These expressions may be used in Equation 33 with the thermodynamic argument of zero net flux in an isothermal enclosure (cf., Reference 5) to obtain expressions for the radiant interchange factors, $\mathcal{J}_{ki}$:

A_k bidirectional: $1 \le k \le N$

$$\mathcal{J}_{ki} = \sum_{j=1}^{N} \left[\bar{\alpha}_{kj} F_{kj} \hat{\delta}_{jk,i} + \sum_{p=N+1}^{M} \hat{\delta}_{jp,i} \left(\sum_{b=N+1}^{M} \bar{\alpha}_{kj(b)} \varphi_{kj(p,b)} \right) \right] + \sum_{j=N+1}^{M} \hat{\delta}_{j,i} \left[\bar{\alpha}_{kj} F_{kj} + \sum_{p=N+1}^{M} \bar{\alpha}_{kj(p)} \varphi_{kj(p)} \right] \quad (51)$$

A_k diffuse-plus-specular: $N+1 \le k \le M$

$$\mathcal{J}_{ki} = \alpha_k \left\{ \sum_{j=1}^{N} \left[F_{kj} \hat{\delta}_{jk,i} + \sum_{p=N+1}^{M} \varphi_{kj(p)} \hat{\delta}_{jp,i} \right] + \sum_{j=N+1}^{M} \varphi_{kj} \hat{\delta}_{j,i} \right\} \quad (52)$$

In both Equations 51 and 52, node A_i is arbitrary and may refer to a bidirectional, diffuse-plus-specular, c~ space node. Thermodynamic arguments (Reference 2) may be invoked to show that reciprocity applies

$$A_k \mathcal{J}_{ki} = A_i \mathcal{J}_{ik} \quad (53)$$

A summation relationship follows from Equation 33 as

$$\sum_{i=1}^{M+1} \mathcal{J}_{ki} = \bar{\epsilon}_k \tag{54}$$

Considerable computational convenience may be achieved from an alternate formulation as follows:

$$J_k = \epsilon_k \sigma T_k^4 + \rho_k^d G_k \tag{55}$$

which can be rearranged as

$$G_k = \left(J_k - \epsilon_k \sigma T_k^4\right) \Big/ \rho_k^d \tag{56}$$

Equation 56, when introduced in the net heat flux expression for a diffuse-specular surface, leads to

$$q_{k,\,net} = \left[\left(\alpha_k + \rho_k^d\right) \epsilon_k \sigma T_k^4 - \alpha_k J_k\right] \Big/ \rho_k^d \tag{57}$$

Equation 48b may be used for J_k, together with the usual thermodynamic arguments to obtain

$$q_{k,\,net} = \sum_{i=1}^{M+1} \sigma \mathcal{J}_{ki} \left(T_k^4 - T_i^4\right)$$

where

$$\mathcal{J}_{ki} = \frac{\alpha_k}{\rho_k^d} \hat{\delta}_{\substack{k \\ i}} \;, \quad \begin{cases} N+1 \le k \le M \\ 1 \le i \le M+1 \end{cases} \tag{58}$$

The computational advantage of using Equation 58 instead of Equation 52 is apparent; a computational, but not conceptual, disadvantage occurs when $\rho_k^d = 0$. The difficulty is not conceptual because $\hat{\delta}_{k_i}$ is directly proportional to ρ_k^d (when $k \neq i$), so that reflectance is eliminated from the formulation. However, it is troublesome to account for this elimination in a computer program. The reciprocity relation remains unchanged, but the summation rule becomes

$$\sum_{i=1}^{M+1} \mathcal{J}_{ki} = \frac{\epsilon_k}{\rho_k^d} \left(1 - \rho_k^m\right) \quad , \quad N+1 \leq k \leq M \tag{59}$$

The radiant interchange factors formulated in Equations 52 and 58 are numerically identical except for $\mathcal{J}_{kk}$. The two different self-absorption factors are related as

$$\mathcal{J}_{kk} \quad = \quad \mathcal{J}_{kk} \quad - \epsilon_k^2 \big/ \rho_k^d \tag{60}$$

Equation 52   Equation 58

These two factors have been c[illegible]l previously and are discussed briefly in Reference 5 and "Comment, [illegible]eference 10.

Equations 51 through 54 and 58 through 60 represent the relationships which account for radiative interchange in a semi-gray, mixed enclosure in which excitation is provided by surface temperature. All these equations reduce to conventional expressions (References 1 and 5) when there are no directional emitting/bidirectional reflecting surfaces in an enclosure.

SOLAR RADIATION INTERCHANGE

Solar energy represents an important radiative input in spacecraft thermal design and many other applications. The use of semi-gray interchange factors (Reference 5) provides a convenient engineering procedure for distinguishing between thermally excited radiation and solar radiation. Thermal radiation is characterized by the gray emittance, ϵ, and Stefan-Boltzmann emissive power, σT^4, of a surface. Solar radiation is characterized by a gray solar absorptance, α^* of a surface, and the solar constant, S. The assumption of a mixed enclosure is more appropriate to the solar waveband than to thermal radiation (Reference 6) insofar as most "engineering surfaces" tend to be specular reflectors of long-wave radiant energy ($\lambda = O(10\mu)$). In the solar waveband ($0 < \lambda < 3\mu$), spacecraft surfaces may be purely specular, diffuse-plus-specular, or bidirectional in reflective character.

The derivation of radiant interchange factors for solar energy is very similar to the development preceding Equation 51. The analysis requires solving for pseudoradiosities and radiosities when 1) the surface properties of interest are r*, α*, ρ*, the solar waveband responses and 2) the excitation at a surface is due to an external collimated input. The governing equation may be written in matrix form analogous to Equation 35:

$$\begin{vmatrix} \hat{D}^*_{kw\,jp} & \hat{D}^*_{k\,j'} \\ \hat{D}^*_{k'\,jp} & \hat{D}^*_{k'\,j'} \end{vmatrix} \cdot \begin{vmatrix} \hat{\mathcal{J}}^*_{kw} \\ \hat{J}^*_{k'} \end{vmatrix} = S \begin{vmatrix} \widehat{\overline{r^*_{skw}\;C_{s,k}}} \\ \widehat{\rho^{*d}_{k'}\;C_{s,k'}} \end{vmatrix} \tag{61}$$

where S is the solar constant, the $C_{s,k}$ are solar incidence factors (Reference 5). $\overline{r}^*_{skw}$ is the solar bidirectional reflectance of surface A_k for radiant energy arriving from the direction of the solar disc and redirected toward A_w, and $\rho^{*d}_{k'}$ is the diffuse component of solar reflectance at $A_{k'}$. So long as the solar direction is assumed fixed relative to the enclosure, $\overline{r}^*_{skw}$ depends on the relative locations of A_k and A_w in the same manner as $\overline{\epsilon}_{kw}$ implied in Equation 35. Similarly the product $(SC_{s,k})$ represents the external excitation analogous to the temperature which is tacit in E_k.

The excitation function $\overline{r^*_{skw}\,C_{sk}}$ represents an inseparable product in the presence of specularly reflecting surfaces in the same sense that $\overline{\alpha_k G_k}$ is inseparable on the left side of Equation 49. It is necessary to account for direct solar irradiation from the direction of the sun and for specularly reflected solar irradiation from the direction of various specular surfaces. The excitation at a bidirectional surface may be represented as

$$\overline{r^*_{skw}\,C_{s,k}} = \underbrace{\overline{r}^*_{skw}\,C_{s,k}}_{\text{Direct}} + \sum_{p=N+1}^{M} \overline{r}^*_{s(p)kw} \left[\underbrace{C_{s(p),k}}_{\text{Single reflection}} + \sum_{b=N+1}^{M} \left(\underbrace{C_{s(p,b),}}_{\text{Double reflection}} + \cdots \right) \right]$$

The solar bidirectional reflectances on the right side of Equation 62 are $\bar{r}^*_{skw}$ for A_k irradiated directly by the solar disc, reflecting energy toward an arbitrary surface A_w, and $\bar{r}^*_{s(p)kw}$ for A_k irradiated by an image of the sun lying behind A_p, reflecting energy toward A_w. The solar incidence factors on the right are zero if the sun cannot be seen either directly (i.e., $C_{s,k} = 0$) or as an image (i.e., $C_{s(p),k} = 0$) from A_k. When the solar disc or its images are visible from A_k, the solar incidence factors are

$$A_k C_{s,k} = A_k^*, \text{ irradiated fraction of } A_k \text{ projected toward sun} \tag{63a}$$

$$A_k C_{s(p),k} \Big/ \rho_p^{*m} = A_k^{*(p)}, \text{ irradiated fraction of } A_k \text{ projected toward } A_p \tag{63b}$$

$$A_k C_{s(p,b),k} \Big/ \rho_p^{*m} \rho_b^{*m} = A_k^{*(p,b)}, \text{ irradiated fraction of } A_k \text{ projected toward } A_{b(p)} \tag{63c}$$

The several areas are shown in Figure 3 for a simple geometry. The figure illustrates several projected areas on A_k irradiated by direct solar flux and by one and two specular reflections in A_p and A_b.

It is convenient, but not imperative, to define a mean value of solar bidirectional reflectance to affect a separation of the left side of Equation 62:

$$\overline{r^*_{skw}} = \left(\overline{r^*_{skw} C_{sk}}\right) \Big/ \overline{C}_{s,k} \tag{64}$$

where the denominator factor $\overline{C}_{s,k}$ is the conventional solar incidence factor is

$$\overline{C}_{s,k} = C_{s,k} + \sum_{p=N+1}^{M} \left[C_{s(p),k} + \sum_{b=N+1}^{M} \left(C_{s(p,b),k} + \ldots \right) \right] \tag{65}$$

The excitation function for a diffuse-plus-specular surface is separable and the incidence factor expression is identical to Equation 63b.

It is possible to use the form of Equation 47 to express a solution for $\mathcal{J}^*_{kw}$ and J^*_k merely by modifying the excitation vector and the δ matrices. That is, in Equation 47, replace the column vector

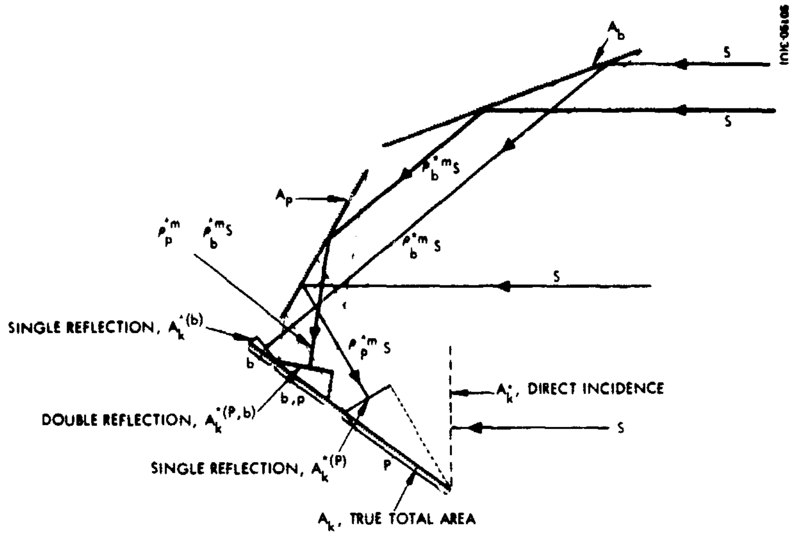

Figure 3. Several Projected Areas in A_k Irradiated by Direct Solar Flux and by One and Two Specular Reflections in A_p and A_b

$$[E_1, \; E_2, \; \ldots \; E_N \;\vdots\; E_{N+1}, \; \ldots \; E_M]^T$$

with the column vector

$$S\,[\overline{C}_{s,\,1}, \; \overline{C}_{s,\,2}, \; \ldots \; \overline{C}_{s,\,N} \;\vdots\; C_{s,\,N+1}, \; \ldots \; C_{s,\,N+2}]^T$$

