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A SEMI-ANALYTIC THEORY FOR THE MOTION OF A LUNAR SATELLITE

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ABSTRACT

A semi-analytical solution to the problem of the motion of a satellite of the moon is presented. The theory is developed to third order, where first order is 10^{-2}. Perturbative effects which are considered include those due to the attraction of the moon, earth, and sun, the non-sphericity of the moon's gravitational field, the oblateness of the earth, coupling of lower-order terms, solar radiation pressure, and physical libration. Short-period terms and those with the period of the moon's longitude are produced by means of von Zeipel's method; it is proposed to obtain the secular perturbations, and those depending only on the argument of perilune, by numerical integration of the equations of motion.

CONTENTS

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INTRODUCTION

The motion of a satellite of the moon, or lunar orbiter, is analyzed. The solution is developed in powers of 10^{-2}—i.e., first order is 10^{-2}, second order is 10^{-4}, etc. The Hamiltonian for the "main problem" consists of zero-, first-, and second-order quantities. The higher-order Hamiltonian is of third order and consists of two parts, the first containing terms generated by coupling of lower-order terms and the second consisting of terms added by considering further perturbing forces, such as solar radiation pressure, physical libration, non-sphericity of the earth's potential field, the attraction of the sun, etc. Additional terms are produced by considering also the eccentricity and inclination of the moon's orbit.

In order to retain the relative orders of the disturbing forces, it is necessary to restrict the semi-major axis of the orbiter to about four moon radii or less. Furthermore, certain restrictions must be made on the eccentricity and inclination in order not to invalidate the solution. Therefore, the following assumptions on the semi-major axis, eccentricity, and inclination are made:

$$a \leq 4 \text{ moon radii}$$

$$.01 < e < .75$$

$$\sin I > .01 .$$

The small parameter of first order is $n^*_{☾}$, which is the mean motion of the moon's mean longitude. Recent determinations of the spherical harmonics of the moon (see References 1, 2, and 3) indicate that the second degree zonal and sectorial harmonic coefficients, together with the third,

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fourth, and fifth degree zonal harmonic coefficients, are all of about the same order of magnitude—namely, 10^{-4}. Therefore, the small parameters of second order are J_2, J_{22}, J_3, J_4, J_5, and $(n_{☾}/n)^2$, where J_2 and J_{22} define the principal part of the oblateness of the moon and $(n_{☾}/n)^2$ is the square of the ratio of the mean motion of the moon to that of the orbiter. The quantities J_3, J_4, and J_5 are higher degree zonal harmonic parameters of the moon. The small parameters of third order are $j_2(n_{☾}/n)^2$, $(n_{\oplus}/n)^2$, $(n_{☾}/n)^2 \sin(i_{☾}/2)$, $(n_{☾}/n)^2 e_{☾}$, σ, $(n_{☾}/n)^3$, and $\alpha n_{\odot}/n$. Here, the parameter j_2 is the principal part of the oblateness of the earth. The quantity $(n_{\oplus}/n)^2$ is the square of the ratio of the mean motion of the earth to that of the orbiter. The fact that the moon's orbital plane is inclined to its equator and the fact that the moon's orbit about the earth is elliptical give rise to the two small parameters of third order $(n_{☾}/n)^2 \sin(i_{☾}/2)$ and $(n_{☾}/n)^2 e_{☾}$, respectively. The radiation pressure gives rise to the third order parameter σ, and $(n_{☾}/n)^3$ is the cube of the ratio of the mean motion of the moon to that of the orbiter. Finally, $\alpha n_{\odot}/n$ is the correction due to physical libration.

The longest meridian of the moon contains the line joining the centers of mass of the earth and the moon. The right-handed, rotating, selenocentric coordinate systems adopted for this problem will then be as follows: The z-axis is the rotational axis of the moon, and the xy-plane is the moon's equatorial plane. The x-axis passes through the moon's longest meridian, and is assumed to rotate with the motion $n^*_{☾}$.

For a semi-major axis of 4 moon radii,

$$\left(\frac{n_{☾}}{n}\right)^2 \simeq 4.8 \times 10^{-4}$$

$$\left(\frac{n_{\oplus}}{n}\right)^2 \simeq 2.7 \times 10^{-6}$$

so that the oblateness terms in the lunar potential and the earth perturbations are both of about the same order. The largest oblateness term of the earth is then of the order of 10^{-6}. The perturbation of the sun is of third order.

Another perturbation to be considered is the effective radiation pressure of the sun, whose strength is about 1×10^{-4} dyne/cm^2. If the area-mass ratio of the orbiter is 1.5×10^{-1} cm^2/g, then the disturbing acceleration due to radiation pressure is also of third order.

EQUATIONS OF MOTION

The first step is to determine the equations of motion for the gravitational fields of the moon as a primary and the earth and sun as perturbations.

The following notation will be used:

subscript 0 : moon

subscript 1 : orbiter

subscript 2 : earth

subscript 3 : sun.

In an inertial system, the equations of motion are

$$m_j \ddot{\rho}_j = \mathrm{grad}_{\rho_j} U \quad (j = 0, 1, 2, 3) \; ,$$

where

$$U = k^2 \left(\frac{m_0 m_1}{r_{01}} + \frac{m_0 m_2}{r_{02}} + \frac{m_0 m_3}{r_{03}} + \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right) , \tag{1}$$

and ρ_j is the radius vector of any one of the four bodies. If ξ_j, η_j, ζ_j are the rectangular inertial coordinates of one of the bodies, then the equations of motion can also be written

$$m_j \ddot{\xi}_j = \frac{\partial U}{\partial \xi_j} \quad (j = 0, 1, 2, 3) \; . \tag{2}$$

Similar expressions hold for η_j and ζ_j.

It is now convenient to refer the orbiter to a moon-centered system, the moon to an earth-centered system, and the sun to a system whose origin is at the center of mass of the earth-moon system, designated by the subscript G. Therefore,

$$x_1 = \xi_1 - \xi_0 \; ,$$

$$x_0 = \xi_0 - \xi_2 \; ,$$

and

$$x_3 = \xi_3 - \xi_G = \xi_3 - \frac{m_0 \xi_0 + m_2 \xi_2}{m_0 + m_2} \; . \tag{3}$$

Similar expressions hold for y_j and z_j.

The equations of motion must be transformed accordingly. The partials in Equation 2 are then computed with respect to the new variables by making use of

$$\frac{\partial U}{\partial \xi_k} = \sum_{j=0}^{3} \frac{\partial U}{\partial x_j} \frac{\partial x_j}{\partial \xi_k} \quad (k = 0, 1, 2, 3) \; .$$

It follows that

$$\frac{\partial U}{\partial \xi_0} = \frac{\partial U}{\partial x_0} - \frac{\partial U}{\partial x_1} - \frac{m_0}{m_0 + m_2}\frac{\partial U}{\partial x_3},$$

$$\frac{\partial U}{\partial \xi_1} = \frac{\partial U}{\partial x_1},$$

$$\frac{\partial U}{\partial \xi_2} = -\frac{\partial U}{\partial x_0} - \frac{m_2}{m_0 + m_2}\frac{\partial U}{\partial x_3},$$

and

$$\frac{\partial U}{\partial \xi_3} = \frac{\partial U}{\partial x_3}. \tag{4}$$

Since

$$\ddot{x}_1 = \ddot{\xi}_1 - \ddot{\xi}_0,$$

it follows that

$$\ddot{x}_1 = \left(\frac{1}{m_1} + \frac{1}{m_0}\right)\frac{\partial U}{\partial x_1} - \frac{1}{m_0}\frac{\partial U}{\partial x_0} + \frac{1}{m_0 + m_2}\frac{\partial U}{\partial x_3}. \tag{5}$$

The force function U must now be expressed in terms of the new coordinates. We have

$$r_{01}^2 = (\xi_1 - \xi_0)^2 + (\eta_1 - \eta_0)^2 + (\zeta_1 - \zeta_0)^2 = x_1^2 + y_1^2 + z_1^2 = r_1^2,$$

$$r_{02}^2 = (\xi_0 - \xi_2)^2 + \cdots = x_0^2 + \cdots = r_0^2,$$

$$\begin{aligned} r_{03}^2 &= (\xi_3 - \xi_0)^2 + \cdots = \left(x_3 - \frac{m_2}{m_0 + m_2}x_0\right)^2 + \cdots \\ &= x_3^2 + y_3^2 + z_3^2 + \left(\frac{m_2}{m_0 + m_2}\right)^2 (x_0^2 + y_0^2 + z_0^2) - \frac{2m_2}{m_0 + m_2}(x_0 x_3 + y_0 y_3 + z_0 z_3) \\ &= r_3^2 + \left(\frac{m_2}{m_0 + m_2}\right)^2 r_0^2 - \frac{2m_2}{m_0 + m_2}\vec{r}_0 \cdot \vec{r}_3. \end{aligned}$$

$$r_{12}^2 = (\xi_1 - \xi_2)^2 + \cdots = (x_1 + x_0)^2 + \cdots = r_1^2 + r_0^2 + 2\vec{r}_1 \cdot \vec{r}_0.$$

$$r_{13}^2 = (\xi_1 - \xi_3)^2 + \cdots = \left[(x_1 - x_3) + \frac{m_2}{m_0 + m_2} x_0\right]^2 + \cdots$$

$$= r_1^2 + r_3^2 - 2\vec{r}_1 \cdot \vec{r}_3 + \left(\frac{m_2}{m_0 + m_2}\right)^2 r_0^2 + \frac{2m_2}{m_0 + m_2} (\vec{r}_0 \cdot \vec{r}_1 - \vec{r}_0 \cdot \vec{r}_3) ,$$

and

$$r_{23}^2 = (\xi_2 - \xi_3)^2 + \cdots = \left(-x_3 - \frac{m_0}{m_0 + m_2} x_0\right)^2 + \cdots$$

$$= r_3^2 + \left(\frac{m_0}{m_0 + m_2}\right)^2 r_0^2 + \frac{2m_0}{m_0 + m_2} \vec{r}_0 \cdot \vec{r}_3 . \tag{6}$$

Since U contains the inverses of these radii, note that

$$\frac{1}{r_{01}} = \frac{1}{r_1} ,$$

$$\frac{1}{r_{02}} = \frac{1}{r_0} ,$$

$$\frac{1}{r_{03}} = \frac{1}{r_3} \left[1 + \left(\frac{m_2}{m_0 + m_2}\right)^2 \left(\frac{r_0}{r_3}\right)^2 - \frac{2m_2}{m_0 + m_2} \frac{r_0}{r_3} \cos S_{03}\right]^{-1/2} ,$$

$$\frac{1}{r_{12}} = \frac{1}{r_0} \left[1 + \left(\frac{r_1}{r_0}\right)^2 + 2 \frac{r_1}{r_0} \cos S_{10}\right]^{-1/2} ,$$

$$\frac{1}{r_{13}} = \frac{1}{r_3} \left[1 + \left(\frac{r_1}{r_3}\right)^2 - 2 \frac{r_1}{r_3} \cos S_{13} + \left(\frac{m_2}{m_0 + m_2}\right)^2 \left(\frac{r_0}{r_3}\right)^2 + \frac{2m_2}{m_0 + m_2} \left(\frac{r_0 r_1}{r_3^2} \cos S_{10} - \frac{r_0}{r_3} \cos S_{03}\right)\right]^{-1/2} ,$$

and

$$\frac{1}{r_{23}} = \frac{1}{r_3} \left[1 + \left(\frac{m_0}{m_0 + m_2}\right)^2 \left(\frac{r_0}{r_3}\right)^2 + \frac{2m_0}{m_0 + m_2} \frac{r_0}{r_3} \cos S_{03}\right]^{-1/2} . \tag{7}$$

The angles used are shown in Figure 1. The angle S_{03} may be expressed in terms of S_{03}', as follows: Since

$$\cos S_{03}' = \frac{\vec{r}_0 \cdot \vec{r}_3\,'}{r_0 r_3'} ,$$

and

$$\frac{m_0}{m_0 + m_2} \vec{r}_0 + \vec{r}_3 = \vec{r}_3\,' ,$$

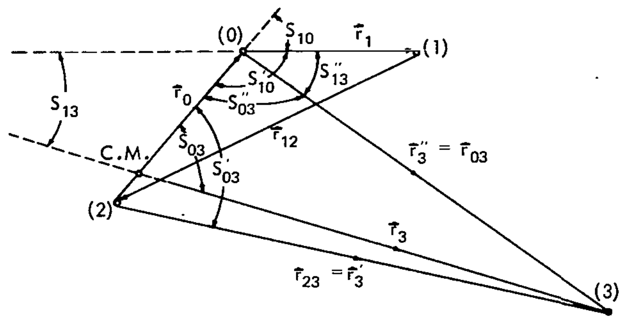

Figure 1—Relative positions of the moon, orbiter, earth, and sun.

then

$$\cos S'_{03} = \frac{\dfrac{m_0}{m_0 + m_2} r_0^2 + r_0 r_3 \cos S_{03}}{r_0 r_3'}$$

$$= \frac{r_0}{r_3'} \cdot \frac{m_0}{m_0 + m_2} + \frac{r_3}{r_3'} \cos S_{03} .$$

Finally,

$$\cos S_{03} = \frac{r_3'}{r_3} \left(\cos S'_{03} - \frac{r_0}{r_3'} \frac{m_0}{m_0 + m_2} \right) .$$

Certainly, to third order, $r_3' \simeq r_3$, so that

$$\cos S_{03} = \cos S'_{03} - \frac{r_0}{r_3} \frac{m_0}{m_0 + m_2} . \tag{8}$$

On the other hand, S_{10} may be replaced by 180°— S'_{10} where S'_{10} is the selenocentric elongation between the earth and the orbiter.

OBLATENESS TERMS

If the plane of reference is the lunar equatorial plane, then the disturbing force per unit mass may be written as

$$U_{OBL.} = \frac{k^2 m_0}{r_1} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{R_{☾}}{r_1} \right)^n J_{nm} P_{nm} (\sin \beta) \cos m (\lambda - \lambda_{nm}) .$$

The only terms to be included are J_{22}, $J_{20} = -J_2$, $J_{30} = -J_3$, $J_{40} = -J_4$, and $J_{50} = -J_5$. Therefore,

$$U_{OBL.} = -\frac{\mu_0}{r_1} \left[\sum_{n=2}^{5} \left(\frac{R_{☾}}{r_1} \right)^n J_n P_n (\sin \beta) - \left(\frac{R_{☾}}{r_1} \right)^2 J_{22} P_{22} (\sin \beta) \cos 2 (\lambda' - \lambda_{22}) \right] . \tag{9}$$

The angle β is the latitude of the orbiter with respect to the equator of the moon, λ' its longitude reckoned from any fixed direction, and λ_{22} the longitude of the moon's longest meridian from the same fixed direction. Note that λ' and β can be expressed in terms of the coordinates x_1, y_1, z_1 of the orbiter. However, λ_{22} will contain the time explicitly, since

$$\lambda_{22} = \lambda_{22}(0) + \gamma_{☾} t ,$$

where $\gamma_{☾}$ is the frequency of rotation of the moon around its axis. Further, if we neglect the physical libration of the moon, the longest meridian is always pointing toward the earth and $\gamma_{☾} \simeq n^*_{☾}$.

CANONICAL EQUATIONS AND GRAVITATIONAL TERMS

Let us choose as canonical variables the Delaunay set

$$L = \sqrt{\mu_0 a} , \qquad l = \text{mean anomaly,}$$

$$G = L\sqrt{1 - e^2} , \qquad \omega = \text{argument of pericenter,}$$

$$H = G \cos I , \qquad \Omega = \text{longitude of ascending node,}$$

where a, e, and I are the semi-major axis, eccentricity, and inclination, respectively, and where $\mu_0 = k^2 m_0$, in which k is the Gaussian constant and m_0 is the mass of the moon. Then the equations of motion become

$$\dot{L} = \frac{\partial \tilde{F}}{\partial l} , \qquad \dot{G} = \frac{\partial \tilde{F}}{\partial \omega} , \qquad \dot{H} = \frac{\partial \tilde{F}}{\partial \Omega} ,$$

$$\dot{l} = -\frac{\partial \tilde{F}}{\partial L} , \qquad \dot{\omega} = -\frac{\partial \tilde{F}}{\partial G} , \qquad \dot{\Omega} = -\frac{\partial \tilde{F}}{\partial H} , \tag{10}$$

where

$$\tilde{F} = \frac{\mu_0^2}{2L^2} + U_{GRAV.} + U_{OBL.} . \tag{11}$$

In Equation 11, $U_{GRAV.}$ is a function which has to satisfy the condition

$$\ddot{x}_1 = \frac{\partial}{\partial x_1}\left(U_{GRAV.} + \frac{\mu_0}{r_1}\right) . \tag{12}$$

The force function U depends on the new variables x through Equations 6. Then,

$$\frac{\partial U}{\partial x_0} = k^2 \left[-m_0 m_2 \frac{x_0}{r_{02}^3} - m_0 m_3 \frac{x_3 - \frac{m_2}{m_0 + m_2} x_0}{r_{03}^3} \left(-\frac{m_2}{m_0 + m_2} \right) - m_1 m_2 \frac{x_1 + x_0}{r_{12}^3} \right.$$

$$\left. - m_1 m_3 \frac{x_1 - x_3 + \frac{m_2}{m_0 + m_2} x_0}{r_{13}^3} \left(\frac{m_2}{m_0 + m_2} \right) - m_2 m_3 \frac{- x_3 - \frac{m_0}{m_0 + m_2} x_0}{r_{23}^3} \left(-\frac{m_0}{m_0 + m_2} \right) \right]$$

$$= \frac{\partial}{\partial x_1} \left[- k^2 m_0 m_2 \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{02}^3} + k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\vec{r}_3 \cdot \vec{r}_1}{r_{03}^3} - k^2 \frac{m_0 m_2^2 m_3}{(m_0 + m_2)^2} \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{03}^3} + k^2 m_1 m_2 \frac{1}{r_{12}} \right.$$

$$\left. + k^2 \frac{m_1 m_2 m_3}{m_0 + m_2} \frac{1}{r_{13}} - k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\vec{r}_3 \cdot \vec{r}_1}{r_{23}^3} - k^2 \frac{m_0^2 m_2 m_3}{(m_0 + m_2)^2} \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{23}^3} \right] \quad (13)$$

and

$$\frac{\partial U}{\partial x_3} = k^2 \left(- m_0 m_3 \frac{x_3 - \frac{m_2}{m_0 + m_2} x_0}{r_{03}^3} + m_1 m_3 \frac{x_1 - x_3 + \frac{m_2}{m_0 + m_2} x_0}{r_{13}^3} + m_2 m_3 \frac{- x_3 - \frac{m_0}{m_0 + m_2} x_0}{r_{23}^3} \right)$$

$$= \frac{\partial}{\partial x_1} \left(- k^2 m_0 m_3 \frac{\vec{r}_3 \cdot \vec{r}_1}{r_{03}^3} + k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{03}^3} \right.$$

$$\left. - k^2 m_1 m_3 \frac{1}{r_{13}} - k^2 m_2 m_3 \frac{\vec{r}_3 \cdot \vec{r}_1}{r_{23}^3} - k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{23}^3} \right) . \quad (14)$$

Then, substitution of Equations 13 and 14 into Equation 5 yields

$$\ddot{x}_1 = \frac{m_1 + m_0}{m_1 m_0} \frac{\partial}{\partial x_1} \left(k^2 \frac{m_0 m_1}{r_{01}} + k^2 m_1 m_2 \frac{1}{r_{12}} + k^2 m_1 m_3 \frac{1}{r_{13}} \right)$$

$$- \frac{1}{m_0} \frac{\partial}{\partial x_1} \left(- k^2 m_0 m_2 \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{02}^3} + k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\vec{r}_3 \cdot \vec{r}_1}{r_{03}^3} - k^2 \frac{m_0 m_2^2 m_3}{(m_0 + m_2)^2} \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{03}^3} + k^2 m_1 m_2 \frac{1}{r_{12}} \right.$$

$$+ k^2 \frac{m_1 m_2 m_3}{m_0 + m_2} \frac{1}{r_{13}} - k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\vec{r}_3 \cdot \vec{r}_1}{r_{23}^3} - k^2 \frac{m_0^2 m_2 m_3}{\left(m_0 + m_2\right)^2} \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{23}^3}\Bigg)$$

$$+ \frac{1}{m_0 + m_2} \frac{\partial}{\partial x_1} \Bigg[- k^2 m_0 m_3 \frac{\vec{r}_3 \cdot \vec{r}_1}{r_{03}^3} + k^2 \frac{m_0 m_2 m_3}{\left(m_0 + m_2\right)} \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{03}^3}$$

$$- k^2 m_1 m_3 \frac{1}{r_{13}} - k^2 m_2 m_3 \frac{\vec{r}_3 \cdot \vec{r}_1}{r_{23}^3} - k^2 \frac{m_0 m_2 m_3}{m_0 + m_2} \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{23}^3}\Bigg]$$

$$= \frac{\partial}{\partial x_1} \left(k^2 \frac{m_0 + m_1}{r_{01}} + k^2 \frac{m_2}{r_{12}} + k^2 \frac{m_3}{r_{13}} + k^2 m_2 \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{02}^3} - k^2 m_3 \vec{r}_1 \cdot \vec{r}_3'' \frac{1}{r_{03}^3} \right) . \tag{15}$$

Therefore, upon comparing Equations 12 and 15 we find

$$U_{\mathrm{GRAV.}} = \frac{k^2 m_1}{r_{01}} + \frac{k^2 m_2}{r_{12}} + \frac{k^2 m_3}{r_{13}} + k^2 m_2 \frac{\vec{r}_0 \cdot \vec{r}_1}{r_{02}^3} - k^2 m_3 \frac{\vec{r}_1 \cdot \vec{r}_3''}{r_{03}^3} , \tag{16}$$

where

$$\vec{r}_3'' = \vec{r}_3 - \frac{m_2}{m_0 + m_2} \vec{r}_0 ,$$

$$r_{01} = r_1 ,$$

$$r_{02} = r_0 ,$$

r_{13} = series in Legendre polynomials,

and

r_{03} = series in Legendre polynomials.

The term $k^2 m_1 / r_{01}$ can, of course, be neglected. Finally, from Equations 9, 11, and 16,

$$\tilde{F} = \frac{\mu_0^2}{2L^2} + \frac{\mu_2}{r_{12}} + \frac{\mu_3}{r_{13}} + \mu_2 \frac{\vec{r}_0 \cdot \vec{r}_1}{r_0^3} - \frac{\mu_3 \vec{r}_1 \cdot \vec{r}_3''}{r_{03}^3}$$

$$+ \frac{\mu_0}{r_1} \left[- \sum_{n=2}^{5} \left(\frac{R_{☾}}{r_1}\right)^n J_n P_n (\sin \beta) + \left(\frac{R_{☾}}{r_1}\right)^2 J_{22} P_{22} (\sin \beta) \cos 2 \left(\lambda' - \lambda_{22}\right) \right] . \tag{17}$$

Now

$$\frac{\vec{r}_0 \cdot \vec{r}_1}{r_0^3} = \frac{-r_0\, r_1 \cos S'_{10}}{r_0^3} = -\frac{r_1}{r_0^2} \cos S'_{10}$$

and

$$\frac{1}{r_{12}} = \frac{1}{r_0} + \frac{r_1}{r_0^2} \cos S'_{10} + \frac{1}{r_0} \sum_{p=2}^{\infty} P_p \left(\cos S'_{10}\right) \left(\frac{r_1}{r_0}\right)^p .$$

Then,

$$\frac{\mu_2}{r_{12}} + \mu_2 \frac{\vec{r}_0 \cdot \vec{r}_1}{r_0^3} = \frac{\mu_2}{r_0} + \frac{\mu_2}{r_0} \sum_{p=2}^{\infty} \left(\frac{r_1}{r_0}\right)^p P_p \left(\cos S'_{10}\right) ,$$

and since μ_2/r_0 is independent of the position of the orbiter, Equation 17 becomes

$$\hat{F} = \frac{\mu_0^2}{2L^2} + \frac{\mu_2}{r_0} \sum_{p=2}^{\infty} \left(\frac{r_1}{r_0}\right)^p P_p \left(\cos S'_{10}\right) + \frac{\mu_3}{r_{13}} - \mu_3 \frac{\vec{r}_1 \cdot \vec{r}_3''}{r_{03}^3}$$

$$+ \frac{\mu_0}{r_1} \left[- \sum_{n=2}^{5} \left(\frac{R_{\mathbb{C}}}{r_1}\right)^n J_n P_n (\sin \beta) + \left(\frac{R_{\mathbb{C}}}{r_1}\right)^2 J_{22} P_{22} (\sin \beta) \cos 2\left(\lambda' - \lambda_{22}\right) \right] . \quad (18)$$

A further simplification can be made if one considers the following expansion:

$$r_{13}^2 = \left[x_1 - \left(x_3 - \frac{m_2}{m_0 + m_2}\, x_0 \right) \right]^2 + \cdots$$

$$= r_1^2 + r_{03}^2 - 2 r_1\, r_{03} \cos S''_{13} = r_{03}^2 \left[1 + \left(\frac{r_1}{r_{03}}\right)^2 - 2 \frac{r_1}{r_{03}} \cos S''_{13} \right] .$$

Then,

$$\frac{1}{r_{13}} = \frac{1}{r_{03}} \sum_{p=0}^{\infty} \left(\frac{r_1}{r_{03}}\right)^p P_p \left(\cos S''_{13}\right)$$

$$= \frac{1}{r_{03}} + \frac{r_1 \cos S''_{13}}{r_{03}^2} + \frac{1}{r_{03}} \sum_{p=2}^{\infty} \left(\frac{r_1}{r_{03}}\right)^p P_p \left(\cos S''_{13}\right) .$$

Also,

$$\frac{\vec{r}_1 \cdot \vec{r}_3''}{r_{03}^3} = \frac{r_1 r_{03} \cos S_{13}''}{r_{03}^3} = \frac{r_1 \cos S_{13}''}{r_{03}^2} .$$

Since r_{03} is independent of x_1, y_1, z_1, Equation 18 becomes

$$\tilde{F} = \frac{\mu_0^2}{2L^2} + \frac{\mu_2}{r_0} \sum_{p=2}^{\infty} \left(\frac{r_1}{r_0}\right)^p P_p \left(\cos S_{10}'\right) + \frac{\mu_3}{r_{03}} \sum_{p=2}^{\infty} \left(\frac{r_1}{r_{03}}\right)^p P_p \left(\cos S_{13}''\right)$$

$$+ \frac{\mu_0}{r_1} \left[- \sum_{n=2}^{5} \left(\frac{R_{☾}}{r_1}\right)^n J_n P_n (\sin \beta) + \left(\frac{R_{☾}}{r_1}\right)^2 J_{22} P_{22} (\sin \beta) \cos 2\left(\lambda' - \lambda_{22}\right) \right] . \quad (19)$$

From now on, the subscript 1 will be omitted.

Including terms of second order, the Hamiltonian becomes

$$\tilde{F} \quad \frac{\mu_0^2}{2L^2} + \frac{\mu_2}{r_0} \left(\frac{r}{r_0}\right)^2 \left[P_2 \left(\cos S_{10}'\right) + \frac{r}{r_0} P_3 \left(\cos S_{10}'\right)\right] + \frac{\mu_3}{r_3} \left(\frac{r}{r_3}\right)^2 P_2 \left(\cos S_{13}''\right)$$

$$+ \frac{\mu_0}{r} \left[- \sum_{n=2}^{5} \left(\frac{R_{☾}}{r}\right)^n J_n P_n (\sin \beta) + \left(\frac{R_{☾}}{r}\right)^2 J_{22} P_{22} (\sin \beta) \cos 2\left(\lambda' - \lambda_{22}\right) \right] . \quad (20)$$

THE ANGLES S_{10}' AND S_{13}''

In view of the form assigned to the oblateness terms, the plane of reference is the equator of the moon. Therefore, the next step is to express the angles S_{10}' and S_{13}'' in terms of the orbital elements of the orbiter, the moon, the earth, and the sun, with respect to that plane. The geometry is shown in Figure 2.

The explicit form of S_{10}' and S_{13}'' requires the solution of two spherical quadrangles. This is now done.

If x, y, z and $x_{\oplus}$, $y_{\oplus}$, $z_{\oplus}$ are the rectangular coordinates of the orbiter and the earth, respectively, and r and $r_{\oplus}$ their selenocentric distances, then

$$\cos S_{10}' = \frac{\vec{r} \cdot \vec{r}_{\oplus}}{r r_{\oplus}} .$$

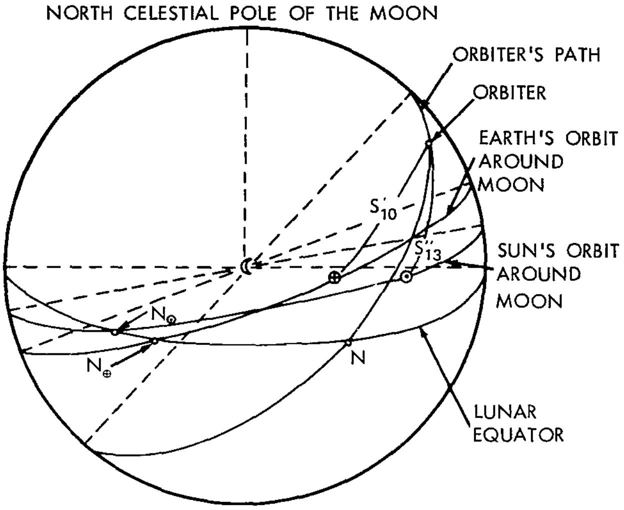

Figure 2—Selenocentric celestial sphere.

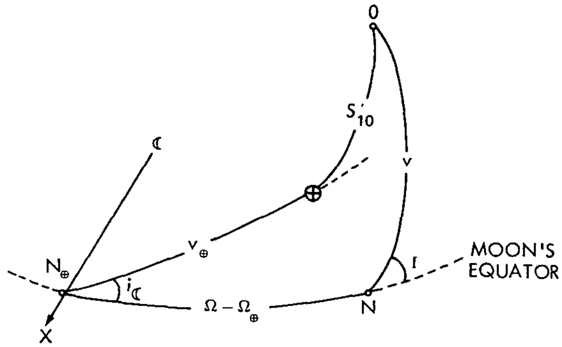

Figure 3—Angular relationships.

Consider

$$\Delta\Omega = \Omega - \Omega_{\oplus} ,$$

$$v = f + \alpha , \quad \text{and} \quad v_{\oplus} = f_{\oplus} + \alpha_{\oplus} .$$

Then, using vector notation, in the equatorial (moon) system indicated in Figure 3,

$$\vec{r}_{\oplus} = r_{\oplus} \begin{pmatrix} \cos v_{\oplus} \\ \sin v_{\oplus} \cos i_{☾} \\ \sin v_{\oplus} \sin i_{☾} \end{pmatrix} . \tag{21}$$

$$\vec{r} = r \begin{pmatrix} (\cos \alpha \cos \Delta\Omega - \sin \alpha \sin \Delta\Omega \cos I) \cos f - (\sin \alpha \cos \Delta\Omega + \cos \alpha \sin \Delta\Omega \cos I) \sin f \\ (\cos \alpha \sin \Delta\Omega + \sin \alpha \cos \Delta\Omega \cos I) \cos f - (\sin \alpha \sin \Delta\Omega - \cos \alpha \cos \Delta\Omega \cos I) \sin f \\ \sin \alpha \sin I \cos f + \cos \alpha \sin I \sin f \end{pmatrix} . \tag{22}$$

This enables us to compute $\cos S_{10}'$.

In exactly the same way, $\cos S_{13}''$ is obtained by substituting $\odot$ for $\oplus$ in the formulas in this section.

THE MAIN PROBLEM

The following approximations are made:

a. The earth's orbit around the moon (or vice versa) lies on the lunar equatorial plane. The error introduced by this approximation is proportional to the sine of half the inclination of the lunar orbit to its equator (~6° 41'), or about 0.06.

b. The orbit of the moon around the earth is circular and the motion uniform. The error is proportional to the moon's eccentricity, or about 0.055.

c. The sun's perturbations are negligible. The relative error for a moderately high satellite is about 0.05.

d. The mean longitude of the earth, $\lambda_\oplus$, is equal to λ_{22}.

