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THE ALGEBRA OF PROBABILITY DENSITY FUNCTIONS

Prepared under Contract No. NAS8-18405

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Computer Systems Branch

For

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER
Huntsville, Alabama

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By

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ABSTRACT

Presented in this report is a set of mathematical rules for the calculation of the probability density function of a variable known as an arbitrary function of one or more random variables.

All proofs have been included.

The applicability of these rules is illustrated in a number of examples.

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THE ALGEBRA OF PROBABILITY DENSITY FUNCTIONS

SUMMARY

Formulae, proofs and examples are given concerning the solution of the following two problems:

1. Let $p(x)$ be the probability density function of a random variable x and $y(x)$ an analytic, uniquely valued and otherwise arbitrary function of x. Find the probability density function of y ;

2. Let $p_k\ (x_k)$, $k = 1,(1),n$, be the probability density functions of a set of n random variables x_k, $k = 1,(1),n$ and $z = \phi\ (x_1, x_2, \ldots x_k)$ and analytic, uniquely valued and other wise arbitrary function of these random variables. Find the probability density function of z.

The Algebra of Probability Density Functions

INTRODUCTION

A large portion of the systems analysis, design and simulation work that is being oriented toward the development of hardware and software criteria for the implementation of the concept of a general purpose spaceborne digital computer, will require extensive use of mathematical techniques based on the development of probability models and the algebraic manipulation of probability density functions. This short paper, which summarizes some of the fundamentals of such techniques, is intended as a reference for various NASA personnel and their subcontractors, with the intent of assisting them in the use of probability density functions. In particular, those persons throughout MSFC, who are associated with the Computation Laboratory Spaceborne Computer Project, are assumed to be the important readers; however, techniques outlined herein are certainly of general applicability to the design of other systems presently being considered by NASA. Some of the information contained herein can be extracted from readily available text material. However, many contributions to the field of statistics by the author are included, because they are believed to represent important techniques which are not widely known. Proofs are included only when they are felt to be necessary or when they contain techniques which are of general interest.

The Transformation Rule

Let $p(x)$ be a probability density function, i.e. $p(x_o)dx$ represents the probability that the random variable x lies between x_o and $x_o + dx$. Let $y = y(x)$ be analytic and uniquely valued, i.e. to each value of x there corresponds one

The validity of (5) is restricted by the following condition:

If $z = z_o = \phi(x_o) =$ constant is chosen such that R is non-empty and the components of x_o, namely $x_{10}, x_{20}, \ldots, x_{no}$, all belong to S except for some x_{ko}, where k is arbitrary but different from j, then the equation

(8) $$z_o = \phi(x_{1,0}, x_{2,0}, \ldots x_{k-1,0}, x_{k,0}, x_{k+1,0}, \ldots x_{n,o})$$ must have no more than one solution in $x_{k,0} \in S$, for all possible choices of k.

The preceeding restriction does not really degrade our solution. In effect, assume (5) to be correct under the restriction imposed. Next, assume that for some value of k, the equation (3) has more than one root belonging to S. Then, it is not difficult to see that formula (5) must be replaced by a summation of the same integral over all solutions $x_{k,0}$. Similarly, if there is more than one value of k yielding multiple roots of (8) belonging to S, the summation must extend over all possible combinations of solutions. Hence, no generality is lost if we prove formula (5) assuming the restriction (8) to be valid.

A general and rather elegant proof of (5) can be obtained using known concepts of differential geometry. It will be complemented later by an elementary, but more restricted alternate proof. Consider the manifold C defined by $z =$ constant and the portion of this manifold which belongs to S, namely R as defined by (6). Let R + dR be a similar manifold, corresponding to $z + dz =$ constant. If we denote by dS the hypervolume between R and R + dR belonging to S, then $p(z)dz$ represents the probability that a point x belongs to dS.

and only one value of y. The converse is not necessarily true; in other words, to each y, there may correspond a set of k values of x, namely

$$x^{(1)} \quad x^{(2)} \quad \dots \quad x^{(k)} \quad .$$

Then the probability density function $p_1(y)$ of y will be given by

$$(1) \qquad p_1(y) = \sum_{i = 1,(1),k} p(x^{(i)}) \left| \left(\frac{dx}{dy}\right)_{x = x^{(i)}} \right| \quad .$$

In particular, if k = 1, we have

$$(2) \qquad p_1(y) = \frac{p(x)}{|y'(x)|}$$

where x is assumed to be expressed as a function of y.