Also, form a column matrix of bidirectional reflectances

$$\hat{\bar{r}}^*_{sj} = \left[\overline{\bar{r}^*_{sj,\,1}} \;\; \overline{\bar{r}^*_{sj,\,2}}, \; \ldots \; \overline{\bar{r}^*_{sjM}} \right]^T$$

and take the products analogous to Equations 44 through 46b, including

$$\hat{\delta}^*_{\substack{kw\\j}} = \left(\hat{b}^*_{\substack{kw\\j}} \right) \cdot \left(\hat{\bar{r}}^*_{sj} \right) = \sum_{p=1}^{M} \overline{\bar{r}^*_{sjp}} \, \beta^*_{\substack{kw,\\jp}} \quad 1 \le j \le N \qquad (66a)$$

$$\hat{\delta}^{*}_{\substack{k' \\ j}} = \left(\hat{b}^{*}_{\substack{k' \\ j}}\right) \cdot \left(\hat{\overline{r}}^{*}_{sj}\right) = \sum_{p=1}^{M} \overline{\overline{r}^{*}_{sjp}} \; \beta^{*}_{\substack{k'w \\ j,p}} \, , \quad 1 \leq j \leq N \tag{66b}$$

$$\hat{\delta}^{*}_{\substack{kw \\ j'}} = \rho^{*d}_{j'} \; \beta^{*}_{\substack{kw \\ j'}} \, , \quad N+1 \leq j' \leq M \tag{66c}$$

$$\hat{\delta}^{*}_{\substack{k' \\ j'}} = \rho^{*d}_{j'} \; \beta^{*}_{\substack{k' \\ j'}} \, , \quad N+1 \leq j' \leq M \tag{66d}$$

The pseudoradiosities and radiosities follow from Equations 47 and 61 through 66d as

$$\mathcal{J}^{*}_{kw} = S \sum_{j=1}^{M} \overline{C}_{sj} \; \hat{\delta}^{*}_{\substack{kw \\ j}} \, , \quad \begin{cases} 1 \leq k \leq N \\ 1 \leq w \leq M \end{cases} \tag{67a}$$

$$J^{*}_{k} = S \sum_{j=1}^{M} \overline{C}_{sj} \; \hat{\delta}^{*}_{\substack{k \\ j}} \, , \qquad N+1 \leq k \leq M \tag{67b}$$

The net heat flux in the solar waveband is the absorbed incident flux because surfaces at structural temperatures do not emit appreciable energy in short wavelengths:

$$q^{*}_{k,\,net} = -\overline{\alpha^{*}_{k} \; G^{*}_{k}} \tag{67c}$$

Th... net solar heat flux must account for both direct and reflected solar energy. Equation 67c is expanded only for the case of a bidirectional surface ($1 \leq k \leq N$) and the final result for a diffuse-plus-specular surface is given without comment:

$$q^{*}_{k,\,net} = -\left\{ S\,\overline{\alpha^{*}_{ks}\, C_{s,\,k}} + \sum_{j=1}^{M} \left[\mathcal{J}^{*}_{jk}\, \overline{\alpha^{*}_{kj}}\, F_{kj} + \sum_{p=N+1}^{M} \mathcal{J}^{*}_{jp} \left(\sum_{b=N+1}^{M} \overline{\alpha^{*}_{kj(b)}}\, \varphi^{*}_{kj(p,\,b)} \right) \right] + \sum_{j=N+1}^{M} J^{*}_{j} \left[\overline{\alpha}^{*}_{kj}\, F_{kj} + \sum_{p=N+1}^{M+1} \overline{\alpha}^{*}_{kj(p)}\, \varphi^{*}_{kj(p)} \right] \right\} \qquad (68)$$

Equations 67a and 67b are introduced in Equation 68 to find

$$q^{*}_{k,\,net} = -S\left\{ \overline{\alpha^{*}_{ks}\, C_{s,\,k}} + \sum_{i=1}^{N} \overline{C}_{s,\,i} \left[\sum_{j=1}^{N} \left(\overline{\alpha}^{*}_{kj}\, F_{kj}\, \hat{\delta}^{*}_{\substack{jk\\ i}} + \sum_{p=N+1}^{M} \hat{\delta}^{*}_{\substack{jp\\ i}} \left\{ \sum_{b=N+1}^{M} \overline{\alpha}^{*}_{kj(b)}\, \varphi^{*}_{kj(p,\,b)} \right\} \right) + \sum_{j=N+1}^{M} \hat{\delta}^{*}_{\substack{j\\ i}} \left(\overline{\alpha}^{*}_{kj}\, F_{kj} + \sum_{p=N+1}^{M} \overline{\alpha}^{*}_{kj(p)}\, \varphi^{*}_{kj(p)} \right) \right] \right\} \qquad (69)$$

By analogy to Equation 51, for:

<u>A_k bidirectional, $1 \le k \le N$:</u>

$$\mathcal{J}^{*}_{ki} = \sum_{j=1}^{N} \left[\overline{\alpha}^{*}_{kj}\, F_{kj}\, \hat{\delta}^{*}_{\substack{jk\\ i}} + \sum_{p=N+1}^{M} \hat{\delta}^{*}_{\substack{jp\\ i}} \left(\sum_{b=N+1}^{M} \alpha^{*}_{kj(b)}\, \varphi^{*}_{kj(p,\,b)} \right) \right] + \sum_{j=N+1}^{M} \hat{\delta}^{*}_{\substack{j\\ i}} \left[\overline{\alpha}^{*}_{kj}\, F_{kj} + \sum_{p=N+1}^{M} \overline{\alpha^{*}_{kj(p)}}\, \varphi^{*}_{kj(p)} \right] \qquad (70)$$

Similarly, it can be found that for:

A_k diffuse-plus-specular, $N+1 \le k \le M$:

$$\mathcal{J}^*_{ki} = \alpha^*_k \left\{ \sum_{j=1}^{N} \left[F_{kj} \widehat{\delta^*_{jk}}_{i} + \sum_{p=N+1}^{M} \varphi^*_{kj(p)} \widehat{\delta^*_{jp}}_{i} \right] + \sum_{j=N+1}^{M} \varphi^*_{kj} \widehat{\delta^*_{j}}_{i} \right\} \quad (71)$$

With the use of Equations 70 and 71, the net solar heat flux may be expressed as

$$q^*_{k,\,net} = -S \left[\overline{\alpha^*_{ks}\, C_{s,\,k}} + \sum_{i=1}^{M} \overline{C}_{s,\,i}\, \mathcal{J}^*_{ki} \right], \quad 1 \le k^* \le M \quad (72)$$

The direct incidence term $\overline{\alpha^*_{ks}\, C_{s,\,k}}$ is inseparable for a bidirectional surface and must be evaluated in the manner of Equation 62 by using $\overline{\alpha}^*_{ks}$, etc., in place of $\overline{r}^*_{skw}$, etc. For a diffuse-plus-specular node, the direct incidence term reduces to $\alpha^*_k\, C_{s,\,k}$.

The alternate formulation equivalent to Equation 58 may be used for diffuse-plus-specular surfaces,

$$\mathcal{J}^*_{ki} = \frac{\alpha^*_k}{\rho^{*d}_k} \widehat{\delta^*_k}_{i}, \quad \left\{ \begin{array}{l} N_T! \le k \le M \\ 1 \le i \le M \end{array} \right. \quad (73)$$

However, the factor $\mathcal{J}^*_{kk}$ from Equation 73 accounts for direct incidence (see "Comments," Reference 10) so that

$$q^*_{k,\,net} = -S \sum_{i=1}^{M} \overline{C}_{s,\,i}\, \mathcal{J}^*_{k,\,i}, \quad N+1 \le k \le M \quad (74)$$

[Equation 73]

A simple proof of this equivalence is presented in Appendix B.

Several fundamental similarities and differences should be observed between the thermal factors $\mathcal{J}_{ki}$ and the solar factors $\mathcal{J}^*_{ki}$. Both factors represent absorption coefficients at the surface A_k for radiant flux which originates or appears to originate at surface A_i. The transfer matrices are identical in form for each factor and differ only insofar as different waveband properties must be used to form the respective matrices. The most fundamental difference between the factors is that thermal energy is emitted by A_i while solar energy is reflected by A_i. This difference is concealed in the factors δ (Equations 44 through 46b) and δ^* (Equations 66a through 66d). An additional difference is that the space node, M+1, plays a vital role in thermal factor applications, but is unimportant for solar factors.

A SECOND-ORDER APPROXIMATION: MEAN-TO-LOCAL INTERCHANGE FACTORS

The computation of radiant interchange factors in an enclosure having real emittance/reflectance surface properties may be accomplished by using the matrix inversion formulation presented previously. If the enclosure contains N nodal surfaces, the transfer matrix is or order $N^2 \times N^2$. By using a sophisticated matrix inversion routine, the Hughes Aircraft Company can deal with an enclosure of maximum size $N \leq 38$. If larger enclosures are encountered, or if computer run time must be minimized, approximate techniques may be used to obtain the nodal radiant interchange factors.

An algorithm is derived below for computing approximate interchange factors in an enclosure which is arbitrarily large. The formulation is an outgrowth of a technique developed by Bobco and his associates (References 13 and 14) in which local radiant equilibrium temperatures are computed from zonal mean values of temperature. The "mean-to-local" formulation is based on the solution of the linear Fredholm integral equation of the second kind by iterative procedures (e.g., successive approximation or successive substitution). The mean-to-local method subdivides a set of nodes (the complete enclosure) into a smaller number of subsets. Radiant interchange factors are computed among pairs of subsets and then these "mean" values are used to obtain "local" interchange factors between any two surfaces in the enclosure.

NODES AND SUBSETS

The problem statement is identical to the description given previously. The analytical approach differs in that a specific number of nodal areas are collected to form several subsets. Nodal areas are designated with Latin indices, and subsets are identified with Greek indices. Any given subset contains nodes of only one directional characteristic, i.e., a bidirectional subset contains only directional emitting and bidirectionally reflecting nodes, while a diffuse/specular subset contains nodes which are diffuse emitters and diffuse-plus-specular reflectors. Within this framework, the location and surface properties of any node are chosen according to convenience (e.g., geometric proximity, similar surface finishes, random), but a nodal surface can occupy only one subset. The first subset is assumed to contain N_1 bidirectional nodes; the n^{th} subset contains N_n bidirectional nodes and

$$\sum_{\gamma=1}^{n} N_\gamma = N, \text{ total number of bidirectional nodes.}$$

The n + 1 subset contains M_{n+1} diffuse/specular nodes, and the last subset contains M_m diffuse/specular nodes so that

$$\sum_{\gamma=n+1}^{m} M_\gamma = M - N, \text{ total number of diffuse/specular nodes.}$$

The single space node, M + 1, is given the subset identification m + 1. The arrangement of nodes and subsets is shown schematically in Figure 4. In summing over the nodes comprising a subset, the notation

$$\sum_{k=1_\gamma}^{N_\gamma} (\quad) \text{ and } \sum_{i=1_\chi}^{M_\chi}$$

will be used to denote that the γ^{th} subset contains N_γ nodes and the χ^{th} subset contains M_χ nodes. N_γ implies a bidirectional subset, and M_χ denotes a diffuse/specular subset.