With these approximations, the precision of the disturbing function is not higher than 10^{-4}. It will be called the "disturbing function of the main problem," and it is given by

$$\tilde{\tilde{F}} = \frac{\mu_0^2}{2L^2} + \frac{n_0^2}{\epsilon} r^2 P_2 \left(\cos \tilde{S}'_{10}\right) + \frac{\mu_0}{r}\left\{\left(\frac{R_{☾}}{r}\right)^2 \left[- J_2 P_2 (\sin\beta) + J_{22} P_{22} (\sin\beta) \cos 2\left(\lambda' - \lambda_\oplus\right)\right] - \left(\frac{R_{☾}}{r}\right)^3 J_3 P_3 (\sin\beta) - \left(\frac{R_{☾}}{r}\right)^4 J_4 P_4 (\sin\beta) - \left(\frac{R_{☾}}{r}\right)^5 J_5 P_5 (\sin\beta)\right\} , \qquad (23)$$

where

$$\epsilon = 1 + \frac{\mu_0}{\mu_2}$$

and where the angle $\tilde{S}'_{10}$ is as shown in Figure 4. Since $i_{☾} = 0$ and $v_\oplus + \Omega_\oplus$ can be replaced by $\lambda_\oplus$, it then follows that

$$\cos \tilde{S}'_{10} = \cos v \cos\left(\Omega - \lambda_\oplus\right) - \sin v \sin\left(\Omega - \lambda_\oplus\right) \cos I = s . \qquad (24)$$

The equations of motion are given by the canonical set

$$\dot{L} = \frac{\partial \tilde{\tilde{F}}}{\partial l} , \qquad \dot{l} = -\frac{\partial \tilde{\tilde{F}}}{\partial L} ,$$

$$\dot{G} = \frac{\partial \tilde{\tilde{F}}}{\partial \omega} , \qquad \dot{\omega} = -\frac{\partial \tilde{\tilde{F}}}{\partial G} ,$$

$$\dot{H} = \frac{\partial \tilde{\tilde{F}}}{\partial \Omega} , \qquad \dot{\Omega} = -\frac{\partial \tilde{\tilde{F}}}{\partial H} . \qquad (25)$$

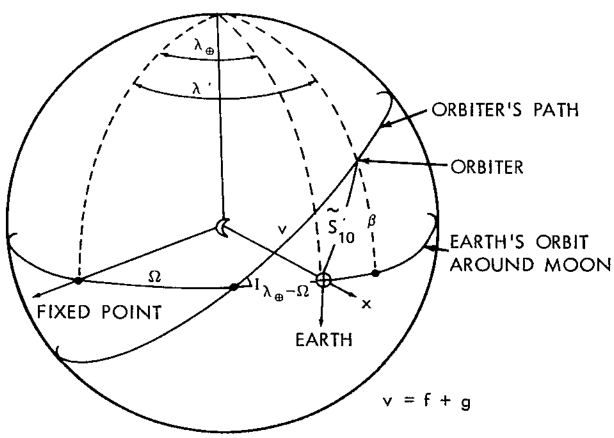

Figure 4—Simplified selenocentric celestial sphere.

Since the variables Ω and $\lambda_\oplus$ appear only in the combination $\Omega - \lambda_\oplus$, where $\lambda_\oplus = n_{☾}^* t + \text{const.}$, the degree of freedom is reduced by one by choosing as a new variable

$$h = \Omega - \lambda_{\oplus} .$$

The Hamiltonian must be modified accordingly, for

$$\dot{h} = \dot{\Omega} - n_{\mathbb{C}}^{*} = -\frac{\partial}{\partial H}\left(\tilde{\tilde{F}} + n_{\mathbb{C}}^{*} H\right)$$

(i.e., upon substitution of $\tilde{\tilde{F}} + n_{\mathbb{C}}^{*} H$ in place of $\tilde{\tilde{F}}$).

The Hamiltonian is still time-dependent through $\lambda_{\oplus}$. Since the longest meridian is always pointing toward the earth, it is possible to choose the rotating system whose x-axis passes through this meridian. The final form of the Hamiltonian for the main problem is therefore

$$F = \frac{\mu_0^2}{2L^2} + n_{\mathbb{C}}^{*} H + \frac{n_{\mathbb{C}}^2 r^2}{\epsilon} P_2(s) + \frac{\mu_0}{r}\left\{\left(\frac{R_{\mathbb{C}}}{r}\right)^2 \left[- J_2 P_2(\sin\beta) + J_{22} P_{22}(\sin\beta)\cos 2\lambda\right]\right.$$
$$\left. - \left(\frac{R_{\mathbb{C}}}{r}\right)^3 J_3 P_3(\sin\beta) - \left(\frac{R_{\mathbb{C}}}{r}\right)^4 J_4 P_4(\sin\beta) - \left(\frac{R_{\mathbb{C}}}{r}\right)^5 J_5 P_5(\sin\beta)\right\} , \quad (26)$$

where

$$\lambda = \lambda' - \lambda_{\oplus} .$$

The equations of motion are (using g in place of ω):

$$\dot{L} = \frac{\partial F}{\partial l} , \qquad \dot{G} = \frac{\partial F}{\partial g} , \qquad \dot{H} = \frac{\partial F}{\partial h} ,$$

$$\dot{l} = -\frac{\partial F}{\partial L} , \qquad \dot{g} = -\frac{\partial F}{\partial G} , \qquad \dot{h} = -\frac{\partial F}{\partial H} . \quad (27)$$

In the discussion that follows, the solution of this system will be given using von Zeipel's method.

DEVELOPMENT OF THE DISTURBING FUNCTION

The notation used will be as shown in Figure 4.

From Equation 24,

$$\cos \tilde{S}_{10}' = s = \cos\left(v + \Omega - \lambda_{\oplus}\right) - 2\sin^2\frac{I}{2}\sin v \sin\left(\lambda_{\oplus} - \Omega\right) ;$$

thus,

$$P_2(s) = \frac{3}{2} s^2 - \frac{1}{2}$$

$$= \frac{3}{2} \Big\{ \cos^2 f \cos(2g + 2h) + \sin^2(g + h)$$

$$- \sin f \cos f \sin(2g + 2h) + 4 \sin^4 \frac{I}{2} \sin^2 h \left[\cos^2 g - \cos^2 f \cos 2g \right.$$

$$\left. + \sin f \cos f \sin 2g \right] + 4 \sin^2 \frac{I}{2} \sin h \left[\sin f \cos f \cos(2g + h) \right.$$

$$\left. - \cos g \sin(g + h) + \cos^2 f \sin(2g + h) \right] \Big\} - \frac{1}{2} \cdot \quad (28)$$

Further,

$$\sin \beta = \sin I \sin v = \sin I \sin(f + g) ,$$

$$\cos \beta \cos(\lambda - h) = \cos v ,$$

$$\cos \beta \sin(\lambda - h) \quad \cos I \sin v ,$$

and therefore

$$\cos \beta \cos \lambda = \cos v \cos h - \cos I \sin v \sin h ,$$

from which

$$P_2(\sin \beta) \quad \frac{3}{2} \sin^2 \beta - \frac{1}{2} \quad \frac{3}{2} \sin^2 I \sin^2(f + g) - \frac{1}{2} ,$$

$$P_{22}(\sin \beta) \cos 2\lambda = 6 \cos^2 \beta \cos^2 \lambda - 3 \cos^2 \beta$$

$$= 6 \left(\zeta^2 \cos^2 f + \chi^2 \sin^2 f + 2\zeta\chi \sin f \cos f \right) - 3 \left(1 - \sin^2 I \sin^2 v \right),$$

$$P_3(\sin \beta) = \frac{5}{2} \sin^3 \beta - \frac{3}{2} \sin \beta = \frac{5}{2} \sin^3 I \sin^3(f + g) - \frac{3}{2} \sin I \sin(f + g) ,$$

$$P_4(\sin \beta) = \frac{35}{8} \sin^4 \beta - \frac{15}{4} \sin^2 \beta + \frac{3}{8} = \frac{35}{8} \sin^4 I \sin^4(f + g) - \frac{15}{4} \sin^2 I \sin^2(f + g) + \frac{3}{8} ,$$

$$P_5(\sin \beta) = \frac{63}{8} \sin^5 \beta - \frac{35}{4} \sin^3 \beta + \frac{15}{8} \sin \beta = \frac{63}{8} \sin^5 I \sin^5(f + g) - \frac{35}{4} \sin^3 I \sin^3(f + g) + \frac{15}{8} \sin I \sin(f + g) , \quad (29)$$

where

$$\zeta = \cos g \cos h - \cos I \sin g \sin h$$

and

$$\chi = -\sin g \cos h - \cos I \cos g \sin h .$$

Therefore, writing $\mu_{☾}$ for μ_0 in Equation 26, we have

$$F = \frac{\mu_{☾}^2}{2L^2} + n_{☾}^* H + \frac{3}{2} \frac{n_{☾}^2 r^2}{\epsilon} \Big\{ \cos^2 f \cos(2g + 2h) + \sin^2(g + h) - \sin f \cos f \sin(2g + 2h)$$

$$+ 4 \sin^4 \frac{I}{2} \sin^2 h \left[\cos^2 g - \cos^2 f \cos 2g + \sin f \cos f \sin 2g\right] + 4 \sin^2 \frac{I}{2} \sin h \Big[\sin f \cos f \cos(2g + h)$$

$$- \cos g \sin(g + h) + \cos^2 f \sin(2g + h)\Big]\Big\} - \frac{1}{2\epsilon} n_{☾}^2 r^2 + \frac{\mu_{☾} R_{☾}^2}{r^3} \Big\{ -\frac{3}{2} J_2 \sin^2 I \sin^2(f + g)$$

$$+ \frac{1}{2} J_2 + J_{22} \left[6\left(\zeta^2 \cos^2 f + \chi^2 \sin^2 f + 2\zeta\chi \sin f \cos f\right) - 3\left(1 - \sin^2 I \sin^2 v\right)\right]\Big\}$$

$$- \frac{\mu_{☾} R_{☾}^3 J_3}{r^4} \left[\frac{5}{2} \sin^3 I \sin^3(f + g) - \frac{3}{2} \sin I \sin(f + g)\right] - \frac{\mu_{☾} R_{☾}^4 J_4}{r^5} \Big[\frac{35}{8} \sin^4 I \sin^4(f + g)$$

$$- \frac{15}{4} \sin^2 I \sin^2(f + g) + \frac{3}{8}\Big] - \frac{\mu_{☾} R_{☾}^5 J_5}{r^6} \Big[\frac{63}{8} \sin^5 I \sin^5(f + g) - \frac{35}{4} \sin^3 I \sin^3(f + g)$$

$$+ \frac{15}{8} \sin I \sin(f + g)\Big] \quad (30)$$

$$= F_0 + F_1 + F_2 ,$$

where

$$F_0 = \frac{\mu_{☾}^2}{2L^2} \qquad \text{(0th order)} ,$$

$$F_1 = n_{☾}^* H \qquad \text{(1st order)} ,$$

and

$$F_2 = F - (F_1 + F_0) \qquad \text{(2nd order)} . \tag{31}$$

ELIMINATION OF SHORT-PERIOD PERTURBATIONS

The terms in F which depend on l will result in short-period perturbations. According to von Zeipel's method, the elimination of these terms corresponds to the solution of the system

$$F_0'(L') = F_0(L') ,$$

$$F_1'(H') = F_1(H') + \frac{\partial S_1}{\partial l} \frac{\partial F_0}{\partial L'} ,$$

$$F_2' = F_2 + \frac{\partial S_1}{\partial h} \frac{\partial F_1}{\partial H'} + \frac{\partial S_2}{\partial l} \frac{\partial F_0}{\partial L'} + \frac{1}{2} \left(\frac{\partial S_1}{\partial l}\right)^2 \frac{\partial^2 F_0}{\partial L'^2} \cdots , \tag{32}$$

where the generating function of the transformation

$$(L, G, H, l, g, h) \rightarrow (L', G', H', l', g', h')$$

is

$$S = L'l + G'g + H'h + S_1 + S_2 + \cdots , \tag{33}$$

and the new Hamiltonian

$$F' = F_0' + F_1' + F_2' + \cdots \tag{34}$$

should be independent of l'.

In order to accomplish this last requirement, a particular solution is given by

$$S_1 = 0 ,$$

$$F_2' = F_{2s} ,$$

and

$$\frac{\partial S_2}{\partial l} \frac{\partial F_0}{\partial L'} = - F_{2p} , \tag{35}$$

where F_{2s} and F_{2p} are, respectively, the parts of F_2 independent of l and dependent on l.

If S_2 is determined from this system, then the relations between old and new variables are

$$l' = l + \frac{\partial S_2}{\partial L'} , \qquad g' = g + \frac{\partial S_2}{\partial G'} , \qquad h' = h + \frac{\partial S_2}{\partial H'} ,$$

$$L = L' + \frac{\partial S_2}{\partial l} , \qquad G = G' + \frac{\partial S_2}{\partial g} , \qquad H = H' + \frac{\partial S_2}{\partial h} . \tag{36}$$

Next, the "secular" part of F_2 is determined. By definition,

$$F_{2s} = \frac{1}{2\pi} \int_0^{2\pi} F_2 \, dl . \tag{37}$$

Using the relations (E = eccentric anomaly)

$$dl = (1 - e \cos E) \, dE ,$$

$$r \cos f = a(\cos E - e) ,$$

$$r \sin f = a\sqrt{1 - e^2} \sin E ,$$

$$r = a(1 - e \cos E) ,$$

$$dl = \frac{r^2}{a^2 \left(1 - e^2\right)^{1/2}} df ,$$

and

$$\frac{1}{r} = \frac{1 + e \cos f}{a\left(1 - e^2\right)} , \tag{38}$$

it follows that

$$F' = \frac{\mu_{\mathbb{C}}^2}{2L'^2} + n_{\mathbb{C}}^* H' + \frac{n_{\mathbb{C}}^2 L'^4}{16 \mu_{\mathbb{C}}^2 \epsilon} \left\{ \left(5 - 3 \frac{G'^2}{L'^2}\right) \left[\left(- 1 + 3 \frac{H'^2}{G'^2}\right) + 3 \left(1 - \frac{H'^2}{G'^2}\right) \cos 2h' \right] \right.$$

$$\left. + 15 \left(1 - \frac{G'^2}{L'^2}\right) \left[\frac{1}{2} \left(1 + \frac{H'}{G'}\right)^2 \cos 2(g' + h') + \left(1 - \frac{H'^2}{G'^2}\right) \cos 2g' + \frac{1}{2} \left(1 - \frac{H'}{G'}\right)^2 \cos 2(g' - h') \right] \right\}$$

$$+ \frac{1}{4} \frac{\mu_{\mathbb{C}}^4}{L'^6} \left(\frac{L'}{G'}\right)^3 \left[- R_{\mathbb{C}}^2 \, J_2 \left(1 - 3 \frac{H'^2}{G'^2}\right) + 6 R_{\mathbb{C}}^2 \, J_{22} \left(1 - \frac{H'^2}{G'^2}\right) \cos 2h' \right]$$

$$- \frac{3}{8} \frac{\mu_{\mathbb{C}}^5 R_{\mathbb{C}}^3 J_3}{L'^8} \left(\frac{L'}{G'}\right)^5 \left(1 - \frac{G'^2}{L'^2}\right)^{1/2} \left(1 - 5\frac{H'^2}{G'^2}\right) \left(1 - \frac{H'^2}{G'^2}\right)^{1/2} \sin g'$$

$$- \frac{3}{128} \frac{\mu_{\mathbb{C}}^6 R_{\mathbb{C}}^4 J_4}{L'^{10}} \left(\frac{L'}{G'}\right)^7 \left[\left(5 - 3\frac{G'^2}{L'^2}\right) \left(3 - 30\frac{H'^2}{G'^2} + 35\frac{H'^4}{G'^4}\right) - 10\left(1 - \frac{G'^2}{L'^2}\right)\left(1 - \frac{H'^2}{G'^2}\right)\left(1 - 7\frac{H'^2}{G'^2}\right)\cos 2g'\right]$$

$$- \frac{5}{256} \frac{\mu_{\mathbb{C}}^7 R_{\mathbb{C}}^5 J_5}{L'^{12}} \left(\frac{L'}{G'}\right)^9 \left(1 - \frac{G'^2}{L'^2}\right)^{1/2} \left(1 - \frac{H'^2}{G'^2}\right)^{1/2} \left[6\left(7 - 3\frac{G'^2}{L'^2}\right)\left(1 - 14\frac{H'^2}{G'^2} + 21\frac{H'^4}{G'^4}\right)\sin g'\right.$$

$$\left. - 7\left(1 - \frac{G'^2}{L'^2}\right)\left(1 - \frac{H'^2}{G'^2}\right)\left(1 - 9\frac{H'^2}{G'^2}\right)\sin 3g'\right] \cdot \quad (39)$$

Also, since

$$\dot{L}' = \frac{\partial F'}{\partial l'} = 0 ,$$

then

$$L' = \text{const.}$$

The generating function S_2 and the short-period perturbations are obtained next. Since

$$\frac{\partial F_0}{\partial L'} = -\frac{\mu_{\mathbb{C}}^2}{L'^3} = -n' ,$$

from Equation 35 it follows that

$$S_2 = \frac{1}{n'} \int F_{2p}\, dl = \frac{1}{n'} \int \left(F_2 - F_{2s}\right) dl . \quad (40)$$

The integration is carried out using E or f as independent variables, again making use of Equations 38. As a result,

$$S_2 = \frac{1}{8} \frac{\mu_{\mathbb{C}} R_{\mathbb{C}}^2 J_2}{n' a'^3 \left(1 - e'^2\right)^{3/2}} \left[-2\left(1 - 3\cos^2 I'\right) B_{2,1} + \left(\sin^2 I'\right) B_{2,2}\right]$$

$$+ \frac{1}{8} \frac{\mu_{\mathbb{C}} R_{\mathbb{C}}^2 J_{22}}{n' a'^3 \left(1 - e'^2\right)^{3/2}} \left[6\left(\sin^2 I'\right) B_{22,1} + \left(1 - \cos I'\right)^2 B_{22,2} + \left(1 + \cos I'\right)^2 B_{22,3}\right]$$

$$- \frac{1}{96} \frac{\mu_{\mathbb{C}} R_{\mathbb{C}}^3 J_3 \sin I'}{n' a'^4 \left(1 - e'^2\right)^{5/2}} \left[3 \left(1 - 5 \cos^2 I'\right) B_{3,1} + \left(\sin^2 I'\right) B_{3,2}\right]$$

$$+ \frac{1}{512} \frac{\mu_{\mathbb{C}} R_{\mathbb{C}}^4 J_4}{n' a'^5 \left(1 - e'^2\right)^{7/2}} \left[- 2 \left(3 - 30 \cos^2 I' + 35 \cos^4 I'\right) B_{4,1}\right.$$

$$\left. + 2 \sin^2 I' \left(1 - 7 \cos^2 I'\right) B_{4,2} - \left(\sin^4 I'\right) B_{4,3}\right]$$

$$- \frac{1}{30720} \frac{\mu_{\mathbb{C}} R_{\mathbb{C}}^5 J_5 \sin I'}{n' a'^6 \left(1 - e'^2\right)^{9/2}} \left[30 \left(1 - 14 \cos^2 I' + 21 \cos^4 I'\right) B_{5,1} + 5 \sin^2 I' \left(1 - 9 \cos^2 I'\right) B_{5,2} - 3 \left(\sin^4 I'\right) B_{5,3}\right]$$

$$+ \frac{1}{384} \frac{n_{\mathbb{C}}^2 a'^2}{\epsilon n'} \left\{12 \left[- 2 \left(1 - 3 \cos^2 I'\right) \left(2 + 3e'^2\right) + 6 \sin^2 I' \left(2 + 3e'^2\right) \cos 2h + 30e'^2 \sin^2 I' \cos 2g\right.\right.$$

$$\left. + 15e'^2 (1 - \cos I')^2 \cos (2g - 2h) + 15e'^2 (1 + \cos I')^2 \cos (2g + 2h)\right] \cdot (E' - l)$$

$$\left. + 4 \left(1 - 3 \cos^2 I'\right) B_{\mathbb{C},1} + 6 \left(\sin^2 I'\right) B_{\mathbb{C},2} + 3 \left(1 - \cos I'\right)^2 B_{\mathbb{C},3} + 3 \left(1 + \cos I'\right)^2 B_{\mathbb{C},4}\right\}, \quad (41)$$

where

$$B_{2,1} = e' \sin f' + (f' - l),$$

$$B_{2,2} = 3e' \sin (f' + 2g) + 3 \sin (2f' + 2g) + e' \sin (3f' + 2g),$$

$$B_{22,1} = 2(f' - l) \cos 2h + e' \sin (f' - 2h) + e' \sin (f' + 2h),$$

$$B_{22,2} = 3e' \sin (f' + 2g - 2h) + 3 \sin (2f' + 2g - 2h) + e' \sin (3f' + 2g - 2h),$$

$$B_{22,3} = 3e' \sin (f' + 2g + 2h) + 3 \sin (2f' + 2g + 2h) + e' \sin (3f' + 2g + 2h),$$

$$B_{3,1} = 12e' (f' - l) \sin g + 3e'^2 \cos (f' - g) - 6 \left(2 + e'^2\right) \cos (f' + g) - 6e' \cos (2f' + g) - e'^2 \cos (3f' + g),$$

$$B_{3,2} = 15e'^2 \cos (f' + 3g) + 30e' \cos (2f' + 3g) + 10 \left(2 + e'^2\right) \cos (3f' + 3g)$$
$$+ 15e' \cos (4f' + 3g) + 3e'^2 \cos (5f' + 3g),$$

$$B_{4,1} = 6 \left(2 + 3e'^2\right) (f' - l) + 9e' \left(4 + e'^2\right) \sin f' + 9e'^2 \sin 2f' + e'^3 \sin 3f',$$

$$B_{4,2} = 60e'^2(f'-l)\cos 2g + 10e'^3\sin(f'-2g) + 30e'\left(4+e'^2\right)\sin(f'+2g) + 20\left(2+3e'^2\right)\sin(2f'+2g)$$

$$+ 10e'\left(4+e'^2\right)\sin(3f'+2g) + 15e'^2\sin(4f'+2g) + 2e'^3\sin(5f'+2g) \; ,$$

$$B_{4,3} = 35e'^3\sin(f'+4g) + 105e'^2\sin(2f'+4g) + 35e'\left(4+e'^2\right)\sin(3f'+4g) + 35\left(2+3e'^2\right)\sin(4f'+4g)$$

$$+ 21e'\left(4+e'^2\right)\sin(5f'+4g) + 35e'^2\sin(6f'+4g) + 5e'^3\sin(7f'+4g) \; ,$$

$$B_{5,1} = 120e'\left(4+3e'^2\right)(f'-l)\sin g - 30\left(8+24e'^2+3e'^4\right)\cos(f'+g) + 60e'^2\left(6+e'^2\right)\cos(f'-g)$$

$$- 60e'\left(4+3e'^2\right)\cos(2f'+g) + 60e'^3\cos(2f'-g)$$

$$- 20e'^2\left(6+e'^2\right)\cos(3f'+g) + 5e'^4\cos(3f'-g) - 30e'^3\cos(4f'+g) - 3e'^4\cos(5f'+g) \; ,$$

$$B_{5,2} = -840e'^3(f'-l)\sin 3g + 420e'^2\left(6+e'^2\right)\cos(f'+3g) - 105e'^4\cos(f'-3g)$$

$$+ 420e'\left(4+3e'^2\right)\cos(2f'+3g) + 70\left(8+24e'^2+3e'^4\right)\cos(3f'+3g) + 210e'\left(4+3e'^2\right)\cos(4f'+3g)$$

$$+ 84e'^2\left(6+e'^2\right)\cos(5f'+3g) + 140e'^3\cos(6f'+3g) + 15e'^4\cos(7f'+3g) \; ,$$

$$B_{5,3} = 315e'^4\cos(f'+5g) + 1260e'^3\cos(2f'+5g) + 420e'^2\left(6+e'^2\right)\cos(3f'+5g)$$

$$+ 630e'\left(4+3e'^2\right)\cos(4f'+5g) + 126\left(8+24e'^2+3e'^4\right)\cos(5f'+5g) + 420e'\left(4+3e'^2\right)\cos(6f'+5g)$$

$$+ 180e'^2\left(6+e'^2\right)\cos(7f'+5g) + 315e'^3\cos(8f'+5g) + 35e'^4\cos(9f'+5g) \; ,$$

$$B_{\mathbb{C},1} = 9e'\left(4+e'^2\right)\sin E' - 9e'^2\sin 2E' + e'^3\sin 3E' \; ,$$

$$B_{\mathbb{C},2} = -9e'\left(4+e'^2\right)\sin(E'-2h) + 9e'^2\sin(2E'-2h) - e'^3\sin(3E'-2h)$$

$$- 9e'\left(4+e'^2\right)\sin(E'+2h) + 9e'^2\sin(2E'+2h) - e'^3\sin(3E'+2h)$$

$$- 15e'\left[\left(2+e'^2\right) - 2\left(1-e'^2\right)^{1/2}\right]\sin(E'-2g) + 3\left[\left(2+e'^2\right) - 2\left(1-e'^2\right)^{1/2}\left(1+e'^2\right)\right]\sin(2E'-2g)$$

$$- e'\left[\left(2-e'^2\right) - 2\left(1-e'^2\right)^{1/2}\right]\sin(3E'-2g) - 15e'\left[\left(2+e'^2\right) + 2\left(1-e'^2\right)^{1/2}\right]\sin(E'+2g)$$

$$+ 3\left[\left(2+e'^2\right) + 2\left(1-e'^2\right)^{1/2}\left(1+e'^2\right)\right]\sin(2E'+2g) - e'\left[\left(2-e'^2\right) + 2\left(1-e'^2\right)^{1/2}\right]\sin(3E'+2g) \; ,$$

$$B_{\mathbb{C},3} = -15e' \left[\left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2}\right] \sin(E' - 2g + 2h) + 3\left[\left(2 + e'^2\right)\right.$$

$$\left. - 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right] \sin(2E' - 2g + 2h) - e' \left[\left(2 - e'^2\right) - 2\left(1 - e'^2\right)^{1/2}\right] \sin(3E' - 2g + 2h)$$

$$- 15e' \left[\left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2}\right] \sin(E' + 2g - 2h) + 3\left[\left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right] \sin(2E' + 2g - 2h)$$

$$- e' \left[\left(2 - e'^2\right) + 2\left(1 - e'^2\right)^{1/2}\right] \sin(3E' + 2g - 2h) \ ,$$

and

$$B_{\mathbb{C},4} = -15e' \left[\left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2}\right] \sin(E' - 2g - 2h) + 3\left[\left(2 + e'^2\right)\right.$$

$$\left. - 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right] \sin(2E' - 2g - 2h) - e' \left[\left(2 - e'^2\right) - 2\left(1 - e'^2\right)^{1/2}\right] \sin(3E' - 2g - 2h)$$

$$- 15e' \left[\left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2}\right] \sin(E' + 2g + 2h) + 3\left[\left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right] \sin(2E' + 2g + 2h)$$

$$- e' \left[\left(2 - e'^2\right) + 2\left(1 - e'^2\right)^{1/2}\right] \sin(3E' + 2g + 2h) \ . \tag{42}$$

Thus, for the short-period terms,

$$S = L' \, l + G' \, g + H' \, h + S_2 \tag{43}$$

to second order.

The short-period perturbations are given by

$$l' = \frac{\partial S}{\partial L'} = l + \frac{\partial S_2}{\partial L'} \ ,$$

$$g' = \frac{\partial S}{\partial G'} = g + \frac{\partial S_2}{\partial G'} \ ,$$

$$h' = \frac{\partial S}{\partial H'} = h + \frac{\partial S_2}{\partial H'} \ ,$$

$$L = \frac{\partial S}{\partial l} = L' + \frac{\partial S_2}{\partial l} \ ,$$

$$G = \frac{\partial S}{\partial g} = G' + \frac{\partial S_2}{\partial g} \ ,$$

and

$$H = \frac{\partial S}{\partial h} = H' + \frac{\partial S_2}{\partial h} . \tag{44}$$

It is easier to compute the partial derivatives with respect to the Keplerian elements a', e', I' and then compute them with respect to L', G', H', as follows:

$$\frac{\partial S_2}{\partial L'} = \frac{\partial S_2}{\partial a'}\frac{\partial a'}{\partial L'} + \frac{\partial S_2}{\partial e'}\frac{\partial e'}{\partial L'} = 2\frac{L'}{\mu_{☾}}\frac{\partial S_2}{\partial a'} + \frac{G'^2}{e' L'^3}\frac{\partial S_2}{\partial e'} ,$$

$$\frac{\partial S_2}{\partial G'} = \frac{\partial S_2}{\partial e'}\frac{\partial e'}{\partial G'} + \frac{\partial S_2}{\partial I'}\frac{\partial I'}{\partial G'} = -\frac{G'}{e' L'^2}\frac{\partial S_2}{\partial e'} + \frac{H'}{G'^2 \sin I'}\frac{\partial S_2}{\partial I'} ,$$

and

$$\frac{\partial S_2}{\partial H'} = \frac{\partial S_2}{\partial I'}\frac{\partial I'}{\partial H'} = -\frac{1}{G' \sin I'}\frac{\partial S_2}{\partial I'} . \tag{45}$$

Furthermore, it is important to note that

$$\frac{\partial f'}{\partial e'} = \left(\frac{a'}{r'} + \frac{L'^2}{G'^2}\right) \sin f' ,$$

$$\frac{\partial E'}{\partial e'} = \frac{a'}{r'} \sin E' ,$$

and

$$\frac{\partial n'}{\partial a'} = -\frac{3}{2}\frac{n'}{a'} . \tag{46}$$

The partial derivative with respect to l does not need to be computed again, since

$$\frac{\partial S_2}{\partial l} = \frac{1}{n'} F_{2p} = \frac{1}{n'} (F - F') .$$