In order to prove (1) and (2), simply observe that the probability of y belonging to (y, y + dy) is given by

$$p_1(y)dy = \sum_{i = 1,(1),k} p(x^{(i)})\, dx$$

$$= \sum_{i = 1,(1),k} p(x^{(i)}) \left| \left(\frac{dx}{dy}\right)_{x = x^{(i)}} \right| dy$$

where the absolute value enters because density functions are positive definite by definition.

As an example, let

$$y = x^2$$

then $x = \pm \sqrt{y}$, i.e. k = 2

and $\frac{dx}{dy} = \pm \frac{1}{2\sqrt{y}}$

hence, by virtue of (1):

$$p_1(y) = \frac{1}{2\sqrt{y}} \left[p(\sqrt{y}) + p(-\sqrt{y}) \right] \quad .$$

If say $x \in (-1,1)$ and $p(x) = \frac{1}{2}$, we obtain $p_1(y) = \frac{1}{2\sqrt{y}}$ with $y \in (0,1)$. If $x \in (0,1)$ and $p(x) = 1$, we have $p(-\sqrt{y}) = 0$ and therefore $p_1(y) = \frac{1}{2\sqrt{y}}$ with $y \in (0,1)$ as before.

The Composition Rule

Let $x = x(x_1, x_2, \ldots, x_n)$ be an n-component vector and assume

(3) $$x_{k1} < x_k < x_{k2} \quad , \quad k = 1,(1),n \quad .$$

The bounds x_{k1} and x_{k2} are not necessarily constant and may depend on all components of x, except x_k. Denote by S the n-dimensional manifold of all points x satisfying (3) and let $p_k(x_k)$, $k = 1, (1), n$, represent the n probability density functions of the variables x_k. Furthermore define

(4) $$z = \phi(x)$$

where z is real and p is analytic and uniquely valued in S. The question is to determine the probability density function p(z) of z.

Under certain restrictions, to be detailed later, the answer is:

(5) $$p(z) = \int_R \frac{\Pi \, p_k(x_k) dx_k}{\left| \frac{\partial \phi}{\partial x_j} \right| dx_j}$$

where the product Π is to be taken over all n values of k, j is an arbitrary index between 1 and n and the integration manifold R is defined by

(6) $$R = S \cap C$$

where C is the manifold defined by

(7) $$\phi(x) = z = \text{constant}$$

which has n-1 dimensions.

Let $d^{n-1}R$ be an infinitesimal portion of R such that

$$dS = \int_R \frac{dz}{|\nabla \phi|} d^{n-1}R \quad .$$

Then, from differential geometry, it is known that

$$d^{n-1}R = \frac{|\nabla \phi|}{\frac{\partial \phi}{\partial x}} \frac{\Pi dx_k}{dx_j}$$

where the product Π extends over all n values of k, and j is an arbitrary index between 1 and n. It follows that

$$dS = \int_R \frac{dz \ \Pi \ dx_k}{\left| \frac{\partial \phi}{\partial x_j} \right| dx_j} \quad .$$

where the integrand is an n-dimensional hypervolume (if $n > 3$; otherwise, a volume, surface or line) having all dimensions infinitesimal. The probability that a point x will belong to this hypervolume is

$$\frac{dz \ \Pi \ p_k(x_k) dx_k}{\left| \frac{\partial \phi}{\partial x_j} \right| dx_j} \quad .$$

If the condition (8) is satisfied, we may now integrate over R to obtain (5), which completes the proof.

Notice that neither C nor R need to be connected. The symbol ∇ stands of course for gradient, which is a vector so that $|\nabla \phi|$ means the magnitude of the vector. The index j must be chosen so that $\frac{\partial \phi}{\partial x_j} \neq 0$.