The general forms of thermally excited nodal pseudoradiosity and radiosity equations are given by Equations 5 and 6, and their solutions are given by Equation 48. The solution implies inverting a matrix of size (M+ 1) x (M+ 1). Postulate that it is possible to define mean values of all properties, geometrical factors, and emissive powers for all subsets so that pseudoradiosity and radiosity expressions for subsets have the form

$$\mathcal{J}_{\gamma\chi} = \epsilon_{\gamma\chi} E_\gamma + \sum_{\zeta=1}^{n} \left[\mathcal{J}_{\zeta\gamma} R_{\zeta\gamma\chi} + \sum_{\nu=n+1}^{m} \mathcal{J}_{\zeta\nu} R_{\zeta(\nu)\gamma\chi} \right]$$

$$+ \sum_{\zeta=n+1}^{m} J_\zeta R'_{\zeta\gamma\chi} , \quad \begin{cases} 1 \leq \gamma \leq n \\ 1 \leq \chi \leq m + 1 \end{cases} \qquad (75)$$

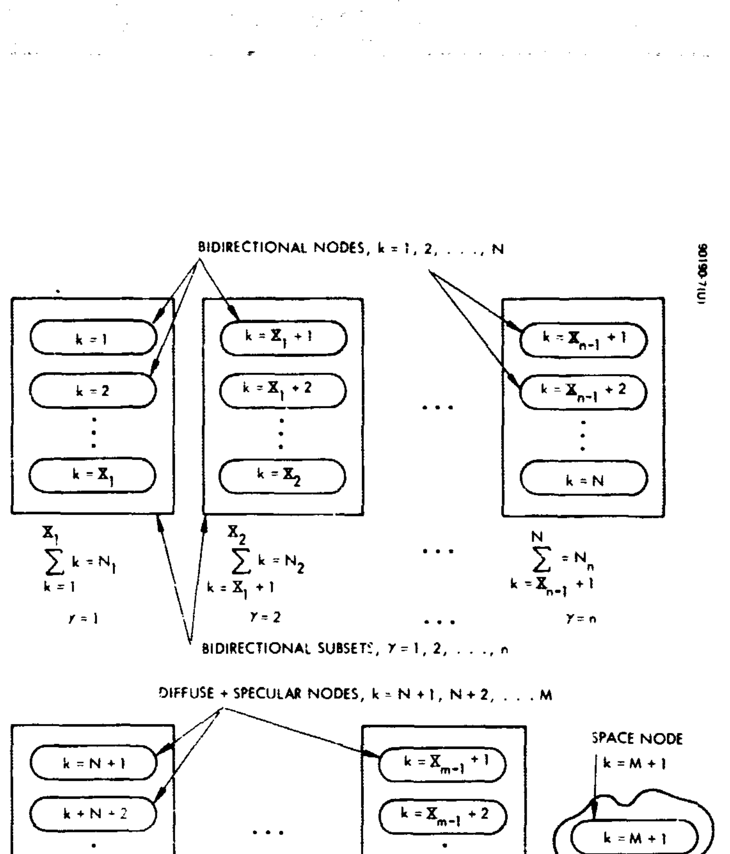

Figure 4. Nodes (Local Areas) and Subsets (Mean Area)

$$J_\gamma = \epsilon_\gamma E_\gamma + \sum_{\zeta=1}^{n} \left[\mathscr{g}_{\zeta\gamma} R_{\zeta\gamma} + \sum_{\nu=n+1}^{m} \mathscr{g}_{\zeta\nu} R_{\zeta(\nu)\gamma} \right]$$

$$+ \sum_{\zeta=n+1}^{m} J_\zeta R'_{\zeta\gamma}, \quad n+1 < \gamma < m+1 \tag{76}$$

where the R's represent products of reflectance and geometrical factors (e.g., $R_{\zeta\gamma\chi} = r_{\zeta\gamma\chi} F_{\gamma\zeta}$). A discussion of appropriate mean values follows the analytical development below. The solution of the linear set implicit in Equations 75 and 76 requires inverting a matrix of order $(m+1) \times (m+1)$ and may be expressed as

$$\mathscr{J}_{\gamma\chi} = \sum_{\zeta=1}^{m+1} E_\zeta \hat{\delta}_{\gamma\chi\,\zeta} \tag{77a}$$

$$J_\gamma = \sum_{\zeta=1}^{m+1} E_\zeta \hat{\delta}_{\gamma\,\zeta} \tag{77b}$$

These subset solutions represent a first order approximation to the nodal solutions of Equations 48. That is, if $k \subset \gamma$ and $i \subset \chi$ then

$$\mathscr{J}_{ki} \approx \mathscr{J}_{\gamma\chi} \tag{78a}$$

$$J_k \approx J_\gamma \tag{78b}$$

The accuracy of the approximations may not be adequate for many applications, but they can be improved without additional matrix inversion.

ANALYTICAL PREVIEW: THERMAL INTERCHANGE

A brief overview of the procedure is given here before discussion of mean values and presentation of algebraic details. The improvement starts by using Equations 77 and 78 iteratively in Equations 5 and 6. The first iteration leads to results of the form

$$\mathscr{J}_{kw} \approx \epsilon_{kw} E_k + \sum_{\zeta=1}^{m+1} E_\zeta \Gamma'_{\zeta kw}$$

$$J_k \approx \epsilon_k E_k + \sum_{\zeta=1}^{m+1} E_\zeta \Gamma'_{\zeta k}$$

The definition of E_ζ must be expressed in terms of the E_j's comprising a subset in the form

$$E_\zeta = \sum_{j=1}^{M_\zeta} E_j K_j$$

where the K's are weighting factors. When this result is introduced in the first iteration, it is possible to obtain

$$\mathcal{J}_{kw} \approx \epsilon_{kw} E_k + \sum_{j=1}^{M+1} E_j \Gamma_{\zeta_j} \gamma_k \xi_w = \sum_{j=1}^{M+1} E_j H_{jkw}$$

$$J_k \approx \epsilon_k E_k + \sum_{j=1}^{M+1} E_j \Gamma_{\zeta_j} \gamma_k = \sum_{j=1}^{M+1} E_j H_{jk}$$

The first iteration expressions are introduced in the $\overline{\alpha_k G_k}$ relations, Equation 34, to obtain a second level of improvement. The $\overline{\alpha_k G_k}$ results, in turn, are used in the net heat flux equations to obtain $\mathcal{F}_{ki}$ expressions as follow:

A_k bidirectional: $1 \le k \le N$

$$\mathcal{F}_{ki} = \sum_{j=1}^{N} \left[\bar{\alpha}_{kj} F_{kj} H_{ijk} + \sum_{p=N+1}^{M+1} \left(\sum_{b=N+1}^{M+1} \bar{\alpha}_{kj(b)} \varphi_{kj(p,b)} \right) H_{ijp} \right]$$

$$+ \sum_{j=N+1}^{M+1} \left(\sum_{b=N+1}^{M+1} \bar{\alpha}_{kj(b)} \varphi_{kj(b)} \right) H_{ij} \qquad (79)$$

A_k diffuse/specular: $N + 1 \leq k \leq M$

$$\mathcal{J}_{ki} = \alpha_k \left\{ \sum_{j=1}^{N} \left[F_{kj} H_{ijk} + \sum_{p=N+1}^{M+1} \varphi_{kj(p)} H_{ijp} \right] + \sum_{j=N+1}^{M+1} \varphi_{kj} H_{ij} \right\} \qquad (80)$$

MEAN VALUES

The mean values of emissive surface properties may be found by using the procedures followed by Bevans and Edwards (Reference 1). The only conceptual difficulty arises in defining the mean value of black body emissive power for a nonisothermal subset. The pseudoradiosity of bidirectional node A_k toward an arbitrary A_i is given by Equation 5. The mean value of pseudoradiosity from the subset $A_Y \supset A_k$ toward $A_X \supset A_i$ is

$$\frac{1}{\pi} \int_{A_Y} \int_{A_X} \mathcal{J}_{ki} K_{ki} \, dA_i \, dA_k = \mathcal{J}_{YX} A_Y F_{YX} \qquad (81)$$

The engineering assumption of uniform pseudoradiosity at a surface simplifies Equation 81 by replacing the integrals with sums

$$\mathcal{J}_{YX} = \frac{\sum_{k=1}^{M_Y} \sum_{i=1}^{M_X (\text{or } N_X)} \mathcal{J}_{ki} A_k F_{ki}}{A_Y F_{YX}} \qquad (82)$$

The mean shape factor is defined in the usual manner as

$$A_Y F_{YX} = \sum_{k=1_Y}^{M_Y} \sum_{i=1_X}^{M_X (\text{or } N_X)} A_k F_{ki} \qquad (83)$$

For convenience, in the remaining discussion A_i will be assumed to be a diffuse/specular node; this choice will simplify the notation of the upper limit of summation over index i.

When the emission term of Equation 5 is used in Equation 81 it yields

$$\epsilon_{\gamma\chi} E_{\gamma} = \frac{\sum_{k=1_{\gamma}}^{N_{\gamma}} \sum_{i=1_{\chi}}^{M_{\chi}} \epsilon_{ki} E_k A_k F_{ki}}{A_{\gamma} F_{\gamma\chi}} \tag{84}$$

Equation 84 defines the mean directional emission from A_{γ} toward A_{χ}, but it does not define either $\epsilon_{\gamma\chi}$ or E_{γ}. If all of the nodes in A_{γ} were at the same temperature, then Equation 84 would define $\epsilon_{\gamma\chi}$ (cf., Reference 1):

$$\epsilon_{\gamma\chi} = \frac{\sum_{k=1_{\gamma}}^{N_{\gamma}} \sum_{i=1_{\chi}}^{M_{\chi}} \epsilon_{ki} A_k F_{ki}}{A_{\gamma} F_{\gamma\chi}} \tag{85}$$

Equation 85 will be taken as the definition of $\epsilon_{\gamma\chi}$. However, if Equations 84 and 85 are used to define E_{γ}, the result would be improper because A_{γ} would appear to have a different black body emissive power for each different A_{χ}. This anomaly may be reconciled by defining a mean value E_{γ} in terms of hemispherical emission:

$$E_{\gamma} = \frac{\sum_{\chi=1}^{m+1} \sum_{k=1_{\gamma}}^{N_{\gamma}} \sum_{i=1_{\chi}}^{M_{\chi}} \epsilon_{ki} A_k F_{ki} E_k}{\sum_{\chi=1}^{m+1} \sum_{k=1_{\gamma}}^{M_{\gamma}} \sum_{i=1_{\chi}}^{M_{\chi}} \epsilon_{ki} A_k F_{ki}} \tag{86}$$

The denominator of Equation 86 defines the hemispherical emittance of A_{γ}:

$$\epsilon_{\gamma} = \frac{\sum_{\chi=1}^{m+1} \sum_{k=1_{\gamma}}^{N_{\gamma}} \sum_{i=1_{\chi}}^{M_{\chi}} \epsilon_{ki} A_k F_{ki}}{A_{\gamma}} = \frac{\sum_{k=1}^{N_{\gamma}} \bar{\epsilon}_k A_k}{A_{\gamma}} \tag{87}$$

Equations 86 and 87 may be combined to obtain

$$E_\gamma = \frac{\sum_{k=1_\gamma}^{M_\gamma} \sum_{i=1}^{M+1} \epsilon_{ki} A_k F_{ki} E_k}{\epsilon_\gamma A_\gamma} = \frac{\sum_{k=1_\gamma}^{M_\gamma} \bar{\epsilon}_k A_k E_k}{\epsilon_\gamma A_\gamma} \tag{88}$$

where use is made of the identity

$$\sum_{\chi=i}^{m+1} \sum_{i=1_\chi}^{M_\chi} [\cdots] = \sum_{i=1}^{M+1} [\cdots]$$

and $\bar{\epsilon}_k$ is the hemispherical emittance of A_k. It should be observed that simple area averaging is adequate for defining any of the mean values E_γ and $\bar{\epsilon}_\gamma$ but not for $\epsilon_{\gamma\chi}$. The emissive power weighting factor is

$$K_j = \bar{\epsilon}_j A_j / \epsilon_{\xi_j} A_{\xi_j} \tag{89}$$

Mean emissive parameters for a diffuse/specular set follow in the same way and lead to the outermost equalities of Equations 87 and 88.

