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SOME RELATIONSHIPS BETWEEN RANDOM AND NON-RANDOM EXPECTANCIES

Prepared under Contract No. NAS8-18405

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COMPUTER SCIENCES CORPORATION
Computer Systems Branch

For

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER
Huntsville, Alabama

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SOME RELATIONSHIPS BETWEEN RANDOM AND NON-RANDOM EXPECTANCIES

By

George C. Marshall Space Flight Center
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ABSTRACT

Several relationships involving random and non-random probabilities corresponding to instantaneous and identical events are established. A conjectured approximation is indicated.

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For

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Computer Systems Division
Computation Laboratory

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SUMMARY

For an instantaneous, repeatable event, let:

$p(t)$ = probability that the event will occur at least once within $(0,t]$ if it occurred at time 0;

$q(t)$ = probability that the event will occur at least once within $(0,t]$ if time 0 is random;

T_1 = average time between consecutive occurrences;

T_2 = average time from random moment to next occurrence.

Then

$$q'(t) = \int_0^\infty \frac{p'(t+\alpha)q'(\tau)}{1-p(\tau)}\, d\tau$$

$$T_1 = \int_0^\infty t p'(t)dt$$

$$T_2 = \int_0^\infty t q'(f)dt$$

and it is conjectured that

$$T_1 \approx \frac{2\alpha+1}{\alpha+1} T_2$$

where α is the stochastic order of the process.

Technical Note

SOME RELATIONSHIPS BETWEEN RANDOM AND NON-RANDOM EXPECTANCIES

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Abstract

Several relationships involving random and non-random probabilities corresponding to instantaneous and identical events are established. A conjectured approximation is indicated.

Introduction

Consider an instantaneous event which occurs generally at irregular time intervals. The purpose of this paper is to establish relationships between:

- The probability that the next occurrence will be at some time t if the time of the last preceding occurrence is known; this is a non-random situation by definition;
- The probability that the next occurrence will be at some time t if the time of the last preceding occurrence is NOT known; this is a random situation by definition;
- The average waiting times to the next occurrence, from the last preceding one or from a random moment.

Such formulae are frequently used in the probabilistic modeling of systems and this note is intended to serve as a standard reference.

Analysis

An instantaneous event occurs at irregular time intervals. It is not known whether the probability to occur at any particular time depends on the times of past occurrences. Let:

$p(t)$ = probability that the event will occur at least once within $(0,t]$ if it occurred at time 0;

$q(t)$ = probability that the event will occur at least once within $(0,t]$ if time 0 is random;

T_1 = average time between consecutive occurrences;

T_2 = average time from random moment to next occurrence.

Assume $0 < t_1 < t_2$ and that the event occurred at $t = 0$. Then, the probability that the next occurrence belongs to $(t_1, t_2]$ is

$$p(t_2) - p(t_1) \quad .$$

If $t_1 = t$ and $t_2 = t + dt$, it follows that the probability for t being the time between consecutive occurrences is

$$p'(t)dt \quad .$$

Hence

$$(1) \qquad T_1 = \int_0^\infty t p'(t)dt$$

and similarly

$$(2) \qquad T_2 = \int_0^\infty t q'(t)dt = \int_0^\infty t f(t)dt$$

where $f(t) = q'(t)$ and, since $q(0) = 0$, we have also

$$(3) \qquad q(t) = \int_0^t f(\tau)d\tau \quad .$$

Assume now that $t = 0$ has been randomly selected and that the last preceding occurrence was at $t = -\tau$ where $\tau > 0$. Then the probability that there was no occurrence in $(-\tau, 0]$ is $1 - p(\tau)$ and therefore, the probability that the next occurrence belongs to $(0, t]$ is

$$\frac{p(t+\tau) - p(\tau)}{1 - p(\tau)} \quad .$$

Since the probability that the last preceding occurrence actually belonged to $(-\tau - d\tau, -\tau]$ is $f(\tau)d\tau$, it follows that

$$(4) \qquad q(t) = \int_0^\infty \frac{p(t+\tau) - p(\tau)}{1 - p(\tau)} f(\tau)d\tau$$

and if $f(\tau)$ is finite for all $\tau \in [0, \infty]$ it is easy to verify that $q(0) = 0$ and $q(\infty) = 1$ as should indeed be the case.

If the function $p(t)$ is differentiable everywhere, then, it follows from (4) that

$$(5) \qquad f(t) = \int_0^\infty \frac{p'(t+\tau)f(\tau)}{1 - p(\tau)} d\tau \quad .$$

This is a homogeneous Fredholm type integral equation of the second kind which, in general, permits the determination of $q(t)$ if $p(t)$ is known. The converse is generally not possible. For the latest numerical procedures, see references [1] and [2].