An alternate and elementary proof, restricted to two dimensions, would be as follows:

Let x and y be two independent random variables satisfying

$$x_1 \leq x \leq x_2$$
$$y_1 \leq y \leq y_2$$

and let S be the (x,y) space defined by these bounds, i.e. generally a surface. If the bounds are constant and the coordinates rectangular, S would be a rectangle. Nevertheless x_1 and/or x_2 may depend on y, and vice-versa, so that S could be any surface, neither necessarily bounded nor necessarily simply connected. Define $p_1(x)$ and $p_2(y)$ as the respective probability densities and

$$z = \phi\ (x,y)$$

where ϕ is analytic and uniquely valued in S. We seek the probability density p(z) of z.

Consider the curve C given by z = constant and let R be the portion of C belonging to S. Similarly, let R + dR be the position of the curve z + dz = constant belonging to S. Denote by dS the area between R and R + dR belonging to S. Then p(z)dz represents the probability that a point (x,y) belongs to dS. In order to calculate this probability, let

$$d^2S = d\ell\ dr$$

where $d\ell$ is an infinitesinal line element of R and dr is the distance between R and R + dR of that point. Now, assuming rectangular coordinates, which does not impair generality, we have

$$d\ell = \sqrt{(dx)^2 + (dy)^2}$$

and

$$dr = \frac{dz}{|\nabla\phi|} .$$

Hence

$$d^2S = \frac{dz}{|\nabla\phi|}\ \sqrt{(dx)^2 + (dy)^2} = dz \qquad \frac{\sqrt{(dx)^2 + (dy)^2}}{\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2}}$$

But on R:

$$dz = 0 = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

i.e.

$$\frac{dy}{dx} = - \frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$$

Substituting in the expression for d^2S, we obtain

$$d^2S = \frac{dzdx}{\left|\frac{\partial \phi}{\partial y}\right|} = \frac{dzdy}{\left|\frac{\partial \phi}{\partial x}\right|}$$

The probability that the point (x,y) will belong to d^2S is now

$$p_1 (x) p_2 (y) d^2S = p_1 (x) p_2 (y) \frac{dy}{\left|\frac{\partial \phi}{\partial x}\right|} dz$$

where the meanings of x and y may be interchanged. If the equations $z_o = \phi(x_o,y)$ and $z_o = \phi(x,y_o)$ are single rooted in S, we integrate the preceding expression over R, obtaining:

(9) $$p(z) = \int_R p_1 (x) p_2 (y) \frac{dy}{\left|\frac{\partial \phi}{\partial x}\right|}$$

where x is of course a function of z and y. The meanings of x and y can again be interchanged. This is the 2-dimensional equivalent of (5). If the equations $z_o = \phi(x_o,y)$ and $z_o = \phi(x,y_o)$ are not single rooted in S, additional terms must be added for all combinations of roots. Of course, $\frac{\partial \phi}{\partial x}$ or $\frac{\partial \phi}{\partial y}$ must be distinct from zero.

In practice, the main difficulty in the use of formula (5) is a merely geometrical one, namely, the determination of the boundaries of the manifold R. In addition, if formula (5) is to be used numerically, straight quadrature procedures may be conducive to serious computational difficulties, in the sense of excessive machine time requirements. The recommended method to circumvent these difficulties will become apparent in the forthcoming examples and it should perhaps be added that the author has successfully used this method for several years in purely numerical and multi-dimensional applications.

The boundary situation can be illustrated with the following simple example. Consider the case of 3 random variables, bounded by constants, so that S is 3-dimensional and actually, assuming a tri-orthogonal coordinate system, a rectangular parallelepiped. In this case, C will be a surface and let us assume that it is a plane so that formula (5) applies without multiple root complications. Then R will be the intersection between this plane and S, that is, anything from a triangle to an irregular hexagon, depending on the position of C.

Notice that, in the particular case when

$$z = \phi\,(x,y) = x + y$$

then formula (9) becomes

$$p(z) = \int_R p_1\,(z - y)\; p_2\,(y)\; dy$$

i.e. the well known and classic convolution formula.

<u>Examples</u>

We start with some simple, 2-dimensional cases.