Thus,

$$\frac{\partial S_2}{\partial a'} = -\frac{3}{16}\frac{n' R_{☾}^2 J_2}{a'\left(1-e'^2\right)^{3/2}}\left[-2\left(1-3\cos^2 I'\right)B_{2,1}+\left(\sin^2 I'\right)B_{2,2}\right]$$

$$-\frac{3}{16}\frac{n' R_{☾}^2 J_{22}}{a'\left(1-e'^2\right)^{3/2}}\left[6\left(\sin^2 I'\right)B_{22,1}+(1-\cos I')^2 B_{22,2}+(1+\cos I')^2 B_{22,3}\right]$$

$$+\frac{5}{192}\frac{n' R_{☾}^3 J_3 \sin I'}{a'^2\left(1-e'^2\right)^{5/2}}\left[3\left(1-5\cos^2 I'\right)B_{3,1}+\left(\sin^2 I'\right)B_{3,2}\right]$$

$$-\frac{7}{1024}\frac{n' R_{☾}^4 J_4}{a'^3\left(1-e'^2\right)^{7/2}}\left[-2\left(3-30\cos^2 I'+35\cos^4 I'\right)B_{4,1}\right.$$

$$\left.+2\sin^2 I'\left(1-7\cos^2 I'\right)B_{4,2}-\left(\sin^4 I'\right)B_{4,3}\right]$$

$$+\frac{9}{61440}\frac{n' R_{☾}^5 J_5 \sin I'}{a'^4\left(1-e'^2\right)^{9/2}}\left[30\left(1-14\cos^2 I'+21\cos^4 I'\right)B_{5,1}\right.$$

$$\left.+5\sin^2 I'\left(1-9\cos^2 I'\right)B_{5,2}-3\left(\sin^4 I'\right)B_{5,3}\right]$$

$$+\frac{7}{768}\frac{n_{☾}^2 a'}{\epsilon n'}\left\{12\left[-2\left(1-3\cos^2 I'\right)\left(2+3e'^2\right)+6\sin^2 I'\left(2+3e'^2\right)\cos 2h\right.\right.$$

$$\left.+30e'^2\sin^2 I'\cos 2g+15e'^2(1-\cos I')^2\cos(2g-2h)+15e'^2(1+\cos I')^2\cos(2g+2h)\right](E'-l)$$

$$\left.+4\left(1-3\cos^2 I'\right)B_{☾,1}+6\left(\sin^2 I'\right)B_{☾,2}+3(1-\cos I')^2 B_{☾,3}+3(1+\cos I')^2 B_{☾,4}\right\}, \quad (47)$$

$$\frac{\partial S_2}{\partial I'} = \frac{1}{4}\frac{n' R_{☾}^2 J_2}{\left(1-e'^2\right)^{3/2}}\sin I'\cos I'\left(-6B_{2,1}+B_{2,2}\right)$$

$$+\frac{1}{4}\frac{n' R_{☾}^2 J_{22}}{\left(1-e'^2\right)^{3/2}}\sin I'\left[6(\cos I')B_{22,1}+(1-\cos I')B_{22,2}-(1+\cos I')B_{22,3}\right]$$

$$-\frac{1}{32}\frac{n' R_{☾}^3 J_3 \cos I'}{a'\left(1-e'^2\right)^{5/2}}\left[\left(11-15\cos^2 I'\right)B_{3,1}+\left(\sin^2 I'\right)B_{3,2}\right]$$

$$+\frac{1}{128}\frac{n' R_{\mathbb{C}}^4 J_4 \sin I' \cos I'}{a'^2 \left(1-e'^2\right)^{7/2}}\left[-10\left(3-7\cos^2 I'\right)B_{4,1}+2\left(4-7\cos^2 I'\right)B_{4,2}-\left(\sin^2 I'\right)B_{4,3}\right]$$

$$-\frac{1}{2048}\frac{n' R_{\mathbb{C}}^5 J_5 \cos I'}{a'^3 \left(1-e'^2\right)^{9/2}}\left[2\left(29-126\cos^2 I'+105\cos^4 I'\right)B_{5,1}\right.$$

$$\left.+\sin^2 I'\left(7-15\cos^2 I'\right)B_{5,2}-\left(\sin^4 I'\right)B_{5,3}\right]$$

$$+\frac{1}{64}\frac{n' a'^2}{\epsilon}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2 \sin I'\left\{12\left[-2\cos I'\left(2+3e'^2\right)+2\cos I'\left(2+3e'^2\right)\cos 2h\right.\right.$$

$$\left.+10e'^2\cos I'\cos 2g+5e'^2\left(1-\cos I'\right)\cos\left(2g-2h\right)-5e'^2\left(1+\cos I'\right)\cos\left(2g+2h\right)\right]\left(E'-l\right)$$

$$\left.+4(\cos I')B_{\mathbb{C},1}+2(\cos I')B_{\mathbb{C},2}+(1-\cos I')B_{\mathbb{C},3}-(1+\cos I')B_{\mathbb{C},4}\right\}, \tag{48}$$

$$\frac{\partial S_2}{\partial e'}=\frac{1}{32}\frac{n' R_{\mathbb{C}}^2 J_2}{\left(1-e'^2\right)^{5/2}}\left\{-2\left(1-3\cos^2 I'\right)\left[12e'\left(f'-l\right)+3\left(4+3e'^2\right)\sin f'+6e'\sin 2f'+e'^2\sin 3f'\right]\right.$$

$$+\sin^2 I'\left[-18e'\sin 2g-3\left(4-7e'^2\right)\sin\left(f'+2g\right)+3e'^2\sin\left(f'-2g\right)+36e'\sin\left(2f'+2g\right)\right.$$

$$\left.\left.+\left(28+11e'^2\right)\sin\left(3f'+2g\right)+18e'\sin\left(4f'+2g\right)+3e'^2\sin\left(5f'+2g\right)\right]\right\}$$

$$+\frac{1}{32}\frac{n' R_{\mathbb{C}}^2 J_{22}}{\left(1-e'^2\right)^{5/2}}\left\{6\sin^2 I'\left[24e'\left(f'-l\right)\cos 2h+3\left(4+3e'^2\right)\sin\left(f'+2h\right)+3\left(4+3e'^2\right)\sin\left(f'-2h\right)\right.\right.$$

$$\left.+6e'\sin\left(2f'+2h\right)+6e'\sin\left(2f'-2h\right)+e'^2\sin\left(3f'+2h\right)+e'^2\sin\left(3f'-2h\right)\right]$$

$$+\left(1-\cos I'\right)^2\left[-18e'\sin\left(2g-2h\right)-3\left(4-7e'^2\right)\sin\left(f'+2g-2h\right)+3e'^2\sin\left(f'-2g+2h\right)\right.$$

$$+36e'\sin\left(2f'+2g-2h\right)+\left(28+11e'^2\right)\sin\left(3f'+2g-2h\right)+18e'\sin\left(4f'+2g-2h\right)$$

$$\left.+3e'^2\sin\left(5f'+2g-2h\right)\right]+\left(1+\cos I'\right)^2\left[-18e'\sin\left(2g+2h\right)-3\left(4-7e'^2\right)\sin\left(f'+2g+2h\right)\right.$$

$$+3e'^2\sin\left(f'-2g-2h\right)+36e'\sin\left(2f'+2g+2h\right)+\left(28+11e'^2\right)\sin\left(3f'+2g+2h\right)$$

$$\left.\left.+18e'\sin\left(4f'+2g+2h\right)+3e'^2\sin\left(5f'+2g+2h\right)\right]\right\}$$

$$- \frac{1}{128} \frac{n' R_{\mathbb{C}}^3 J_3 \sin I'}{a' \left(1 - e'^2\right)^{7/2}} \Big\{ \left(1 - 5\cos^2 I'\right) \Big[48\left(1 + 4e'^2\right)(f' - l) \sin g + 24\left(2 + e'^2\right) \cos 2g$$

$$+ 42e' \left(2 + e'^2\right) \cos(f' - g) - 72e' \left(4 + e'^2\right) \cos(f' + g) + 24e'^2 \cos(2f' - g) - 24\left(3 + 5e'^2\right) \cos(2f' + g)$$

$$+ 3e'^3 \cos(3f' - g) - 2e' \left(34 + 9e'^2\right) \cos(3f' + g) - 24e'^2 \cos(4f' + g) - 3e'^3 \cos(5f' + g) \Big]$$

$$+ \sin^2 I' \Big[- 40e'^2 \cos 3g - 5e'^3 \cos(f' - 3g) - 10e' \left(6 - 5e'^2\right) \cos(f' + 3g) - 40\left(1 - 3e'^2\right) \cos(2f' + 3g)$$

$$+ 40e' \left(4 + e'^2\right) \cos(3f' + 3g) + 20\left(5 + 6e'^2\right) \cos(4f' + 3g) + 2e' \left(54 + 11e'^2\right) \cos(5f' + 3g)$$

$$+ 40e'^2 \cos(6f' + 3g) + 5e'^3 \cos(7f' + 3g) \Big] \Big\}$$

$$+ \frac{1}{2048} \frac{n' R_{\mathbb{C}}^4 J_4}{a'^2 \left(1 - e'^2\right)^{9/2}} \Big\{ - 2\left(3 - 30\cos^2 I' + 35\cos^4 I'\right) \Big[120e' \left(4 + 3e'^2\right)(f' - l) + 30\left(8 + 36e'^2 + 5e'^4\right) \sin f'$$

$$+ 240e' \left(1 + e'^2\right) \sin 2f' + 5e'^2 \left(24 + 5e'^2\right) \sin 3f' + 30e'^3 \sin 4f' + 3e'^4 \sin 5f' \Big]$$

$$+ 4\sin^2 I' \left(1 - 7\cos^2 I'\right) \Big[120e' \left(2 + 5e'^2\right)(f' - l) \cos 2g - 20e' \left(14 + 5e'^2\right) \sin 2g + 5e'^2 \left(48 + 19e'^2\right) \sin(f' - 2g)$$

$$+ 10\left(8 + 144e'^2 + 23e'^4\right) \sin(f' + 2g) + 200e' \left(4 + 3e'^2\right) \sin(2f' + 2g) + 50e'^3 \sin(2f' - 2g)$$

$$+ 30\left(8 + 24e'^2 + 3e'^4\right) \sin(3f' + 2g) + 5e'^4 \sin(3f' - 2g) + 10e' \left(34 + 25e'^2\right) \sin(4f' + 2g)$$

$$+ e'^2 \left(192 + 31e'^2\right) \sin(5f' + 2g) + 50e'^3 \sin(6f' + 2g) + 5e'^4 \sin(7f' + 2g) \Big]$$

$$- \sin^4 I' \Big[- 350e'^3 \sin 4g + 35e'^4 \sin(f' - 4g) - 35e'^2 \left(24 - 13e'^2\right) \sin(f' + 4g) - 280e' \left(4 - 5e'^2\right) \sin(2f' + 4g)$$

$$- 70\left(8 - 36e'^2 - 7e'^4\right) \sin(3f' + 4g) + 700e' \left(4 + 3e'^2\right) \sin(4f' + 4g) + 14\left(104 + 252e'^2 + 29e'^4\right) \sin(5f' + 4g)$$

$$+ 280e' \left(8 + 5e'^2\right) \sin(6f' + 4g) + 5e'^2 \left(264 + 37e'^2\right) \sin(7f' + 4g) + 350e'^3 \sin(8f' + 4g)$$

$$+ 35e'^4 \sin(9f' + 4g) \Big] \Big\}$$

$$- \frac{1}{8192} \frac{n' R_{\mathbb{C}}^5 J_5 \sin I'}{a'^3 \left(1 - e'^2\right)^{11/2}} \Big\{ 2\left(1 - 14\cos^2 I' + 21\cos^4 I'\right) \Big[480\left(4 + 41e'^2 + 18e'^4\right)(f' - l) \sin g$$

$$+ 120\left(8 + 16e'^2 + 3e'^4\right) \cos g - 1800e'\left(8 + 12e'^2 + e'^4\right) \cos(f' + g) + 15e'\left(336 + 848e'^2 + 85e'^4\right) \cos(f' - g)$$

$$- 120\left(16 + 98e'^2 + 39e'^4\right) \cos(2f' + g) + 660e'^2\left(4 + 3e'^2\right) \cos(2f' - g) - 5e'\left(624 + 1072e'^2 + 95e'^4\right) \cos(3f' + g)$$

$$+ 40e'^3\left(23 + 4e'^2\right) \cos(3f' - g) - 60e'^2\left(38 + 21e'^2\right) \cos(4f' + g) + 180e'^4 \cos(4f' - g)$$

$$\left. - 24e'^3\left(37 + 5e'^2\right) \cos(5f' + g) + 15e'^5 \cos(5f' - g) - 180e'^4 \cos(6f' + g) - 15e'^5 \cos(7f' + g)\right]$$

$$+ \sin^2 I'\left(1 - 9\cos^2 I'\right)\left[- 3360e'^2\left(1 + 2e'^2\right)(f' - l) \sin 3g - 140e'^2\left(32 + 9e'^2\right) \cos 3g\right.$$

$$+ 105e'\left(16 + 208e'^2 + 25e'^4\right) \cos(f' + 3g) - 840e'^3\left(3 + e'^2\right) \cos(f' - 3g) + 9240e'^2\left(2 + e'^2\right) \cos(2f' + 3g)$$

$$- 420e'^4 \cos(2f' - 3g) + 1400e'\left(8 + 12e'^2 + e'^4\right) \cos(3f' + 3g) - 35e'^5 \cos(3f' - 3g)$$

$$+ 840\left(4 + 19e'^2 + 7e'^4\right) \cos(4f' + 3g) + 21e'\left(304 + 432e'^2 + 35e'^4\right) \cos(5f' + 3g)$$

$$+ 140e'^2\left(36 + 17e'^2\right) \cos(6f' + 3g) + 120e'^3\left(17 + 2e'^2\right) \cos(7f' + 3g) + 420e'^4 \cos(8f' + 3g)$$

$$\left. + 35e'^5 \cos(9f' + 3g)\right] - \sin^4 I'\left[- 756e'^4 \cos 5g - 504e'^3\left(5 - 2e'^2\right) \cos(f' + 5g)\right.$$

$$- 63e'^5 \cos(f' - 5g) - 1260e'^2\left(4 - 3e'^2\right) \cos(2f' + 5g) - 105e'\left(48 - 80e'^2 - 13e'^4\right) \cos(3f' + 5g)$$

$$504\left(4 - 25e'^2 - 15e'^4\right) \cos(4f' + 5g) + 1512e'\left(8 + 12e'^2 + e'^4\right) \cos(5f' + 5g)$$

$$+ 168\left(32 + 130e'^2 + 45e'^4\right) \cos(6f' + 5g) + 45e'\left(240 + 304e'^2 + 23e'^4\right) \cos(7f' + 5g)$$

$$+ 1260e'^2\left(7 + 3e'^2\right) \cos(8f' + 5g) + 56e'^3\left(65 + 7e'^2\right) \cos(9f' + 5g) + 756e'^4 \cos(10f' + 5g)$$

$$\left.\left. + 63e'^5 \cos(11f' + 5g)\right]\right\}$$

$$+ \frac{1}{128} \frac{n' a^2}{\epsilon} \left(\frac{n_{☾}}{n'}\right)^2 \left\{24e'\left[- 2\left(1 - 3\cos^2 I'\right) + 6 \sin^2 I' \cos 2h + 10 \sin^2 I' \cos 2g\right.\right.$$

$$\left. + 5(1 - \cos I')^2 \cos(2g - 2h) + 5(1 + \cos I')^2 \cos(2g + 2h)\right](E' - l)$$

$$+ 4\left(1 - 3\cos^2 I'\right)\left[3\left(4 + 3e'^2\right) \sin E' - 6e' \sin 2E' + e'^2 \sin 3E'\right]$$

$$+\ 2\sin^2 I' \Big[-9\left(4+3e'^2\right)\sin(E'-2h)+18e'\sin(2E'-2h)-3e'^2\sin(3E'-2h)$$

$$-\ 9\left(4+3e'^2\right)\sin(E'+2h)+18e'\sin(2E'+2h)-3e'^2\sin(3E'+2h)$$

$$-\ 15\left\{\left(2+3e'^2\right)-2\left(1-e'^2\right)^{1/2}+2e'^2\left(1-e'^2\right)^{-1/2}\right\}\sin(E'-2g)$$

$$+\ 6e'\left\{1-2\left(1-e'^2\right)^{1/2}+\left(1-e'^2\right)^{-1/2}\left(1+e'^2\right)\right\}\sin(2E'-2g)$$

$$-\ \left\{\left(2-3e'^2\right)-2\left(1-e'^2\right)^{1/2}+2e'^2\left(1-e'^2\right)^{-1/2}\right\}\sin(3E'-2g)$$

$$-\ 15\left\{\left(2+3e'^2\right)+2\left(1-e'^2\right)^{1/2}-2e'^2\left(1-e'^2\right)^{-1/2}\right\}\sin(E'+2g)$$

$$+\ 6e'\left\{1+2\left(1-e'^2\right)^{1/2}-\left(1-e'^2\right)^{-1/2}\left(1+e'^2\right)\right\}\sin(2E'+2g)$$

$$-\ \left\{\left(2-3e'^2\right)+2\left(1-e'^2\right)^{1/2}-2e'^2\left(1-e'^2\right)^{-1/2}\right\}\sin(3E'+2g)\Big]$$

$$+\ (1-\cos I')^2\Big[-15\left\{\left(2+3e'^2\right)-2\left(1-e'^2\right)^{1/2}+2e'^2\left(1-e'^2\right)^{-1/2}\right\}\sin(E'-2g+2h)$$

$$+\ 6e'\left\{1-2\left(1-e'^2\right)^{1/2}+\left(1-e'^2\right)^{-1/2}\left(1+e'^2\right)\right\}\sin(2E'-2g+2h)$$

$$-\ \left\{\left(2-3e'^2\right)-2\left(1-e'^2\right)^{1/2}+2e'^2\left(1-e'^2\right)^{-1/2}\right\}\sin(3E'-2g+2h)$$

$$-\ 15\left\{\left(2+3e'^2\right)+2\left(1-e'^2\right)^{1/2}-2e'^2\left(1-e'^2\right)^{-1/2}\right\}\sin(E'+2g-2h)$$

$$+\ 6e'\left\{1+2\left(1-e'^2\right)^{1/2}-\left(1-e'^2\right)^{-1/2}\left(1+e'^2\right)\right\}\sin(2E'+2g-2h)$$

$$-\ \left\{\left(2-3e'^2\right)+2\left(1-e'^2\right)^{1/2}-2e'^2\left(1-e'^2\right)^{-1/2}\right\}\sin(3E'+2g-2h)\Big]$$

$$+\ (1+\cos I')^2\Big[-15\left\{\left(2+3e'^2\right)-2\left(1-e'^2\right)^{1/2}+2e'^2\left(1-e'^2\right)^{-1/2}\right\}\sin(E'-2g-2h)$$

$$+\ 6e'\left\{1-2\left(1-e'^2\right)^{1/2}+\left(1-e'^2\right)^{-1/2}\left(1+e'^2\right)\right\}\sin(2E'-2g-2h)$$

$$-\ \left\{\left(2-3e'^2\right)-2\left(1-e'^2\right)^{1/2}+2e'^2\left(1-e'^2\right)^{-1/2}\right\}\sin(3E'-2g-2h)$$

$$-\ 15\left\{\left(2+3e'^2\right)+2\left(1-e'^2\right)^{1/2}-2e'^2\left(1-e'^2\right)^{-1/2}\right\}\sin(E'+2g+2h)$$

$$+ 6e' \left\{1 + 2\left(1 - e'^2\right)^{1/2} - \left(1 - e'^2\right)^{-1/2} \left(1 + e'^2\right)\right\} \sin(2E' + 2g + 2h)$$

$$- \left\{\left(2 - 3e'^2\right) + 2\left(1 - e'^2\right)^{1/2} - 2e'^2 \left(1 - e'^2\right)^{-1/2}\right\} \sin(3E' + 2g + 2h)\Big]\Bigg\}$$

$$+ \frac{1}{384} \frac{n' a^2}{\epsilon} \left(\frac{n_{\mathbb{C}}}{n'}\right)^2 \Bigg\{12 \Big[-2\left(1 - 3\cos^2 I'\right)\left(2 + 3e'^2\right) + 6 \sin^2 I' \left(2 + 3e'^2\right) \cos 2h$$

$$+ 30e'^2 \sin^2 I' \cos 2g + 15e'^2 (1 - \cos I')^2 \cos(2g - 2h) + 15e'^2 (1 + \cos I')^2 \cos(2g + 2h)\Big]$$

$$+ 12\left(1 - 3\cos^2 I'\right) \left[3e'\left(4 + e'^2\right) \cos E' - 6e'^2 \cos 2E' + e'^3 \cos 3E'\right]$$

$$+ 18 \sin^2 I' \Big[-3e' \left(4 + e'^2\right) \cos(E' - 2h) + 6e'^2 \cos(2E' - 2h) - e'^3 \cos(3E' - 2h)$$

$$- 3e' \left(4 + e'^2\right) \cos(E' + 2h) + 6e'^2 \cos(2E' + 2h) - e'^3 \cos(3E' + 2h)$$

$$- 5e' \left\{\left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2}\right\} \cos(E' - 2g)$$

$$+ 2 \left\{\left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right\} \cos(2E' - 2g)$$

$$- e' \left\{\left(2 - e'^2\right) - 2\left(1 - e'^2\right)^{1/2}\right\} \cos(3E' - 2g)$$

$$- 5e' \left\{\left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2}\right\} \cos(E' + 2g)$$

$$+ 2 \left\{\left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right\} \cos(2E' + 2g)$$

$$- e' \left\{\left(2 - e'^2\right) + 2\left(1 - e'^2\right)^{1/2}\right\} \cos(3E' + 2g)\Big]$$

$$+ 9(1 - \cos I')^2 \Big[-5e' \left\{\left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2}\right\} \cos(E' - 2g + 2h)$$

$$+ 2 \left\{\left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right\} \cos(2E' - 2g + 2h)$$

$$- e' \left\{\left(2 - e'^2\right) - 2\left(1 - e'^2\right)^{1/2}\right\} \cos(3E' - 2g + 2h)$$

$$- 5e' \left\{\left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2}\right\} \cos(E' + 2g - 2h)$$

$$+ 2 \left\{\left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right\} \cos(2E' + 2g - 2h)$$

$$- e' \left\{\left(2 - e'^2\right) + 2\left(1 - e'^2\right)^{1/2}\right\} \cos(3E' + 2g - 2h)\Big]$$

$$+ 9(1 + \cos I')^2 \Big[- 5e' \left\{\left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2}\right\} \cos(E' - 2g - 2h)$$

$$+ 2 \left\{\left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right\} \cos(2E' - 2g - 2h)$$

$$- e' \left\{\left(2 - e'^2\right) - 2\left(1 - e'^2\right)^{1/2}\right\} \cos(3E' - 2g - 2h)$$

$$- 5e' \left\{\left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2}\right\} \cos(E' + 2g + 2h)$$

$$+ 2 \left\{\left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right\} \cos(2E' + 2g + 2h)$$

$$- e' \left\{\left(2 - e'^2\right) + 2\left(1 - e'^2\right)^{1/2}\right\} \cos(3E' + 2g + 2h)\Big]\Big\} \frac{a'}{r'} \sin E', \quad (49)$$

$$\frac{\partial S_2}{\partial g} = \frac{1}{4} \frac{n' R_{\mathbb{C}}^2 J_2}{\left(1 - e'^2\right)^{3/2}} \sin^2 I' \left[3e' \cos(f' + 2g) + 3\cos(2f' + 2g) + e' \cos(3f' + 2g)\right]$$

$$+ \frac{1}{4} \frac{n' R_{\mathbb{C}}^2 J_{22}}{\left(1 - e'^2\right)^{3/2}} \Big\{(1 - \cos I')^2 \left[3e' \cos(f' + 2g - 2h) + 3\cos(2f' + 2g - 2h) + e' \cos(3f' + 2g - 2h)\right]$$

$$+ (1 + \cos I')^2 \left[3e' \cos(f' + 2g + 2h) + 3\cos(2f' + 2g + 2h) + e' \cos(3f' + 2g + 2h)\right]\Big\}$$

$$- \frac{1}{32} \frac{n' R_{\mathbb{C}}^3 J_3 \sin I}{a' \left(1 - e'^2\right)^{5/2}} \Big\{\left(1 - 5\cos^2 I'\right) \Big[12e' (f' - l) \cos g + 3e'^2 \sin(f' - g) + 6\left(2 + e'^2\right) \sin(f' + g)$$

$$+ 6e' \sin(2f' + g) + e'^2 \sin(3f' + g)\Big] - \sin^2 I' \Big[15e'^2 \sin(f' + 3g) + 30e' \sin(2f' + 3g)$$

$$+ 10\left(2 + e'^2\right) \sin(3f' + 3g) + 15e' \sin(4f' + 3g) + 3e'^2 \sin(5f' + 3g)\Big]\Big\}$$

$$+ \frac{1}{128} \frac{n' R_{\mathbb{C}}^4 J_4 \sin^2 I}{a'^2 \left(1 - e'^2\right)^{7/2}} \Big\{\left(1 - 7\cos^2 I'\right) \Big[- 60e'^2 (f' - l) \sin 2g - 10e'^3 \cos(f' - 2g) + 30e' \left(4 + e'^2\right) \cos(f' + 2g)$$

$$+ 20\left(2 + 3e'^2\right) \cos(2f' + 2g) + 10e' \left(4 + e'^2\right) \cos(3f' + 2g) + 15e'^2 \cos(4f' + 2g) + 2e'^3 \cos(5f' + 2g)\Big]$$

$$- \sin^2 I' \Big[35e'^3 \cos(f' + 4g) + 105e'^2 \cos(2f' + 4g) + 35e' \left(4 + e'^2\right) \cos(3f' + 4g)$$

$$+ 35\left(2 + 3e'^2\right) \cos(4f' + 4g) + 21e'\left(4 + e'^2\right) \cos(5f' + 4g) + 35e'^2 \cos(6f' + 4g) + 5e'^3 \cos(7f' + 4g)\Big]\Big\}$$

$$- \frac{1}{2048} \frac{n' R_{\mathbb{C}}^5 J_5 \sin I'}{a'^3 \left(1 - e'^2\right)^{9/2}} \Big\{ 2\left(1 - 14\cos^2 I' + 21\cos^4 I'\right) \Big[120e'\left(4 + 3e'^2\right)(f' - l) \cos g$$

$$+ 30\left(8 + 24e'^2 + 3e'^4\right) \sin(f' + g) + 60e'^2\left(6 + e'^2\right) \sin(f' - g) + 60e'\left(4 + 3e'^2\right) \sin(2f' + g)$$

$$+ 60e'^3 \sin(2f' - g) + 20e'^2\left(6 + e'^2\right) \sin(3f' + g) + 5e'^4 \sin(3f' - g) + 30e'^3 \sin(4f' + g)$$

$$+ 3e'^4 \sin(5f' + g)\Big] - \sin^2 I'\left(1 - 9\cos^2 I'\right) \Big[840e'^3 (f' - l) \cos 3g + 420e'^2\left(6 + e'^2\right) \sin(f' + 3g)$$

$$+ 105e'^4 \sin(f' - 3g) + 420e'\left(4 + 3e'^2\right) \sin(2f' + 3g) + 70\left(8 + 24e'^2 + 3e'^4\right) \sin(3f' + 3g)$$

$$+ 210e'\left(4 + 3e'^2\right) \sin(4f' + 3g) + 84e'^2\left(6 + e'^2\right) \sin(5f' + 3g) + 140e'^3 \sin(6f' + 3g)$$

$$+ 15e'^4 \sin(7f' + 3g)\Big] + \sin^4 I' \Big[315e'^4 \sin(f' + 5g) + 1260e'^3 \sin(2f' + 5g)$$

$$+ 420e'^2\left(6 + e'^2\right) \sin(3f' + 5g) + 630e'\left(4 + 3e'^2\right) \sin(4f' + 5g) + 126\left(8 + 24e'^2 + 3e'^4\right) \sin(5f' + 5g)$$

$$+ 420e'\left(4 + 3e'^2\right) \sin(6f' + 5g) + 180e'^2\left(6 + e'^2\right) \sin(7f' + 5g) + 315e'^3 \sin(8f' + 5g)$$

$$+ 35e'^4 \sin(9f' + 5g)\Big]\Big\}$$

$$+ \frac{1}{64} \frac{n' a'^2}{\epsilon} \left(\frac{n_{\mathbb{C}}}{n'}\right)^2 \Big\{ - 60e'^2 \Big[2\sin^2 I' \sin 2g + (1 - \cos I')^2 \sin(2g - 2h)$$

$$+ (1 + \cos I')^2 \sin(2g + 2h)\Big] (E' - l) + 2\sin^2 I' \Big[15e' \Big\{ \left(2 + e'^2\right)$$

$$- 2\left(1 - e'^2\right)^{1/2} \Big\} \cos(E' - 2g) - 3 \Big\{ \left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right) \Big\} \cos(2E' - 2g)$$

$$+ e' \Big\{ \left(2 - e'^2\right) - 2\left(1 - e'^2\right)^{1/2} \Big\} \cos(3E' - 2g)$$

$$- 15e' \Big\{ \left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2} \Big\} \cos(E' + 2g)$$

$$+ 3 \Big\{ \left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right) \Big\} \cos(2E' + 2g)$$

$$- e' \Big\{ \left(2 - e'^2\right) + 2\left(1 - e'^2\right)^{1/2} \Big\} \cos(3E' + 2g)\Big]$$

$$+ (1 - \cos I')^2 \Big[15e' \Big\{ (2 + e'^2) - 2 (1 - e'^2)^{1/2} \Big\} \cos (E' - 2g + 2h)$$

$$- 3 \Big\{ (2 + e'^2) - 2 (1 - e'^2)^{1/2} (1 + e'^2) \Big\} \cos (2E' - 2g + 2h)$$

$$+ e' \Big\{ (2 - e'^2) - 2 (1 - e'^2)^{1/2} \Big\} \cos (3E' - 2g + 2h)$$

$$- 15e' \Big\{ (2 + e'^2) + 2 (1 - e'^2)^{1/2} \Big\} \cos (E' + 2g - 2h)$$

$$+ 3 \Big\{ (2 + e'^2) + 2 (1 - e'^2)^{1/2} (1 + e'^2) \Big\} \cos (2E' + 2g - 2h)$$

$$- e' \Big\{ (2 - e'^2) + 2 (1 - e'^2)^{1/2} \Big\} \cos (3E' + 2g - 2h) \Big]$$

$$+ (1 + \cos I')^2 \Big[15e' \Big\{ (2 + e'^2) - 2 (1 - e'^2)^{1/2} \Big\} \cos (E' - 2g - 2h)$$

$$- 3 \Big\{ (2 + e'^2) - 2 (1 - e'^2)^{1/2} (1 + e'^2) \Big\} \cos (2E' - 2g - 2h)$$

$$+ e' \Big\{ (2 - e'^2) - 2 (1 - e'^2)^{1/2} \Big\} \cos (3E' - 2g - 2h)$$

$$- 15e' \Big\{ (2 + e'^2) + 2 (1 - e'^2)^{1/2} \Big\} \cos (E' + 2g + 2h)$$

$$+ 3 \Big\{ (2 + e'^2) + 2 (1 - e'^2)^{1/2} (1 + e'^2) \Big\} \cos (2E' + 2g + 2h)$$

$$- e' \Big\{ (2 - e'^2) + 2 (1 - e'^2)^{1/2} \Big\} \cos (3E' + 2g + 2h) \Big] \Big\} \quad (50)$$

and

$$\frac{\partial S_2}{\partial h} = \frac{1}{4} \frac{n' R_{\mathbb{C}}^2 J_{22}}{(1 - e'^2)^{3/2}} \Big\{ 6 \sin^2 I' \left[- 2(f' - l) \sin 2h - e' \cos (f' - 2h) + e' \cos (f' + 2h) \right]$$

$$- (1 - \cos I')^2 \left[3e' \cos (f' + 2g - 2h) + 3 \cos (2f' + 2g - 2h) + e' \cos (3f' + 2g - 2h) \right]$$

$$+ (1 + \cos I')^2 \left[3e' \cos (f' + 2g + 2h) + 3 \cos (2f' + 2g + 2h) + e' \cos (3f' + 2g + 2h) \right] \Big\}$$

$$+ \frac{1}{192} \frac{n' a'^2}{\epsilon} \left(\frac{n_{\mathbb{C}}}{n'} \right)^2 \Big\{ 36 \Big[- 2 \sin^2 I' (2 + 3e'^2) \sin 2h + 5e'^2 (1 - \cos I')^2 \sin (2g - 2h)$$

$$- 5e'^2 (1 + \cos I')^2 \sin (2g + 2h)\Big] (E' - l)$$

$$+ 6 \sin^2 I' \Big[9e' \left(4 + e'^2\right) \cos (E' - 2h) - 9e'^2 \cos (2E' - 2h) + e'^3 \cos (3E' - 2h)$$

$$- 9e' \left(4 + e'^2\right) \cos (E' + 2h) + 9e'^2 \cos (2E' + 2h) - e'^3 \cos (3E' + 2h)\Big]$$

$$+ 3(1 - \cos I')^2 \Big[- 15e' \left\{\left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2}\right\} \cos (E' - 2g + 2h)$$

$$+ 3 \left\{\left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right\} \cos (2E' - 2g + 2h)$$

$$- e' \left\{\left(2 - e'^2\right) - 2 \left(1 - e'^2\right)^{1/2}\right\} \cos (3E' - 2g + 2h)$$

$$+ 15e' \left\{\left(2 + e'^2\right) + 2 \left(1 - e'^2\right)^{1/2}\right\} \cos (E' + 2g - 2h)$$

$$- 3 \left\{\left(2 + e'^2\right) + 2 \left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right\} \cos (2E' + 2g - 2h)$$

$$+ e' \left\{\left(2 - e'^2\right) + 2 \left(1 - e'^2\right)^{1/2}\right\} \cos (3E' + 2g - 2h)\Big]$$

$$+ 3(1 + \cos I')^2 \Big[15e' \left\{\left(2 + e'^2\right) - 2 \left(1 - e'^2\right)^{1/2}\right\} \cos (E' - 2g - 2h)$$

$$- 3 \left\{\left(2 + e'^2\right) - 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right\} \cos (2E' - 2g - 2h)$$

$$+ e' \left\{\left(2 - e'^2\right) - 2 \left(1 - e'^2\right)^{1/2}\right\} \cos (3E' - 2g - 2h)$$

$$- 15e' \left\{\left(2 + e'^2\right) + 2 \left(1 - e'^2\right)^{1/2}\right\} \cos (E' + 2g + 2h)$$

$$+ 3 \left\{\left(2 + e'^2\right) + 2\left(1 - e'^2\right)^{1/2} \left(1 + e'^2\right)\right\} \cos (2E' + 2g + 2h)$$

$$- e' \left\{\left(2 - e'^2\right) + 2 \left(1 - e'^2\right)^{1/2}\right\} \cos (3E' + 2g + 2h)\Big]\Bigg\} \cdot \quad (51)$$

If

$$\Delta L = L - L' ,$$

$$\Delta G = G - G' ,$$

and

$$\Delta H = H - H' ,$$

the short-period perturbations in the Keplerian elements a, e, I are obtained by

$$a = a' + 2a' \frac{\Delta L}{L'} ,$$

$$e = e' + \frac{1 - e'^2}{e'} \left(\frac{\Delta L}{L'} - \frac{\Delta G}{G'} \right) ,$$

and

$$I = I' + \cot I' \left(\frac{\Delta G}{G'} - \frac{\Delta H}{H'} \right) , \tag{52}$$

and, in the expressions for ΔL, ΔG, ΔH, the variables l, g, h may be replaced by the variables l', g', h' with an error of the fourth order.

THE LONG-PERIOD TERMS; ELIMINATION OF h′

At this stage, the Hamiltonian $F' = F_0' + F_1' + F_2'$ depends on the variables g', h', L', G', and H'; and L' is a constant with respect to time. The next step in von Zeipel's method consists of the elimination of h' and g'. The elimination of h' is performed by means of a generating function

$$S' = L'' l' + G'' g' + H'' h' + S_1' + S_2' + \cdots \tag{53}$$

of a canonical transformation from (l', g', h', L', G', H') to $(l'', g'', h'', L'', G'', H'')$. The new Hamiltonian

$$F'' = F_0'' + F_1'' + F_2'' + \cdots \tag{54}$$

should be independent of h''.