A pragmatic approach [is] utilized to define the reflective components of subset emergent fluxes. It [is] postulated that the initial mean values $\mathcal{J}_{\gamma\chi}$ and J_γ may be reasonably crude estimates so long as energy conservation principles are retained. An attractive assumption is to assume that all nodal surfaces are diffusely emitting and reflecting to solve for J_k, $k = 1, \cdots, M$; then let $\mathcal{J}_{ki} = J_k F_{ki}$ for the first iteration. Unfortunately, this approach does not decrease the size of the transfer matrix appreciably if N is small while M is very large. Instead, it is postulated that all $\mathcal{J}_{ki} = \mathcal{J}_{\gamma\chi}$ for $k \subset \gamma$ and $i \subset \chi$ and all $J_k = J_\gamma$, $k \subset \gamma$ on the right hand side of Equations 75 and 76. This ad hoc assumption is introduced after using Equation 82 on both sides of a subset emergent flux relation and collecting all coefficients (sums of products of reflectances and geometry factors) of radiosities and pseudoradiosities. Use is made also of the general form of the defining equations for emergent fluxes for the third approximation given in Reference 1. Each nodal area is summed over all incident directions and then over the emergent directions implied by $\mathcal{J}_{\gamma\chi}$ and Equation 82. As a result, the R-factors of Equations 75 and 76 are found as follow:

$$R_{\zeta\gamma\chi} = \frac{\sum_{j=1_\zeta}^{N_\zeta} \sum_{k=1_\gamma}^{N_\gamma} \sum_{k'=1_\gamma}^{N_\gamma} \sum_{i=1_\chi}^{M_\chi} r_{jk'i} A_{k'} F_{k'j} A_i F_{ik}}{A_\gamma A_\chi F_{\chi\gamma}} \tag{89a}$$

$$R_{\zeta(\nu)\gamma\chi} = \frac{\sum_{p=1_\nu}^{M_\nu} \sum_{k=1_\gamma}^{M_\gamma} \sum_{k'=1_\gamma}^{N_\gamma} \sum_{i=1_\chi}^{M_\chi} \sum_{b=N+1}^{M} r_{j(b)k'i} A_{k'} \varphi_{k'j(p,b)} A_i F_{ik}}{A_\gamma A_\chi F_{\chi\gamma}} \tag{89b}$$

$$R'_{\zeta\gamma\chi} = \frac{\sum_{j=1_\zeta}^{M_\zeta} \sum_{k=1_\gamma}^{N_\gamma} \sum_{k'=1_\gamma}^{N_\gamma} \sum_{i=1_\chi}^{M_\chi} \sum_{b=N+1}^{M} r_{j(b)k'i} A_{k'} \varphi_{k'j(b)} A_i F_{ik}}{A_\gamma A_\chi F_{\chi\gamma}} \tag{89c}$$

$$R_{\zeta\gamma} = \frac{1}{A_\gamma} \sum_{j=1_\zeta}^{N_\zeta} \sum_{k=1_\gamma}^{M_\gamma} \left[\rho_k^d A_k F_{kj} + \sum_{p=1_\nu}^{M_\nu} \rho_k^d A_k \varphi_{kj(p)} \right] \tag{89d}$$

$$R_{\zeta(\nu)\gamma} = \frac{1}{A_\gamma} \sum_{j=1_\zeta}^{N_\zeta} \sum_{k=1_\gamma}^{M_\gamma} \sum_{p=1_\nu}^{M_\nu} \rho_k^d A_k \varphi_{kj(p)} \tag{89e}$$

$$R'_{\zeta\gamma} = \frac{1}{A_\gamma} \sum_{j=1_\zeta}^{M_\zeta} \sum_{k=1_\gamma}^{M_\gamma} \rho_k^d A_k \varphi_{kj} \tag{89f}$$

The index k' is used in Equations 89a through 89c as a dummy for k during the sum over incident directions. In Equation 89c, the direct view term, $F_{k'j}$, must be included in $\varphi_{k'j(b)}$. The use of Equations 89 eliminates the need for defining separate mean values of $\overline{r}_{\zeta\gamma\chi}$, $\overline{r}_{\zeta(\nu)\gamma\chi}$, etc., because the R's may be introduced in a transfer matrix directly.

The definitions of mean values and R's is heuristic rather than rigorous. In this regard, it is possible that other definitions may be obtained which either improve the accuracy of the mean value solutions or simplify the R-algorithms or both. A variety of numerical experiments should be performed to assess the several possibilities before the present definitions are accepted as final.

MEAN-TO-LOCAL PSEUDORADIOSITY AND RADIOSITY

There is a direct one-to-one correspondence between the node-to node transfer matrix elements given by Equations 38 and the subset-to-subset elements. The R's must be used instead of reflectance-geometry products. For example, instead of Equation 38d, the element is

$$D_{\substack{\gamma' \\ \zeta\nu'}} = -\left[R_{\zeta\gamma'}\,\delta^{\gamma'}_{\nu'} + R_{\zeta(\nu')\gamma'}\,(1 - \delta^{\gamma'}_{\nu'})\right] \tag{90}$$

The subset solutions have already been indicated in Equations 77.

The nodal excitation is recovered in Equations 77 with the use of Equation 88:

$$\mathcal{J}_{\gamma\chi} = \sum_{\zeta=1}^{m+1} E_\zeta\,\hat{\delta}_{\substack{\gamma\chi \\ \zeta}} = \sum_{\zeta=1}^{m+1}\ \sum_{j=1_\zeta}^{N_\zeta\ (\text{or } M_\zeta)} E_j\,\frac{\bar{\epsilon}_j A_j}{\epsilon_{\zeta j} A_{\zeta j}}\,\hat{\delta}_{\substack{\gamma\chi \\ \zeta j}}$$

$$= \sum_{j=1}^{M+1} E_j\,\Gamma_{j\gamma\chi} \tag{91a}$$

$$J_\gamma = \sum_{\zeta=1}^{m+1} E_\zeta\,\hat{\delta}_{\substack{\gamma \\ \zeta}} = \sum_{\zeta=1}^{m+1}\ \sum_{j=1_\zeta}^{N_\zeta\ (\text{or } M_\zeta)} E_j\,\frac{\epsilon_j A_j}{\epsilon_{\zeta j} A_{\zeta j}}\,\hat{\delta}_{\substack{\gamma \\ \zeta j}}$$

$$= \sum_{j=1}^{m+1} E_j\,\Gamma_{j\gamma} \tag{91b}$$

where the Γ-factors are introduced for notational convenience,

$$\Gamma_{j\gamma\chi} = \frac{\bar{\epsilon}_j A_j}{\epsilon_{\zeta_j} A_{\zeta_j}} \hat{\delta}^{\gamma\chi}_{\zeta_j} \tag{92a}$$

$$\Gamma_{j\gamma} = \frac{\epsilon_j A_j}{\epsilon_{\zeta_j} A_{\zeta_j}} \hat{\delta}^{\gamma}_{\zeta_j} \tag{92b}$$

The subset pseudoradiosity $\mathcal{J}_{\zeta_j \gamma_k} \approx \mathcal{J}_{jk}$ and subset radiosity $J_{\zeta_j} \approx J_j$ $J_{\zeta_j} \approx J_j$ are introduced in Equations 5 and 6 to obtain iterated solutions for $\mathcal{J}_{kw}$ and J_k. The order of summation must be rearranged to permit solutions expressed as products of excitation at various nodes, E_i, and influence coefficients. The iterated solutions take the following forms:

$$\mathcal{J}_{kw} = \bar{\epsilon}_{kw} E_k + \sum_{i=1}^{M+1} E_i \left\{ \sum_{j=1}^{N} \left[\bar{r}_{jk} \; F_{kj} \Gamma_{i\zeta_j\gamma_k} + \sum_{p=N+1}^{M+1} \Gamma_{i\zeta_j\nu_p} \left(\sum_{b=N+1}^{M} \bar{r}_{j(b)kw} \varphi_{kj(p,b)} \right) \right] + \sum_{j=N+1}^{M+1} \Gamma_{i\zeta_j} \left(\bar{r}_{jkw} F_{kj} + \sum_{p=N+1}^{M+1} \bar{r}_{j(p)kw} \varphi_{kj(p)} \right) \right\} \tag{93}$$

$$J_k = \epsilon_k E_k + \sum_{i=1}^{M+1} E_i \rho_k^d \left\{ \sum_{j=1}^{N} \left[F_{kj} \Gamma_{i\zeta_j\gamma_k} + \sum_{p=N+1}^{M+1} \Gamma_{i\zeta_j\nu_p} \varphi_{kj(p)} \right] + \sum_{j=N+1}^{M+1} \varphi_{kj} \Gamma_{i\zeta_j} \right\} \tag{94}$$

Equations 93 and 94 may be abbreviated as

$$\mathcal{J}_{kw} = \sum_{i=1}^{M+1} E_i H_{ikw} \tag{95a}$$

$$J_k = \sum_{i=1}^{M+1} E_i H_{ik} \quad (95b)$$

where

$$H_{ikw} = \epsilon_{ki} \delta_k^i + [\text{coefficient of } E_i \text{ in } (93)] \quad (96a)$$

$$H_{ik} = \epsilon_k \delta_k^i + [\text{coefficient of } E_i \text{ in } (94)] \quad (96b)$$

The iterated solutions of Equations 95 are used to construct absorbed flux expressions, $\overline{\alpha_k G_k}$, in Equations 34. The absorbed flux is used in the net heat flux, Equation 33, and the usual thermodynamic arguments are invoked to define script-F factors which are given by Equations 79 and 80. These script-F factors satisfy the summation relation given by Equation 54.

SOLAR INTERCHANGE FACTORS

The computation of radiation interchange factors for a mixed enclosure excited by a plane wave of solar energy may profit from the mean-to-local concept. The derivation is similar to the case of mean-to-local thermal excitation in that mean values must be defined for a variety of parameters, and a subset transfer matrix is inverted to obtain a first estimate of solar radiosity or pseudoradiosity at a surface. This estimate is improved iteratively and used in nodal expressions of net solar heat flux to obtain an expression equivalent to Equation 72.

The subset emergent flux equations are similar to Equations 75 and 76 with two exceptions: 1) the excitation terms are proportional to the product of solar flux, S, and incidence factors $C_{s,k}$, rather than emissive power, E, and 2) solar waveband properties must be used to construct the R's. This discussion will stress the development of subset excitation functions and present final results. Intermediate steps may be inferred from the preceding mean-to-local derivation.

The emergent flux relations for subsets may be written as

$$\mathcal{J}^*_{\gamma\chi} = S\, r^*_{s,\gamma,\chi} Z_{s,\gamma} + \overline{\rho^*_{\gamma,\chi} G^*_\gamma} \quad (97)$$

$$J^*_\gamma = S\, \rho^{*d}_\gamma Z_{s,\gamma} + \rho^{*d}_\gamma G^*_\gamma \quad (98)$$

where $Z_{s,\gamma}$ represents a subset solar incidence factor and $\overline{\rho^*_{\gamma\chi} G^*_\gamma}$ is an inseparable product formed by the reflective sum of Equation 75 and using solar properties instead of thermal properties. Equations 82 and 83 may be used to obtain

$$r^*_{s,\gamma,\chi} = \frac{\sum_{k=1_\gamma}^{N_\gamma} \sum_{i=1_\chi}^{M_\chi} r^*_{ski} A_k F_{ki}}{A_\gamma F_{\gamma\chi}} \tag{99}$$

a result analogous to Equation 85. The analogy may be extended to the subset incidence factor to obtain

$$Z_{s,\gamma} = \frac{1}{\rho^*_{\gamma,s} A_\gamma} \sum_{\chi=1}^{m+1} \sum_{k=1_\gamma}^{N_\gamma} \sum_{i=1_\chi}^{M_\chi} \overline{r^*_{ski} C_{sk}} A_k F_{ki}$$

$$= \frac{1}{\rho^*_{\gamma,s} A_\gamma} \sum_{k=1_\gamma}^{N_\gamma} \overline{C}_{s,k} \rho^*_{k,s} A_k \tag{100}$$

where $\overline{r^*_{ski} C_{sk}}$ is given by Equation 62 and use is made of Equations 64 and 65 to obtain Equation 100. The directional reflectance is

$$\rho^*_{\gamma,s} = \frac{\sum_{\chi=1}^{m+1} \sum_{k=1_\gamma}^{N_\gamma} \sum_{i=1_\chi}^{M_\chi} r^*_{ski} A_k F_{ki}}{A_\gamma} = \frac{\sum_{k=1_\gamma}^{N_\gamma} \rho^*_{k,s} A_k}{A_\gamma} \tag{101}$$