PROBLEM 1

Given $x \in (0,1) \quad , \quad p_1\,(x) = 1$

$y \in (0,1) \quad , \quad p_2\,(y) = 1$

$z = x + y$

Find p (z)

Using (9)

$$p\,(z) = \int_R dy$$

Now, if $z \in (0,1)$ then $y \in (0,z)$

and if $z \in (1,2)$ then $y \in (z-1,1)$

Hence

$$p(z) = \int_0^z dy = z \quad \text{if } 0 \leqq z \leqq 1$$

and

$$p(z) = \int_{z-1}^1 dy = 2 - z \text{ if } 1 \leqq z \leqq 2$$

Verification:

$$\int_0^2 p(z)\,dz = \int_0^1 zdz + \int_1^2 (2-z)\,dz = 1$$

PROBLEM 2

Given

$$x \in (0,1) \quad , \quad p_1(x) = 1$$

$$y \in (0,1) \quad , \quad p_2(y) = 1$$

$$z = x - y$$

find $p(z)$.

We have

$$p(z) = \int_R dy$$

Now, if $z \in (-1,0)$, then $y \in (-z,1)$

and if $z \in (0,1)$, then $y \in (0,1-z)$

Hence

$$p(z) = \int_{-z}^1 dy = 1 + z \quad \text{if } -1 \leqq z \leqq 0$$

and

$$p(z) = \int_0^{1-z} dy = 1 - z \quad \text{if } 0 \leqq z \leqq 1$$

Verification:

$$\int_{-1}^{1} p(z)dz = \int_{-1}^{0} (1 + z)\, dz + \int_{0}^{1} (1 - z)\, dz = 1$$

The next problem is 3-dimensional.

PROBLEM 3

Given $a \in (0,1)$, $p_1(a) = 1$

$b \in (0,1)$, $p_2(b) = 1$

$c \in (0,1)$, $p_3(c) = 1$

$z = b^2 - 4ac$

find $p(z)$

In this case, $z \in (-4,1)$. There is no multiple root problem, since the second root in b would be negative and thus does not belong to S. Application of formula (5) would require the determination of the intersection R between the unit cube representing S and the surface $b^2 - 4ac = z =$ constant. This is rather tedious and can be circumvented as follows:

Substitute $x_1 = b^2$. Then, since $b \in (0,1)$, we can apply formula (2) so that

$$p_4(x_1) = \frac{1}{2b} = \frac{1}{2\sqrt{x_1}}$$

The problem is now reformulated in the form

$$z = x_1 - 4ac \quad .$$

Next, we substitute $x_2 = 4ac$. Then, by virtue of formula (9), we have

$$p_5(x_2) = \int_R \frac{da}{4a}$$

where R is the intersection between the proper unit square and the hyperbola $4ac = x_2 =$ constant. Hence, for all values of x_2, a varies from 1 to $\frac{x_2}{4}$

and therefore

$$p_5(x_2) = -\tfrac{1}{4} \int_1^{x_2/4} \frac{da}{a} = -\tfrac{1}{4} \ln \frac{x_2}{4} .$$

$$= \tfrac{1}{2} \ln 2 - \tfrac{1}{4} \ln x_2 .$$

The problem is now reformulated as

$$z = x_1 - x_2$$

where

$$x_1 \in (0,1) \quad , \quad p_4(x_1) = \frac{1}{2\sqrt{x_1}} .$$

$$x_2 \in (0,4) \quad , \quad p_5(x_2) = \tfrac{1}{2} \ln 2 - \tfrac{1}{4} \ln x_2$$

and therefore, by virtue of formula (9):

$$p(z) = \tfrac{1}{4} \ln 2 \int_R \frac{dx_2}{\sqrt{z + x}} - \frac{1}{8} \int_R \frac{\ln x_2}{\sqrt{z + x_2}} dx_2$$

where R is the intersection between the rectangle formed by the values of x_1 and x_2 and the straight line $z = x_1 - x_2 =$ constant. Hence the following integration limits apply:

If $z \in (-4,-3)$, $x_2 \in (-z,4)$

If $z \in (-3,0)$, $x_2 \in (-z,1-z)$

If $z \in (0,1)$, $x_2 \in (0,1-z)$.