In order to simplify the formulas, the following symbols are introduced:

$$\mu \equiv \mu_{\mathbb{C}}, \quad b \equiv R_{\mathbb{C}} ,$$

$$\eta' = \frac{G'}{L'} = \sqrt{1 - e'^2} .$$

and

$$\theta' = \frac{H'}{G'} = \cos I' .$$

Then, from Equation 39,

$$
\begin{aligned}
F' = {} & \frac{\mu^2}{2L'^2} + n_{\mathbb{C}}^* H' + \frac{n_{\mathbb{C}}^2 L'^4}{16\mu^2 \epsilon} \Big\{ \left(5 - 3\eta'^2\right) \left[\left(-1 + 3\theta'^2\right) + 3\left(1 - \theta'^2\right) \cos 2h' \right] \\
& + 15\left(1 - \eta'^2\right) \left[\frac{1}{2} (1 + \theta')^2 \cos 2(g' + h') + \left(1 - \theta'^2\right) \cos 2g' + \frac{1}{2} (1 - \theta')^2 \cos 2(g' - h') \right] \Big\} \\
& + \frac{1}{4} \frac{\mu^4}{L'^6 \eta'^3} \left[- b^2 J_2 \left(1 - 3\theta'^2\right) + 6\, b^2 J_{22} \left(1 - \theta'^2\right) \cos 2h' \right] \\
& - \frac{3}{8} \frac{\mu^5 b^3 J_3}{L'^8 \eta'^5} \left(1 - \eta'^2\right)^{1/2} \left(1 - \theta'^2\right)^{1/2} \left(1 - 5\theta'^2\right) \sin g' \\
& - \frac{3}{128} \frac{\mu^6 b^4 J_4}{L'^{10} \eta'^7} \left[\left(5 - 3\eta'^2\right) \left(3 - 30\theta'^2 + 35\theta'^4\right) - 10\left(1 - \eta'^2\right) \left(1 - \theta'^2\right) \left(1 - 7\theta'^2\right) \cos 2g' \right] \\
& - \frac{5}{256} \frac{\mu^7 b^5 J_5}{L'^{12} \eta'^9} \left(1 - \eta'^2\right)^{1/2} \left(1 - \theta'^2\right)^{1/2} \Big[6\left(7 - 3\eta'^2\right) \left(1 - 14\theta'^2 + 21\theta'^4\right) \sin g' \\
& \qquad - 7\left(1 - \eta'^2\right) \left(1 - \theta'^2\right) \left(1 - 9\theta'^2\right) \sin 3g' \Big] . \qquad (55)
\end{aligned}
$$

The elimination of h' depends on the solution of the system

$$
\begin{aligned}
F_0''(L'') &= F_0'(L'') = \frac{\mu^2}{2L''^2} , \\
F_1''(H'') &= F_1'(H'') = n_{\mathbb{C}}^* H'' , \\
F_2'' &= F_2' + \frac{\partial S_1'}{\partial h'} \frac{\partial F_1'}{\partial H''} , \\
& \cdots\cdots\cdots
\end{aligned}
\qquad (56)
$$

and, since S_1' does not depend on l',

$$L'' = L' .$$

As before, a particular solution is given by

$$F_2'' = F_{2s}' = \text{part of } F_2' \text{ independent of } h'$$

and

$$n_{\mathbb{C}}^* \frac{\partial S_1'}{\partial h'} = -F_{2p}' = -\left(F_2' - F_{2s}'\right) . \tag{57}$$

Then it is found that

$$F_2'' = \frac{n_{\mathbb{C}}^2 L'^4}{16\mu^2 \epsilon} \left[\left(5 - 3\eta''^2\right)\left(-1 + 3\theta''^2\right)\right.$$

$$\left. + 15\left(1 - \eta''^2\right)\left(1 - \theta''^2\right) \cos 2g''\right] - \frac{1}{4} b^2 J_2 \frac{\mu^4}{L'^6} \frac{1}{\eta''^3} \left(1 - 3\theta''^2\right)$$

$$- \frac{3}{8} b^3 J_3 \frac{\mu^5}{L'^8 \eta''^5} \left(1 - \eta''^2\right)^{1/2} \left(1 - \theta''^2\right)^{1/2} \left(1 - 5\theta''^2\right) \sin g$$

$$- \frac{3}{128} b^4 J_4 \frac{\mu^6}{L'^{10} \eta''^7} \left[\left(5 - 3\eta''^2\right)\left(3 - 30\theta''^2 + 35\theta''^4\right) - 10\left(1 - \eta''^2\right)\left(1 - \theta''^2\right)\left(1 - 7\theta''^2\right) \cos 2g''\right]$$

$$- \frac{5}{256} b^5 J_5 \frac{\mu^7}{L'^{12} \eta''^9} \left(1 - \eta''^2\right)^{1/2} \left(1 - \theta''^2\right)^{1/2} \left[6\left(7 - 3\eta''^2\right)\left(1 - 14\theta''^2 + 21\theta''^4\right) \sin g''\right.$$

$$\left. - 7\left(1 - \eta''^2\right)\left(1 - \theta''^2\right)\left(1 - 9\theta''^2\right) \sin 3g''\right] . \tag{58}$$

where

$$\eta'' = \frac{G''}{L'}$$

and

$$\theta'' = \frac{H''}{G''} .$$

Furthermore,

$$n_{\mathbb{C}}^* S_1' = -\frac{1}{16} \frac{\mu^2}{\epsilon L'^2} \left(\frac{n_{\mathbb{C}}}{n'}\right)^2 \left\{\frac{3}{2} \left(5 - 3\eta''^2\right) \left(1 - \theta''^2\right) \sin 2h' \right.$$

$$\left. + \frac{15}{4} \left(1 - \eta''^2\right) \left[(1 + \theta'')^2 \sin 2(g' + h') - (1 - \theta'')^2 \sin 2(g' - h')\right]\right\}$$

$$- \frac{3}{4} n'^2 b^2 J_{22} \frac{1}{\eta''^3} \left(1 - \theta''^2\right) \sin 2h' \ . \tag{59}$$

The long-period perturbations depending on the motion of the node are given by

$$L' = L'' + \frac{\partial S_1'}{\partial l'} = L'' \ , \tag{60}$$

$$G' = G'' + \frac{\partial S_1'}{\partial g'}$$

$$= G'' \left\{1 - \frac{15}{32\epsilon} \left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right) \frac{1 - \eta''^2}{\eta''} \left[(1 + \theta'')^2 \cos 2(g'' + h'') - (1 - \theta'')^2 \cos 2(g'' - h'')\right]\right\} , \tag{61}$$

$$H' = H'' + \frac{\partial S_1'}{\partial h'}$$

$$= H'' \left\{1 - \frac{3}{16\epsilon} \left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right) \frac{1}{\eta'' \theta''} \left[\left(5 - 3\eta''^2\right) \left(1 - \theta''^2\right) \cos 2h'' + \frac{5}{2} \left(1 - \eta''^2\right) \left\{(1 + \theta'')^2 \cos 2(g'' + h'') \right.\right.\right.$$

$$\left.\left.\left. + (1 - \theta'')^2 \cos 2(g'' - h'')\right\}\right] - \frac{3}{2} \frac{n'}{H''} \left(\frac{n'}{n_{\mathbb{C}}^*}\right) b^2 J_{22} \frac{1 - \theta''^2}{\eta''^3} \cos 2h''\right\} , \tag{62}$$

$$l' = l'' - \frac{\partial S_1'}{\partial L'}$$

$$= l'' + \frac{1}{16\epsilon} \left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right) \left\{3\left(10 - 3\eta''^2\right) \left(1 - \theta''^2\right) \sin 2h'' + 15\left(1 - \frac{1}{2} \eta''^2\right) \left[(1 + \theta'')^2 \sin 2(g'' + h'') \right.\right.$$

$$\left.\left. - (1 - \theta'')^2 \sin 2(g'' - h'')\right]\right\} - \frac{9}{4} \frac{n'^2}{n_{\mathbb{C}}^*} b^2 J_{22} \frac{1 - \theta''^2}{\eta''^3 L'} \sin 2h'' \ , \tag{63}$$

$$g' = g'' - \frac{\partial S_1'}{\partial G''}$$

$$= g'' + \frac{1}{16\epsilon}\left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right)\frac{1}{\eta''}\left[3\left(5\theta''^2 - 3\eta''^2\right)\sin 2h'' - \frac{15}{2}(1+\theta'')\left(\eta''^2 + \theta''\right)\sin 2(g''+h'')\right.$$

$$\left. + \frac{15}{2}(1-\theta'')\left(\eta''^2 - \theta''\right)\sin 2(g''-h'')\right] - \frac{3}{4}\frac{n'^2}{n_{\mathbb{C}}^*} b^2 J_{22}\frac{3-5\theta''^2}{L'\eta''^4}\sin 2h'' , \quad (64)$$

and

$$h' = h'' - \frac{\partial S_1'}{\partial H''}$$

$$= h'' + \frac{1}{16\epsilon}\left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right)\frac{1}{\eta''}\left\{3\theta''\left(3\eta''^2 - 5\right)\sin 2h'' + \frac{15}{2}\left(1-\eta''^2\right)\left[(1+\theta'')\sin 2(g''+h'')\right.\right.$$

$$\left.\left. + (1-\theta'')\sin 2(g''-h'')\right]\right\} - \frac{3}{2}\frac{n'^2}{n_{\mathbb{C}}^*} b^2 J_{22}\frac{\theta''}{L'\eta''^4}\sin 2h'' . \quad (65)$$

It is important to note that these terms are factored by first-order factors. Then, a second-order theory produces first-order long-period terms. It is necessary to go to third order to obtain second-order long-period terms. This is done next.

THE SECOND-ORDER LONG-PERIOD TERMS AND ELIMINATION OF h' AND THE TIME

The small parameters of third order are as follows:

$j_2\left(\frac{n_{\mathbb{C}}}{n}\right)^2$	(earth's oblateness)
$\left(\frac{n_{\oplus}}{n}\right)^2$	(sun's perturbation)
$\left(\frac{n_{\mathbb{C}}}{n}\right)^2 \sin\frac{i_{\mathbb{C}}}{2}$	(correction due to the inclination of the moon's orbit to its equator)
σ	(solar radiation)

$\left(\frac{n_{☾}}{n}\right)^2 e_{☾}$ (eccentricity of the moon's orbit)

$\left(\frac{n_{☾}}{n}\right)^3$ (earth's perturbations—third Legendre polynomial)

and

$\frac{a n_{\odot}}{n}$ (correction due to physical libration)

Since short-period perturbations of the third order will be neglected, it is understood that all terms of third order which depend on l will be ignored. Suppose that F_3 is the third-order part of the Hamiltonian and F_3' the part free of short-period terms. The Hamiltonian will be time-dependent through the longitudes of the earth and of the sun. This fact introduces one more degree of freedom in the problem, and it is necessary to introduce the time as a canonical variable. This is done through use of the following equations ($\tau = t +$ const.):

$$\frac{d\tau}{dt} = -\frac{\partial \mathfrak{F}}{\partial T} = 1 \tag{66}$$

and

$$\frac{dT}{dt} = \frac{\partial \mathfrak{F}}{\partial \tau} = \frac{\partial \mathfrak{F}}{\partial t} = \frac{\partial F_3}{\partial t} , \tag{67}$$

so that

$$\mathfrak{F} = F - T \tag{68}$$

and

$$T = -F_3 + \text{const.} \tag{69}$$

This additional complication does not affect the development of the theory up to this point, if one sets

$$\frac{\partial S_2'}{\partial \tau} = 0 . \tag{70}$$

The next step is then to eliminate h' to third order to obtain second-order long-period perturbations depending on the argument h'.

The orders of magnitude are as follows:

$$\mathfrak{F}_0' = F_0' - T = \frac{\mu^2}{2L'} - T$$

$$\mathfrak{F}_1' = F_1' = n_{\mathbb{C}}^* H'$$

$$\mathfrak{F}_2' = F_2'$$

and

$$\mathfrak{F}_3' = F_3' , \tag{71}$$

where T has been incorporated into $\mathfrak{F}_0'$. The von Zeipel differential equation of the third order is then

$$\mathfrak{F}_3' + \frac{\partial S_1'}{\partial h'} \frac{\partial \mathfrak{F}_2'}{\partial H''} + \frac{\partial S_1'}{\partial g'} \frac{\partial \mathfrak{F}_2'}{\partial G''} + \frac{\partial S_2'}{\partial h'} \frac{\partial \mathfrak{F}_1'}{\partial H''} + \frac{\partial S_3'}{\partial \tau} \frac{\partial \mathfrak{F}_0'}{\partial T} = \mathfrak{F}_3'' + \frac{\partial S_1'}{\partial G''} \frac{\partial \mathfrak{F}_2''}{\partial g'} , \tag{72}$$

or

$$F_3' + \frac{\partial S_1'}{\partial h'} \frac{\partial F_2'}{\partial H''} + \frac{\partial F_2'}{\partial G''} \frac{\partial S_1'}{\partial g'} + n_{\mathbb{C}}^* \frac{\partial S_2'}{\partial h'} - \frac{\partial S_3'}{\partial \tau} = F_3'' + \frac{\partial F_2''}{\partial g'} \frac{\partial S_1'}{\partial G''} . \tag{73}$$

Since we had supposed that $\partial S_2'/\partial \tau = 0$ in the previous order evaluation, a particular solution of the third-order equation is found by taking

$$F_3'' = F_{3s}' + \left(\frac{\partial S_1'}{\partial h'} \frac{\partial F_2'}{\partial H''}\right)_s + \left(\frac{\partial F_2'}{\partial G''} \frac{\partial S_1'}{\partial g'}\right)_s - \left(\frac{\partial F_2''}{\partial g'} \frac{\partial S_1'}{\partial G''}\right)_s , \tag{74}$$

where the subscript $_s$ designates non-dependence upon h'' and τ. Further,

$$n_{\mathbb{C}}^* \frac{\partial S_2'}{\partial h'} = - F_{3ph'}' - \left(\frac{\partial S_1'}{\partial h'} \frac{\partial F_2'}{\partial H''}\right)_{ph'} - \left(\frac{\partial F_2'}{\partial G''} \frac{\partial S_1'}{\partial g'}\right)_{ph'} + \left(\frac{\partial F_2''}{\partial g'} \frac{\partial S_1'}{\partial G''}\right)_{ph'} , \tag{75}$$

and

$$\frac{\partial S_3'}{\partial \tau} = F_{3p\tau}' + \left(\frac{\partial S_1'}{\partial h'} \frac{\partial F_2'}{\partial H''}\right)_{p\tau} + \left(\frac{\partial F_2'}{\partial G''} \frac{\partial S_1'}{\partial g'}\right)_{p\tau} - \left(\frac{\partial F_2''}{\partial g'} \frac{\partial S_1'}{\partial G'}\right)_{p\tau} , \tag{76}$$

where the subscript $_{ph'}$ indicates the inclusion of all trigonometric terms with arguments of the form $jh' + kg'$, where j and k are integers ($j \neq 0$), and the subscript $_{p\tau}$ indicates the inclusion of all trigonometric terms with arguments of the form $j\tau + kh' + mg'$, where j, k, and m are integers ($j \neq 0$).

Then S_2' will be obtained as a periodic function independent of the time, and S_3' as a periodic function dependent on time.

Since the functions F_{2p}', F_2', S_1', and F_2'' are known, the only parts that have to be computed are those corresponding to the third-order terms of the Hamiltonian (F_3').

THIRD-ORDER TERMS GENERATED BY COUPLING OF SECOND-ORDER TERMS

It was found (from Equations 55 and 58) that

$$F_2' = \frac{1}{16\epsilon}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2 n' L' \Big\{\left(5 - 3\eta'^2\right)\left[\left(-1 + 3\theta'^2\right) + 3\left(1 - \theta'^2\right)\cos 2h'\right]$$

$$+ 15\left(1 - \eta'^2\right)\left[\frac{1}{2}(1 + \theta')^2 \cos 2(g' + h') + \left(1 - \theta'^2\right)\cos 2g' + \frac{1}{2}(1 - \theta')^2 \cos 2(g' - h')\right]\Big\}$$

$$+ \frac{1}{4}\frac{\mu^4}{L'^6\,\eta'^3}\left[-b^2 J_2\left(1 - 3\theta'^2\right) + 6b^2 J_{22}\left(1 - \theta'^2\right)\cos 2h'\right] - \frac{3}{8}\frac{\mu^5 b^3 J_3}{L'^8\,\eta'^5}\left(1 - \eta'^2\right)^{1/2}\left(1 - \theta'^2\right)^{1/2}\left(1 - 5\theta'^2\right)\sin g'$$

$$- \frac{3}{128}\frac{\mu^6 b^4 J_4}{L'^{10}\,\eta'^7}\left[\left(5 - 3\eta'^2\right)\left(3 - 30\theta'^2 + 35\theta'^4\right) - 10\left(1 - \eta'^2\right)\left(1 - \theta'^2\right)\left(1 - 7\theta'^2\right)\cos 2g'\right]$$

$$- \frac{5}{256}\frac{\mu^7 b^5 J_5}{L'^{12}\,\eta'^9}\left(1 - \eta'^2\right)^{1/2}\left(1 - \theta'^2\right)^{1/2}\left[6\left(7 - 3\eta'^2\right)\left(1 - 14\theta'^2 + 21\theta'^4\right)\sin g' - 7\left(1 - \eta'^2\right)\left(1 - \theta'^2\right)\left(1 - 9\theta'^2\right)\sin 3g'\right], \quad (77)$$

$$F_2'' = \frac{1}{16\epsilon}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2 n' L' \left[\left(5 - 3\eta''^2\right)\left(-1 + 3\theta''^2\right) + 15\left(1 - \eta''^2\right)\left(1 - \theta''^2\right)\cos 2g''\right]$$

$$- \frac{1}{4} b^2 J_2 \frac{\mu^4}{L'^6}\frac{1}{\eta''^3}\left(1 - 3\theta''^2\right) - \frac{3}{8} b^3 J_3 \frac{\mu^5}{L'^8\,\eta''^5}\left(1 - \eta''^2\right)^{1/2}\left(1 - \theta''^2\right)^{1/2}\left(1 - 5\theta''^2\right)\sin g''$$

$$- \frac{3}{128} b^4 J_4 \frac{\mu^6}{L'^{10}\,\eta''^7}\left[\left(5 - 3\eta''^2\right)\left(3 - 30\theta''^2 + 35\theta''^4\right) - 10\left(1 - \eta''^2\right)\left(1 - \theta''^2\right)\left(1 - 7\theta''^2\right)\cos 2g''\right]$$

$$- \frac{5}{256} b^5 J_5 \frac{\mu^7}{L'^{12}\,\eta''^9}\left(1 - \eta''^2\right)^{1/2}\left(1 - \theta''^2\right)^{1/2}\left[6\left(7 - 3\eta''^2\right)\left(1 - 14\theta''^2 + 21\theta''^4\right)\sin g''\right.$$

$$\left. - 7\left(1 - \eta''^2\right)\left(1 - \theta''^2\right)\left(1 - 9\theta''^2\right)\sin 3g''\right], \quad (78)$$

and $n_{☾}^{*} S_1'$ is given by Equation 59. In the coupling terms, the variables in F_2' can be double-primed with an error that is of the fourth order.

The partials needed are given below.

$$
\begin{aligned}
\frac{\partial F_2'}{\partial H''} = {} & \frac{3}{16\epsilon} n' \left(\frac{n_{☾}}{n'}\right)^2 \frac{1}{\eta'' \theta''} \Big\{ 2\left(5 - 3\eta''^2\right)\theta''^2 \left(1 - \cos 2h''\right) \\
& + 5\left(1 - \eta''^2\right)\theta'' \Big[\left(1 + \theta''\right) \cos 2(g'' + h'') - 2\theta'' \cos 2g'' \\
& - \left(1 - \theta''\right) \cos 2(g'' - h'')\Big]\Big\} + \frac{3}{2} b^2 n'^2 \frac{\theta''}{L' \eta''^4} \left(J_2 - 2J_{22} \cos 2h''\right) \\
& + \frac{3}{8} b^3 J_3 \frac{\mu n'^2 \theta'' \left(11 - 15\theta''^2\right) \left(1 - \eta''\right)^{1/2}}{L'^3 \eta''^6 \left(1 - \theta''^2\right)^{1/2}} \sin g'' \\
& + \frac{15}{32} b^4 J_4 \frac{\mu^2 n'^2 \theta''}{L'^5 \eta''^8} \Big[\left(5 - 3\eta''^2\right)\left(3 - 7\theta''^2\right) - 2\left(1 - \eta''^2\right)\left(4 - 7\theta''^2\right) \cos 2g''\Big] \\
& + \frac{15}{256} b^5 J_5 \frac{\mu^3 n'^2 \theta'' \left(1 - \eta''^2\right)^{1/2}}{L'^7 \eta''^{10} \left(1 - \theta''^2\right)^{1/2}} \Big[2\left(7 - 3\eta''^2\right)\left(29 - 126\theta''^2 + 105\theta''^4\right) \sin g'' \\
& - 7\left(1 - \eta''^2\right)\left(1 - \theta''^2\right)\left(7 - 15\theta''^2\right) \sin 3g''\Big] , \qquad (79)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F_2'}{\partial G''} = {} & \frac{3n'}{16\epsilon} \left(\frac{n_{☾}}{n'}\right)^2 \frac{1}{\eta''} \Big[2\left(\eta''^2 - 5\theta''^2\right) + 2\left(5\theta''^2 - 3\eta''^2\right) \cos 2h'' \\
& - 5(1 + \theta'')\left(\eta''^2 + \theta''\right) \cos 2(g'' + h'') + 10\left(\theta''^2 - \eta''^2\right) \cos 2g'' \\
& + 5(1 - \theta'')\left(\theta'' - \eta''^2\right) \cos 2(g'' - h'')\Big] + \frac{3}{4} \frac{n'^2 b^2}{L' \eta''^4} \Big[J_2 \left(1 - 5\theta''^2\right) - 2J_{22} \left(3 - 5\theta''^2\right) \cos 2h''\Big] \\
& + \frac{3}{8} b^3 J_3 \frac{\mu n'^2}{L'^3 \eta''^6 \left(1 - \eta''^2\right)^{1/2} \left(1 - \theta''^2\right)^{1/2}} \Big[5 - 41\theta''^2 + 40\theta''^4 - \eta''^2 \left(4 - 35\theta''^2 + 35e''^4\right)\Big] \sin g'' \\
& + \frac{15}{128} b^4 J_4 \frac{\mu^2 n'^2}{L'^5 \eta''^8} \Big\{21 - 270\theta''^2 + 385\theta''^4 - 9\eta''^2 \left(1 - 14\theta''^2 + 21\theta''^4\right) \\
& - 2\Big[7 - 72\theta''^2 + 77\theta''^4 - \eta''^2 \left(5 - 56\theta''^2 + 63\theta''^4\right)\Big] \cos 2g''\Big\}
\end{aligned}
$$

$$+ \frac{15}{256} b^5 J_5 \frac{\mu^3 n'^2}{L'^7 \eta''^{10} \left(1 - \eta''^2\right)^{1/2} \left(1 - \theta''^2\right)^{1/2}} \left\{2\left[7\left(9 - 164\theta''^2 + 441\theta''^4 - 294\theta''^6\right)\right.\right.$$

$$\left. - \eta''^2 \left(77 - 1445\theta''^2 + 3955\theta''^4 - 2667\theta''^6\right) + 3\eta''^4 \left(6 - 119\theta''^2 + 336\theta''^4 - 231\theta''^6\right)\right] \sin g''$$

$$\left. - 7\left(1 - \eta''^2\right) \left(1 - \theta''^2\right) \left[3 - 37\theta''^2 + 42\theta''^4 - \eta''^2 \left(2 - 27\theta''^2 + 33\theta''^4\right)\right] \sin 3g''\right\} , \tag{80}$$

$$\frac{\partial F_2''}{\partial g'} = - \frac{15}{8\epsilon} n' L' \left(\frac{n_{\mathbb{C}}}{n'}\right)^2 \left(1 - \eta''^2\right)\left(1 - \theta''^2\right) \sin 2g''$$

$$- \frac{3}{8} b^3 J_3 \frac{\mu n'^2 \left(1 - \eta''^2\right)^{1/2} \left(1 - \theta''^2\right)^{1/2} \left(1 - 5\theta''^2\right)}{L'^2 \eta''^5} \cos g''$$

$$- \frac{15}{32} b^4 J_4 \frac{\mu^2 n'^2 \left(1 - \eta''^2\right) \left(1 - \theta''^2\right) \left(1 - 7\theta''^2\right)}{L'^4 \eta''^7} \sin 2g''$$

$$- \frac{15}{256} b^5 J_5 \frac{\mu^3 n'^2 \left(1 - \eta''^2\right)^{1/2} \left(1 - \theta''^2\right)^{1/2}}{L'^6 \eta''^9} \left[2\left(7 - 3\eta''^2\right) \left(29 - 126\theta''^2 + 105\theta''^4\right) \cos g''\right.$$

$$\left. - 7\left(1 - \eta''^2\right) \left(1 - \theta''^2\right) \left(7 - 15\theta''^2\right) \cos 3g''\right] , \tag{81}$$

$$\frac{\partial S_1'}{\partial h'} = \frac{1}{n_{\mathbb{C}}^*} F_{2p}' - \frac{3L'}{32\epsilon} \left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right)\left\{2\left(5 - 3\eta''^2\right) \left(1 - 3\theta''^2\right) \cos 2h''\right.$$

$$\left. + 5\left(1 - \eta''^2\right) \left[(1 + \theta'')^2 \cos 2(g'' + h'') + (1 - \theta'')^2 \cos 2(g'' - h'')\right]\right\}$$

$$- \frac{3}{2} \frac{n'^2}{n_{\mathbb{C}}^*} b^2 J_{22} \frac{1 - \theta''^2}{\eta''^3} \cos 2h'' , \tag{82}$$

$$\frac{\partial S_1'}{\partial g'} = - \frac{15}{32\epsilon} \left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right) \left(1 - \eta''^2\right) L' \left[(1 + \theta'')^2 \cos 2(g'' + h'') - (1 - \theta'')^2 \cos 2(g'' - h'')\right] , \tag{83}$$

and

$$\frac{\partial S_1'}{\partial G''} = - \frac{3}{32\epsilon} \left(\frac{n_{\mathbb{C}}^2}{n_{\mathbb{C}}^* n'}\right) \frac{1}{\eta} \left[2\left(5\theta''^2 - 3\eta''^2\right) \sin 2h'' - 5(1 + \theta'')\left(\eta''^2 + \theta''\right) \sin 2(g'' + h'')\right.$$

$$+ 5(1 - \theta'')\left(\eta''^2 - \theta''\right) \sin 2(g'' - h'')\Big] - \frac{3}{4}\,\frac{n'^2\, b^2\, J_{22}}{n^*_{\mathbb{C}}\,\eta''^4\, L'}\left(-3 + 5\theta''^2\right) \sin 2h'' \,. \qquad (84)$$

From these equations and Equations 74 and 75, it can be shown that

$$S_2' \text{ (coupling)} = \frac{9}{4096\epsilon^2}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2 \left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2 \frac{L'}{\eta''}\Big\{-32\theta''\left(1 - \theta''^2\right)\eta''^2\left(15 - 17\eta''^2\right) \sin 2h''$$

$$- 4\theta''\left(1 - \theta''^2\right)\left(5 - 3\eta''^2\right)^2 \sin 4h''$$

$$+ 80(1 + \theta'')^2 (2 - 3\theta'')\eta''^2\left(1 - \eta''^2\right) \sin(2g'' + 2h'')$$

$$+ 80(1 - \theta'')^2 (2 + 3\theta'')\eta''^2\left(1 - \eta''^2\right) \sin(2g'' - 2h'')$$

$$+ 10(1 + \theta'')^2 (1 - \theta'')\left(1 - \eta''^2\right)\left[5(1 - \theta'') - 6\eta''^2\right] \sin(2g'' + 4h'')$$

$$+ 10(1 - \theta'')^2 (1 + \theta'')\left(1 - \eta''^2\right)\left[5(1 + \theta'') - 6\eta''^2\right] \sin(2g'' - 4h'')$$

$$+ 25(1 + \theta'')^3\left(1 - \eta''^2\right)\left[(1 - \theta'') - 2\eta''^2\right] \sin(4g'' + 4h'')$$

$$+ 25(1 - \theta'')^3\left(1 - \eta''^2\right)\left[(1 + \theta'') - 2\eta''^2\right] \sin(4g'' - 4h'')\Big\}$$

$$+ \frac{9}{256\epsilon}\, n'\, \eta''^{-4}\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2 J_2\, b^2 \Big[4\theta''\left(1 - \theta''^2\right)\left(5 - 3\eta''^2\right) \sin 2h''$$

$$+ 5(1 + \theta'')^2\left(1 + 2\theta'' - 5\theta''^2\right)\left(1 - \eta''^2\right) \sin(2g'' + 2h'')$$

$$+ 5(1 - \theta'')^2\left(1 - 2\theta'' - 5\theta''^2\right)\left(1 - \eta''^2\right) \sin(2g'' - 2h'')\Big]$$

$$- \frac{9}{512\epsilon}\, n'\, \eta''^{-4}\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)^2 J_{22}\, b^2 \Big\{-8\theta''\left(1 - \theta''^2\right)\left(5 - 3\eta''^2\right)\left(2 \sin 2h'' - \sin 4h''\right)$$

$$- 20(1 + \theta'')^2 (1 - \theta'')(3 - 5\theta'')\left(1 - \eta''^2\right) \sin(2g'' + 2h'')$$

$$- 20(1 - \theta'')^2 (1 + \theta'')(3 + 5\theta'')\left(1 - \eta''^2\right) \sin(2g'' - 2h'')$$

$$+\ 5(1+\theta'')^2\,(1-\theta'')\,(1+5\theta'')\,\left(1-\eta''^2\right)\sin(2g''+4h'')$$

$$+\ 5(1-\theta'')^2\,(1+\theta'')\,(1-5\theta'')\,\left(1-\eta''^2\right)\sin(2g''-4h'')\Big\}$$

$$+\frac{9}{8}\left(\frac{n'}{n_{\mathbb{C}}^{*}}\right)^2 n'^2\,\frac{\theta''\left(1-\theta''^2\right)}{L'\,\eta''^7}\,J_2\,J_{22}\,b^4\sin 2h''-\frac{9}{16}\left(\frac{n'}{n_{\mathbb{C}}^{*}}\right)^2 n'^2\,\frac{\theta''\left(1-\theta''^2\right)}{L'\,\eta''^7}\,J_{22}^2\,b^4\sin 4h''$$

$$+\frac{9}{1024\epsilon}\,\frac{\mu n'\,\left(1-\eta''^2\right)^{1/2}\left(1-\theta''^2\right)^{1/2}}{L'^2\,\eta''^6}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^{*}}\right)^2 J_3\,b^3\Big\{(1+\theta'')\Big[25\left(1-\theta''-\theta''^2-7\theta''^3\right)$$

$$-\ \eta''^2\left(9+15\theta''+15\theta''^2-175\theta''^3\right)\Big]\cos(g''+2h'')$$

$$+\,(1-\theta'')\Big[25\left(1+\theta''-\theta''^2+7\theta''^3\right)$$

$$-\ \eta''^2\left(9-15\theta''+15\theta''^2+175\theta''^3\right)\Big]\cos(g''-2h'')$$

$$-\ 25\left(1-\eta''^2\right)(1+\theta'')^2\left(1+2\theta''-7\theta''^2\right)\cos(3g''+2h'')$$

$$-\ 25\left(1-\eta''^2\right)(1-\theta'')^2\left(1-2\theta''-7\theta''^2\right)\cos(3g''-2h'')\Big\}$$

$$+\ \frac{9}{128}\left(\frac{n'}{n_{\mathbb{C}}^{*}}\right)^2\,\frac{\mu n'^2\left(1-\eta''^2\right)^{1/2}\left(1-\theta''^2\right)^{1/2}}{L'^3\,\eta''^9}\,J_3\,J_{22}\,b^5\Big[(1+\theta'')\left(3-25\theta''+5\theta''^2+25\theta''^3\right)\cos(g''+2h'')$$

$$+\ (1-\theta'')\left(3+25\theta''+5\theta''^2-25\theta''^3\right)\cos(g''-2h'')\Big]$$

$$+\ \frac{45}{8192\epsilon}\,\frac{n'^2}{L'\,\eta''^8}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^{*}}\right)^2 J_4\,b^4\Big\{8\theta''\left(1-\theta''^2\right)\Big[35-10\eta''^2\left(2+7\theta''^2\right)-3\eta''^4\left(1-14\theta''^2\right)\Big]\sin 2h''$$

$$+\ \left(1-\eta''^2\right)(1+\theta'')^2\Big[35\left(3+4\theta''-30\theta''^2-20\theta''^3+51\theta''^4\right)$$

$$-\ 3\eta''^2\left(11+36\theta''-126\theta''^2-196\theta''^3+315\theta''^4\right)\Big]\sin(2g''+2h'')$$

$$+\ \left(1-\eta''^2\right)(1-\theta'')^2\Big[35\left(3-4\theta''-30\theta''^2+20\theta''^3+51\theta''^4\right)$$