For a subset of diffuse/specular surfaces, Equation 100 becomes

$$Z_{s,\gamma} = \frac{\sum_{k=1_\gamma}^{M_\gamma} \rho^{*d}_k \overline{C}_{s,k} A_k}{\rho^{*d}_\gamma A_\gamma}, \quad n+1 \le \gamma \le m \tag{102}$$

and

$$\rho_{\gamma}^{*d} = \frac{1}{A_{\gamma}} \sum_{k=1_{\gamma}}^{M_{\gamma}} \rho_k^{*d} A_k \tag{103}$$

Matrix inversion yields solutions

$$\mathscr{J}^{*}_{\gamma,\chi} = S \sum_{\zeta=1}^{m} Z_{s,\zeta} \hat{\delta}^{*}_{\substack{\gamma\chi \\ \zeta}} = S \sum_{j=1}^{M} \overline{C}_{s,j} \frac{\rho^{*}_{j,s} A_j}{\rho^{*}_{\zeta,s} A_{\zeta}} \hat{\delta}^{*}_{\substack{\gamma\chi \\ \zeta_j}} \tag{104}$$

$$J^{*}_{\gamma} = S \sum_{\zeta=1}^{m} Z_{s,\zeta} \hat{\delta}^{*}_{\substack{\gamma \\ \zeta}} = S \sum_{j=1}^{M} \overline{C}_{s,j} \frac{\rho^{*}_{j,s} A_j}{\rho^{*}_{\zeta,s} A_{\zeta}} \hat{\delta}^{*}_{\substack{\gamma \\ \zeta_j}} \tag{105}$$

where Equations 100 and 102 are used to obtain the last expressions in Equations 104 and 105. The coefficients of $\overline{C}_{s,j}$ in Equations 104 and 105 represent the $\Gamma^{*}_{j\gamma\chi}$ and $\Gamma^{*}_{j\gamma}$ analogous to Equations 92a and 92b, respectively, and $\rho^{*}_{j,s} = \rho^{*d}_{j}$ when $N+1 \le j \le M$. The iteration which recovers the nodal radiosity uses the approximations $\mathscr{J}^{*}_{\gamma_k\chi_i} \approx \mathscr{J}^{*}_{ki}$ and $J^{*}_{\gamma} \approx J^{*}_{k}$ of Equations 104 and 105 to construct nodal equations:

$$\mathscr{J}^{*}_{ki} = S\, \overline{r^{*}_{ski}\, C_{sk}} + \sum_{j=1}^{N} \left[r^{*}_{jki} F_{kj} \mathscr{J}^{*}_{\zeta_j\gamma_k} + \sum_{p=N+1}^{M} \mathscr{J}^{*}_{\zeta_j\nu_p} \left(\sum_{b=1}^{M} r^{*}_{j(b)ki} \varphi^{*}_{kj(p,b)} \right) \right]$$

$$+ \sum_{j=N+1}^{M} J^{*}_{\zeta_j} \left(r^{*}_{jki} F_{kj} + \sum_{p=N+1}^{M} r^{*}_{j(p)ki} \varphi^{*}_{kj(p)} \right) \tag{106}$$

$$J^{*}_{k} = S \rho^{*d}_{k} \overline{C}_{s,k} + \rho^{*d}_{k} \left\{ \sum_{j=N+1}^{N} \left[\mathscr{J}^{*}_{\zeta_j\gamma_k} F_{kj} + \sum_{p=N+1}^{M} \mathscr{J}^{*}_{\zeta_j\nu_p} \varphi^{*}_{kj(p)} \right] \right.$$

$$\left. + \sum_{j=1} J^{*}_{\zeta_j} \varphi^{*}_{kj} \right\} \tag{107}$$

The iterated solutions are applied to solar net heat flux expressions (cf., Equation 68) and after some rearrangement, a form identical to Equation 72 is obtained where the $\mathcal{J}^*_{ki}$ are given below:

A_k bidirectional: $1 \le k \le N$

$$\mathcal{J}^*_{ki} = \sum_{j=1}^{N} \left[\overline{\alpha}^*_{kj} \, F_{kj} \, H^*_{ij} \qquad \sum_{p=N+1}^{M} \left(\sum_{b=N+1}^{M} \overline{\alpha}^*_{kj(b)} \, \varphi^*_{kj(p,b)} \right) H^*_{ijp} \right]$$

$$+ \sum_{j=N+1}^{M} \left(\sum_{b=N+1}^{M} \overline{\alpha}^*_{kj(b)} \, \varphi^*_{kj(b)} \right) H^*_{ij} \tag{108}$$

A_k diffuse/specular: $N+1 \le k \le M$

$$\mathcal{J}^*_{ki} = \alpha^*_k \left\{ \sum_{j=1}^{N} \left[F_{kj} \, H^*_{ijk} + \sum_{p=N+1}^{M} \varphi^*_{kj(p)} \, H^*_{ijp} \right] + \sum_{j-N+1}^{M} \varphi^*_{kj} \, H^*_{ij} \right\} \tag{109}$$

The H*'s are given explicitly for the sake of completeness:

$$H^*_{ijk} = \overline{r}^*_{sjk} \, \delta^i_j + \sum_{w=1}^{N} \left[\overline{r}^*_{wjk} \, F_{jw} \, \Gamma^*_{i\xi_w \zeta_j} + \sum_{p=N+1}^{M} \left(\sum_{b=N+1}^{M} \overline{r}^*_{w(b)jk} \varphi_{j,w(b)} \right) \Gamma^*_{i\xi_w \nu_p} \right.$$

$$\left. + \sum_{w=N+1}^{M} \left(\sum_{p=N+1}^{M} r^*_{w(p)jk} \, \varphi^*_{jw(b)} \right) \Gamma^*_{i\xi_w} \right. \tag{110}$$

$$H^*_{ij} = \rho^{*d}_j \, \delta^i_j + \rho^{*d}_j \left\{ \sum_{w=1}^{N} \left[F_{jw} \, \Gamma^*_{i\xi_w \zeta_j} + \sum_{p=N+1}^{M} \varphi^*_{jw(b)} \, \Gamma_{i\xi_w \nu_p} \right] \right.$$

$$\left. + \sum_{w=N+1}^{M} \varphi^*_{jw} \, \Gamma^*_{i\xi_w} \right\} \tag{111}$$

COMMENTS ON THE MEAN-TO-LOCAL ALGORITHM

The advantage of using the mean-to-local concept for computing radiation interchange factors is associated with inverting the transfer matrix. If an enclosure of 10^3 nodes is postulated, the computer time required for matrix inversion becomes the dominant factor in obtaining the interchange factors. If the matrix size can be reduced from $10^3 \times 10^3$ to $10^2 \times 10^2$, the matrix inversion time may be decreased by three orders of magnitude. However, it must not be construed that the time for computing all interchange factors is reduced by three orders of magnitude. The economy of matrix inversion is partly overcome by multiplication and summation requirements explicit in the R-factors and the iteration.

The mean-to-local algorithm is necessarily less accurate than the complete nodal inversion procedure. The assumption that all nodes in a subset have the same magnitude of pseudoradiosity or radiosity provides a guideline for selecting the nodes in any given subset for the purpose of enhancing accuracy. Intuition suggests that subsets should contain nodes which are both in close geometric proximity and have similar surface finishes. The geometric stipulation provides a meaningful physical interpretation for nodal and subset shape-factors and directional emission and reflection. Similarity of surface finishes strengthens the nodal and subset emission/reflection equivalence and suggests that equilibrium temperatures of nodes and subset will be comparable. The mean-to-local results reported in References 13 and 14 indicate that the greatest loss of accuracy will occur when two nodes in different subsets are closely coupled by a direct view while the subset coupling is relatively weak (i.e., $F_{ki} >> F_{\gamma\chi}$).

A variety of computational experiments should be performed to define the domain of utility for the mean-to-local concept. Tradeoffs between running time and accuracy should be established in terms of the total number of nodes in an enclosure, the number of nodes in each subset, and influence of geometry and surface properties.

IMPLEMENTATION OF RADIATION INTERCHANGE COMPUTATIONS

The node-to-node thermal and solar interchange factor equations have been programmed in Fortran V for digital computation for both direct inversion and mean-to-local algorithms. A complete description of the program is beyond the scope of this report, and only a brief discussion of the program capability and options will be given here. A detailed account of the program, including several sample problems, is presented in Reference 15.

The general program computes thermal and solar radiation interchange factors for enclosures composed of any combination of the following surface types:

1) Diffuse (diffuse emittance and reflectance)

2) Diffuse plus specular (diffuse emittance and reflectance having diffuse and specular components)

3) Specular (diffuse emittance and specular reflectance)

4) Bidirectional (directional emittance and bidirectional reflectance)

The number of surfaces that can be accommodated by the program is limited by storage requirements and the size of the transfer matrix. Storage capabilities permit up to 1500 surfaces in the enclosure, while the largest transfer matrix can contain 1500 by 1500 elements. The size of the transfer matrix, which depends upon the number and types of surfaces in the enclosure, can be computed by

$$NM + M - N = \text{number of rows} = \text{number of columns} \leq 1500$$

where N is the number of bidirectional surfaces and M is the total number of surfaces. Thus, the maximum number of surfaces in enclosures without bidirectional surfaces (N = 0) is 1500, while fully bidirectional enclosures (N = M) are limited to 38.

In addition to the standard node-to-node computation, a mean-to-local approximation is available in the program. This option utilizes groups of surfaces and mean values for a major portion of the calculations. The elements of the transfer matrix are group terms, thereby permitting a significant reduction in matrix size and time for inversion.

INPUTS AND OPTIONS

The minimum input data required to run the program consist of a unique identification number for each surface, surface property, and geometry data defining the shape and location of each surface in the enclosure. If the surface is not bidirectional, the diffuse emittance (ϵ) and reflectance (ρ^d) are input, and the specular reflectance component (ρ^s) is computed and stored.

$$\rho^s = 1 - \epsilon - \rho^d$$

Bidirectional surface data consist of bidirectional reflectance ($r_{jkw}(\theta_1, \theta_2, \varphi)$) and hemispherical-directional reflectance ($\rho(\theta_1)$). The former is input as a function of incidence angle (θ_1), reflected angle (θ_2), and the azimuthal angle between the two (φ), while the latter is specified as a function of incidence angle. Surface geometry and location data are essentially the same as that required by CONFAC (Reference 16) which is utilized as a subroutine to compute form factors between surfaces. In addition to the standard CONFAC surfaces, internal and external surfaces of hollow cones and cylinders can be employed for surface simulation. Surface data input can be minimized by use of the NODE option, which permanently discretizes a surface into planar subsurfaces. This subroutine is also employed to temporarily discretize nonplanar surfaces (i) into planes (i$'$) as required for CONFAC viewing surfaces. All the planar form factors F_{ij} are then summed over i$'$, and the total form factor F_{ij} is utilized in subsequent computations.

Three groups of options are available which reduce run time, modify program constants, or select alternate computational paths. The first group consists of KNOWN F and EQUIVALENT AF input blocks. If some, or all, of the form factors are known, either from previous runs of this program or from other sources, they can be input directly, thereby reducing the number of CONFAC computations to be performed. Often the presence of equivalent AFs due to geometrical symmetry within the enclosure can be established by inspection. When this situation exists, the user may input all the pairs of surfaces with the same (but unknown) value of AF into the EQUIVALENT block. CONFAC is used to compute the form factor between one pair of these surfaces; this factor is then assigned to each of the remaining pairs of surfaces in the equivalent group.

The second set of options permits modification of two program constants: the CONFAC mapping grid size and the allowable tolerance on the conservation checks (AF and Aφ). If the grid size is not specified by the user, a 6 by 6 grid is employed, and the user may select mapping grids in multiples of 6. The standard conservation tolerance is ±0.05A; this can be modified to any value desired by the user.