The average times T_1 and T_2 satisfy the inequality

$$(6) \qquad T_1 \geq T_2 \geq \frac{T_1}{2} \quad .$$

In order to prove this, notice that the inequality $T_1 \geq T_2$ is obvious, since T_2 can never be greater than T_1 and $T_1 = T_2$ is possible only if the sequence has no memory, that is if the probability of the event to occur at any particular time is independent of all preceding occurrence times. In this case it is well known that $p(t)$ must have the form $p(t) = 1- e^{-\alpha t}$ where α is a positive constant and the reader may easily verify with the aid of (5) that this yields $f(t) = p'(t) = \alpha\ e^{-\alpha t}$, i.e. $q(t) = p(t)$. Thus, it remains to be proven that $2T_2 \geq T_1$.

The smallest possible value of T_2 occurs if $p(t)$ has the form

$$\begin{aligned} p(t) &= 0 \quad , \quad t \in [0, t_o) \\ &= 1 \quad , \quad t \in [t_o, \infty) \end{aligned}$$

where $t_o > 0$. In this case, the time of the next occurrence may be considered to depend on the times of ALL preceding occurrences. Furthermore, $p'(t)$ will be an impulse function, so that

$$\int_o^\infty tp'(t)dt = t_o = T_1 \quad .$$

Now, for all $p(t)$, it is true that

$$T_2 = \int_o^\infty tdt \int_o^\infty \frac{p'(t+\tau)f(\tau)}{1-p(\tau)} d\tau$$

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[1] Bellman, R.E. : A New Approach to the Numerical Solution of a Class of Linear and Nonlinear Integral Equations of the Fredholm Type. Proc. Nat. Acad. of Science, 1965, pp. 1501-1503.

[2] Day, J.T. : Note on the Numerical Solution of Integro-Differential Equations. Computer Journal, 9,4, (Feb. 67), pp. 394-395.

$$= \int_0^\infty \frac{f(\tau)d\tau}{1 - p(\tau)} \quad \int_0^\infty tp'(t + \tau)dt$$

$$= \int_0^\infty \frac{f(\tau)}{1 - p(\tau)} \quad \left[T_1 - \int_0^\infty tp'(t)dt - \tau(1 - p(\tau))\right] \, d\tau$$

$$= T_1 \int_0^\infty \frac{f(\tau)}{1 - p(\tau)} \, d\tau - \int_0^\infty \frac{f(\tau)\tau}{1 - p(\tau)} \, d \int_0^\tau tp'(t) \, dt - T_2 \quad .$$

Hence,

$$2T_2 = \int_0^\infty \frac{f(\tau)d}{1 - p(\tau)} \quad \int_\tau^\infty tp'(t)dt$$

$$(7) \qquad = \int_0^\infty tp'(t)dt \int_0^t \frac{f(\tau)\,d\tau}{1 - p(\tau)} \quad .$$

But in this case, for $t \in (0,t_o]$, $f(t)$ satisfies

$$f(t) = f(t_o - t)$$

and for $t = 0$ or $t \geq t_o$, we have $f(t) = 0$. Hence, by virtue of (7),

$$2T_2 = t_o \int_0^{t_o} \frac{f(\tau)d\tau}{1 - p(\tau)} = T_1 \int_0^{t_o} f(\tau)d\tau = T_1 \int_0^\infty f(\tau)d = T_1$$

which completes the proof of (6).

Let α be the order of the process in the sense that the probability of the event to occur at any time t depends on the times of the last preceding occurrences. Thus $\alpha = 0$ means "no memory" while $\alpha = \infty$ means that the times of all occurrences are fixed. Then, it is conjectured that if $p(t)$ is reasonably smooth, we have the approximation

$$(8) \qquad T_1 \approx \frac{2\alpha + 1}{\alpha + 1} T_2 \quad .$$

For example, let

$$p(t) = t \quad , \quad 0 \leq t \leq 1$$
$$\quad = 1 \quad , \quad t \geq 1 \quad .$$

Then, the reader may easily verify with the aid of (5) that

$$f(t) = 2(1 - t) \quad , \quad 0 \leq t \leq 1$$
$$= 0 \quad , \quad t \geq 1$$

and

$$q(t) = t(2 - t) \quad , \quad 0 \leq t \leq 1$$
$$= 1 \quad , \quad t \geq 1$$

Using (1) and (2), this yields

$$T_1 = 1/2$$

and

$$T_2 = 1/3$$

so that

$$T_1 = (3/2)T_2 \quad .$$

Since this process corresponds to $\alpha = 1$, this same result can be duplicated with the aid of formula (8). Still, in general (8) is only an approximation.

REFERENCES

Cramer, H. : Mathematical Methods of Statistics. Princeton University Press, Princeton, N. J. 1951.

Feller, W. : Probability Theory and its Applications. John Wiley and Sons Inc., New York, N. Y. 1957.

Kendall, M. G. and Stuart, A. : The Advanced Theory of Statistics, Vol. I, Distribution Theory. Charles Griffin and Company, Ltd., London, England 1958.