$$-\ 3\eta''^2\left(11-36\theta''-126\theta''^2+196\theta''^3+315\theta''^4\right)\Big]\sin(2g''-2h'')$$

$$- 35\left(1-\eta''^2\right)^2 \left(1-\theta''^2\right) \Big[(1+\theta'')^2 \left(1+2\theta''-9\theta''^2\right) \sin(4g''+2h'')$$

$$+ (1-\theta'')^2 \left(1-2\theta''-9\theta''^2\right) \sin(4g''-2h'')\Big]\Big\}$$

$$+ \frac{45}{512}\left(\frac{n'}{n_{\mathbb{C}}^*}\right)^2 \frac{n'^3\left(1-\theta''^2\right)}{L'^2\,\eta''^{11}} J_4 J_{22}\, b^6 \Big[4\left(5-3\eta''^2\right)\theta''\left(3-7\theta''^2\right)\sin 2h''$$

$$+ \left(1-\eta''^2\right)(1+\theta'')\left(3-19\theta''-7\theta''^2+35\theta''^3\right)\sin(2g''+2h'')$$

$$+ \left(1-\eta''^2\right)(1-\theta'')\left(3+19\theta''-7\theta''^2-35\theta''^3\right)\sin(2g''-2h'')\Big]$$

$$+ \frac{45}{32768\epsilon} \frac{n'^3\left(1-\eta''^2\right)^{1/2}\left(1-\theta''^2\right)^{1/2}}{\mu\eta''^{10}} \left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)^2 J_5\, b^5 \Big\{(1+\theta'')\Big[35\left(15+41\theta''-262\theta''^2\right.$$

$$\left.-346\theta''^3+519\theta''^4+321\theta''^5\right)+14\eta''^2\left(259-255\theta''-566\theta''^2+1710\theta''^3-525\theta''^4-1455\theta''^5\right)$$

$$-\eta''^4\left(1559-1015\theta''-5446\theta''^2+7350\theta''^3+735\theta''^4-5775\theta''^5\right)\Big]\cos(g''+2h'')$$

$$+(1-\theta'')\Big[35\left(15-41\theta''-262\theta''^2+346\theta''^3+519\theta''^4-321\theta''^5\right)$$

$$+14\eta''^2\left(259+255\theta''-566\theta''^2-1710\theta''^3-525\theta''^4+1455\theta''^5\right)$$

$$-\eta''^4\left(1559+1015\theta''-5446\theta''^2-7350\theta''^3+735\theta''^4+5775\theta''^5\right)\Big]\cos(g''-2h'')$$

$$-14\left(1+\theta''^2\right)\Big[5\left(9-7\theta''-119\theta''^2+15\theta''^3+174\theta''^4\right)-\eta''^2\left(179-14\theta''-1320\theta''^2+30\theta''^3+1605\theta''^4\right)$$

$$+3\eta''^4\left(18+7\theta''-135\theta''^2-15\theta''^3+165\theta''^4\right)\Big]\cos(3g'+2h'')$$

$$-14(1-\theta'')^2\Big[5\left(9+7\theta''-119\theta''^2-15\theta''^3+174\theta''^4\right)$$

$$-\eta''^2\left(179+14\theta''-1320\theta''^2-30\theta''^3+1605\theta''^4\right)$$

$$+3\eta''^4\left(18-7\theta''-135\theta''^2+15\theta''^3+165\theta''^4\right)\Big]\cos(3g''-2h'')$$

$$+105\left(1-\eta''^2\right)\left(1-\theta''^2\right)\Big[1-9\theta''^2-\eta''^2\left(3-11\theta''^2\right)\Big]$$

$$\times\Big[(1+\theta'')^2\cos(5g''+2h'')+(1-\theta'')^2\cos(5g''-2h'')\Big]\Big\}$$

$$
+ \frac{45}{4096}\left(\frac{n'}{n^*_{\mathbb{C}}}\right)^2 n'^4 \frac{\left(1-\eta''^2\right)^{1/2}\left(1-\theta''^2\right)^{1/2}}{\mu L'\,\eta''^{13}} J_5 J_{22}\, b^7 \Big\{ 2\left(7-3\eta''^2\right)\left(29-126\theta''^2+105\theta''^4\right)\Big[(1+\theta'')(3-5\theta'')\cos(g''+2h'')
$$

$$
+ (1-\theta'')(3+5\theta'')\cos(g''-2h'')\Big] - 7\left(1-\eta''^2\right)\left(1-\theta''^2\right)\left(7-15\theta''^2\right)\Big[(1+\theta'')(3-5\theta'')\cos(3g''+2h'')
$$

$$
+ (1-\theta'')(3+5\theta'')\cos(3g''-2h'')\Big]\Big\}, \tag{85}
$$

$$
F_3''\,(\text{coupling}) = \frac{9}{128}\left(\frac{n_{\mathbb{C}}}{n'}\right)^3\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)\frac{n'^2}{\epsilon^2}\, a'^2 \sqrt{1-e''^2}\cos I''\Big[\left(2+33e''^2\right)
$$

$$
- \left(2-17e''^2\right)\cos^2 I'' + 15e''^2\sin^2 I''\cos 2g''\Big]
$$

$$
+ \frac{9}{32}\left(\frac{n_{\mathbb{C}}}{n'}\right)\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right) n'^2 \frac{\sin^2 I''\cos I''}{\epsilon\left(1-e''^2\right)^2} J_{22}\, b^2\left[2\left(2+3e''^2\right)+15e''^2\cos 2g''\right]
$$

$$
+ \frac{9}{4}\left(\frac{n'}{n^*_{\mathbb{C}}}\right) n'^2 \frac{\sin^2 I\,\cos I}{a'^2\left(1-e^2\right)^{7/2}} J_{22}^2\, b^4, \tag{86}
$$

and, of course,

$$
S_3'\,(\text{coupling}) = 0. \tag{87}
$$

It is important to note that S_2' (coupling) is made up of second-order terms, although it is obtained by multiplying a second-order quantity by a first-order quantity.

The next few sections are devoted to the computation of F_3'. From that computation various additional parts of F_3'', S_3', and S_2' are then derived.

THE RADIATION PRESSURE

For this computation, the shadow effect will be neglected. Its inclusion would require, of course, the introduction of a shadow function due both to the moon and to the earth.

Consider σ to be the absolute value of the acceleration of the orbiter arising from the solar radiation. Then, the disturbing function for the radiation pressure will be given by (see the earlier section, "The Angles S_{10}' and S_{13}''")

$$
R_\sigma = -\sigma r \cos S_{13}'', \tag{88}
$$

or, neglecting the inclination of the lunar equator to the sun's orbit,

$$
R_\sigma = -\sigma r\left[\cos(f+g)\cos\left(\Omega-\lambda_\odot\right) - \sin(f+g)\sin\left(\Omega-\lambda_\odot\right)\cos I\right]. \tag{89}
$$

The elimination of short-period terms gives

$$R_\sigma' = \frac{3}{2}\sigma a' e' \left[\cos^2 \frac{I'}{2} \cos\left(g' + h' + \lambda_\oplus - \lambda_\odot\right) + \sin^2 \frac{I'}{2} \cos\left(g' - h' + \lambda_\odot - \lambda_\oplus\right)\right] , \qquad (90)$$

where the node Ω was written in terms of the canonical variable h. There is no secular contribution from this effect, that is, F_{3s}' (radiation) $= R_{\sigma s}' = 0$. On the other hand, since there are no terms strictly independent of time, the contribution to S_2' is zero, or

$$S_2' \text{ (radiation) } = 0 . \qquad (91)$$

Therefore,

$$\frac{\partial S_3' \text{ (radiation)}}{\partial \tau} = F_{3p\tau}' \text{ (radiation) } = R_\sigma' ,$$

or

$$S_3' \text{ (radiation) } = \frac{3}{2} \frac{\sigma a'' e''}{n_{\mathbb{C}}^* - n_\oplus^*} \left[\cos^2 \frac{I'}{2} \sin\left(g' + h' + \lambda_\oplus - \lambda_\odot\right) - \sin^2 \frac{I''}{2} \sin\left(g'' - h' + \lambda_\odot - \lambda_\oplus\right)\right] , \qquad (92)$$

where

$$\lambda_\odot = n_\oplus^* \tau + \text{const.} ,$$

$$\lambda_\oplus = n_{\mathbb{C}}^* \tau + \text{const.} ,$$

and $n_\oplus^*$ and $n_{\mathbb{C}}^*$ are, respectively, the mean motion in longitude of the sun around the moon and of the moon around the earth. Of course, $n_{\mathbb{C}}^* - n_\odot^* \simeq n_{\mathbb{C}}^*$ is a good approximation.

THE SECOND LEGENDRE POLYNOMIAL FOR THE SUN'S GRAVITATIONAL PERTURBATIONS

The most important term in the disturbing function due to the sun's gravity is

$$\frac{\mu_3}{r_3} \left(\frac{r}{r_3}\right)^2 P_2 \left(\cos S_{13}''\right) , \qquad \text{(see Equation 20)}$$

where, for the computation of S''_{13}, the sun can be considered to move along the moon's equator. Thus,

$$\cos S''_{13} = \cos(f+g)\cos\left(\Omega-\lambda_\odot\right) - \sin(f+g)\sin\left(\Omega-\lambda_\odot\right)\cos I \ .$$

On the other hand, neglecting the mass of the moon,

$$n_\oplus^2\, a_3^3 = k^2\left(m_\odot + m_\oplus\right) = k^2 m_\odot\left(1+\frac{m_\oplus}{m_\odot}\right),$$

and since the ratio $m_\oplus/m_\odot$ can be neglected,

$$n_\oplus^2\, a_3^3 = \mu_3 \ .$$

Furthermore, if the eccentricity of the earth orbit is neglected,

$$\begin{aligned}
F_3(\text{sun}) &= n_\oplus^2\, r^2 P_2\left(\cos S''_{13}\right) \\
&= \frac{1}{2} n_\oplus^2\, r^2\left(3\cos^2 S''_{13} - 1\right) \\
&= \frac{3}{2} n_\oplus^2\, r^2\Big[\cos^2(f+g)\cos^2\left(\Omega-\lambda_\odot\right) - 2\sin(f+g)\cos(f+g)\sin\left(\Omega-\lambda_\odot\right)\cos\left(\Omega-\lambda_\odot\right)\cos I \\
&\qquad + \sin^2(f+g)\sin^2\left(\Omega-\lambda_\odot\right)\cos^2 I\Big] - \frac{1}{2} n_\oplus^2\, r^2 \ . \qquad (93)
\end{aligned}$$

Rewritten, Equation 93 becomes

$$\begin{aligned}
F_3(\text{sun}) &= \frac{3}{2} n_\oplus^2\, r^2\Big[\left(\cos^2 f\cos^2 g + \sin^2 f\sin^2 g\right)\cos^2\left(\Omega-\lambda_\odot\right) \\
&\quad - 2\left(-\sin^2 f\sin g\cos g + \cos^2 f\sin g\cos g\right)\sin\left(\Omega-\lambda_\odot\right)\cos\left(\Omega-\lambda_\odot\right)\cos I \\
&\quad + \left(\sin^2 f\cos^2 g + \cos^2 f\sin^2 g\right)\sin^2\left(\Omega-\lambda_\odot\right)\cos^2 I\Big] - \frac{1}{2} n_\oplus^2\, r^2 + \frac{3}{2} n_\oplus^2\, r^2\sin f\cos f\cdot Q \ , \qquad (94)
\end{aligned}$$

where Q is independent of f. It follows that

$$\begin{aligned}
F_3'(\text{sun}) &= \frac{1}{32} n_\oplus^2\, a''^2\Big[-2\left(2+3e''^2\right)\left(1-3\cos^2 I''\right) + 30e''^2\sin^2 I''\cos 2g'' \\
&\quad + 6\left(2+3e''^2\right)\sin^2 I''\cos 2\left(h''+\lambda_\oplus-\lambda_\odot\right) + 15e''^2\left(1+\cos I''\right)^2\cos 2\left(g''+h''+\lambda_\oplus-\lambda_\odot\right) \\
&\quad + 15e''^2\left(1-\cos I''\right)^2\cos 2\left(g''-h''-\lambda_\oplus+\lambda_\odot\right)\Big] \ . \qquad (95)
\end{aligned}$$

The secular contribution is

$$F_3''\ (\text{sun}) = \frac{1}{16}\, n_\oplus^2\, a''^2 \left[-\left(2 + 3e''^2\right)\left(1 - 3\cos^2 I''\right) + 15e''^2\left(1 - \cos^2 I''\right)\cos 2g''\right] . \tag{96}$$

There is no contribution to S_2', and the contribution to S_3' is

$$S_3'\ (\text{sun}) = \frac{3}{64}\, \frac{n_\oplus^2\, a''^2}{n_{☾}^* - n_\oplus^*} \left[2\left(2 + 3e''^2\right)\sin^2 I''\, \sin 2\left(h'' + \lambda_\oplus - \lambda_\odot\right)\right.$$

$$\left. + 5e''^2 (1 + \cos I'')^2 \sin 2\left(h'' + g'' + \lambda_\oplus - \lambda_\odot\right) + 5e''^2 (1 - \cos I'')^2 \sin 2\left(h'' - g'' + \lambda_\oplus - \lambda_\odot\right)\right] . \tag{97}$$

THE THIRD LEGENDRE POLYNOMIAL FOR THE EARTH'S GRAVITATIONAL PERTURBATIONS

From Equation 20,

$$F_3\ (\text{earth}) = \frac{\mu_2}{r_0}\left(\frac{r}{r_0}\right)^3 P_3\left(\cos S_{10}'\right) . \tag{98}$$

Now,

$$n_{☾}^2\, a_{☾}^3 = k^2\left(m_\oplus + m_{☾}\right) = k^2\, m_\oplus\left(1 + \frac{m_{☾}}{m_\oplus}\right) = \mu_2\, \epsilon ,$$

so that, considering the moon's orbit to be circular,

$$F_3\ (\text{earth}) = \frac{n_{☾}^2\, a_{☾}^3}{\epsilon}\, \frac{r^3}{a_{☾}^4}\, P_3\left(\cos S_{10}'\right) = \frac{n_{☾}^2}{\epsilon\, a_{☾}}\, r^3\, P_3\left(\cos S_{10}'\right) = \frac{1}{2}\, \frac{n_{☾}^2}{\epsilon\, a_{☾}}\, r^3\left(5\cos^3 S_{10}' - 3\cos S_{10}'\right) . \tag{99}$$

If

$$A_\oplus = \frac{1}{2}\left[(1 + \cos I)\cos(g + h) + (1 - \cos I)\cos(g - h)\right]$$

and

$$B_\oplus = \frac{1}{2}\left[(1 + \cos I)\sin(g + h) + (1 - \cos I)\sin(g - h)\right] ,$$

then

$$\cos S_{10}' = A_\oplus \cos f - B_\oplus \sin f ,$$

so that

$$F_3\,(\text{earth}) = \frac{1}{2}\frac{n_{\mathbb{C}}^2}{\epsilon\, a_{\mathbb{C}}}\, r^3 \Big[5\big(A_\oplus^3 \cos^3 f - 3A_\oplus^2 B_\oplus \cos^2 f \sin f$$

$$+ 3A_\oplus B_\oplus^2 \cos f \sin^2 f - B_\oplus^3 \sin^3 f\big) - 3\big(A_\oplus \cos f - B_\oplus \sin f\big)\Big] \cdot \quad (100)$$

It follows that

$$F_3'\,(\text{earth}) = -\frac{5}{16}\frac{n_{\mathbb{C}}^2\, a'^3\, e'}{\epsilon\, a_{\mathbb{C}}}\left\{5\left[\left(3 + 4e'^2\right) A_\oplus'^3 + 3A_\oplus' B_\oplus'^2 \left(1 - e'^2\right)\right] - 3A_\oplus' \left(4 + 3e'^2\right)\right\}, \quad (101)$$

where

$$A_\oplus'^3 = \frac{1}{32}\Big[3\left(1-\theta'^2\right)(1-\theta')\cos(g'-3h') + (1-\theta')^3\cos(3g'-3h')$$

$$+ 3\left(1-\theta'^2\right)(1-\theta')\cos(3g'-h') + 3(1-\theta')\left(3+2\theta'+3\theta'^2\right)\cos(g'-h')$$

$$+ 3(1+\theta')\left(3-2\theta'+3\theta'^2\right)\cos(g'+h') + 3\left(1-\theta'^2\right)(1+\theta')\cos(3g'+h')$$

$$+ 3\left(1-\theta'^2\right)(1+\theta')\cos(g'+3h') + (1+\theta')^3\cos(3g'+3h')\Big],$$

$$A_\oplus' = \frac{1}{2}(1+\theta')\cos(g'+h') + \frac{1}{2}(1-\theta')\cos(g'-h'),$$

and

$$A_\oplus' B_\oplus'^2 = \frac{1}{32}\Big[(1+\theta')\left(3-2\theta'+3\theta'^2\right)\cos(g'+h')$$

$$+ (1-\theta')\left(3+2\theta'+3\theta'^2\right)\cos(g'-h') - 3(1-\theta')\left(1-\theta'^2\right)\cos(3g'-h')$$

$$+ (1-\theta')\left(1-\theta'^2\right)\cos(g'-3h') - (1-\theta')^3\cos(3g'-3h')$$

$$- (1+\theta')^3\cos(3g'+3h') - 3(1+\theta')\left(1-\theta'^2\right)\cos(3g'+h')$$

$$+ (1+\theta')\left(1-\theta'^2\right)\cos(g'+3h')\Big].$$

The contributions to F_3'' and to S_3' are both zero. The contribution to S_2' is then given by

$$S_2'\,(\text{earth}) = -\frac{1}{n_{\mathbb{C}}^*}\int F_3'\,(\text{earth})\, dh',$$

or

$$S_2'\,(\text{earth}) = \frac{5}{1536}\left(\frac{n_☾}{n_☾^*}\right)\frac{n_☾\, a''^3 e''}{a_☾\, \epsilon}\Big[-9(1+\cos I'')\left(1+10\cos I''-15\cos^2 I''\right)\left(4+3e''^2\right)\sin(g''+h'')$$

$$+9(1-\cos I'')\left(1-10\cos I''-15\cos^2 I''\right)\left(4+3e''^2\right)\sin(g''-h'')$$

$$+15\sin^2 I''(1+\cos I'')\left(4+3e''^2\right)\sin(g''+3h'')-15\sin^2 I''(1-\cos I'')\left(4+3e''^2\right)\sin(g''-3h'')$$

$$+315\sin^2 I''(1+\cos I'')e''^2\sin(3g''+h'')-315\sin^2 I''(1-\cos I'')e''^2\sin(3g''-h'')$$

$$+35(1+\cos I'')^3 e''^2\sin(3g''+3h'')-35(1-\cos I'')^3 e''^2\sin(3g''-3h'')\Big]\,. \quad (102)$$

THE ECCENTRICITY OF THE MOON'S ORBIT

The correction for the eccentricity of the moon's orbit is given by

$$F_3\left(e_☾\right) = \frac{n_☾^2}{\epsilon}\left[\left(\frac{a_☾}{r_☾}\right)^3 - 1\right] r^2 P_2\left(\cos\tilde{S}_{10}\right)\,, \quad (103)$$

or, keeping only the first power of $e_☾$,

$$F_3\left(e_☾\right) = \frac{3}{2}\, e_☾ \frac{n_☾^2}{\epsilon}\, r^2 \cos l_\oplus P_2\left(\cos\tilde{S}_{10}\right)$$

$$= \frac{3}{4}\frac{e_☾ n_☾^2}{\epsilon}\, r^2\left(3\cos^2\tilde{S}_{10}-1\right)\cos l_\oplus\,, \quad (104)$$

where the mean anomaly $l_\oplus$ is $l_\oplus = n_☾ t + \text{const}$. Then,

$$F_3'\left(e_☾\right) = \frac{3}{64}\frac{n_☾^2 a'^2}{\epsilon}\, e_☾\Big[-2\left(2+3e'^2\right)\left(1-3\cos^2 I'\right)+30e'^2\sin^2 I'\cos 2g'+6\left(2+3e'^2\right)\sin^2 I'\cos 2h'$$

$$+15e'^2(1+\cos I')^2\cos(2g'+2h')+15e'^2(1-\cos I')^2\cos(2g'-2h')\Big]\cos l_\oplus\,. \quad (105)$$

There are no contributions to S_2' or F_3''. The contribution to S_3' is

$$S_3'\left(e_☾\right) = \frac{3}{64}\frac{n_☾\, a''^2 e_☾}{\epsilon}\Big[-2\left(2+3e''^2\right)\left(1-3\cos^2 I''\right)+30e''^2\sin^2 I''\cos 2g''+6\left(2+3e''^2\right)\sin^2 I''\cos 2h''$$

$$+15e''^2(1+\cos I'')^2\cos(2g''+2h'')+15e''^2(1-\cos I'')^2\cos(2g''-2h'')\Big]\sin l_\oplus\,. \quad (106)$$

It is worthwhile to note that, although this result is a part of a third-order generating function, its actual order is $(n_{\mathbb{C}}/n)\, e_{\mathbb{C}}$, which is considered to be a second-order quantity, because a small divisor $n_{\mathbb{C}}$ is introduced through the integration.

THE INCLINATION OF THE MOON'S ORBIT TO ITS EQUATOR

The expression for the earth's perturbation is

$$\frac{n_{\mathbb{C}}^2 r^2}{\epsilon} P_2\left(\cos S_{10}'\right) ,$$

where $\cos S_{10}' = K\left(i_{\mathbb{C}}\right)$. By a Taylor expansion, if $\sin i_{\mathbb{C}}$ is small,

$$\cos S_{10}' \simeq K(0) + \left[\frac{\partial K\left(i_{\mathbb{C}}\right)}{\partial\left(\sin i_{\mathbb{C}}\right)}\right]_{i_{\mathbb{C}}=0} \sin i_{\mathbb{C}} = \cos \tilde{S}_{10}' + \left[\frac{\partial \cos S_{10}'}{\cos i_{\mathbb{C}}\, \partial i_{\mathbb{C}}}\right]_{i_{\mathbb{C}}=0} \sin i_{\mathbb{C}} . \qquad (107)$$

Thus the correction to be introduced is

$$F_3\left(i_{\mathbb{C}}\right) = \frac{3n_{\mathbb{C}}^2 r^2}{\epsilon} \left[\frac{\partial \cos S_{10}'}{\cos i_{\mathbb{C}}\, \partial i_{\mathbb{C}}}\right]_{i_{\mathbb{C}}=0} \sin i_{\mathbb{C}} \cos \tilde{S}_{10} . \qquad (108)$$

It is easily shown that the bracketed quantity in Equation 108 is

$$\sin I \sin v_{\oplus} \sin(f + g) .$$

Thus,

$$F_3\left(i_{\mathbb{C}}\right) = 3\frac{n_{\mathbb{C}}^2 r^2}{\epsilon} \cos \tilde{S}_{10}' \sin(f+g) \sin v_{\oplus} \sin I \sin i_{\mathbb{C}} . \qquad (109)$$

Now,

$$\cos \tilde{S}_{10}' \sin(f+g) = \sin(f+g)\cos(f+g)\cos h - \sin^2(f+g)\sin h \cos I .$$

Therefore, the part of $F_3\left(i_{\mathbb{C}}\right)$ which is free from short-period terms is

$$F_3'\left(i_{\mathbb{C}}\right) = \frac{3n_{\mathbb{C}}^2 a'^2}{8\epsilon} \sin v_{\oplus} \sin I' \sin i_{\mathbb{C}} \Big[-2\left(2+3e'^2\right)\cos I' \sin h'$$

$$+ 5e'^2(1+\cos I')\sin(2g'+h') + 5e'^2(1-\cos I')\sin(2g'-h')\Big] . \qquad (110)$$

There is no contribution to F_3'' and S_2'. The contribution to S_3' is

$$S_3'\left(i_{\mathbb{C}}\right) = -\frac{3n_{\mathbb{C}}^2 a''^2}{8\epsilon\left(n_{\mathbb{C}} + N_{\omega_{\mathbb{C}}}\right)} \cos v_\oplus \sin I' \sin i_{\mathbb{C}} \Big[-2\left(2+3e''^2\right)\cos I'' \sin h''$$

$$+ 5e''^2\left(1+\cos I''\right)\sin\left(2g''+h''\right) + 5e''^2\left(1-\cos I''\right)\sin\left(2g''-h''\right)\Big] , \qquad (111)$$

where $N_{\omega_{\mathbb{C}}}$ is given by $\omega_{\mathbb{C}} = N_{\omega_{\mathbb{C}}} t + \text{const}$. Again, this is a second-order contribution, since $n_{\mathbb{C}}^2 \big/ \left(n_{\mathbb{C}} + N_{\omega_{\mathbb{C}}}\right) \simeq n_{\mathbb{C}}$.

THE NON-SPHERICITY OF THE POTENTIAL FIELD OF THE EARTH

Because of the fact that the earth's equator is not the reference plane, the form of the disturbing function due to the zonal harmonic coefficient j_2 of the earth will be derived from basic relations.

In Figure 5, E is the earth, M the moon, and S the orbiter. The disturbing function for the motion of S is given by

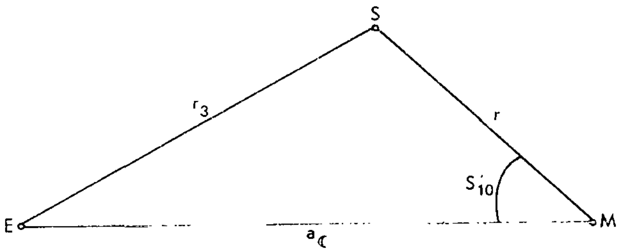

Figure 5–Earth-moon-orbiter configuration.

$$F_\oplus = \frac{k^2 m_\oplus}{r_3'}\left[1 - \frac{R_\oplus^2 j_2}{r_3'^2} P_2(\sin\varphi)\right] , \qquad (112)$$

where $R_\oplus$ is the equatorial radius of the earth and φ the latitude of the orbiter with respect to the earth's equator. The part $k^2 m_\oplus / r_3'$ has already been taken into account, and the rest is a third-order quantity. Therefore,

$$F_3(\oplus) = -\frac{k^2 m_\oplus}{r_3'^3} R_\oplus^2 j_2 P_2(\sin\varphi) . \qquad (113)$$

If the terms $\left(a/a_{\mathbb{C}}\right) j_2$ and $e_{\mathbb{C}}\, j_2$ are neglected, it follows that

$$F_3(\oplus) = -\frac{n_{\mathbb{C}}^2 R_\oplus^2 j_2}{\epsilon} P_2(\sin\varphi) , \qquad (114)$$

assuming $a_{\mathbb{C}} / r_3' = 1$. The angle φ must now be expressed in terms of the orbital elements referred to the lunar equator.

In Figure 6 the geometry of the problem is given. It follows that

$$\sin\varphi = \left(\sin I \cos I_\oplus - \cos\psi \cos I \sin I_\oplus\right)\sin(f+g) - \sin I_\oplus \sin\psi \cos(f+g) .$$

Then, if

$$\bar{A} = \sin I \cos I_{\oplus} - \cos\psi \cos I \sin I_{\oplus}$$

and

$$\bar{B} = -\sin I_{\oplus} \sin\psi ,$$

we have

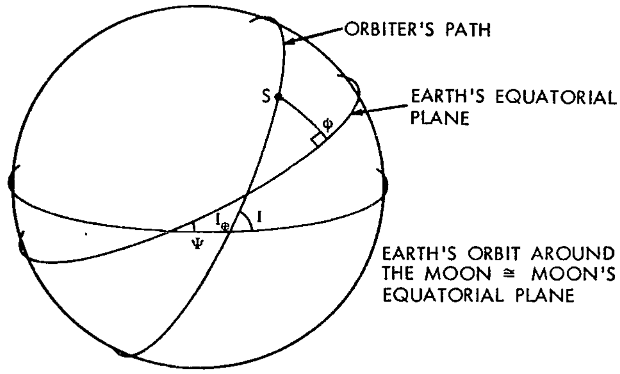

Figure 6—Planar configurations.

$$F_3(\oplus) = -\frac{n_{☾}^2 R_{\oplus}^2 j_2}{\epsilon}\left[\frac{3}{4}\left(\bar{A}^2+\bar{B}^2\right) - \frac{1}{2} + \frac{3}{4}\left(\bar{B}^2-\bar{A}^2\right)\cos 2(f+g) + \frac{3}{2}\bar{A}\bar{B}\sin 2(f+g)\right] . \tag{115}$$

Using the relation

$$dl = df + 2\sum_{k=1}^{\infty} k\beta^k\left(\frac{1}{k} + \sqrt{1-e^2}\right)\cos kf(-1)^k\, df ,$$

where

$$\beta = \frac{1}{e}\left(1-\sqrt{1-e^2}\right) ,$$

elimination of short-period terms from Equation 115 yields

$$F_3'(\oplus) = -\frac{n_{☾}^2 R_{☾}^2 j_2}{\epsilon}\left\{\frac{3}{4}\left(A^2+B^2\right) - \frac{1}{2} + \frac{3}{2}\beta'^2\left(\frac{1}{2}+\sqrt{1-e'^2}\right)\left[\left(\bar{B}^2-\bar{A}^2\right)\cos 2g' + 2\bar{A}\bar{B}\sin 2g'\right]\right\} . \tag{116}$$

where

$$\beta' = \beta(e') .$$

Furthermore, since $\psi = h + \lambda_{\oplus} - \bar{\Omega}'$, where $\bar{\Omega}'$ is the longitude of the descending node of the moon's equator on the earth's equator, we have

$$\begin{aligned}\bar{B}^2 + \bar{A}^2 = &\left(\sin^2 I' \cos^2 I_{\oplus} + \frac{1}{2}\cos^2 I' \sin^2 I_{\oplus} + \frac{1}{2}\sin^2 I_{\oplus}\right)\\ &- \frac{1}{2}\sin^2 I_{\oplus}\sin^2 I' \cos 2\left(h' + \lambda_{\oplus} - \bar{\Omega}'\right)\\ &- \frac{1}{2}\sin 2I_{\oplus}\sin 2I' \cos\left(h' + \lambda_{\oplus} - \bar{\Omega}'\right)\end{aligned}$$

$$\bar{B}^2 - \bar{A}^2 = \left(\frac{1}{2}\sin^2 I_\oplus - \sin^2 I'\cos^2 I_\oplus - \frac{1}{2}\cos^2 I'\sin^2 I_\oplus\right)$$

$$- \frac{1}{2}\sin^2 I_\oplus\left(1 + \cos^2 I'\right)\cos 2\left(h' + \lambda_\oplus - \bar{\Omega}'\right)$$

$$+ \frac{1}{2}\sin 2I'\sin 2I_\oplus \cos\left(h' + \lambda_\oplus - \bar{\Omega}'\right) ,$$

and

$$\bar{A}\bar{B} = \frac{1}{2}\sin^2 I_\oplus \cos I'\sin 2\left(h' + \lambda_\oplus - \bar{\Omega}'\right) - \frac{1}{2}\sin 2I_\oplus \sin I'\sin\left(h' + \lambda_\oplus - \bar{\Omega}'\right) .$$

Thus the contribution to S_2' is zero, and

$$F_3''(\oplus) = -\frac{1}{8}\frac{n_{☾}^2 j_2 R_\oplus^2}{\epsilon}\Big[-\left(1 - 3\cos^2 I''\right)$$

$$+ 3\beta''^2\left(1 + 2\sqrt{1 - e''^2}\right)\sin^2 I''\cos 2g''\Big]\left(1 - 3\cos^2 I_\oplus\right) . \qquad (117)$$

The contribution to S_3' will be given by the integration of

$$\frac{\partial S_3'(\oplus)}{\partial\tau} = -\frac{3}{16}\frac{n_{☾}^2 j_2 R_\oplus^2 \sin I_\oplus}{\epsilon}\Big\{-2\sin I''\left[\sin I''\sin I_\oplus \cos 2\left(h' + \lambda_\oplus - \bar{\Omega}'\right) + 4\cos I''\cos I_\oplus \cos\left(h' + \lambda_\oplus - \bar{\Omega}'\right)\right]$$

$$+ \beta''^2\left(1 + 2\sqrt{1 - e''^2}\right)\Big[-(1 + \cos I\ )^2\sin I_\oplus \cos 2\left(h' + g' + \lambda_\oplus - \bar{\Omega}'\right)$$

$$-(1 - \cos I'')^2\sin I_\oplus \cos 2\left(h' - g' + \lambda_\oplus - \bar{\Omega}'\right) + 4\sin I''(1 + \cos I'')\cos I_\oplus \cos\left(h' + 2g' + \lambda_\oplus - \bar{\Omega}'\right)$$

$$- 4\sin I'(1 - \cos I'')\cos I_\oplus \cos\left(h' - 2g' + \lambda_\oplus - \bar{\Omega}'\right)\Big]\Big\} . \qquad (118)$$

Since

$$\frac{\partial}{\partial\tau}\left(\lambda_\oplus - \bar{\Omega}'\right) = n_{☾}^* - N_{\Omega ☾} ,$$

we have

$$S_3'(\oplus) = -\frac{3}{32}\frac{n_{☾}^2 j_2 R_\oplus^2 \sin I_\oplus}{\epsilon\left(n_{☾}^* - N_{\Omega ☾}\right)}\Big\{-2\sin I''\left[\sin I''\sin I_\oplus \sin 2\left(h'' + \lambda_\oplus - \bar{\Omega}'\right) + 8\cos I''\cos I_\oplus \sin\left(h'' + \lambda_\oplus - \bar{\Omega}'\right)\right]$$

$$+\beta''^2\left(1+2\sqrt{1-e''^2}\right)\left[-(1+\cos I'')^2 \sin I_\oplus \sin 2\left(h''+g''+\lambda_\oplus-\bar{\Omega}'\right)\right.$$

$$-(1-\cos I'')^2 \sin I_\oplus \sin 2\left(h''-g''+\lambda_\oplus-\bar{\Omega}'\right)+8\sin I''(1+\cos I'')\cos I_\oplus \sin\left(h''+2g''+\lambda_\oplus-\bar{\Omega}'\right)$$

$$\left.\left.-8\sin I''(1-\cos I'')\cos I_\oplus \sin\left(h''-2g''+\lambda_\oplus-\bar{\Omega}'\right)\right]\right\} \cdot \quad (119)$$

PHYSICAL LIBRATION, AND THE PRECESSION OF THE LUNAR EQUATOR

Physical libration causes a periodic oscillation in the position of the lunar surface, thereby creating a small angular displacement between the principal lunar meridian and the earth-moon line of centers. The largest contribution to this displacement is given by

$$\chi = \alpha \sin \ell_\odot + \chi_0 \qquad \text{(see Reference 4, p. 316)} \qquad (120)$$

where

$$\alpha = -59'' = -2.86 \times 10^{-4} \text{ rad},$$

$$\chi_0 = \text{constant dependent upon the initial time,}$$

and

$$\ell_\odot = \text{mean anomaly of the sun.}$$

Let $\xi\,\eta\,\zeta$ and $x\,y\,z$ be selenocentric, equatorial coordinate systems with the ξ-axis directed toward the earth and the x-axis passing through the principal meridian (see Figure 7). Then,

$$\ddot{\xi} = \frac{\partial F}{\partial \xi}, \qquad \ddot{\eta} = \frac{\partial F}{\partial \eta},$$

$$x = \xi \cos\chi + \eta \sin\chi,$$

and

$$y = -\xi \sin\chi + \eta \cos\chi. \qquad (121)$$

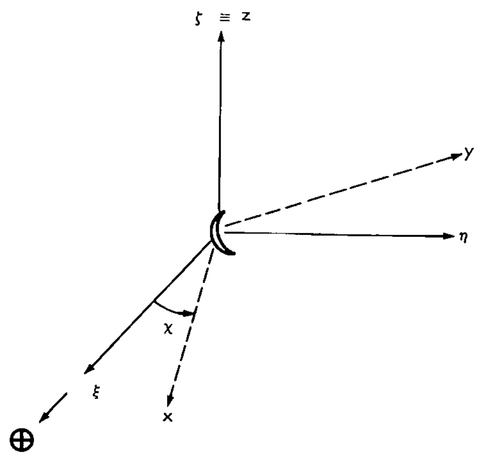

Figure 7—Selenocentric, equatorial coordinate systems.