Operation of the standard program provides thermal interchange factors computed on a nodal basis. Solar interchange factors are computed by using the SUN option. If F's are required for both spectral regions, the NEW PROPERTIES option can be employed. This option permits multiple runs, only the first of which requires a full set of input data. The form factors and surface areas from the first run are saved and utilized in subsequent cases. Thus, a NEW PROPERTIES run for another spectral region would require only new property data for the surfaces.

If the enclosure consists of a large number of surfaces, the user may elect to exercise the MEAN option, which utilizes a mean-to-local approximation for the computation of interchange factors. A major portion of the computations is performed with mean values of groups of surfaces, which can greatly reduce the size of the transfer matrix and the time to invert it. The number of groups is user specified.

A SPACE option is available if the enclosure does not contain bidirectional surfaces. This option employs the conservation equations to obtain form factors (F) and exchange factors (φ) to space, thereby eliminating the need to input property and surface data for space nodes. Systems with bidirectional surfaces require the same quantity of input for space as for any other surface.

The presence of bidirectional and/or specular surfaces requires that all surfaces within the enclosure have unobstructed views to all other surfaces. Blocking can be accommodated only for fully diffuse enclosures. If blocking occurs or is suspected, BLOCKING data specifying the viewer, target, blockers, and CONFAC bridgelines must be included in the input.

OUTPUTS

Output is printed in the following sequence:

1) Card images of the input data
2) Sorted input data
3) F_{ij} and A_iF_{ij} for $i \leq j$
4) AF matrix
5) $A_iF_{ij(k)}$ and $F_{ij(k)}$
6) $A\varphi$ matrix
7) Transfer matrix

8) Inverted transfer matrix

9) Nodal areas

10) AF matrix

Storage requirements are minimized by printing the results as they are computed. Since the AF, $A\varphi$, and AF matrices are symmetrical, only the diagonal and half of each matrix is printed out.

PROGRAM FLOW

The general program consists of five primary functional modules (Figure 5), the first of which is the FILE MODULE which reads, manipulates, sorts, and stores the input data for subsequent computations. Surface generation, transformation, and discretization are performed as required, and the pertinent data are stored in the SURFACE DATA, KNOWN VALUE, EQUIVALENCE, and (if the enclosure is fully diffuse) BLOCKING files. If the enclosure contains surfaces with components of specular reflectance, surface geometry data of first order images are computed with the image generator and stored.

Next, the AF MODULE utilizes the file data to form the diagonal and upper half of the A_iF_{ij} matrix. Surfaces I and J are selected in an orderly manner, and their surface records in the FILE MODULE are searched for flags indicating that KNOWN F or EQUIVALENT AF data are available. If such flags are found, the data are extracted from the appropriate file and stored in the proper location of the AF matrix; then the next surface J is called. Absence of such data necessitates the use of CONFAC to compute the desired A_iF_{ij} term. For fully diffuse enclosures, the FILE MODULE is scanned for BLOCKING flags, the presence of which indicates that multisurface computations are to be performed. As the elements of the AF matrix are computed, a conservation check is made to insure that

$$\sum_J A_iF_{ij} = A_i \pm 0.05\, A_i$$

This computation can be employed either as a conservation check, or as a means to compute the form factor to a "space" node which may be difficult to define geometrically. All the A_iF_{ij} terms and areas A_i are printed for the user's reference. If the conservation check is not satisfied, error messages that include the summation ΣAF are also printed and the sun is aborted.

If the conservation criterion is satisfied, the FILE MODULE is searched for image data, the presence of which dictates the use of the

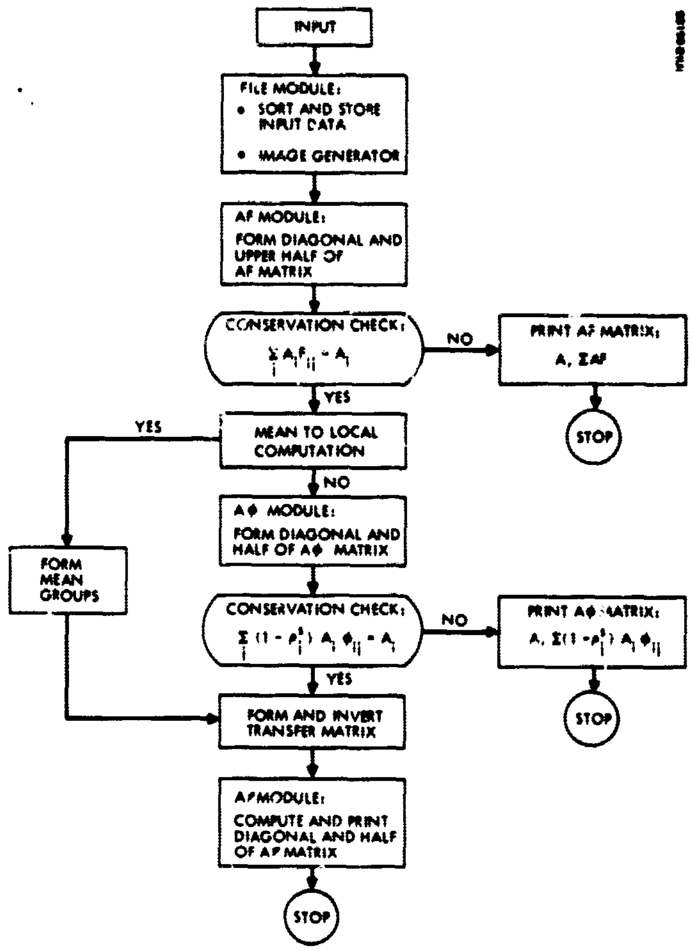

Figure 5. General Flow Chart for Computation of Radiation Interchange Factors

Aφ MODULE to compute and form the $A_i \Phi_{ij}$ matrix. A modified form of CONFAC is employed to compute the form factors from a surface (i) to the image of a surface (j) in a mirror (k) $F_{ij(k)}$. A logical procedure that eliminates the need for bridgelines in CONFAC multisurface calculations has been developed and incorporated in SILFAC. While the Aφ matrix is being formed, a conservation check is made:

$$\sum_{J} (1 - \rho_j^s) A_i \varphi_{ij} = A_i \pm 0.05 A_i$$

If the conservation is satisfied, or if there were no images, the transfer matrix is formed and inverted in the AF MODULE. The elements of the inverted matrix are then employed to compute the diagonal and half of the symmetrical AF matrix.

Use of the mean-to-local option employs the fifth primary module of the program. This module forms groups of individual surfaces and computes mean properties for each group. The elements of the transfer matrix are computed for groups of surfaces, thereby reducing the size of the matrix. After inverting the transfer matrix, the node-to-node AF terms are computed with nodal and mean parameters.

The same computational sequences are performed for thermal and solar interchange factors; the primary difference between the two lies in the property values used.

SUMMARY

The script-F factor developed in the text lends itself to nodal analysis used for thermal design problems. All the algorithms presented here are based on inverting a transfer matrix that contains various products of surface properties and geometrical factors as input elements. Ten expressions are presented for the consideration of potential users:

$\mathcal{F}_{ki}$				
Equation Number	Application			Comments
	A_k	A_i	Waveband	
51	Bidirect	Arbitrary	Thermal	Exact node-to-node
52	Diffuse/ specular	Arbitrary	Thermal	Exact node-to-node
58	Diffuse/ specular	Arbitrary	Thermal	Exact alternate form
70	Bidirect	Arbitrary	Solar	Exact node-to-node
71	Diffuse/ specular	Arbitrary	Solar	Exact node-to-node
73	Diffuse/ specular	Arbitrary	Solar	Exact alternate form
79	Bidirect	Arbitrary	Thermal	Mean-to-local approximation
80	Diffuse/ specular	Arbitrary	Thermal	Mean-to-local approximation
108	Bidirect	Arbitrary	Solar	Mean-to-local approximation
109	Diffuse/ specular	Arbitrary	Solar	Mean-to-local approximation

With the exception of Equations 58 and 73, all these equations have a common form:

$$\mathcal{F}_{ki} = \sum_{j=1}^{N} \left[\alpha_{kj} F_{kj} X_{ijk} + \sum_{p=N+1}^{M+1} \left(\sum_{b=N+1}^{M+1} \Phi_{kj(p,b)} \right) X_{ijp} \right]$$

$$+ \sum_{j=N+1}^{M+1} \left(\sum_{b=N+1}^{M+1} \Phi_{kj(b)} \right) X_{ij}$$

In the exact formulations, the X's are proportional to submatrices which fill the inverted transfer matrix. In the mean-to-local formulations, the X's are polynomials containing submatrices of a smaller inverse transfer matrix based on the subset mean value approximation. The Φ's depend on surface properties and geometrical factors. The expression above serves for A_k being bidirectional or diffuse/specular and for thermal and solar wavebands when appropriate surface properties are used for the Φ's and X's. Equations 58 and 73 are more succinct, but are limited to A_k being non-black with a strong diffuse component of reflectance. These latter constraints make it difficult to incorporate the brief expressions in large, general purpose computer programs.

These equations have been coded in Fortran V for digital computation. The program will compute script-F factors for any combination of the following surface types:

1) Diffuse emission and diffuse reflection

2) Diffuse emission and diffuse-plus-specular reflection

3) Diffuse emission and diffuse/specular reflection

4) Directional emission and bidirectional reflection

The number of nodes the program can accommodate is limited by a transfer matrix which holds 1500 by 1500 elements. If all nodes are bidirectional, and direct inversion is used, the maximum is $\sqrt{1500} \approx 38$. If no nodes are bidirectional using direct inversion, the maximum number of nodes is 1500. The use of the mean-to-local option can increase the maximum number of nodes by an order of magnitude.

The computer program requires input of surface property data (emittance, diffuse reflectance, directional reflectance, and bidirectional reflectance) for the waveband of interest (thermal or solar). Node geometry data are similar to that required for CONFAC, which is used as a subroutine.

Printed outputs of the computer program include the following:

1) Input data

2) Nodal shape factors

3) Transfer matrix

4) Inverted transfer matrix

5) Nodal areas

6) AF matrix

The printout is minimized by using reciprocity wherever appropriate.

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APPENDIX A. BIDIRECTIONAL REFLECTANCE OF DIFFUSE-PLUS-SPECULAR SURFACES

Experimental evidence (Reference 6) suggests that thermal control surfaces used in aerospace systems have reflective characteristics which are diffuse-plus-specular ($\rho = \rho^d + \rho^m$) for incident thermal radiation and either diffuse-plus-specular or bidirectional for incident solar radiation. Analytical procedures and algorithms are well defined for the case of diffuse-plus-specular surfaces, while additional studies are required for bidirectional surfaces. In order to verify and de-bug computer programs based on bidirectionally reflecting enclosures, it will be convenient to make a comparison between an enclosure formulated as a diffuse-plus-specular model and the same enclosure formulated as a bidirectional model. However, a nodal formulation is required to relate diffuse and specular components of reflectance to the bidirectional reflectance.

Bevans and Edwards (Reference 1) developed a "one bounce" approximation to represent bidirectional reflectance in terms of diffuse and specular components:

$$r_{j,k,w} = \rho_k^d + \rho_k^m \frac{F_{w,j(k)}}{F_{wk} F_{kj}} \qquad \text{(A-1)}$$

It will be shown below that the "one bounce" relationship is exact, rather than approximate, so long as the pseudoradiosity equation given in Reference 1 is modified to take proper account of specular reflections.

ANALYSIS

A four surface enclosure is shown in Figure A-1; for purposes of discussion, it will be assumed that A_1 and A_2 reflect in an arbitrary manner with no specular component, while A_3 and A_4 have reflectances $\rho_k = \rho_k^d + \rho_k^m$, k = 3, 4, respectively. The image method is used below with the concept of pseudoradiosity to construct expressions for bidirectional reflectances in terms of diffuse and specular components.