For reasons of briefness, we shall calculate the result only for positive z, i.e. in this case, $z \in (0,1)$. We obtain

$$\int_R \frac{dx_2}{\sqrt{z + x_2}} = \int_o^{1-z} \frac{dx}{\sqrt{z + x}} = 2(1-\sqrt{z})$$

$$\int_R \frac{\ln x_2}{\sqrt{z + x_2}} dx_2 = \int_o^{1-z} \frac{\ln dx}{\sqrt{z + x}} = 2\ln(1-z) - 4(1-\sqrt{z}) -$$

$$-2\sqrt{z}\ \ln\left(4z\ \frac{1-\sqrt{z}}{1+\sqrt{z}}\right) .$$

Hence

$$p(z) = \tfrac{1}{2}(1-\sqrt{z})(1 + \ln 2) - \tfrac{1}{4}\ln(1 - z) + \frac{\sqrt{z}}{4}\ \ln\left(4z\ \frac{1-\sqrt{z}}{1+\sqrt{z}}\right)$$

for $z \in (0,1)$.

In particular:

$p(0) = \tfrac{1}{2}(1 + \ln 2) \approx .8466 \ldots.$

$p(\tfrac{1}{2}) \approx .2321 \ldots.$

$p(1) = 0$.

The reader may easily find the probability density $p(z)$ for the remaining range of z.

The preceding problem establishes the guidelines for a general procedure to circumvent the difficulty of determining the manifold R when the latter has more than one dimension.

The following problem sheds some light on the variety of possible applications of the preceding theory.

PROBLEM 4

Given the second order equation

$$ax^2 + bx + c = 0$$

where a, b, c $\in$ (0,1) with uniform distribution, what is the probability that the equation has real roots?

The results from the preceding problem can obviously be applied here. The roots are real if $b^2 - 4ac \geqq 0$ and therefore, if p(z) is taken from Problem (3), the answer is

$$P = \int_0^1 p(z)dz \quad .$$

Using the fact that

$$\int_0^1 \sqrt{z} \ln \left(4z \frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right) dz = - \frac{10}{9} \quad .$$

we find

$$P = \frac{1}{36} (5 + 6 \ln 2) \approx .2544 \ldots$$

This result has been found independently by S. Karlin of Stanford University. The preceding problem is more than just academic; in fact, it is of considerable interest in certain aspects of logical computer design. This problem was solved, several years ago, at the request of an engineering staff which was reluctant to accept the answer. Subsequently, a Monte-Carlo type determination of P was programmed by simply random generating three numbers a, b and c in (0,1) and checking the sign of $b^2 - 4ac$. This program ran in 1964 on a Burroughs B-5000 computer and confirmed the theoretical result beyond reasonable doubt.

The following example illustrates a simple application in the analysis of the probabilistic behavior of information systems.

PROBLEM 5

Assume that the sole purpose of a system is to perform the matrix multiplication $M = AB$, each multiplication process starting instantly after the preceding one has terminated. The matrices A and B have dimensions $m \times n$ and $n \times \ell$ respectively and these dimensions are considered to be discrete random variables with known density functions $p_1(m)$, $p_2(n)$, $p_3(\ell)$, and fixed greatest lower and least upper bounds. Notice that since m, n and ℓ are integers, the density functions are not analytical. Several questions may be posed, such as:

I. If the computer has sufficient capacity to handle all possible sizes, what is the average number of multiplication programs executed per unit of time?

II. Same question if some limit is imposed on m, n and ℓ.

III. If some limit is imposed on m, n and ℓ, what is the percentage of multiplication requests which will be satisfied?

In order to answer question I, it is necessary to find first the probability density function p(z) of execution time z. Now, it is known from computer theory, that very approximately

$$z = k\, m\, n\, \ell$$

where k is a constant which may be assumed known, depending on the type of computer and the choice of the time unit. Formula (5) must be applied now, taking into consideration that m, n and ℓ are integers, so that integrals become summations. Following the process outlined in problem (3), we define

$$x = n\ell$$

and obtain the corresponding density function in the form

$$p_4(x) = \sum p_2(n)p_3(\tfrac{x}{n})\,\frac{1}{n}$$

where the summation extends over all values of n such that the inequalities

$$n_1 \leqq n \leqq n_2$$

$$\ell_1 \leqq \frac{x}{n} \leqq \ell_2$$

are both satisfied, where (n_1,n_2) and (ℓ_1,ℓ_2) are the greatest lower and least upper bounds of n and ℓ respectively and therefore known.