Thus,

$$\ddot{x} = \ddot{\xi}\cos\chi + \ddot{\eta}\sin\chi + 2\alpha n_{\odot}\cos\ell_{\odot}(\dot{\eta}\cos\chi - \dot{\xi}\sin\chi)$$

$$- \alpha n_{\odot}^{2}\sin\ell_{\odot}(\eta\cos\chi - \xi\sin\chi) - \alpha^{2}n_{\odot}^{2}\cos^{2}\ell_{\odot}(\eta\sin\chi + \xi\cos\chi) \; , \quad (122)$$

where $n_{\odot}$ = mean motion of the sun's mean anomaly. But

$$\dot{\eta}\cos\chi - \dot{\xi}\sin\chi = \alpha n_{\odot} x\cos\ell_{\odot} + \dot{y}$$

and

$$\ddot{\xi}\cos\chi + \ddot{\eta}\sin\chi = \frac{\partial F}{\partial x} \; , \quad (123)$$

so Equation 122 becomes

$$\ddot{x} = \frac{\partial F}{\partial x} + 2\alpha n_{\odot}\cos\ell_{\odot}\left(\alpha n_{\odot} x\cos\ell_{\odot} + \dot{y}\right) - \alpha n_{\odot}^{2}\sin\ell_{\odot}(\eta\cos\chi - \xi\sin\chi) - \alpha^{2}n_{\odot}^{2}x\cos^{2}\ell_{\odot} \, . \quad (124)$$

If one neglects terms higher than third-order, Equation 124 reduces to

$$\ddot{x} = \frac{\partial F}{\partial x} + 2\alpha n_{\odot}\dot{y}\cos\ell_{\odot} \, . \quad (125)$$

A similar computation yields

$$\ddot{y} = \frac{\partial F}{\partial y} - 2\alpha n_{\odot}\dot{x}\cos\ell_{\odot} \, . \quad (126)$$

Thus,

$$\ddot{x} = \frac{\partial \bar{F}}{\partial x} \; , \qquad \ddot{y} = \frac{\partial \bar{F}}{\partial y} \; , \quad (127)$$

where

$$\bar{F} = F + 2\alpha n_{\odot}(x\dot{y} - \dot{x}y)\cos\ell_{\odot}$$

$$= F + 2\alpha n_{\odot} H\cos\ell_{\odot} \, . \quad (128)$$

Hence, the addition to the Hamiltonian of the problem is

$$F_3\,(\mathrm{lib.}) = 2\alpha n_{\odot} H \cos \ell_{\odot} \,. \tag{129}$$

Then,

$$F_{3_s}\,(\mathrm{lib.}) \equiv 0$$

$$F_{3ph'}\,(\mathrm{lib.}) \equiv 0$$

$$F_{3p\tau}\,(\mathrm{lib.}) = 2\alpha n_{\odot} H \cos \ell_{\odot} \,, \tag{130}$$

and therefore

$$\frac{\partial S_3'\,(\mathrm{lib.})}{\partial \tau} = 2\alpha n_{\odot} H \cos \ell_{\odot} \Longrightarrow S_3'\,(\mathrm{lib.}) = 2\alpha H \sin \ell_{\odot} \,. \tag{131}$$

Now, the precession in the lunar equator is created by the small angle of inclination (~1°32!1) between the lunar equatorial plane and the ecliptic. This results in the regression of the equatorial node in the ecliptic. However, the motion of the node is already implicit in the $n_{\mathbb{C}}^*$—hence no additional corrections need be made.

COMPLETE "SECULAR" THIRD-ORDER HAMILTONIAN

The complete secular third-order Hamiltonian is given by

$$F_3'' = F_3''\,(\mathrm{coupling}) + F_3''\,(\mathrm{sun}) + F_3''\,(\oplus) \,, \tag{132}$$

where F_3'' (coupling) is given by Equation 86, F_3'' (sun) by Equation 96, and F_3'' ($\oplus$) by Equation 117.

THE DETERMINING FUNCTION FOR LONG-PERIOD TERMS

Additional long-period perturbations depending on the motion of the node, the motion of the earth, and the motion of the sun will be given through the determining functions S_2' and S_3'. The determining functions S_2' and S_3' are

$$S_2' = S_2'\,(\mathrm{coupling}) + S_2'\,(\mathrm{earth}) \tag{133}$$

and

$$S_3' = S_3'(\text{radiation}) + S_3'(\text{sun}) + S_3'\left(e_{☾}\right) + S_3'\left(i_{☾}\right) + S_3'(\oplus) + S_3'(\text{libration}) \; ; \tag{134}$$

the various functions are defined by Equations 85, 92, 97, 102, 106, 111, 119, and 131.

In the next section, the partial derivatives needed in order to find additional perturbations in the canonical elements are given.

LONG-PERIOD PERTURBATIONS OF SECOND ORDER

The perturbations of long period are obtained from

$$l'' = \frac{\partial S}{\partial L''} = l' + \frac{\partial S_1'}{\partial L''} + \frac{\partial S_2'}{\partial L''} + \frac{\partial S_3'}{\partial L''}, \qquad L' = \frac{\partial S}{\partial l'} = L'',$$

$$g'' = \frac{\partial S}{\partial G''} = g' + \frac{\partial S_1'}{\partial G''} + \frac{\partial S_2'}{\partial G''} + \frac{\partial S_3'}{\partial G''}, \qquad G' = \frac{\partial S}{\partial g'} = G'' + \frac{\partial S_1'}{\partial g'} + \frac{\partial S_2'}{\partial g'} + \frac{\partial S_3'}{\partial g'},$$

$$h'' = \frac{\partial S}{\partial H''} = h' + \frac{\partial S_1'}{\partial H''} + \frac{\partial S_2'}{\partial H''} + \frac{\partial S_3'}{\partial H''}, \qquad H' = \frac{\partial S}{\partial h'} = H'' + \frac{\partial S_1'}{\partial h'} + \frac{\partial S_2'}{\partial h'} + \frac{\partial S_3'}{\partial h'} . \tag{135}$$

The terms corresponding to S_1' have already been obtained. Next the partial derivatives of S_2' and S_3' with respect to a'', e'', I'', g', and h' are computed. For convenience, however, the primes have been dropped in this section.

The "mean" mean motion n' (called n) depends on a' (called a) through the relation

$$n^2 = \frac{\mu}{a^3},$$

and

$$\frac{\partial S}{\partial L} = 2\sqrt{\frac{a}{\mu}}\,\frac{\partial S}{\partial a} + \frac{1 - e^2}{e\sqrt{\mu a}}\,\frac{\partial S}{\partial e},$$

$$\frac{\partial S}{\partial G} = -\frac{1}{e}\sqrt{\frac{1 - e^2}{\mu a}}\,\frac{\partial S}{\partial e} + \frac{\cot I}{\sqrt{\mu a\left(1 - e^2\right)}}\,\frac{\partial S}{\partial I},$$

and

$$\frac{\partial S}{\partial H} = - \frac{1}{\sin I \sqrt{\mu a \left(1 - e^2\right)}} \frac{\partial S}{\partial I} . \tag{136}$$

The following definitions are made:

$$C_1 = \left(1 - e^2\right)\left(2 - 17e^2\right) \sin^2 I \cos I$$

$$C_2 = \left(2 + 3e^2\right)^2 \sin^2 I \cos I$$

$$C_3 = e^2\left(1 - e^2\right)(1 + \cos I)^2 (2 - 3\cos I)$$

$$C_4 = e^2\left(1 - e^2\right)(1 - \cos I)^2 (2 + 3\cos I)$$

$$C_5 = e^2 \sin^2 I (1 + \cos I)\left[6e^2 - (1 + 5\cos I)\right]$$

$$C_6 = e^2 \sin^2 I (1 - \cos I)\left[6e^2 - (1 - 5\cos I)\right]$$

$$C_7 = e^2 (1 + \cos I)^3 \left[2e^2 - (1 + \cos I)\right]$$

$$C_8 = e^2 (1 - \cos I)^3 \left[2e^2 - (1 - \cos I)\right]$$

$$C_9 = \left(2 + 3e^2\right) \sin^2 I \cos I$$

$$C_{10} = e^2 (1 + \cos I)^2 \left(1 + 2\cos I - 5\cos^2 I\right)$$

$$C_{11} = e^2 (1 - \cos I)^2 \left(1 - 2\cos I - 5\cos^2 I\right)$$

$$C_{12} = \left(2 + 3e^2\right) \cos I$$

$$C_{13} = e^2 (1 + \cos I)(3 - 5\cos I)$$

$$C_{14} = e^2 (1 - \cos I)(3 + 5\cos I)$$

$$C_{15} = e^2 (1 + \cos I)(1 + 5\cos I)$$

$$C_{16} = e^2 (1 - \cos I)(1 - 5\cos I)$$

$$C_{17} = (1 + \cos I)\left[25\left(1 - \cos I - \cos^2 I - 7\cos^3 I\right) - \left(1 - e^2\right)\left(9 + 15\cos I + 15\cos^2 I - 175\cos^3 I\right)\right]$$

$$C_{18} = (1 - \cos I)\left[25\left(1 + \cos I - \cos^2 I + 7\cos^3 I\right) - \left(1 - e^2\right)\left(9 - 15\cos I + 15\cos^2 I + 175\cos^3 I\right)\right]$$

$$C_{19} = e^2 (1 + \cos I)^2 \left(1 + 2 \cos I - 7 \cos^2 I\right)$$

$$C_{20} = e^2 (1 - \cos I)^2 \left(1 - 2 \cos I - 7 \cos^2 I\right)$$

$$C_{21} = (1 + \cos I) \left(3 - 25 \cos I + 5 \cos^2 I + 25 \cos^3 I\right)$$

$$C_{22} = (1 - \cos I) \left(3 + 25 \cos I + 5 \cos^2 I - 25 \cos^3 I\right)$$

$$C_{23} = \sin^2 I \cos I \left[35 - 10\left(1 - e^2\right)\left(2 + 7 \cos^2 I\right) - 3\left(1 - e^2\right)^2 \left(1 - 14 \cos^2 I\right)\right]$$

$$C_{24} = e^2 (1 + \cos I)^2 \left[35\left(3 + 4 \cos I - 30 \cos^2 I - 20 \cos^3 I + 51 \cos^4 I\right) - 3\left(1 - e^2\right)\left(11 + 36 \cos I - 126 \cos^2 I - 196 \cos^3 I + 315 \cos^4 I\right)\right]$$

$$C_{25} = e^2 (1 - \cos I)^2 \left[35\left(3 - 4 \cos I - 30 \cos^2 I + 20 \cos^3 I + 51 \cos^4 I\right) - 3\left(1 - e^2\right)\left(11 - 36 \cos I - 126 \cos^2 I + 196 \cos^3 I + 315 \cos^4 I\right)\right]$$

$$C_{26} = e^4 \sin^2 I (1 + \cos I)^2 \left(1 + 2 \cos I - 9 \cos^2 I\right)$$

$$C_{27} = e^4 \sin^2 I (1 - \cos I)^2 \left(1 - 2 \cos I - 9 \cos^2 I\right)$$

$$C_{28} = \left(2 + 3e^2\right) \cos I \left(3 - 7 \cos^2 I\right)$$

$$C_{29} = e^2 (1 + \cos I) \left(3 - 19 \cos I - 7 \cos^2 I + 35 \cos^3 I\right)$$

$$C_{30} = e^2 (1 - \cos I) \left(3 + 19 \cos I - 7 \cos^2 I - 35 \cos^3 I\right)$$

$$C_{31} = (1 + \cos I) \left[35\left(15 + 41 \cos I - 262 \cos^2 I - 346 \cos^3 I + 519 \cos^4 I + 321 \cos^5 I\right) + 14\left(1 - e^2\right)\left(259 - 255 \cos I - 566 \cos^2 I + 1710 \cos^3 I - 525 \cos^4 I - 1455 \cos^5 I\right) - \left(1 - e^2\right)^2 \left(1559 - 1015 \cos I - 5446 \cos^2 I + 7350 \cos^3 I + 735 \cos^4 I - 5775 \cos^5 I\right)\right]$$

$$C_{32} = (1 - \cos I) \left[35\left(15 - 41 \cos I - 262 \cos^2 I + 346 \cos^3 I + 519 \cos^4 I - 321 \cos^5 I\right) + 14\left(1 - e^2\right)\left(259 + 255 \cos I - 566 \cos^2 I - 1710 \cos^3 I - 525 \cos^4 I + 1455 \cos^5 I\right) - \left(1 - e^2\right)^2 \left(1559 + 1015 \cos I - 5446 \cos^2 I - 7350 \cos^3 I + 735 \cos^4 I + 5775 \cos^5 I\right)\right]$$

$$C_{33} = (1+\cos I)^2 \Big[5\left(9 - 7\cos I - 119\cos^2 I + 15\cos^3 I + 174\cos^4 I\right)$$

$$- \left(1-e^2\right)\left(179 - 14\cos I - 1320\cos^2 I + 30\cos^3 I + 1605\cos^4 I\right)$$

$$+ 3\left(1-e^2\right)^2 \left(18 + 7\cos I - 135\cos^2 I - 15\cos^3 I + 165\cos^4 I\right)\Big]$$

$$C_{34} = (1-\cos I)^2 \Big[5\left(9 + 7\cos I - 119\cos^2 I - 15\cos^3 I + 174\cos^4 I\right)$$

$$- \left(1-e^2\right)\left(179 + 14\cos I - 1320\cos^2 I - 30\cos^3 I + 1605\cos^4 I\right)$$

$$+ 3\left(1-e^2\right)^2 \left(18 - 7\cos I - 135\cos^2 I + 15\cos^3 I + 165\cos^4 I\right)\Big]$$

$$C_{35} = e^2 \sin^2 I (1+\cos I)^2 \Big[1 - 9\cos^2 I - \left(1-e^2\right)\left(3 - 11\cos^2 I\right)\Big]$$

$$C_{36} = e^2 \sin^2 I (1-\cos I)^2 \Big[1 - 9\cos^2 I - \left(1-e^2\right)\left(3 - 11\cos^2 I\right)\Big]$$

$$C_{37} = \left(4+3e^2\right)(1+\cos I)(3-5\cos I)\left(29 - 126\cos^2 I + 105\cos^4 I\right)$$

$$C_{38} = \left(4+3e^2\right)(1-\cos I)(3+5\cos I)\left(29 - 126\cos^2 I + 105\cos^4 I\right)$$

$$C_{39} = e^2 \sin^2 I (1+\cos I)(3-5\cos I)\left(7 - 15\cos^2 I\right)$$

$$C_{40} = e^2 \sin^2 I (1-\cos I)(3+5\cos I)\left(7 - 15\cos^2 I\right) \tag{137}$$

Then, the partial derivatives are as follows:

$$\frac{\partial S_2' \,(\text{coupling})}{\partial a} = \frac{63}{8192\epsilon^2}\left(\frac{n_{☾}}{n}\right)^2 \left(\frac{n_{☾}}{n_{☾}^*}\right)^2 \frac{na}{\left(1-e^2\right)^{1/2}} \Big[32\,C_1 \sin 2h - 4C_2 \sin 4h + 80C_3 \sin(2g+2h)$$

$$+ 80C_4 \sin(2g-2h) + 10C_5 \sin(2g+4h) + 10C_6 \sin(2g-4h) + 25C_7 \sin(4g+4h) + 25C_8 \sin(4g-4h)\Big]$$

$$- \frac{27}{512\epsilon}\left(\frac{n_{☾}}{n_{☾}^*}\right)^2 J_2 b^2 \frac{n}{a\left(1-e^2\right)^2} \Big[4C_9 \sin 2h + 5C_{10} \sin(2g+2h) + 5C_{11} \sin(2g-2h)\Big]$$

$$+ \frac{27}{1024\epsilon}\left(\frac{n_{☾}}{n_{☾}^*}\right)^2 J_{22} b^2 \frac{n \sin^2 I}{a\left(1-e^2\right)^2} \Big[-8C_{12}(2\sin 2h - \sin 4h) - 20C_{13} \sin(2g+2h)$$

$$- 20C_{14} \sin(2g-2h) + 5C_{15} \sin(2g+4h) + 5C_{16} \sin(2g-4h)\Big]$$

$$- \frac{117}{16}\left(\frac{n}{n_{\mathbb{C}}^*}\right)^2 J_2 J_{22} b^4 \frac{n \sin^2 I \cos I}{a^3\left(1-e^2\right)^{7/2}} \sin 2h + \frac{117}{32}\left(\frac{n}{n_{\mathbb{C}}^*}\right)^2 J_{22}^2 b^4 \frac{n \sin^2 I \cos I}{a^3\left(1-e^2\right)^{7/2}} \sin 4h$$

$$- \frac{45}{2048\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)^2 J_3 b^3 \frac{ne \sin I}{a^2\left(1-e^2\right)^3}\Big[C_{17}\cos(g+2h) + C_{18}\cos(g-2h) - 25C_{19}\cos(3g+2h) - 25C_{20}\cos(3g-2h)\Big]$$

$$- \frac{135}{256}\left(\frac{n}{n_{\mathbb{C}}^*}\right)^2 J_3 J_{22} b^5 \frac{ne \sin I}{a^4\left(1-e^2\right)^{9/2}}\Big[C_{21}\cos(g+2h) + C_{22}\cos(g-2h)\Big]$$

$$- \frac{315}{16384\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)^2 J_4 b^4 \frac{n}{a^3\left(1-e^2\right)^4}\Big[8C_{23}\sin 2h + C_{24}\sin(2g+2h) + C_{25}\sin(2g-2h)$$

$$- 35C_{26}\sin(4g+2h) - 35C_{27}\sin(4g-2h)\Big]$$

$$- \frac{765}{1024}\left(\frac{n}{n_{\mathbb{C}}^*}\right)^2 J_4 J_{22} b^6 \frac{n \sin^2 I}{a^5\left(1-e^2\right)^{11/2}}\Big[4C_{28}\sin 2h + C_{29}\sin(2g+2h) + C_{30}\sin(2g-2h)\Big]$$

$$- \frac{405}{65536\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)^2 J_5 b^5 \frac{ne \sin I}{a^4\left(1-e^2\right)^5}\Big[C_{31}\cos(g+2h) + C_{32}\cos(g-2h) - 14C_{33}\cos(3g+2h)$$

$$- 14C_{34}\cos(3g-2h) + 105C_{35}\cos(5g+2h) + 105C_{36}\cos(5g-2h)\Big]$$

$$- \frac{855}{8192}\left(\frac{n}{n_{\mathbb{C}}^*}\right)^2 J_5 J_{22} b^7 \frac{ne \sin I}{a^6\left(1-e^2\right)^{13/2}}\Big[2C_{37}\cos(g+2h) + 2C_{38}\cos(g-2h)$$

$$- 7C_{39}\cos(3g+2h) - 7C_{40}\cos(3g-2h)\Big] \quad (138)$$

$$\frac{\partial S_2'\,(\text{coupling})}{\partial I} = \frac{9}{2048\epsilon^2}\left(\frac{n_{\mathbb{C}}}{n}\right)^2\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)^2 \frac{na^2 \sin I}{\sqrt{1-e^2}}\Big\{-16\left(1-3\cos^2 I\right)\left(1-e^2\right)\left(2-17e^2\right)\sin 2h$$

$$+ 2\left(1-3\cos^2 I\right)\left(2+3e^2\right)^2 \sin 4h - 40(1+\cos I)(1-9\cos I)\, e^2\left(1-e^2\right)\sin(2g+2h)$$

$$+ 40(1-\cos I)(1+9\cos I)\, e^2\left(1-e^2\right)\sin(2g-2h) + 10(1+\cos I)\, e^2\Big[(3-5\cos I)(1+2\cos I)$$

$$- 3e^2(1-3\cos I)\Big]\sin(2g+4h) - 10(1-\cos I)\, e^2\Big[(3+5\cos I)(1-2\cos I)$$

$$- 3e^2 (1 + 3\cos I)\Big] \sin(2g - 4h) + 25(1 + \cos I)^2 e^2 \left[2(1 + \cos I) - 3e^2\right] \sin(4g + 4h)$$

$$- 25(1 - \cos I)^2 e^2 \left[2(1 - \cos I) - 3e^2\right] \sin(4g - 4h)\Big\}$$

$$+ \frac{9}{64\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)^2 J_2 b^2 \frac{n \sin I}{\left(1 - e^2\right)^2}\Big[- \left(1 - 3\cos^2 I\right)\left(2 + 3e^2\right)\sin 2h$$

$$- 5(1 + \cos I)\left(1 - \cos I - 5\cos^2 I\right) e^2 \sin(2g + 2h) + 5(1 - \cos I)\left(1 + \cos I - 5\cos^2 I\right) e^2 \sin(2g - 2h)\Big]$$

$$- \frac{9}{256\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)^2 J_{22} b^2 \frac{n \sin I}{\left(1 - e^2\right)^2}\Big[4\left(1 - 3\cos^2 I\right)\left(2 + 3e^2\right)(2\sin 2h - \sin 4h)$$

$$- 20(1 + \cos I)\left(1 + 7\cos I - 10\cos^2 I\right) e^2 \sin(2g + 2h)$$

$$+ 20(1 - \cos I)\left(1 - 7\cos I - 10\cos^2 I\right) e^2 \sin(2g - 2h)$$

$$- 5(1 + \cos I)(1 + 2\cos I)(3 - 5\cos I) e^2 \sin(2g + 4h)$$

$$+ 5(1 - \cos I)(1 - 2\cos I)(3 + 5\cos I) e^2 \sin(2g - 4h)\Big]$$

$$- \frac{9}{8}\left(\frac{n}{n_{\mathbb{C}}^*}\right)^2 J_2 J_{22} b^4 \frac{n \sin I\left(1 - 3\cos^2 I\right)}{a^2\left(1 - e^2\right)^{7/2}} \sin 2h + \frac{9}{16}\left(\frac{n}{n_{\mathbb{C}}^*}\right)^2 J_{22}^2 b^4 \frac{n \sin I\left(1 - 3\cos^2 I\right)}{a^2\left(1 - e^2\right)^{7/2}} \sin 4h$$

$$+ \frac{9}{1024\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)^2 J_3 b^3 \frac{ne}{a\left(1 - e^2\right)^3}\Big\{(1 + \cos I)\Big[25\cos I\left(5 + 19\cos I + 3\cos^2 I - 35\cos^3 I\right)$$

$$+ \left(1 - e^2\right)\left(24 + 27\cos I - 555\cos^2 I - 235\cos^3 I + 875\cos^4 I\right)\Big] \cos(g + 2h)$$

$$+ (1 - \cos I)\Big[25\cos I\left(5 - 19\cos I + 3\cos^2 I + 35\cos^3 I\right)$$

$$- \left(1 - e^2\right)\left(24 - 27\cos I - 555\cos^2 I + 235\cos^3 I + 875\cos^4 I\right)\Big] \cos(g - 2h)$$

$$+ 25e^2 (1 + \cos I)^2\left(4 - 13\cos I - 22\cos^2 I + 35\cos^3 I\right)\cos(3g + 2h)$$

$$- 25e^2 (1 - \cos I)^2\left(4 + 13\cos I - 22\cos^2 I - 35\cos^3 I\right)\cos(3g - 2h)\Big\}$$

$$+\frac{9}{128}\left(\frac{n}{n_{\mathbb{C}}^{*}}\right)^{2} J_3 J_{22} b^5 \frac{ne}{a^3\left(1-e^2\right)^{9/2}}\left[(1+\cos I)\left(22+21\cos I-155\cos^2 I-5\cos^3 I+125\cos^4 I\right)\cos(g+2h)\right.$$

$$\left.-(1-\cos I)\left(22-21\cos I-155\cos^2 I+5\cos^3 I+125\cos^4 I\right)\cos(g-2h)\right]$$

$$-\frac{45}{4096\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^{*}}\right)^{2} J_4 b^4 \frac{n\sin I}{a^2\left(1-e^2\right)^4}\left\{4\left[35\left(1-3\cos^2 I\right)-10\left(1-e^2\right)\left(2+15\cos^2 I-35\cos^4 I\right)\right.\right.$$

$$\left.-3\left(1-e^2\right)^2\left(1-45\cos^2 I+70\cos^4 I\right)\right]\sin 2h+e^2(1+\cos I)\left[35\left(5-24\cos I-90\cos^2 I+52\cos^3 I+153\cos^4 I\right)\right.$$

$$\left.-3\left(1-e^2\right)\left(29-72\cos I-546\cos^2 I+140\cos^3 I+945\cos^4 I\right)\right]\sin(2g+2h)$$

$$-e^2(1-\cos I)\left[35\left(5+24\cos I-90\cos^2 I-52\cos^3 I+153\cos^4 I\right)\right.$$

$$\left.-3\left(1-e^2\right)\left(29+72\cos I-546\cos^2 I-140\cos^3 I+945\cos^4 I\right)\right]\sin(2g-2h)$$

$$-35e^4\left[(1+\cos I)^2\left(2-9\cos I-14\cos^2 I+27\cos^3 I\right)\sin(4g+2h)\right.$$

$$\left.\left.-(1-\cos I)^2\left(2+9\cos I-14\cos^2 I+27\cos^3 I\right)\sin(4g-2h)\right]\right\}$$

$$-\frac{45}{512}\left(\frac{n}{n_{\mathbb{C}}^{*}}\right)^{2} J_4 J_{22} b^6 \frac{n\sin I}{a^4\left(1-e^2\right)^{11/2}}\left[4\left(2+3e^2\right)\left(3-30\cos^2 I+35\cos^4 I\right)\sin 2h\right.$$

$$-2e^2(1+\cos I)\left(8+21\cos I-87\cos^2 I-35\cos^3 I+105\cos^4 I\right)\sin(2g+2h)$$

$$\left.+2e^2(1-\cos I)\left(8-21\cos I-87\cos^2 I+35\cos^3 I+105\cos^4 I\right)\sin(2g-2h)\right]$$

$$+\frac{45}{32768\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^{*}}\right)^{2} J_5 b^5 \frac{ne}{a^3\left(1-e^2\right)^5}\left\{-(1+\cos I)\left[35\left(56-513\cos I-1423\cos^2 I+2778\cos^3 I+3854\cos^4 I\right.\right.\right.$$

$$\left.-2793\cos^5 I-2247\cos^6 I\right)+14\left(1-e^2\right)\left(4-1905\cos I+5329\cos^2 I+1874\cos^3 I\right.$$

$$\left.-16350\cos^4 I+1695\cos^5 I+10185\cos^6 I\right)-\left(1-e^2\right)^2\left(544-15025\cos I+19649\cos^2 I\right.$$

$$\left.\left.+32074\cos^3 I-64890\cos^4 I-10185\cos^5 I+40425\cos^6 I\right)\right]\cos(g+2h)$$

$$+ (1 - \cos I) \Big[35 \big(56 + 513 \cos I - 1423 \cos^2 I - 2778 \cos^3 I$$

$$+ 3854 \cos^4 I + 2793 \cos^5 I - 2247 \cos^6 I \big) + 14 \left(1 - e^2\right) \left(4 + 1905 \cos I + 5329 \cos^2 I - 1874 \cos^3 I\right.$$

$$\left. - 16350 \cos^4 I - 1695 \cos^5 I + 10185 \cos^6 I\right) - \left(1 - e^2\right)^2 \left(544 + 15025 \cos I + 19649 \cos^2 I - 32074 \cos^3 I\right.$$

$$\left. - 64890 \cos^4 I + 10185 \cos^5 I + 40425 \cos^6 I\right) \Big] \cos (g - 2h)$$

$$+ 14(1 + \cos I)^2 \Big[5 \left(11 - 279 \cos I - 165 \cos^2 I + 1321 \cos^3 I + 258 \cos^4 I - 1218 \cos^5 I\right)$$

$$- \left(1 - e^2\right) \left(344 - 3205 \cos I - 2494 \cos^2 I + 13080 \cos^3 I + 3030 \cos^4 I - 11235 \cos^5 I\right)$$

$$+ 3 \left(1 - e^2\right)^2 \left(43 - 310 \cos I - 343 \cos^2 I + 1305 \cos^3 I + 420 \cos^4 I - 1155 \cos^5 I\right) \Big] \cos (3g + 2h)$$

$$- 14(1 - \cos I)^2 \Big[5 \left(11 + 279 \cos I - 165 \cos^2 I - 1321 \cos^3 I + 258 \cos^4 I + 1218 \cos^5 I\right)$$

$$- \left(1 - e^2\right) \left(344 + 3205 \cos I - 2494 \cos^2 I - 13080 \cos^3 I + 3030 \cos^4 I + 11235 \cos^5 I\right)$$

$$+ 3 \left(1 - e^2\right)^2 \left(43 + 310 \cos I - 343 \cos^2 I - 1305 \cos^3 I + 420 \cos^4 I + 1155 \cos^5 I\right) \Big] \cos (3g - 2h)$$

$$- 105 e^2 \sin^2 I (1 + \cos I)^2 \Big[2 - 23 \cos I - 18 \cos^2 I + 63 \cos^3 I$$

$$- \left(1 - e^2\right) \left(6 - 37 \cos I - 22 \cos^2 I + 77 \cos^3 I\right) \Big] \cos (5g + 2h)$$

$$+ 105 e^2 \sin^2 I (1 - \cos I)^2 \Big[2 + 23 \cos I - 18 \cos^2 I - 63 \cos^3 I$$

$$- \left(1 - e^2\right) \left(6 + 37 \cos \theta - 22 \cos^2 I - 77 \cos^3 I\right) \Big] \cos (5g - 2h) \Big\}$$

$$+ \frac{45}{4096} \left(\frac{n}{n_{\mathbb{C}}^*}\right)^2 J_5 J_{22} b^7 \frac{ne}{a^5 \left(1 - e^2\right)^{13/2}} \Big\{ 2 \left(4 + 3e^2\right) \Big[(1 + \cos I) \big(58 + 1075 \cos I - 1947 \cos^2 I$$