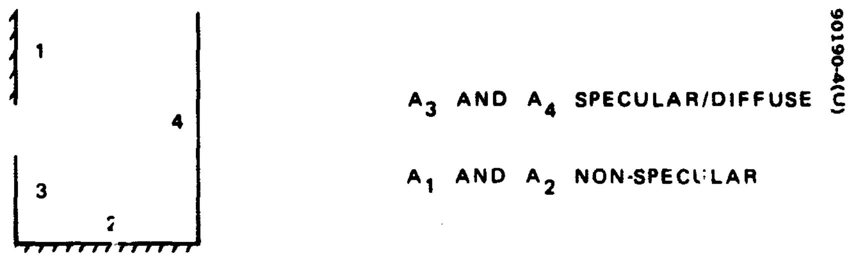

Figure A-1. Hypothetical Enclosure

Conservation of energy suggests that the direct irradiation of a surface A_k by a directional flux from a surface A_j may be expressed as

$$A_k \, G_k^{(j)} = \mathcal{J}_{jk} \, A_j \, F_{jk} \tag{A-2}$$

This equation will be used to construct $\mathcal{J}_{jk}$ when A_j is a specular-plus-diffuse surface. Rearranging Equation A-2 yields

$$\mathcal{J}_{jk} = \frac{A_k}{A_j \, F_{jk}} \, G_k^{(j)} \tag{A-3}$$

The direct irradiation, $G_k^{(j)}$, will consist of an emission component, a diffuse reflection component, and a specular reflection component. When the construction of $\mathcal{J}_{jk}$ is completed, the result will be compared to Bevans and Edwards (Reference 1) third approximation:

$$\mathcal{J}_{jk} = \epsilon_{jk} \, E_k + \sum_{p=1}^{M} \bar{r}_{pjk} \, \mathcal{J}_{pj} \, F_{jp} \tag{A-4}$$

However, Equation A-4 does not account for the possibility of multiple specular reflections in the surface A_j. A more appropriate development for a mixed enclosure (i.e., bidirectional and diffuse-plus-specular) is given in the introduction to this document, but it requires some modification before being applied to the problem at hand, i.e., treating a mixed enclosure as a bidirectional enclosure.

The geometry of Figure A-1 will be used to obtain $\mathcal{F}_{41}$ by constructing $G_1^{(4)}$. A straightforward development of the specular irradiation of A_1 may be achieved with the use of the image method. Figure A-2 shows the images seen from A_1 and which appear to lie behind A_4. A straight line drawn from the lower edge of A_1 through the first image of A_3 in A_4 (i. e., $A_{3(4)}$ is labelled as an image terminator because an observer on A_1 can not see any higher order images which lie above the line. A_1 is assumed to be nonspecular so that it "sees" itself only as the single image $A_{1(4)}$. Inasmuch as all the images lie behind A_4, they may be accounted as part of $\mathcal{F}_{41}$. The accounting details follow:

<u>A_1 to A_1:</u> The only contribution is

$$\rho_4^m A_{1(4)} F_{1(4),1} \mathcal{F}_{14} \tag{A-5}$$

<u>A_2 to A_1:</u> Both $\mathcal{F}_{23}$ and $\mathcal{F}_{24}$ contribute to the specular irradiation of A_1. The first, third, and fifth images of A_2 originate with $\mathcal{F}_{24}$:

$$\mathcal{F}_{24}\left[\rho_4^m A_{2(4)} F_{2(4),1} + \rho_4^{2m} \rho_3^m A_{2(4^2,3)} F_{2(4^2,3),1} + \rho_4^{3m} \rho_3^{2m} A_{2(4^3,3^2)} F_{2(4^3,3^2),1}\right] \tag{A-6}$$

The second and fourth images start with a flux from A_2 to A_3, and the contribution to A_1 is

$$\mathcal{F}_{23}\left[\rho_3^m \rho_4^m A_{2(3,4)} F_{2(3,4),1} + \rho_3^{2m} \rho_4^{2m} A_{2(3^2,4^2)} F_{2(3^2,4^2),1}\right] \tag{A-7}$$

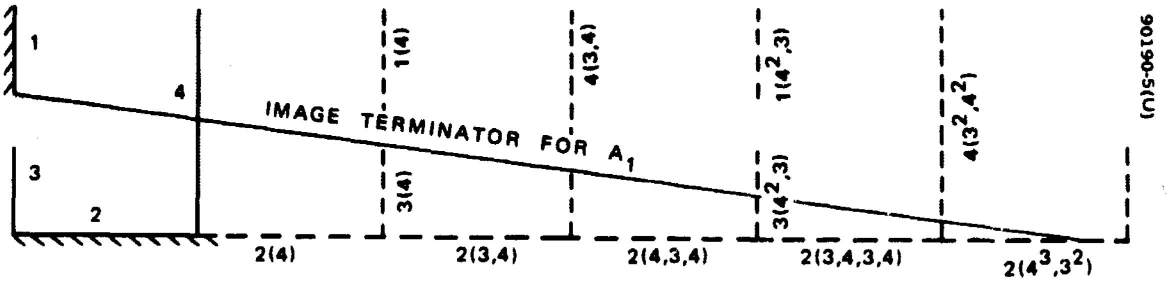

Figure A-2. Images Lying Behind A_4

$\underline{A_3 \text{ to } A_1}$: Only two images of A_3 can be seen from parts of A_1. The contribution of both is due to $\mathscr{J}_{34}$

$$\mathscr{J}_{34}\left[\rho_4^m A_{3(4)} F_{3(4),1} + \rho_4^{2m} \rho_3^m A_{3(4^2,3)} F_{3(4^2,3),1}\right] \quad \text{(A-8)}$$

$\underline{A_4 \text{ to } A_1}$: Two partial views of A_4 contribute to A_1; however, the images originate with the flux from A_4 to A_3.

$$\mathscr{J}_{43}\left[\rho_3^m \rho_4^m A_{4(3,4)} F_{4(3,4),1} + \rho_3^{2m} \rho_4^{2m} A_{4(3^2,4^2)} F_{4(3^2,4^2),1}\right] \quad \text{(A-9)}$$

The specular irradiation of A_1 "arising" at A_4 is then

$$\left[A_1 G_1^{(4)}\right]_{spec} = (A5) + (A6) + (A7) + (A8) + (A9) \quad \text{(A-10)}$$

The nonspecular irradiation of A_1 "arising" at A_4 consists of a directional emission component, $\epsilon_{41} E_4$, and a diffuse reflection component, $\rho_4^d G_4$

$$\left[A_1 G_1^{(4)}\right]_{nonspecular} = A_4 F_{4,1}\left[\epsilon_{4,1} E_4 + \rho_4^d G_4\right] \qquad \text{(A-11)}$$

The irradiation of A_4 may be expressed as

$$G_k = \sum_{j=1}^{4} G_k^{(j)} \qquad \text{(A-12)}$$

where $k = 4$ is the case of interest.

Figure A-3 shows the enclosure and images which may be seen from some portion of A_4. Since both A_3 and A_4 have specular components of reflectance, an infinitude of images of A_2, A_3, and A_4 may be seen from some regions of A_4. The irradiation of A_4 is expressed as

$$G_4 = \mathcal{J}_{14}\left[F_{4,1} + \rho_4^m \rho_3^m F_{4,1(4,3)}\right] + \mathcal{J}_{23}\sum_{j=1}^{\infty} \rho_3^m \left[\rho_3^m \rho_4^m\right]^{j-1} F_{4,2(3^j,4^{j-1})}$$

$$+ \mathcal{J}_{24}\sum_{j=0}^{\infty}\left[\rho_3^m \rho_4^m\right]^j F_{4,2(3^j,4^j)} + \mathcal{J}_{34}\sum_{j=0}^{\infty}\left[\rho_3^m \rho_4^m\right]^j F_{4,3(3^j,4^j)}$$

$$+ \mathcal{J}_{43}\sum_{j=1}^{\infty} \rho_3^m \left[\rho_3^m \rho_4^m\right]^{j-1} F_{4,4(3^j,4^{j-1})} \qquad \text{(A-13)}$$

With the help of Equations A-10 and A-11, the pseudoradiosity of Equation A-3 becomes

$$\mathcal{J}_{41} = \epsilon_{41} E_4 + \rho_4^d G_4 + \frac{\left[A_1 G_1^{(4)}\right]_{spec}}{A_4 F_{4,1}} \qquad \text{(A-14)}$$

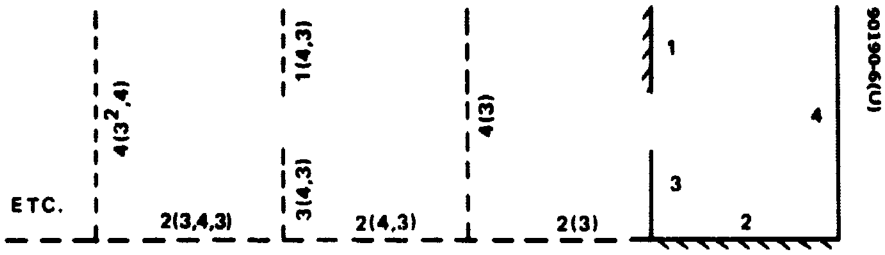

Figure A-3. Images Lying Behind A_3

Equation A-14 may be recast in the form

$$\mathcal{J}_{41} = \epsilon_{41} E_4 + \sum_{p=1}^{4} \sum_{w=1}^{4} \mathcal{J}_{pw} \mathcal{C}_{p(w), 4, 1} \tag{A-15}$$

where the pseudoradiosities, $\mathcal{J}_{pw}$, and the coefficients $\mathcal{C}_{p(w), 4, 1}$ follow from Equations A-5 through A-9 and Equation A-13; the non-zero factors are identified explicitly in Table A-1. The reciprocity relationship

$$A_{k(w)} F_{k(w), j} = A_j F_{j, k(w)} \tag{A-16}$$

is used to simplify the $\mathcal{C}_{p(w), 4, 1}$ expressions. Before proceeding, it should be observed that the formulation summarized in Table A-1 was achieved without introducing the concept of bidirectional reflectance.

The image method is used next with the modified third approximation (Equation A-4) to construct a second expression for $\mathcal{J}_{41}$ which uses the concept of bidirectional reflectance, $r_{w, 4, 1}$. Figure A-2 serves as the basis for the development. The irradiation of A_4 is given by Equation A-13. The contribution of each term to $\mathcal{J}_{41}$ may be found by multiplying each term by an appropriate bidirectional reflectance. The construction is summarized below:

<u>A_1 as a source:</u>

$$\mathcal{J}_{14} \left\{ r_{1, 4, 1} F_{4, 1} + r_{1(4, 3), 4, 1} \left[\rho_4^m \rho_3^m F_{4, 1(4, 3)} \right] \right\} \tag{A-17}$$

TABLE A-1. REFLECTIVE COMPONENTS OF $\mathcal{J}_{41}$ IN EQUATION A-15

$\mathcal{J}_{pw}$	$\mathcal{C}_{p(w),4,1}$
$\mathcal{J}_{14}$	$\rho_4^d \left[F_{4,1} + \rho_4^m \rho_3^m F_{4,1(4,3)}\right] + \frac{\rho_4^m}{F_{1,4}} \left[F_{1,1(4)}\right]$
$\mathcal{J}_{23}$	$\rho_4^d \left[\rho_3^m \sum_{j=1}^{\infty} \left(\rho_3^m \rho_4^m\right)^{j-1} F_{4,2(3^j,4^{j-1})}\right] + \frac{\rho_4^m}{F_{1,4}} \left[\rho_3^m F_{1,2(3,4)} + \rho_3^{2m} \rho_4^m F_{1,2(3^2,4^2)}\right]$
$\mathcal{J}_{24}$	$\rho_4^d \left[\sum_{j=0}^{\infty} \left(\rho_3^m \rho_4^m\right)^{j} F_{4,2(3^j 4^j)}\right] + \frac{\rho_4^m}{F_{1,4}} \left[F_{1,2(4)} + \rho_4^m \rho_3^m F_{1,2(4^2,3)} + \rho_4^{2m} \rho_3^{2m} F_{1,2(4^3,3^2)}\right]$
$\mathcal{J}_{34}$	$\rho_4^d \left[\sum_{j=0}^{\infty} \left(\rho_3^m \rho_4^m\right)^{j} F_{4,3(3^j,4^j)}\right] + \frac{\rho_4^m}{F_{1,4}} \left[F_{1,3(4)} + \rho_4^m \rho_3^m F_{1,3(4^2,3)}\right]$
$\mathcal{J}_{43}$	$\rho_4^d \left[\rho_3^m \sum_{j=1}^{\infty} \left(\rho_3^m \rho_4^m\right)^{j} F_{4,4(3^j,4^{j-1})}\right] + \frac{\rho_4^m}{F_{1,4}} \left[\rho_3^m F_{1,4(3,4)} + \rho_3^{2m} \rho_4^m F_{1,4(3^2,4^2)}\right]$

A comparison of Equation A-17 and the coefficient of $\mathcal{J}_{14}$ in Table A-1 suggests

$$r_{1,4,1} = \rho_4^d + \rho_4^m \frac{F_{1,1(4)}}{F_{1,4} F_{4,1}} \qquad \text{(A-18)}$$

$$r_{1(4,3),4,1} = \rho_4^d \qquad \text{(A-19)}$$

Equation A-19 serves to point out that A_1 does not see any higher order images of itself in A_4.