The density function p(z) can now be obtained in the form

$$p(z) = \frac{1}{k} \sum p_1(m)p_4\left(\frac{z}{km}\right)\frac{1}{m}$$

where the summation extends over all values of m such that the inequalities

$$m_1 \leqq m \leqq m_2$$

$$\ell_1 n_1 \leqq \frac{z}{km} \leqq \ell_2 n_2$$

are both satisfied, where m_1 and m_2 are respectively the greatest lower and least upper bounds of m. It follows that the average execution time will be

$$T = \int_{kn_1m_1\ell_1}^{kn_2m_2\ell_2} zp(z)dz$$

and it should be noticed that the preceding quadrature can be performed numerically WITHOUT truncation error, since the number of distinct values of zp(z) is finite, namely

$$(m_2 - m_1 + 1)(n_2 - n_1 + 1)(\ell_2 - \ell_1 + 1) \quad .$$

The average number of executions per unit of time will be

$$\frac{1}{T}$$

and this answers question I.

Assume now that limits m_o n_o and ℓ_o are imposed on m, n and ℓ. This means that whenever $m > m_o$ or $n > n_o$ or $\ell > \ell_o$, the program is rejected, one possible reason being that the computer does not have the necessary capability. These limits must of course be smaller than the corresponding least upper bounds. The probability that a request will be executed is

$$p = \left(\sum_{m = m_1}^{m_o} p_1(m) \right) \left(\sum_{n = n_1}^{n_o} p_2(n) \right) \left(\sum_{\ell = \ell_1}^{\ell_o} p_1(\ell) \right)$$

and 100 P% is the answer to question III.

In order to answer question II, we assume rejection time to be negligible. Next, we calculate the probability density function $q_o(z)$ of execution time z , subject to the condition that an execution actually takes place. This function $q_o(z)$ is obtained following exactly the scheme conducive to p(z), the only differences being the substitutions.

m_2 by m_o

n_2 by n_o

p_2 by p_o

$p_1(m)$ by $\frac{1}{P_1}$ $p_1(m)$

$$p_2(n) \quad \text{by} \quad \frac{1}{P_2} p_2(n)$$

$$p_3(\ell) \quad \text{by} \quad \frac{1}{P_3} p_3(\ell) \quad .$$

where

$$P_1 = \sum_{m = m_1}^{m_o} p_1(m)$$

$$P_2 = \sum_{n = n_1}^{n_o} p_2(n)$$

and

$$P_3 = \sum_{\ell = \ell_1}^{\ell_o} p_3(\ell) \quad .$$

Once $q_o(z)$ has been obtained, the new probability density function $q(z)$ of execution time z will be

$$q(z) = Pq_o(z) \quad .$$

The new average execution time is

$$T_o = \int_{kn_1m_1\ell_1}^{kn_om_o\ell_o} zq(z)dz$$

and the average number of executions per unit of time is

$$\frac{1}{T_o} \quad .$$

This answers question II.

Notice that if we change the rejection criterion, such as limiting the value of the product $m\ n\ \ell$, the technique for answering question II and III must be redesigned. Nevertheless, the composition rule i.e. formula (5), continues to be useful as a basic tool.

Another important application of the preceding theory lies in the field of error analysis. The standard formulation of a typical problem could be as follows:

An information system has a number of inputs, each one affected by an error with a known probability density. What are the probability densities of the different output errors? Although the literature contains a number of methods for solving such problems, the transformation and composition rules offer the noteworthy advantage of being able to handle any input distribution, any type of error such as random, bias, etc. and not to require the system to be analytical, that is, not necessarily differentiable with respect to all input variables. Such error analysis has been efficiently computerized in the past.

CONCLUSIONS AND RECOMMENDATIONS

The preceding procedures are claimed to be superior to approximate methods presently in use. Their computerized implementation is recommended.

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