$$- 3402 \cos^3 I + 5460 \cos^4 I + 2415 \cos^5 I - 3675 \cos^6 I \big) \cos (g + 2h)$$

$$- (1 - \cos I) \left(58 - 1075 \cos I - 1947 \cos^2 I + 3402 \cos^3 I\right.$$

$$\left. + 5460 \cos^4 I - 2415 \cos^5 I - 3675 \cos^6 I\right) \cos (g - 2h)\Big]$$

$$- 7e^2 \sin^2 I \Big[(1 + \cos I) \left(14 + 209 \cos I - 355 \cos^2 I\right.$$

$$\left. - 345 \cos^3 I + 525 \cos^4 I\right) \cos (3g + 2h) - (1 - \cos I) \left(14 - 209 \cos I - 355 \cos^2 I\right.$$

$$\left. + 345 \cos^3 I + 525 \cos^4 I\right) \cos (3g - 2h)\Big]\Big\} , \quad (139)$$

$$\frac{\partial S_2' \text{(coupling)}}{\partial e} = \frac{9}{4096 \epsilon^2} \left(\frac{n_{☾}}{n}\right)^2 \left(\frac{n_{☾}}{n_{☾}^*}\right)^2 \frac{n a^2 e}{\left(1 - e^2\right)^{3/2}} \Big\{- 96 \sin^2 I \cos I \left(12 - 17e^2\right) \left(1 - e^2\right) \sin 2h$$

$$- 4 \sin^2 I \cos I \left(2 + 3e^2\right) \left(14 - 9e^2\right) \sin 4h + 80 (1 + \cos I)^2 (2 - 3 \cos I) \left(2 - 3e^2\right) \left(1 - e^2\right) \sin (2g + 2h)$$

$$+ 80 (1 - \cos I)^2 (2 + 3 \cos I) \left(2 - 3e^2\right) \left(1 - e^2\right) \sin (2g - 2h)$$

$$+ 10 \sin^2 I (1 + \cos I) \left[6e^2 \left(4 - 3e^2\right) - \left(2 - e^2\right) (1 + 5 \cos I)\right] \sin (2g + 4h)$$

$$+ 10 \sin^2 I (1 - \cos I) \left[6e^2 \left(4 - 3e^2\right) - \left(2 - e^2\right) (1 - 5 \cos I)\right] \sin (2g - 4h)$$

$$+ 25 (1 + \cos I)^3 \left[2e^2 \left(4 - 3e^2\right) - \left(2 - e^2\right) (1 + \cos I)\right] \sin (4g + 4h)$$

$$+ 25 (1 - \cos I)^3 \left[2e^2 \left(4 - 3e^2\right) - \left(2 - e^2\right) (1 - \cos I)\right] \sin (4g - 4h)\Big\}$$

$$+ \frac{9}{128 \epsilon} \left(\frac{n_{☾}}{n_{☾}^*}\right)^2 J_2 b^2 \frac{ne}{\left(1 - e^2\right)^3} \Big[4 \sin^2 I \cos I \left(7 + 3e^2\right) \sin 2h$$

$$+ 5 (1 + \cos I)^2 \left(1 + 2 \cos I - 5 \cos^2 I\right)\left(1 + e^2\right) \sin (2g + 2h)$$

$$+ 5 (1 - \cos I)^2 \left(1 - 2 \cos I - 5 \cos^2 I\right) \left(1 + e^2\right) \sin (2g - 2h)\Big]$$

$$- \frac{9}{256\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^{*}}\right)^{2} J_{22}\, b^{2} \frac{ne \sin^{2} I}{\left(1 - e^{2}\right)^{3}} \Big[- 8 \cos I \left(7 + 3e^{2}\right) (2 \sin 2h - \sin 4h)$$

$$- 20(1 + \cos I)(3 - 5 \cos I)\left(1 + e^{2}\right) \sin (2g + 2h)$$

$$- 20(1 - \cos I)(3 + 5 \cos I)\left(1 + e^{2}\right) \sin (2g - 2h)$$

$$+ 5(1 + \cos I)(1 + 5 \cos I)\left(1 + e^{2}\right) \sin (2g + 4h)$$

$$+ 5(1 - \cos I)(1 - 5 \cos I)\left(1 + e^{2}\right) \sin (2g - 4h)\Big]$$

$$+ \frac{63}{8}\left(\frac{n}{n_{\mathbb{C}}^{*}}\right)^{2} J_{2} J_{22}\, b^{4} \frac{ne \sin^{2} I \cos I}{a^{2}\left(1 - e^{2}\right)^{9/2}} \sin 2h - \frac{63}{16}\left(\frac{n}{n_{\mathbb{C}}^{*}}\right)^{2} J_{22}^{2}\, b^{4} \frac{ne \sin^{2} I \cos I}{a^{2}\left(1 - e^{2}\right)^{9/2}} \sin 4h$$

$$+ \frac{9}{1024\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^{*}}\right)^{2} J_{3}\, b^{3} \frac{n \sin I}{a\left(1 - e^{2}\right)^{4}} \Big\{(1 + \cos I)\Big[150\left(1 - \cos I - \cos^{2} I - 7 \cos^{3} I\right)$$

$$- \left(1 - e^{2}\right)\left(161 - 65 \cos I - 65 \cos^{2} I - 1575 \cos^{3} I\right) + 3\left(1 - e^{2}\right)^{2}\left(9 + 15 \cos I\right.$$

$$\left. + 15 \cos^{2} I - 175 \cos^{3} I\right)\Big] \cos (g + 2h) + (1 - \cos I)\Big[150\left(1 + \cos I - \cos^{2} I + 7 \cos^{3} I\right)$$

$$- \left(1 - e^{2}\right)\left(161 + 65 \cos I - 65 \cos^{2} I + 1575 \cos^{3} I\right)$$

$$+ 3\left(1 - e^{2}\right)^{2}\left(9 - 15 \cos I + 15 \cos^{2} I + 175 \cos^{3} I\right)\Big] \cos (g - 2h)$$

$$- 75e^{2}\left(1 + e^{2}\right)\Big[(1 + \cos I)^{2}\left(1 + 2 \cos I - 7 \cos^{2} I\right) \cos (3g + 2h)$$

$$+ (1 - \cos I)^{2}\left(1 - 2 \cos I - 7 \cos^{2} I\right) \cos (3g - 2h)\Big]\Big\}$$

$$+ \frac{9}{128}\left(\frac{n}{n_{\mathbb{C}}^{*}}\right)^{2} J_{3} J_{22}\, b^{5} \frac{n\left(1 + 8e^{2}\right) \sin I}{a^{3}\left(1 - e^{2}\right)^{11/2}} \Big[(1 + \cos I)\left(3 - 25 \cos I + 5 \cos^{2} I\right.$$

$$\left. + 25 \cos^{3} I\right) \cos (g + 2h) + (1 - \cos I)\left(3 + 25 \cos I + 5 \cos^{2} I - 25 \cos^{3} I\right) \cos (g - 2h)\Big]$$

$$+ \frac{45}{2048\epsilon} \left(\frac{n_{☾}}{n_{☾}^{*}}\right)^2 J_4 b^4 \frac{ne}{a^2(1-e^2)^5} \Big\{ 8 \sin^2 I \cos I \Big[70 - 15(1-e^2)(2 + 7\cos^2 I)$$

$$- 3(1-e^2)^2 (1 - 14\cos^2 I) \Big] \sin 2h$$

$$+ (1 + \cos I)^2 \Big[70 (3 + 4\cos I - 30 \cos^2 I - 20 \cos^3 I + 51 \cos^4 I)$$

$$- 3(1-e^2)(69 + 124 \cos I - 714 \cos^2 I - 644 \cos^3 I + 1365 \cos^4 I)$$

$$+ 3(1-e^2)^2 (11 + 36 \cos I - 126 \cos^2 I - 196 \cos^3 I + 315 \cos^4 I) \Big] \sin(2g + 2h)$$

$$+ (1 - \cos I)^2 \Big[70 (3 - 4\cos I - 30 \cos^2 I + 20 \cos^3 I + 51 \cos^4 I)$$

$$- 3(1-e^2)(69 - 124 \cos I - 714 \cos^2 I + 644 \cos^3 I + 1365 \cos^4 I)$$

$$+ 3(1-e^2)^2 (11 - 36 \cos I - 126 \cos^2 I + 196 \cos^3 I + 315 \cos^4 I) \Big] \sin(2g - 2h)$$

$$- 35e^2 (1 + e^2) \sin^2 I \Big[(1 + \cos I)^2 (1 + 2\cos I - 9\cos^2 I) \sin(4g + 2h)$$

$$+ (1 - \cos I)^2 (1 - 2\cos I - 9 \cos^2 I) \sin(4g - 2h) \Big] \Big\}$$

$$+ \frac{45}{512} \left(\frac{n}{n_{☾}^{*}}\right)^2 J_4 J_{22} b^6 \frac{ne \sin^2 I}{a^4(1-e^2)^{13/2}} \Big\{ 4(28 + 27e^2) \cos I (3 - 7\cos^2 I) \sin 2h$$

$$+ (2 + 9e^2) \Big[(1 + \cos I)(3 - 19 \cos I - 7\cos^2 I + 35 \cos^3 I) \sin(2g + 2h)$$

$$+ (1 - \cos I)(3 + 19\cos I - 7\cos^2 I - 35 \cos^3 I) \sin(2g - 2h) \Big] \Big\}$$

$$+ \frac{45}{32768\epsilon} \left(\frac{n_{☾}}{n_{☾}^{*}}\right)^2 J_5 b^5 \frac{n \sin I}{a^3(1-e^2)^6} \Big\{ (1 + \cos I) \Big[350 (15 + 41 \cos I - 262 \cos^2 I$$

$$- 346 \cos^3 I + 519 \cos^4 I + 321 \cos^5 I)$$

$$+ 7(1-e^2)(3469 - 5925 \cos I + 2734 \cos^2 I$$

$$+ 42930 \cos^3 I - 31755 \cos^4 I - 37725 \cos^5 I)$$

$$- 8\left(1 - e^2\right)^2 \left(4342 - 3885 \cos I - 11018 \cos^2 I + 26460 \cos^3 I\right.$$

$$\left.- 5880 \cos^4 I - 22155 \cos^5 I\right) + 5\left(1 - e^2\right)^3 \left(1559 - 1015 \cos I - 5446 \cos^2 I\right.$$

$$\left.\left.+ 7350 \cos^3 I + 735 \cos^4 I - 5775 \cos^5 I\right)\right] \cos (g + 2h)$$

$$+ (1 - \cos I) \left[350\left(15 - 41 \cos I - 262 \cos^2 I + 346 \cos^3 I + 519 \cos^4 I - 321 \cos^5 I\right)\right.$$

$$+ 7\left(1 - e^2\right) \left(3469 + 5925 \cos I + 2734 \cos^2 I - 42930 \cos^3 I\right.$$

$$\left.- 31755 \cos^4 I + 37725 \cos^5 I\right) - 8\left(1 - e^2\right)^2 \left(4342 + 3885 \cos I\right.$$

$$\left.- 11018 \cos^2 I - 26460 \cos^3 I - 5880 \cos^4 I + 22155 \cos^5 I\right)$$

$$+ 5\left(1 - e^2\right)^3 \left(1559 + 1015 \cos I - 5446 \cos^2 I - 7350 \cos^3 I\right.$$

$$\left.\left.+ 735 \cos^4 I + 5775 \cos^5 I\right)\right] \cos (g - 2h)$$

$$- 14(1 + \cos I)^2 \left[50\left(9 - 7 \cos I - 119 \cos^2 I + 15 \cos^3 I + 174 \cos^4 I\right)\right.$$

$$- \left(1 - e^2\right) \left(1837 - 427 \cos I - 15915 \cos^2 I + 915 \cos^3 I + 20670 \cos^4 I\right)$$

$$+ \left(1 - e^2\right)^2 \left(1577 + 28 \cos I - 11670 \cos^2 I - 60 \cos^3 I + 14205 \cos^4 I\right)$$

$$\left.- 15\left(1 - e^2\right)^3 \left(18 + 7 \cos I - 135 \cos^2 I - 15 \cos^3 I + 165 \cos^4 I\right)\right] \cos (3g + 2h)$$

$$- 14(1 - \cos I)^2 \left[50\left(9 + 7 \cos I - 119 \cos^2 I - 15 \cos^3 I + 174 \cos^4 I\right)\right.$$

$$- \left(1 - e^2\right) \left(1837 + 427 \cos I - 15915 \cos^2 I - 915 \cos^3 I + 20670 \cos^4 I\right)$$

$$+ \left(1 - e^2\right)^2 \left(1577 - 28 \cos I - 11670 \cos^2 I + 60 \cos^3 I + 14205 \cos^4 I\right)$$

$$- 15\left(1 - e^2\right)^3 \left(18 - 7\cos I - 135\cos^2 I + 15\cos^3 I + 165\cos^4 I\right)\Big] \cos(3g - 2h)$$

$$+ 105e^2 \sin^2 I \Big[10\left(1 - 9\cos^2 I\right) - \left(1 - e^2\right)\left(31 - 151\cos^2 I\right)$$

$$+ 5\left(1 - e^2\right)^2\left(3 - 11\cos^2 I\right)\Big]\Big[(1 + \cos I)^2 \cos(5g + 2h) + (1 - \cos I)^2 \cos(5g - 2h)\Big]\Big\}$$

$$+ \frac{45}{4096}\left(\frac{n}{n_{☾}^*}\right)^2 J_5 J_{22} b^7 \frac{n \sin I}{a^5\left(1 - e^2\right)^{15/2}} \Big\{2\left(4 + 57e^2 + 30e^4\right)\left(29 - 126\cos^2 I\right.$$

$$\left. + 105\cos^4 I\right)\Big[(1 + \cos I)(3 - 5\cos I)\cos(g + 2h) + (1 - \cos I)(3 + 5\cos I)\cos(g - 2h)\Big]$$

$$- 7e^2\left(3 + 10e^2\right)\sin^2 I\left(7 - 15\cos^2 I\right)\Big[(1 + \cos I)(3 - 5\cos I)\cos(3g + 2h)$$

$$+ (1 - \cos I)(3 + 5\cos I)\cos(3g - 2h)\Big]\Big\}, \quad (140)$$

$$\frac{\partial S_2'(\text{coupling})}{\partial g} = \frac{45}{1024\epsilon^2}\left(\frac{n_{☾}}{n}\right)^2\left(\frac{n_{☾}}{n_{☾}^*}\right)^2 \frac{na^2}{\left(1 - e^2\right)^{1/2}} \Big[8C_3 \cos(2g + 2h) + 8C_4 \cos(2g - 2h)$$

$$+ C_5 \cos(2g + 4h) + C_6 \cos(2g - 4h) + 5C_7 \cos(4g + 4h) + 5C_8 \cos(4g - 4h)\Big]$$

$$+ \frac{45}{128\epsilon}\left(\frac{n_{☾}}{n_{☾}^*}\right)^2 J_2 b^2 \frac{n}{\left(1 - e^2\right)^2}\Big[C_{10}\cos(2g + 2h) + C_{11}\cos(2g - 2h)\Big]$$

$$- \frac{45}{256\epsilon}\left(\frac{n_{☾}}{n_{☾}^*}\right)^2 J_{22} b^2 \frac{n \sin^2 I}{\left(1 - e^2\right)^2}\Big[-4C_{13}\cos(2g + 2h) - 4C_{14}\cos(2g - 2h)$$

$$+ C_{15}\cos(2g + 4h) + C_{16}\cos(2g - 4h)\Big]$$

$$- \frac{9}{1024\epsilon}\left(\frac{n_{☾}}{n_{☾}^*}\right)^2 J_3 b^3 \frac{ne \sin I}{a\left(1 - e^2\right)^3}\Big[C_{17}\sin(g + 2h) + C_{18}\sin(g - 2h)$$

$$- 75C_{19}\sin(3g + 2h) - 75C_{20}\sin(3g - 2h)\Big]$$

$$- \frac{9}{128}\left(\frac{n}{n_{☾}^*}\right)^2 J_3 J_{22} b^5 \frac{ne \sin I}{a^3\left(1 - e^2\right)^{9/2}}\Big[C_{21}\sin(g + 2h) + C_{22}\sin(g - 2h)\Big]$$

$$+ \frac{45}{4096\epsilon}\left(\frac{n_{☾}}{n_{☾}^{*}}\right)^{2} J_4 b^4 \frac{n}{a^2\left(1-e^2\right)^4}\Big[C_{24}\cos(2g+2h) + C_{25}\cos(2g-2h)$$

$$- 70C_{26}\cos(4g+2h) - 70C_{27}\cos(4g-2h)\Big]$$

$$+ \frac{45}{256}\left(\frac{n}{n_{☾}^{*}}\right)^{2} J_4 J_{22} b^6 \frac{n\sin^2 I}{a^4\left(1-e^2\right)^{11/2}}\Big[C_{29}\cos(2g+2h) + C_{30}\cos(2g-2h)\Big]$$

$$- \frac{45}{32768\epsilon}\left(\frac{n_{☾}}{n_{☾}}\right)^{2} J_5 b^5 \frac{ne\sin I}{a^3\left(1-e^2\right)^5}\Big[C_{31}\sin(g+2h) + C_{32}\sin(g-2h) - 42C_{33}\sin(3g+2h)$$

$$- 42C_{34}\sin(3g-2h) + 525C_{35}\sin(5g+2h) + 525C_{36}\sin(5g-2h)\Big]$$

$$- \frac{45}{4096}\left(\frac{n}{n_{☾}^{*}}\right)^{2} J_5 J_{22} b^7 \frac{ne\sin I}{a^5\left(1-e^2\right)^{13\cdot 2}}\Big[2C_{37}\sin(g+2h) + 2C_{38}\sin(g-2h)$$

$$- 21C_{39}\sin(3g+2h) - 21C_{40}\sin(3g-2h)\Big] \quad . \qquad (141)$$

$$\frac{\partial S_2'(\text{coupling})}{\partial h} = -\frac{9}{1024\epsilon^2}\left(\frac{n_{☾}}{n}\right)^{2}\left(\frac{n_{☾}}{n_{☾}^{*}}\right)^{2}\frac{na^2}{\left(1-e^2\right)^{1/2}}\Big[16C_1\cos 2h - 4C_2\cos 4h + 40C_3\cos(2g+2h)$$

$$- 40C_4\cos(2g-2h) + 10C_5\cos(2g+4h) - 10C_6\cos(2g-4h)$$

$$+ 25C_7\cos(4g+4h) - 25C_8\cos(4g-4h)\Big]$$

$$+ \frac{9}{128\epsilon}\left(\frac{n_{☾}}{n_{☾}^{*}}\right)^{2} J_2 b^2 \frac{n}{\left(1-e^2\right)^2}\Big[4C_9\cos 2h + 5C_{10}\cos(2g+2h) - 5C_{11}\cos(2g-2h)\Big]$$

$$- \frac{9}{128\epsilon}\left(\frac{n_{☾}}{n_{☾}^{*}}\right)^{2} J_{22} b^2 \frac{n\sin^2 I}{\left(1-e^2\right)^2}\Big[-8C_{12}(\cos 2h - \cos 4h) - 10C_{13}\cos(2g+2h) + 10C_{14}\cos(2g-2h)$$

$$+ 5C_{15}\cos(2g+4h) - 5C_{16}\cos(2g-4h)\Big]$$

$$+\frac{9}{4}\left(\frac{n}{n_{\mathbb{C}}^{*}}\right)^{2} J_2 J_{22} b^4 \frac{n \sin^2 I \cos I}{a^2 (1-e^2)^{7/2}} \cos 2h - \frac{9}{4}\left(\frac{n}{n_{\mathbb{C}}^{*}}\right)^{2} J_{22}^2 b^4 \frac{n \sin^2 I \cos I}{a^2 (1-e^2)^{7/2}} \cos 4h$$

$$-\frac{9}{512\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^{*}}\right)^{2} J_3 b^3 \frac{ne \sin I}{a(1-e^2)^3}\left[C_{17} \sin(g+2h) - C_{18} \sin(g-2h) - 25C_{19} \sin(3g+2h) + 25C_{20} \sin(3g-2h)\right]$$

$$-\frac{9}{64}\left(\frac{n}{n_{\mathbb{C}}^{*}}\right)^{2} J_3 J_{22} b^5 \frac{ne \sin I}{a^3 (1-e^2)^{9/2}}\left[C_{21} \sin(g+2h) - C_{22} \sin(g-2h)\right]$$

$$+\frac{45}{4096\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^{*}}\right)^{2} J_4 b^4 \frac{n}{a^2(1-e^2)^4}\left[8C_{23} \cos 2h + C_{24} \cos(2g+2h) - C_{25} \cos(2g-2h)\right.$$

$$\left. - 35C_{26} \cos(4g+2h) + 35C_{27} \cos(4g-2h)\right]$$

$$+\frac{45}{256}\left(\frac{n}{n_{\mathbb{C}}^{*}}\right)^{2} J_4 J_{22} b^6 \frac{n \sin^2 I}{a^4 (1-e^2)^{11/2}}\left[4C_{28} \cos 2h + C_{29} \cos(2g+2h) - C_{30} \cos(2g-2h)\right]$$

$$-\frac{45}{16384\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^{*}}\right)^{2} J_5 b^5 \frac{ne \sin I}{a^3(1-e^2)^5}\left[C_{31} \sin(g+2h) - C_{32} \sin(g-2h) - 14C_{33} \sin(3g+2h)\right.$$

$$\left. + 14C_{34} \sin(3g-2h) + 105C_{35} \sin(5g+2h) - 105C_{36} \sin(5g-2h)\right]$$

$$-\frac{45}{2048}\left(\frac{n}{n_{\mathbb{C}}^{*}}\right)^{2} J_5 J_{22} b^7 \frac{ne \sin I}{a^5(1-e^2)^{13/2}}\left[2C_{37} \sin(g+2h) - 2C_{38} \sin(g-2h)\right.$$

$$\left. - 7C_{39} \sin(3g+2h) + 7C_{40} \sin(3g-2h)\right] . \quad \textbf{(142)}$$

$$\frac{\partial S_2'(\text{earth})}{\partial a} = \frac{5}{512\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^{*}}\right)\frac{n_{\mathbb{C}} a^2 e}{a_{\mathbb{C}}}\left[-9(1+\cos I)\left(1+10\cos I - 15\cos^2 I\right)\left(4+3e^2\right)\sin(g+h)\right.$$

$$+ 9(1-\cos I)\left(1-10\cos I - 15\cos^2 I\right)\left(4+3e^2\right)\sin(g-h)$$

$$+ 15\sin^2 I(1+\cos I)\left(4+3e^2\right)\sin(g+3h)$$

$$- 15\sin^2 I(1-\cos I)\left(4+3e^2\right)\sin(g-3h)$$

$$+ 315 \sin^2 I (1 + \cos I) e^2 \sin(3g + h)$$

$$- 315 \sin^2 I (1 - \cos I) e^2 \sin(3g - h)$$

$$+ 35(1 + \cos I)^3 e^2 \sin(3g + 3h) - 35(1 - \cos I)^3 e^2 \sin(3g - 3h)\Big] , \quad (143)$$

$$\frac{\partial S_2' (\text{earth})}{\partial I} = \frac{5}{512\epsilon} \left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right) \frac{n_{\mathbb{C}}\, a^3 e \sin I}{a_{\mathbb{C}}} \Big[3\left(11 - 10 \cos I - 45 \cos^2 I\right) \left(4 + 3e^2\right) \sin(g + h)$$

$$+ 3\left(11 + 10 \cos I - 45 \cos^2 I\right) \left(4 + 3e^2\right) \sin(g - h)$$

$$- 5(1 + \cos I)(1 - 3 \cos I) \left(4 + 3e^2\right) \sin(g + 3h)$$

$$- 5(1 - \cos I)(1 + 3 \cos I) \left(4 + 3e^2\right) \sin(g - 3h)$$

$$- 105(1 + \cos I)(1 - 3 \cos I) e^2 \sin(3g + h)$$

$$- 105(1 - \cos I)(1 + 3 \cos I) e^2 \sin(3g - h)$$

$$- 35(1 + \cos I)^2 e^2 \sin(3g + 3h) - 35(1 - \cos I)^2 e^2 \sin(3g - 3h)\Big] , \quad (144)$$

$$\frac{\partial S_2' (\text{earth})}{\partial e} = \frac{5}{512\epsilon} \left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right) \frac{n_{\mathbb{C}}\, a^3}{a_{\mathbb{C}}} \Big[- 3(1 + \cos I) \left(1 + 10 \cos I - 15 \cos^2 I\right) \left(4 + 9e^2\right) \sin(g + h)$$

$$+ 3(1 - \cos I) \left(1 - 10 \cos I - 15 \cos^2 I\right) \left(4 + 9e^2\right) \sin(g - h)$$

$$+ 5 \sin^2 I (1 + \cos I) \left(4 + 9e^2\right) \sin(g + 3h)$$

$$- 5 \sin^2 I (1 - \cos I) \left(4 + 9e^2\right) \sin(g - 3h)$$

$$+ 315 \sin^2 I (1 + \cos I) e^2 \sin(3g + h)$$

$$- 315 \sin^2 I (1 - \cos I) e^2 \sin(3g - h)$$

$$+ 35(1 + \cos I)^3 e^2 \sin(3g + 3h) - 35(1 - \cos I)^3 e^2 \sin(3g - 3h)\Big] , \quad (145)$$

$$\frac{\partial S_2'(\text{earth})}{\partial g} = \frac{5}{512\epsilon}\left(\frac{n_{☾}}{n_{☾}^*}\right)\frac{n_{☾}a^3 e}{a_{☾}}\Big[-3(1+\cos I)\left(1+10\cos I-15\cos^2 I\right)\left(4+3e^2\right)\cos(g+h)$$

$$+3(1-\cos I)\left(1-10\cos I-15\cos^2 I\right)\left(4+3e^2\right)\cos(g-h)$$

$$+5\sin^2 I(1+\cos I)\left(4+3e^2\right)\cos(g+3h)$$

$$-5\sin^2 I(1-\cos I)\left(4+3e^2\right)\cos(g-3h)$$

$$+315\sin^2 I(1+\cos I)e^2\cos(3g+h)$$

$$-315\sin^2 I(1-\cos I)e^2\cos(3g-h)$$

$$+35(1+\cos I)^3 e^2\cos(3g+3h)-35(1-\cos I)^3 e^2\cos(3g-3h)\Big], \quad (146)$$

$$\frac{\partial S_2'(\text{earth})}{\partial h} = \frac{5}{512\epsilon}\left(\frac{n_{☾}}{n_{☾}^*}\right)\frac{n_{☾}a^3 e}{a_{☾}}\Big[-3\left(1+\cos I\right)\left(1+10\cos I-15\cos^2 I\right)\left(4+3e^2\right)\cos(g+h)$$

$$-3(1-\cos I)\left(1-10\cos I-15\cos^2 I\right)\left(4+3e^2\right)\cos(g-h)$$

$$+15\sin^2 I(1+\cos I)\left(4+3e^2\right)\cos(g+3h)$$

$$+15\sin^2 I(1-\cos I)\left(4+3e^2\right)\cos(g-3h)$$

$$+105\sin^2 I(1+\cos I)e^2\cos(3g+h)$$

$$+105\sin^2 I(1-\cos I)e^2\cos(3g-h)$$

$$+35(1+\cos I)^3 e^2\cos(3g+3h)+35(1-\cos I)^3 e^2\cos(3g-3h)\Big], \quad (147)$$

$$\frac{\partial S_3'(\text{radiation})}{\partial a} = \frac{3}{2}\frac{\sigma e}{n_{☾}^*-n_{\oplus}^*}\left[\cos^2\frac{I}{2}\sin\left(h+g+\lambda_{\oplus}-\lambda_{\odot}\right)+\sin^2\frac{I}{2}\sin\left(h-g+\lambda_{\oplus}-\lambda_{\odot}\right)\right], \quad (148)$$

$$\frac{\partial S_3'(\text{radiation})}{\partial I} = \frac{3}{4}\frac{\sigma a e\sin I}{n_{☾}^*-n_{\oplus}^*}\left[-\sin\left(h+g+\lambda_{\oplus}-\lambda_{\odot}\right)+\sin\left(h-g+\lambda_{\oplus}-\lambda_{\odot}\right)\right], \quad (149)$$

$$\frac{\partial S_3'\,(\text{radiation})}{\partial e} = \frac{3}{2}\,\frac{\sigma a}{n_{☾}^* - n_{\oplus}^*}\left[\cos^2\frac{I}{2}\,\sin\left(h + g + \lambda_{\oplus} - \lambda_{\odot}\right) + \sin^2\frac{I}{2}\,\sin\left(h - g + \lambda_{\oplus} - \lambda_{\odot}\right)\right]\ , \tag{150}$$

$$\frac{\partial S_3'\,(\text{radiation})}{\partial g} = \frac{3}{2}\,\frac{\sigma a e}{n_{☾}^* - n_{\oplus}^*}\left[\cos^2\frac{I}{2}\,\cos\left(h + g + \lambda_{\oplus} - \lambda_{\odot}\right) - \sin^2\frac{I}{2}\,\cos\left(h - g + \lambda_{\oplus} - \lambda_{\odot}\right)\right]\ , \tag{151}$$

$$\frac{\partial S_3'\,(\text{radiation})}{\partial h} = \frac{3}{2}\,\frac{\sigma a e}{n_{☾}^* - n_{\oplus}^*}\left[\cos^2\frac{I}{2}\,\cos\left(h + g + \lambda_{\oplus} - \lambda_{\odot}\right) + \sin^2\frac{I}{2}\,\cos\left(h - g + \lambda_{\oplus} - \lambda_{\odot}\right)\right]\ , \tag{152}$$

$$\frac{\partial S_3'\,(\text{sun})}{\partial a} = \frac{3}{32}\,\frac{n_{\oplus}^2\, a}{n_{☾}^* - n_{\oplus}^*}\Big[2\left(2 + 3e^2\right)\,\sin^2 I\,\sin 2\left(h + \lambda_{\oplus} - \lambda_{\odot}\right)$$

$$+ 5e^2\,(1 + \cos I)^2\,\sin 2\left(h + g + \lambda_{\oplus} - \lambda_{\odot}\right) + 5e^2\,(1 - \cos I)^2\,\sin 2\left(h - g + \lambda_{\oplus} - \lambda_{\odot}\right)\Big]\ , \tag{153}$$

$$\frac{\partial S_3'\,(\text{sun})}{\partial I} = \frac{3}{32}\,\frac{n_{\oplus}^2\, a^2\,\sin I}{n_{☾}^* - n_{\oplus}^*}\Big[2\left(2 + 3e^2\right)\,\cos I\,\sin 2\left(h + \lambda_{\oplus} - \lambda_{\odot}\right)$$

$$- 5e^2\,(1 + \cos I)\,\sin 2\left(h + g + \lambda_{\oplus} - \lambda_{\odot}\right) + 5e^2\,(1 - \cos I)\,\sin 2\left(h - g + \lambda_{\oplus} - \lambda_{\odot}\right)\Big]\ , \tag{154}$$

$$\frac{\partial S_3'\,(\text{sun})}{\partial e} = \frac{3}{32}\,\frac{n_{\oplus}^2\, a^2\, e}{n_{☾}^* - n_{\oplus}^*}\Big[6\,\sin^2 I\,\sin 2\left(h + \lambda_{\oplus} - \lambda_{\odot}\right)$$

$$+ 5(1 + \cos I)^2\,\sin 2\left(h + g + \lambda_{\oplus} - \lambda_{\odot}\right) + 5(1 - \cos I)^2\,\sin 2\left(h - g + \lambda_{\oplus} - \lambda_{\odot}\right)\Big]\ , \tag{155}$$

$$\frac{\partial S_3'\,(\text{sun})}{\partial g} = \frac{15}{32}\,\frac{n_{\oplus}^2\, a^2\, e^2}{n_{☾}^* - n_{\oplus}^*}\left[(1 + \cos I)^2\cos\,2\left(h + g + \lambda_{\oplus} - \lambda_{\odot}\right) - (1 - \cos I)^2\cos\,2\left(h - g + \lambda_{\oplus} - \lambda_{\odot}\right)\right]\ , \tag{156}$$

$$\frac{\partial S_3'\,(\text{sun})}{\partial h} = \frac{3}{32}\,\frac{n_{\oplus}^2\, a^2}{n_{☾}^* - n_{\oplus}^*}\Big[2\left(2 + 3e^2\right)\,\sin^2 I\,\cos 2\left(h + \lambda_{\oplus} - \lambda_{\odot}\right)$$

$$+ 5e^2\,(1 + \cos I)^2\,\cos 2\left(h + g + \lambda_{\oplus} - \lambda_{\odot}\right) + 5e^2\,(1 - \cos I)^2\,\cos 2\left(h - g + \lambda_{\oplus} - \lambda_{\odot}\right)\Big]\ , \tag{157}$$