<u>A_2 as a source:</u>

$$\mathcal{J}_{23} \left\{ \rho_3^m \sum_{j=1}^{\infty} r_{2(3^j,4^{j-1})} \left[\left(\rho_3^m \rho_4^m \right)^{j-1} F_{4,2(3^j,4^{j-1})} \right] \right\}$$

$$+ \mathcal{J}_{24} \left\{ \sum_{j=0}^{\infty} r_{2(3^j,4^j),4,1} \left[(\rho_3 \rho_4)^j F_{4,2(3^j,4^j)} \right] \right\} \qquad \text{(A-20)}$$

Referring to Table A-1, it follows that

$$r_{2(3^j,4^{j-1}),4,1} = \rho_4^d + \rho_4^m \frac{F_{1,2(3^j,4^j)}}{F_{1,4} F_{4,2(3^j,4^{j-1})}} , \qquad j = 1, 2 \qquad \text{(A-21)}$$

$$r_{2(3^j,4^{j-1}),4,1} = \rho_4^d , \qquad j \geq 3 \qquad \text{(A-22)}$$

$$r_{2(3^j, 4^j), 4, 1} = \rho_4^d + \rho_4^m \frac{F_{1, 2(4^{j+1}, 3^j)}}{F_{1, 4} F_{4, 2(3^j, 4^j)}} \quad , \quad j = 0, 1, 2 \qquad \text{(A-23)}$$

$$r_{2(3^j, 4^j), 4, 1} = \rho_4^d \quad , \quad j \geq 3 \qquad \text{(A-24)}$$

Similar one-bounce expressions for $r_{w, 4, 1}$ arise from considering A_3 and A_4 as sources. The results are given below:

<u>A_3 as a source:</u>

$$r_{3(3^j, 4^j), 4, 1} = \rho_4^d + \rho_4^m \frac{F_{1, 3(4^{j+1}, 3^j)}}{F_{1, 4} F_{4, 3(3^j, 4^j)}} \quad , \quad j = 0, 1 \qquad \text{(A-25)}$$

$$r_{3(3^j, 4^j), 4, 1} = \rho_4^d \quad , \quad j \geq 2 \qquad \text{(A-26)}$$

<u>A_4 as a source:</u>

$$r_{4(3^j, 4^{j-1}), 4, 1} = \rho_4^d + \rho_4^m \frac{F_{1, 4(3^j, 4^j)}}{F_{1, 4} F_{4, 4(3^j, 4^{j-1})}} \quad , \quad j = 1, 2 \qquad \text{(A-27)}$$

$$r_{4(3^j, 4^{j-1}), 4, 1} = \rho_4^d \quad , \quad j \geq 3 \qquad \text{(A-28)}$$

The expression for $\mathcal{J}_{41}$ which results from Equations A-4 and A-21 through A-28 is

$$\mathcal{J}_{41} = \varepsilon_{41} E_4 + \mathcal{J}_{14} \left[\sum_{j=0}^{1} r_{1(4^j, 3^j), 4, 1} \varphi_{4, 1(4^j, 3^j)} \right]$$

$$+ \mathcal{J}_{23} \left[\sum_{j=1}^{\infty} r_{2(3^j, 4^{j-1}), 4, 1} \varphi_{4, 2(3^j, 4^{j-1})} \right]$$

$$+ \mathcal{J}_{24} \left[\sum_{j=0}^{\infty} r_{2(3^j, 4^j), 4, 1} \varphi_{4, 2(3^j, 4^j)} \right]$$

$$+ \mathcal{J}_{34} \left[\sum_{j=0}^{\infty} r_{3(3^j, 4^j), 4, 1} \varphi_{4, 3(3^j, 4^j)} \right]$$

$$+ \mathcal{J}_{43} \left[\sum_{j=1}^{\infty} r_{4(3^j, 4^{j-1}), 4, 1} \varphi_{4, 4(3^j, 4^{j-1})} \right] \qquad \text{(A-29)}$$

Equation A-29 contains $\mathcal{J}_{23}$ and $\mathcal{J}_{43}$, terms which would not appear in the primitive expression, Equation A-4. The exchange factors are

$$\varphi_{4, 1(4^j, 3^j)} = \left[\rho_4^m \rho_3^m \right]^j F_{4, 1(4^j, 3^j)} \qquad \text{(A-30a)}$$

$$\varphi_{4, 2(3^j, 4^{j-1})} = \rho_3^m \left[\rho_3^m \rho_4^m \right]^{j-1} F_{4, 2(3^j, 4^{j-1})} \qquad \text{(A-30b)}$$

$$\varphi_{4, 2(3^j, 4^j)} = \left[\rho_3^m \rho_4^m \right]^j F_{4, 2(3^j, 4^j)} \qquad \text{(A-30c)}$$

$$\varphi_{4,\,3(3^j,\,4^j)} = \left[\rho_3^m\ \rho_4^m\right]^j F_{4,\,3(3^j,\,4^j)} \qquad \text{(A-30d)}$$

$$\varphi_{4,\,4(3^j,\,4^{j-1})} = \rho_3^m \left[\rho_3^m\ \rho_4^m\right] F_{4,4(3^j,\,4^j)} \qquad \text{(A-30e)}$$

This definition of $\bar{r}$'s and φ's separates surface properties so that $\bar{r}_{jkw}$ depends on ρ_k^d, ρ_k^m, and geometry, only; properties of other surfaces are associated with the φ's.

It is possible to generalize these results to a limited extent, but a perfectly general expression must be somewhat vague because the geometry and hence images cannot be enumerated. The expression following represents a first step toward generality for an enclosure of M surfaces:

$$\mathcal{J}_{kj} = \epsilon_{kj} E_k + \sum_{w=1}^{M} \left\{ \mathcal{J}_{wk} \sum_x r_{w(y^x),\,k,\,j}\ \varphi_{k,\,w(y^x)} + \sum_{p=1}^{M} \mathcal{J}_{wp} \left(1 - \delta_k^p\right) \sum_z r_{w(y^z),\,k,\,j}\ \varphi_{k,\,w(y^z)} \right\} \qquad \text{(A-31)}$$

The summation over w accounts for contributions from all surfaces which A_k sees directly; the summation over p accounts for contributions from surfaces which A_k sees as images only. The summation over x and z refer to the number of reflections (i.e., images) required to reach A_k from A_w or A_p, respectively. The index y stands for the various surfaces which create images and the Kronecker delta, δ_k^p, is used to eliminate duplication.

The generalization for bidirectional reflectance in terms of diffuse and specular components may be expressed as

$$r_{w(y^x),\,k,\,j} = \rho_k^d + \rho_k^m \frac{F_{j,\,w(y^{x-1})}}{F_{j,\,k}\ F_{k,\,w(y^x)}} \qquad \text{(A-32)}$$

When $F_{j,w(y^{x-1})} = 0$, the bidirectional reflectance reduces to the diffuse component only in Equation 32.

This generalization of the Bevans and Edwards one-bounce approximation shows how it is possible to separate the properties of a given reflecting surface and exchange factors which usually contain reflective properties of different surfaces. It is possible to use Equation A-4 to obtain an equivalent expression of pseudoradiosity; however, alternate definitions of r_{jkw} would contain properties of surfaces other than A_k. Additionally, care would be required to account for the case where $F_{jp} = 0$, but an image of A_p can be seen from A_j.

APPENDIX B. ALTERNATE EXPRESSIONS FOR $\mathcal{J}^*_{kk}$: DIFFUSE-SPECULAR SURFACES ONLY

Two different expressions are available for the self-absorption factor, $\mathcal{J}^*_{kk}$ of diffuse-plus-specular surfaces. One of these accounts for reflected solar energy only, $\mathcal{J}^*_{kk}(I)$; the other accounts for direct plus reflected solar energy, $\mathcal{J}^*_{kk}(II)$. The two factors given in Equations 71 and 73, respectively, are repeated below:

$$\mathcal{J}^*_{kk}(I) = \alpha^*_k \left\{ \sum_{j=1}^{N} \left[F_{kj}\, \hat{\delta}^{*}_{jk\,k} + \sum_{p=N+1}^{M} \varphi^*_{kj(p)}\, \hat{\delta}^{*}_{jp\,k} \right] + \sum_{j=N+1}^{M} \varphi^*_{kj}\, \hat{\delta}^{*}_{j\,k} \right\} \tag{B-1}$$

$$\mathcal{J}^*_{kk}(II) = \frac{\alpha^*_k}{\rho^{*d}_k}\, \hat{\delta}^{*}_{k\,k} \tag{B-2}$$

The relationship between these factors and the direct plus reflected incidence feature of $\mathcal{J}^*_{kk}(II)$ can be shown by considering the following case. A mixed enclosure of M surface, contains N bidirectional surfaces and M - N diffuse-plus-specular surfaces. Only one surface, A_k, is directly irradiated so that $C_{s,j} = C_{s,k}\,\delta^k_j$, where $N+1 \leq k \leq M$. The general expression for the radiosity of this surface is

$$J^*_k = \rho^{*d}_k \left\{ \underbrace{SC_{s,k}}_{\text{direct}} + \underbrace{\sum_{j=1}^{N} \left[\mathcal{S}^*_{jk} F_{kj} + \sum_{p=N+1}^{M} \mathcal{S}_{jp}\, \varphi^*_{kj(p)} \right] + \sum_{j=N+1}^{M} J_j \varphi^*_{kj}}_{\text{reflected}} \right\} \tag{B-3}$$

Equations 66 reduce to the form

$$\mathcal{J}^*_{jk} = SC_{sk}\,\hat{\delta}^*_{\substack{jk\\k}},\quad \mathcal{J}^*_{jp} = SC_{sk}\,\hat{\delta}^*_{\substack{jp\\k}},\quad J^*_j = SC_{s,k}\,\hat{\delta}^*_{\substack{j\\k}} \tag{B-4}$$

Equations B-4 may be introduced in Equation B-3 and, after eliminating SC_{sk} from both sides, obtain

$$\hat{\delta}^*_{\substack{k\\k}} = \rho_k^{*d}\left\{\underbrace{1}_{\text{direct}} + \underbrace{\sum_{j=1}^{N}\left[\hat{\delta}^*_{\substack{jk\\k}}F_{kj} + \sum_{p=N+1}^{M}\hat{\delta}^*_{\substack{jp\\k}}\varphi^*_{kj(p)}\right] + \sum_{j=N+1}^{M}\hat{\delta}^*_{\substack{j\\k}}\varphi^*_{kj}}_{\text{reflected}}\right\} \tag{B-5}$$

Multiply both sides of Equation B-5 by α_k^*/ρ_k^{*d} and use Equations B-1 and B-2 to find

$$\mathcal{J}^*_{kk}(II) = \underbrace{\alpha^*_k}_{\text{direct}} + \underbrace{\mathcal{J}^*_{kk}(I)}_{\text{reflected}} \tag{B-6}$$

It should be observed that this result is different from the thermal case shown as Equation 60 because the definitions of the $\hat{\delta}$'s and $\hat{\delta}^*$'s are different.