$$\frac{\partial S_3'\,(e_{☾})}{\partial a} = \frac{3}{32\epsilon}\,n_{☾}\,e_{☾}\,a\Big[-2\left(2 + 3e^2\right)\,\left(1 - 3\cos^2 I\right) + 30e^2\,\sin^2 I\,\cos 2g$$

$$+ 6\left(2 + 3e^2\right) \sin^2 I \cos 2h + 15e^2 (1 + \cos I)^2 \cos (2g + 2h)$$

$$+ 15e^2 (1 - \cos I)^2 \cos (2g - 2h)\Big] \sin l_{\oplus} \,, \qquad \textbf{(158)}$$

$$\frac{\partial S_3'\left(e_{☾}\right)}{\partial I} = \frac{9}{32\epsilon} n_{☾} e_{☾} a^2 \sin I \Big[- 2\left(2 + 3e^2\right) \cos I + 10e^2 \cos I \cos 2g$$

$$+ 2\left(2 + 3e^2\right) \cos I \cos 2h - 5e^2 (1 + \cos I) \cos (2g + 2h)$$

$$+ 5e^2 (1 - \cos I) \cos (2g - 2h)\Big] \sin l_{\oplus} \,, \qquad \textbf{(159)}$$

$$\frac{\partial S_3'\left(e_{☾}\right)}{\partial e} = \frac{9}{32\epsilon} n_{☾} e_{☾} a^2 e \Big[- 2(1 - 3\cos^2 I) + 10 \sin^2 I \cos 2g + 6 \sin^2 I \cos 2h$$

$$+ 5(1 + \cos I)^2 \cos (2g + 2h) + 5(1 - \cos I)^2 \cos (2g - 2h)\Big] \sin l_{\oplus} \,, \qquad \textbf{(160)}$$

$$\frac{\partial S_3'\left(e_{☾}\right)}{\partial g} = - \frac{45}{32\epsilon} n_{☾} e_{☾} a^2 e^2 \Big[2 \sin^2 I \sin 2g + (1 + \cos I)^2 \sin (2g + 2h)$$

$$+ (1 - \cos I)^2 \sin (2g - 2h)\Big] \sin l_{\oplus} \,, \qquad \textbf{(161)}$$

$$\frac{\partial S_3'\left(e_{☾}\right)}{\partial h} = - \frac{9}{32\epsilon} n_{☾} e_{☾} a^2 \Big[2\left(2 + 3e^2\right) \sin^2 I \sin 2h + 5e^2 (1 + \cos I)^2 \sin (2g + 2h)$$

$$- 5e^2 (1 - \cos I)^2 \sin (2g - 2h)\Big] \sin l_{\oplus} \,, \qquad \textbf{(162)}$$

$$\frac{\partial S_3'\left(i_{☾}\right)}{\partial a} = - \frac{3}{4\epsilon} \frac{n_{☾}^2 a \sin I \sin i_{☾}}{n_{☾} + N_{\omega ☾}} \Big[- 2\left(2 + 3e^2\right) \cos I \sin h$$

$$+ 5e^2 (1 + \cos I) \sin (2g + h) + 5e^2 (1 - \cos I) \sin (2g - h)\Big] \cos v_{\oplus} \,, \qquad \textbf{(163)}$$

$$\frac{\partial S_3'\left(i_{☾}\right)}{\partial I} = - \frac{3}{8\epsilon} \frac{n_{☾}^2 a^2 \sin i_{☾}}{n_{☾} + N_{\omega ☾}} \Big[2\left(2 + 3e^2\right)\left(1 - 2\cos^2 I\right) \sin h$$

$$- 5e^2 (1 + \cos I)(1 - 2\cos I) \sin (2g + h) + 5e^2 (1 - \cos I)(1 + 2\cos I) \sin (2g - h)\Big] \cos v_{\oplus} \,, \qquad \textbf{(164)}$$

$$\frac{\partial S_3'\left(i_☾\right)}{\partial e} = -\frac{3}{4\epsilon}\,\frac{n_☾^2\, a^2\, e \sin I \sin i_☾}{n_☾ + N_{\omega ☾}}\Big[-6 \cos I \sin h + 5(1+\cos I)\sin(2g+h)$$

$$+\,5(1-\cos I)\sin(2g-h)\Big]\cos v_{\oplus}\ , \tag{165}$$

$$\frac{\partial S_3'\left(i_☾\right)}{\partial g} = -\frac{15}{4\epsilon}\,\frac{n_☾^2\, a^2\, e^2 \sin I \sin i_☾}{n_☾ + N_{\omega ☾}}\Big[(1+\cos I)\cos(2g+h) + (1-\cos I)\cos(2g-h)\Big]\cos v_{\oplus}\ , \tag{166}$$

$$\frac{\partial S_3'\left(i_☾\right)}{\partial h} = -\frac{3}{8\epsilon}\,\frac{n_☾^2\, a^2 \sin I \sin i_☾}{n_☾ + N_{\omega ☾}}\Big[-2\left(2+3e^2\right)\cos I \cos h$$

$$+\,5e^2(1+\cos I)\cos(2g+h) - 5e^2(1-\cos I)\cos(2g-h)\Big]\cos v_{\oplus}\ , \tag{167}$$

$$\frac{\partial S_3'(\oplus)}{\partial a} = 0\ , \tag{168}$$

$$\frac{\partial S_3'(\oplus)}{\partial I} = -\frac{3}{16\epsilon}\,\frac{n_☾^2\, j_2\, R_{\oplus}^2 \sin I_{\oplus}}{n_☾^{*} - N_{\Omega ☾}}\Big\{-2\Big[\sin I \cos I \sin I_{\oplus} \sin 2\left(h+\lambda_{\oplus}-\bar{\Omega}'\right)$$

$$-\,4\left(1-2\cos^2 I\right)\cos I_{\oplus} \sin\left(h+\lambda_{\oplus}-\bar{\Omega}'\right)\Big]$$

$$+\,\beta^2\left(1+2\sqrt{1-e^2}\right)\Big[\sin I(1+\cos I)\sin I_{\oplus}\sin 2\left(h+g+\lambda_{\oplus}-\bar{\Omega}'\right)$$

$$-\,\sin I(1-\cos I)\sin I_{\oplus}\sin 2\left(h-g+\lambda_{\oplus}-\bar{\Omega}'\right)$$

$$-\,4(1+\cos I)(1-2\cos I)\cos I_{\oplus}\sin\left(h+2g+\lambda_{\oplus}-\bar{\Omega}'\right)$$

$$-\,4(1-\cos I)(1+2\cos I)\cos I_{\oplus}\sin\left(h-2g+\lambda_{\oplus}-\bar{\Omega}'\right)\Big]\Big\}\ , \tag{169}$$

$$\frac{\partial S_3'(\oplus)}{\partial e} = -\frac{3}{16\epsilon}\,\frac{n_☾^2\, j_2\, R_{\oplus}^2\, \beta^2 \sin I_{\oplus}\left(2+\sqrt{1-e^2}\right)}{e\left(n_☾^{*} - N_{\Omega ☾}\right)}\Big[-(1+\cos I)^2 \sin I_{\oplus}\sin 2\left(h+g+\lambda_{\oplus}-\bar{\Omega}'\right)$$

$$-\,(1-\cos I)^2 \sin I_{\oplus}\sin 2\left(h-g+\lambda_{\oplus}-\bar{\Omega}'\right)$$

$$+ 8 \sin I(1 + \cos I) \cos I_{\oplus} \sin\left(h + 2g + \lambda_{\oplus} - \bar{\Omega}'\right)$$

$$- 8 \sin I(1 - \cos I) \cos I_{\oplus} \sin\left(h - 2g + \lambda_{\oplus} - \bar{\Omega}'\right)\Big] \ , \quad \textbf{(170)}$$

$$\frac{\partial S_3'(\oplus)}{\partial g} = - \frac{3}{16\epsilon} \frac{n_{\mathbb{C}}^2 j_2 R_{\oplus}^2 \beta^2 \sin I_{\oplus} \left(1 + 2\sqrt{1 - e^2}\right)}{e\left(n_{\mathbb{C}}^* - N_{\Omega\mathbb{C}}\right)} \Big[- (1 + \cos I)^2 \sin I_{\oplus} \cos 2\left(h + g + \lambda_{\oplus} - \bar{\Omega}'\right)$$

$$+ (1 - \cos I)^2 \sin I_{\oplus} \cos 2\left(h - g + \lambda_{\oplus} - \bar{\Omega}'\right)$$

$$+ 8 \sin I(1 + \cos I) \cos I_{\oplus} \cos\left(h + 2g + \lambda_{\oplus} - \bar{\Omega}'\right)$$

$$+ 8 \sin I(1 - \cos I) \cos I_{\oplus} \cos\left(h - 2g + \lambda_{\oplus} - \bar{\Omega}'\right)\Big] \ , \quad \textbf{(171)}$$

$$\frac{\partial S_3'(\oplus)}{\partial h} = - \frac{3}{16\epsilon} \frac{n_{\mathbb{C}}^2 j_2 R_{\oplus}^2 \sin I_{\oplus}}{n_{\mathbb{C}}^* - N_{\Omega\mathbb{C}}} \Big\{- 2 \sin I \Big[\sin I \sin I_{\oplus} \cos 2\left(h + \lambda_{\oplus} - \bar{\Omega}'\right)$$

$$+ 4 \cos I \cos I_{\oplus} \cos\left(h + \lambda_{\oplus} - \bar{\Omega}'\right)\Big]$$

$$+ \beta^2 \left(1 + 2\sqrt{1 - e^2}\right) \Big[- (1 + \cos I)^2 \sin I_{\oplus} \cos 2\left(h + g + \lambda_{\oplus} - \bar{\Omega}'\right)$$

$$- (1 - \cos I)^2 \sin I_{\oplus} \cos 2\left(h - g + \lambda_{\oplus} - \bar{\Omega}'\right)$$

$$+ 4 \sin I(1 + \cos I) \cos I_{\oplus} \cos\left(h + 2g + \lambda_{\oplus} - \bar{\Omega}'\right)$$

$$- 4 \sin I(1 - \cos I) \cos I_{\oplus} \cos\left(h - 2g + \lambda_{\oplus} - \bar{\Omega}'\right)\Big]\Big\} \ , \quad \textbf{(172)}$$

$$\frac{\partial S_3'(\text{lib.})}{\partial a} = 0 \ , \quad \textbf{(173)}$$

$$\frac{\partial S_3'(\text{lib.})}{\partial I} = - 2\alpha G \sin I \sin \ell_{\odot} \ , \quad \textbf{(174)}$$

$$\frac{\partial S_3'(\text{lib.})}{\partial e} = 0 \ , \quad \textbf{(175)}$$

$$\frac{\partial S_3'(\text{lib.})}{\partial g} = 0 \ , \quad \textbf{(176)}$$

and

$$\frac{\partial S_3'(\mathrm{lib.})}{\partial h} = 0 . \tag{177}$$

SECULAR PERTURBATIONS AND PERTURBATIONS DEPENDING STRICTLY ON g″

At this stage of the problem, the remaining part of the Hamiltonian is

$$\mathfrak{F}'' = \mathfrak{F}_0'' + \mathfrak{F}_1'' + \mathfrak{F}_2'' + \mathfrak{F}_3'' , \tag{178}$$

where

$$\mathfrak{F}_0'' = \frac{\mu^2}{2L'^2} - T = \text{const.} ,$$

$$\mathfrak{F}_1'' = n_{\mathbb{C}}^* H' = \text{const.} ,$$

$$\mathfrak{F}_2'' = F_2''(-, g'', -, L', G'', H'') ,$$

and

$$\mathfrak{F}_3'' = F_3''(-, g'', -, L', G'', H) ,$$

and L', T, and H'' are constants in time. T is effectively a third-order quantity; but in the manipulation of von Zeipel's method its contributions are of the zero order, for it is a variable itself, independent of all the others.

It is easy to see that the variable part of the Hamiltonian $\mathfrak{F}''$, which will generate the equations of motion, is factored by small parameters. Therefore, a method of successive approximations such as von Zeipel's cannot be applied. The system of differential equations produced by $\mathfrak{F}''$ must be integrated directly or by using some other kind of approximation.

In a previous version of this theory (Reference 5), the system of differential equations

$$\dot{G}'' = \frac{\partial F''}{\partial g''} ,$$

$$\dot{l}'' = -\frac{\partial F''}{\partial L'} ,$$

$$\dot{g}'' = -\frac{\partial F''}{\partial G''} ,$$

and

$$\dot{h}'' = -\frac{\partial F''}{\partial H''},$$

where $F'' = \mathfrak{F}_0'' + F_1'' + F_2''$, was integrated by making use of a method involving elliptic integrals. This method has appeared in Reference 6, where it was applied to a different problem. In the previous version, it was assumed that J_3, J_4, J_5, and all other spherical harmonics of the moon were of third order or smaller, making the method involving elliptic integrals applicable. Since these and other spherical harmonics of the moon are probably not that small (Reference 3, for example), this method is no longer applicable. Thus it is suggested that the equations of motion for the secular Hamiltonian F'' be integrated numerically. We have

$$F'' = F_0'' + F_1'' + F_2'' + F_3'', \tag{179}$$

where

$$F_0'' = \frac{\mu^2}{2L'^2},$$

$$F_1'' = n_{\mathbb{C}}^* H''.$$

F_2'' is given by Equation 78, and F_3'' is given by Equation 132. The system of equations to be integrated numerically is as follows:

$$\dot{G}'' = \frac{\partial F''}{\partial g''} = \frac{\partial F_2''}{\partial g''} + \frac{\partial F_3''}{\partial g''},$$

$$\dot{l} = -\frac{\partial F''}{\partial L''} = n' - \frac{\partial F_2''}{\partial L''} - \frac{\partial F_3''}{\partial L''},$$

$$\dot{g}'' = -\frac{\partial F''}{\partial G''} = -\frac{\partial F_2''}{\partial G''} - \frac{\partial F_3''}{\partial G''},$$

and

$$\dot{h}'' = -\frac{\partial F''}{\partial H''} = -n_{\mathbb{C}}^* - \frac{\partial F_2''}{\partial H''} - \frac{\partial F_3''}{\partial H''}. \tag{180}$$

Again, it is more convenient to compute the partials with respect to Keplerian elements, and use the expressions

$$\frac{\partial F''}{\partial L''} = 2\sqrt{\frac{a''}{\mu}}\,\frac{\partial F''}{\partial a''} + \frac{1-e''^2}{e''\sqrt{\mu a''}}\,\frac{\partial F''}{\partial e''} ,$$

$$\frac{\partial F''}{\partial G''} = -\frac{1}{e''}\sqrt{\frac{1-e''^2}{\mu a''}}\,\frac{\partial F''}{\partial e''} + \frac{\cot I''}{\sqrt{\mu a''}\,\sqrt{1-e''^2}}\,\frac{\partial F''}{\partial I''} ,$$

and

$$\frac{\partial F''}{\partial H''} = -\frac{1}{\sin I''\sqrt{\mu a''}\,\sqrt{1-e''^2}}\,\frac{\partial F''}{\partial I''} .$$

Thus,

$$\begin{aligned}
\frac{\partial F_2''}{\partial a''} + \frac{\partial F_3''}{\partial a''} = {} & \frac{1}{8\epsilon}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2 n'\,\frac{\sqrt{\mu}}{a''}\left[-\left(2+3e''^2\right)\left(1-3\cos^2 I''\right) + 15e''^2\sin^2 I''\cos 2g''\right] \\
& + \frac{3}{4}\,b^2 J_2\,\frac{\mu\left(1-3\cos^2 I''\right)}{a''^4\left(1-e''^2\right)^{3/2}} + \frac{3}{2}\,b^3 J_3\,\frac{\mu e''\sin I''\left(1-5\cos^2 I''\right)}{a''^5\left(1-e''^2\right)^{5/2}}\sin g'' \\
& + \frac{15}{128}\,b^4 J_4\,\frac{\mu}{a''^6\left(1-e''^2\right)^{7/2}}\left[\left(2+3e''^2\right)\left(3-30\cos^2 I''+35\cos^4 I''\right)\right. \\
& \left. - 10e''^2\sin^2 I''\left(1-7\cos^2 I''\right)\cos 2g''\right] + \frac{15}{128}\,b^5 J_5\,\frac{\mu e''\sin I''}{a''^7\left(1-e''^2\right)^{9/2}}\left[6\left(4+3e''^2\right)\left(1-14\cos^2 I''\right.\right. \\
& \left.\left. + 21\cos^4 I''\right)\sin g'' - 7e''^2\sin^2 I''\left(1-9\cos^2 I''\right)\sin 3g''\right] \\
& + \frac{63}{256\epsilon^2}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)\left(\frac{n_{\mathbb{C}}}{n'}\right)^3 n'^2 a''\left(1-e''^2\right)^{1/2}\cos I''\left[\left(2+33e''^2\right)-\left(2-17e''^2\right)\cos^2 I'' + 15e''^2\sin^2 I''\cos 2g''\right] \\
& - \frac{27}{64\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)\left(\frac{n_{\mathbb{C}}}{n'}\right) b^2 J_{22}\,\frac{n'^2\sin^2 I''\cos I''}{a''\left(1-e''^2\right)^2}\left[2\left(2+3e''^2\right)+15e''^2\cos 2g''\right] \\
& - \frac{117}{8}\left(\frac{n'}{n_{\mathbb{C}}^*}\right) b^4 J_{22}^2\,\frac{n'^2\sin^2 I''\cos I''}{a''^3\left(1-e''^2\right)^{7/2}} + \frac{1}{8}\,n_{\oplus}^2 a''\left[-\left(2+3e''^2\right)\left(1-3\cos^2 I''\right)+15e''^2\sin^2 I''\cos 2g''\right] ,
\end{aligned} \tag{181}$$

$$
\begin{aligned}
\frac{\partial F_2''}{\partial e''} + \frac{\partial F_3''}{\partial e''} = {} & \frac{3}{8\epsilon}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2 n'^2 a''^2 e'' \left[-\left(1 - 3\cos^2 I''\right) + 5\sin^2 I'' \cos 2g''\right] \\
& - \frac{3}{4} b^2 J_2 \frac{\mu e''\left(1 - 3\cos^2 I''\right)}{a''^3\left(1 - e''^2\right)^{5/2}} - \frac{3}{8} b^3 J_3 \frac{\mu\left(1 + 4e''^2\right) \sin I''\left(1 - 5\cos^2 I''\right)}{a''^4\left(1 - e''^2\right)^{7/2}} \sin g'' \\
& - \frac{15}{128} b^4 J_4 \frac{\mu e''}{a''^5\left(1 - e''^2\right)^{9/2}} \Big[\left(4 + 3e''^2\right)\left(3 - 30\cos^2 I'' + 35\cos^4 I''\right) \\
& \qquad - 2\left(2 + 5e''^2\right)\sin^2 I''\left(1 - 7\cos^2 I''\right)\cos 2g''\Big] \\
& - \frac{15}{256} b^5 J_5 \frac{\mu \sin I''}{a''^6\left(1 - e''^2\right)^{11/2}} \Big[2\left(4 + 41e''^2 + 18e''^4\right)\left(1 - 14\cos^2 I'' + 21\cos^4 I''\right) \sin g'' \\
& \qquad - 7e''^2\left(1 + 2e''^2\right)\sin^2 I''\left(1 - 9\cos^2 I''\right)\sin 3g''\Big] \\
& + \frac{9}{128\epsilon^2}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)\left(\frac{n_{\mathbb{C}}}{n'}\right)^3 \frac{n'^2 a''^2 e'' \cos I''}{\left(1 - e''^2\right)^{1/2}} \Big[64 - 99e''^2 + 3\left(12 - 17e''^2\right)\cos^2 I'' \\
& \qquad + 15\left(2 - 3e''^2\right)\sin^2 I'' \cos 2g''\Big] \\
& + \frac{9}{16\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)\left(\frac{n_{\mathbb{C}}}{n'}\right) b^2 J_{22} \frac{n'^2 e'' \sin^2 I'' \cos I''}{\left(1 - e''^2\right)^3} \left[2\left(7 + 3e''^2\right) + 15\left(1 + e''^2\right)\cos 2g''\right] \\
& + \frac{63}{4}\left(\frac{n'}{n_{\mathbb{C}}^*}\right) b^4 J_{22}^2 \frac{n'^2 e'' \sin^2 I'' \cos I''}{a''^2\left(1 - e''^2\right)^{9/2}} - \frac{3}{8} n_{\oplus}^2 a''^2 e''\left(1 - 3\cos^2 I'' - 5\sin^2 I'' \cos 2g''\right) \\
& - \frac{3}{4\epsilon} j_2 R_{\oplus}^2 \frac{n_{\mathbb{C}}^2\left(1 - 3\cos^2 I_{\oplus}\right)\beta''^2\left[2 + \left(1 - e''^2\right)^{1/2}\right]\sin^2 I''}{e''} \cos 2g'' \; , \qquad (182)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F_2''}{\partial I''} + \frac{\partial F_3''}{\partial I''} = {} & \frac{3}{8\epsilon}\left(\frac{n_{\mathbb{C}}}{n'}\right)^2 n'^2 a''^2 \sin I'' \cos I'' \left[-\left(2 + 3e''^2\right) + 5e''^2 \cos 2g''\right] \\
& - \frac{3}{2} b^2 J_2 \frac{\mu \sin I'' \cos I''}{a''^3\left(1 - e''^2\right)^{3/2}} - \frac{3}{8} b^3 J_3 \frac{\mu e'' \cos I''\left(11 - 15\cos^2 I''\right)}{a''^4\left(1 - e''^2\right)^{5/2}} \sin g'' \\
& - \frac{15}{32} b^4 J_4 \frac{\mu \sin I'' \cos I''}{a''^5\left(1 - e''^2\right)^{7/2}} \left[\left(2 + 3e''^2\right)\left(3 - 7\cos^2 I''\right) - 2e''^2\left(4 - 7\cos^2 I''\right)\cos 2g''\right]
\end{aligned}
$$

$$- \frac{15}{256} b^5 J_5 \frac{\mu e'' \cos I''}{a''^6 \left(1 - e''^2\right)^{9/2}} \left[2\left(4 + 3e''^2\right)\left(29 - 126 \cos^2 I'' + 105 \cos^4 I''\right) \sin g''\right.$$

$$\left. - 7e''^2 \sin^2 I'' \left(7 - 15 \cos^2 I''\right) \sin 3g''\right]$$

$$- \frac{9}{128\epsilon^2} \left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)\left(\frac{n_{\mathbb{C}}}{n'}\right)^3 n'^2 a''^2 \left(1 - e''^2\right)^{1/2} \sin I'' \left[2 + 33e''^2 - 3\left(2 - 17e''^2\right) \cos^2 I''\right.$$

$$\left. + 15e''^2 \left(1 - 3 \cos^2 I''\right) \cos 2g''\right]$$

$$- \frac{9}{32\epsilon}\left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)\left(\frac{n_{\mathbb{C}}}{n'}\right) b^2 J_{22} \frac{n'^2 \sin I'' \left(1 - 3 \cos^2 I''\right)}{\left(1 - e''^2\right)^2} \left[2\left(2 + 3e''^2\right) + 15e''^2 \cos 2g''\right]$$

$$- \frac{9}{4} \left(\frac{n'}{n_{\mathbb{C}}^*}\right) b^4 J_{22}^2 \frac{n'^2 \sin I'' \left(1 - 3 \cos^2 I''\right)}{a''^2 \left(1 - e''^2\right)^{7/2}} + \frac{3}{8} n_{\oplus}^2 a''^2 \sin I'' \cos I'' \left[- \left(2 + 3e''^2\right) + 5e''^2 \cos 2g''\right]$$

$$+ \frac{3}{4\epsilon} j_2 R_{\oplus}^2 n_{\mathbb{C}}^2 \left(1 - 3 \cos^2 I_{\oplus}\right) \sin I'' \cos I'' \left\{1 - \beta''^2 \left[1 + 2\left(1 - e''^2\right)^{1/2}\right] \cos 2g''\right\} , \quad \textbf{(183)}$$

and

$$\frac{\partial F_2''}{\partial g''} + \frac{\partial F_3''}{\partial g''} = - \frac{15}{8\epsilon} \left(\frac{n_{\mathbb{C}}}{n'}\right)^2 n'^2 a''^2 e''^2 \sin^2 I'' \sin 2g'' - \frac{3}{8} b^3 J_3 \frac{\mu e'' \sin I'' \left(1 - 5 \cos^2 I''\right)}{a''^4 \left(1 - e''^2\right)^{5/2}} \cos g''$$

$$- \frac{15}{32} b^4 J_4 \frac{\mu e''^2 \sin^2 I'' \left(1 - 7 \cos^2 I''\right)}{a''^5 \left(1 - e''^2\right)^{7/2}} \sin 2g''$$

$$- \frac{15}{256} b^5 J_5 \frac{\mu e'' \sin I''}{a''^6 \left(1 - e''^2\right)^{9/2}} \left[2\left(4 + 3e''^2\right)\left(1 - 14 \cos^2 I'' + 21 \cos^4 I''\right) \cos g''\right.$$

$$\left. - 7e''^2 \sin^2 I'' \left(1 - 9 \cos^2 I''\right) \cos 3g''\right]$$

$$- \frac{135}{64\epsilon^2} \left(\frac{n_{\mathbb{C}}}{n_{\mathbb{C}}^*}\right)\left(\frac{n_{\mathbb{C}}}{n'}\right)^3 n'^2 a''^2 e''^2 \left(1 - e''^2\right)^{1/2} \sin^2 I'' \cos I'' \sin 2g''$$

$$- \frac{135}{16\epsilon}\left(\frac{n_{\mathbb{C}}}{n^*_{\mathbb{C}}}\right)\left(\frac{n_{\mathbb{C}}}{n'}\right) b^2 J_{22} \frac{n'^2 e''^2 \sin^2 I'' \cos I''}{\left(1 - e''^2\right)^2} \sin 2g'' - \frac{15}{8} n_{\oplus}^2 a''^2 e''^2 \sin^2 I'' \sin 2g''$$

$$+ \frac{3}{4\epsilon} j_2 R_{\oplus}^2 n_{\mathbb{C}}^2 \left(1 - 3\cos^2 I_{\oplus}\right)\beta''^2 \left[1 + 2\left(1 - e''^2\right)^{1/2}\right] \sin^2 I'' \sin 2g'' \; . \qquad (184)$$

SUMMARY OF THE DEVELOPMENT

The short-period terms are given by

$$l = l' - \frac{\partial S_2}{\partial L'} = l' + \Delta l \; ,$$

$$g = g' - \frac{\partial S_2}{\partial G'} = g' + \Delta g \; ,$$

$$h = h' - \frac{\partial S_2}{\partial H'} = h' + \Delta h \; ,$$

$$L = L' + \frac{\partial S_2}{\partial l} = L' + \Delta L \; ,$$

$$G = G' + \frac{\partial S_2}{\partial g} = G' + \Delta G \; ,$$

and

$$H = H' + \frac{\partial S_2}{\partial h} = H' + \Delta H \; . \qquad (185)$$

Long-period terms are obtained from

$$l' = l'' - \frac{\partial S_1'}{\partial L''} - \frac{\partial S_2'}{\partial L''} - \frac{\partial S_3'}{\partial L''} = l'' + \Delta l' \; ,$$

$$g' = g'' - \frac{\partial S_1'}{\partial G''} - \frac{\partial S_2'}{\partial G''} - \frac{\partial S_3'}{\partial G'} = g' + \Delta g' \; ,$$

$$h' = h'' - \frac{\partial S_1'}{\partial H''} - \frac{\partial S_2'}{\partial H'} - \frac{\partial S_3'}{\partial H'} = h'' + \Delta h' \; ,$$

$$L' = L'' \; ,$$

$$G' = G'' + \frac{\partial S_1'}{\partial g'} + \frac{\partial S_2'}{\partial g'} + \frac{\partial S_3'}{\partial g'} = G'' + \Delta G' ,$$

and

$$H' = H'' + \frac{\partial S_1'}{\partial h'} + \frac{\partial S_2'}{\partial h'} + \frac{\partial S_3'}{\partial h'} = H'' + \Delta H' . \tag{186}$$

Secular perturbations and perturbations depending only on g'' result from the integration of Equations 180:

$$\Delta l'' = \int\left(n' - \frac{\partial F_2''}{\partial L''} - \frac{\partial F_3''}{\partial L''}\right) dt = l'' - l_0'' ,$$

$$\Delta g'' = -\int\left(\frac{\partial F_2''}{\partial G''} + \frac{\partial F_3''}{\partial G''}\right) dt = g'' - g_0'' ,$$

$$\Delta h'' = -\int\left(n_{\mathbb{C}}^* + \frac{\partial F_2''}{\partial H''} + \frac{\partial F_3''}{\partial H''}\right) dt = h'' - h_0'' ,$$

$$\Delta L'' = 0 ,$$

$$\Delta G'' = \int\left(\frac{\partial F_2''}{\partial g''} + \frac{\partial F_3''}{\partial g''}\right) dt = G'' - G_0'' ,$$

and

$$\Delta H'' = 0 . \tag{187}$$

Then, the perturbations in the Keplerian elements a, e, and I are given by

$$a = a' + 2a' \frac{\Delta L}{L'}$$

and

$$a' = a'' = \frac{L''^2}{\mu} , \tag{188}$$

$$e = e' + \frac{1-e'^2}{e'}\left(\frac{\Delta L}{L'} - \frac{\Delta G}{G'}\right) ,$$

$$e' = e'' - \frac{1-e''^2}{e''}\frac{\Delta G'}{G''} - \frac{1-e''^2}{2e''^3}\left(\frac{\Delta G'}{G''}\right)^2 ,$$

$$e'' = e_0'' - \frac{1-e_0''^2}{e_0''}\frac{\Delta G''}{G_0''} ,$$

and

$$e_0'' = 1 - \frac{G_0''^2}{L''^2} , \qquad (189)$$

$$I = I' + \cot I'\left(\frac{\Delta G}{G'} - \frac{\Delta H}{H'}\right) ,$$

$$I' = I'' + \cot I''\left(\frac{\Delta G'}{G''} - \frac{\Delta H'}{H''}\right) + \frac{\cos I''}{\sin^3 I''}\left(\frac{\Delta G'}{G''}\right)\left(\frac{\Delta H'}{H''}\right) - \frac{1}{2}\,\frac{\cos I''\left(1+\sin^2 I''\right)}{\sin^3 I''}\left(\frac{\Delta G'}{G''}\right)^2 - \frac{1}{2}\cot^3 I''\left(\frac{\Delta H'}{H''}\right)^2 ,$$

$$I'' = I_0'' + \cot I_0''\,\frac{\Delta G''}{G_0''} ,$$

and

$$I_0'' = \cos^{-1}\left(\frac{H''}{G_0''}\right) . \qquad (190)$$

POSITION AND VELOCITY: $e \neq 0$; $I \neq 0°$, $180°$

From the elements L, G, H, l, g, h, one can obtain the coordinates and the components of velocity as follows.

Obtain a, e, and I from the following equations:

$$a = \frac{L^2}{\mu} ,$$

$$e = \sqrt{1 - G^2/L^2} ,$$

and

$$I = \arc\cos\frac{H}{G} \qquad (0° < I < 180°) .$$

Now solve Kepler's equation

$$E - e \sin E = l$$

to obtain E.

Then compute r from

$$r = a(1 - e \cos E) .$$

Next obtain f from

$$\cos f = \frac{a}{r} (\cos E - e)$$

and

$$\sin f = \frac{a}{r} \frac{G}{L} \sin E .$$

Now compute

$$A_x = a(\cos g \cos h - \sin g \sin h \cos I) ,$$

$$B_x = a \frac{G}{L} (- \sin g \cos h - \cos g \sin h \cos I) ,$$

$$A_y = a(\cos g \sin h + \sin g \cos h \cos I) ,$$

$$B_y = a \frac{G}{L} (- \sin g \sin h + \cos g \cos h \cos I) ,$$

$$A_z = a \sin g \sin I ,$$

and

$$B_z = a \frac{G}{L} \cos g \sin I .$$

Then,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A_x & B_x & 0 \\ A_y & B_y & 0 \\ A_z & B_z & 0 \end{pmatrix} \begin{pmatrix} \cos E - e \\ \sin E \\ 0 \end{pmatrix} .$$

Since the system is rotating,

$$\dot{A}_x = n_{\mathbb{C}}^* A_y \ ,$$

$$\dot{B}_x = n_{\mathbb{C}}^* B_y \ ,$$

$$\dot{A}_y = - n_{\mathbb{C}}^* A_x \ ,$$

and

$$\dot{B}_y = - n_{\mathbb{C}}^* B_x \ .$$

Therefore,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = n_{\mathbb{C}}^* \begin{pmatrix} A_y & B_y & 0 \\ -A_x & -B_x & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos E - e \\ \sin E \\ 0 \end{pmatrix}$$

$$+ \frac{a}{r} n \begin{pmatrix} A_x & B_x & 0 \\ A_y & B_y & 0 \\ A_z & B_z & 0 \end{pmatrix} \begin{pmatrix} - \sin E \\ \cos E \\ 0 \end{pmatrix} \ .$$

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