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ON THE CONSTRUCTION OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH QUASIPERIODICAL COEFFICIENTS USING THE METHOD OF ACCELERATED CONVERGENCE

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SUMMARY

This paper constitutes very cumbersome computations with the view of construction of solutions of linear differential equations with quasiperiodical coefficients using the method of the so called "accelerated convergence". This work is based on a large number of Russian and Ukrainian works on various problems of quasiperiodical solutions of differential equations.

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1. Let us consider a system of differential equations

$$\frac{dx}{dt} = [A + P(\omega t)]x, \tag{1}$$

where A is a constant, $P(\omega t)$ are quasiperiodical n-dimensional matrices, $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ is the basis of matrix $P(\omega t)$, $x = (x_1, \ldots, x_n)$ is n-dimensonal vector, t is the time. Moreover, we shall assume that matrix $P(\omega t)$ is small.

As is well known, the question of construction of solutions and reducibility of equations of form (1.1), and also of more general form was considered in the works by Artem'yev [1], Yerugin [9], Shtokalo [12], Kolmogorov [10], Belagi [5], Gel'man [7 - 8], Adrianova [3], Mitropol'skiy [11], Blinova [6] and others. It amounts to the reduction of system (1.1) to a system of differential equations with constant coefficients. As to the latter, so far it constitutes an unresolved problem in all its entirety.

We shall consider in the present work the question of seeking a solution to system (1.1) by way of constructing the reducing matrix., whereupon for ts construction we shall utilize the method of Newton-type accelerated convergence well developed and very successfully applied in the works by Kolmogorov [10], Arnol'd [2] and Bogolyubov [4].

As is done in work [11], it is appropriate to introduce into the system of Eqs.(1.1) the corrections $\xi = \xi_{ij}$ (i, j = 1, 2, ..., n), and consider in its place the following system of differential equations:

$$\frac{dx}{dt} = Ax + [P(\omega t, \xi) + \xi]\, x, \tag{1.2}$$

where $A + \xi = B$, A is the matrix of the system of linear differential equations with constant coefficients, which we shall obtain after transformation of the system of Eqs.(1.1), $P(\phi, \xi)$ is a matrix, periodical with respect to $\phi = (\phi_1, \ldots, \phi_m)$ with period 2π, and analytical relative to complex arguments ϕ, ξ region

$$|\operatorname{Im} \varphi| \leqslant \varrho, \qquad |\xi| \leqslant \sigma,$$

$$|\xi| = \sum_{i,j=1}^{n} |\xi_{ij}|. \tag{1.3}$$

Let us set ourselves the following problem: to find such an analytical transformation with respect to

$$x = \Phi(\phi)y, \quad (\phi = \omega t), \tag{1.4}$$

where $\Phi(\phi)$ is a matrix, periodical with respect to ϕ with period 2π, and such $\xi = \xi^{(\infty)}$ that at $\xi = \xi^{(\infty)}$ system (1.2) be reduced to a linear one with the constant coefficients

$$\frac{dy}{dt} = Ay \tag{1.5}$$

Then, integrating system (1.5), we shall obtain the general solution of system (1.2) in the form

$$x = \Phi(\omega t)\, e^{At} x_0. \tag{1.6}$$

The solution of this problem constitutes the corollary of the results obtained in the work [11]; however, in our opinion, it is of interest to conduct a more detailed operation in the considered case of Eq.(1.2) also; this being in connection with the special actuality of the problem on reducibility of equations with quasiperiodical coefficients. At the same time, by the course of the solution of the problem, we shall resolve not only the question of reducibility of system (1.2), but also ascertain the form of this system's solution and obtain a method of construction of the reducing matrix $\Phi(\omega t)$.

2. Before we pass to the demonstration of the basic theorems for the system (1.2), let us pause at deduction of certain estimates, required in the following.

We shall denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of matrix A and assume that for certain constants $\varepsilon > 0$, $d > 0$, the following inequalities:

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(2.1)

are fulfilled for any integral values of vectors $k = (k_1, \ldots, k_m)$, where $(k\omega) = k_1\omega_1 + \ldots + k_m\omega_m$, $|k| = \Sigma |k_j|$, $i = \sqrt{-1}$.

We shall show that at fulfillment of condition (2,1), the equation

$$i(k\omega)Y_k = AY_k - Y_kA + P_k, \tag{2.2}$$

where

$$P_k = \frac{1}{(2\pi)^m} \int_0^{2\pi} \ldots \int_0^{2\pi} P(\varphi) e^{-i(k\varphi)} d\varphi, \qquad |k| > 0, \tag{2.3}$$

is resolvable relative to Y_k for $|k| > 0$ and the estimate

$$|Y_k| < c_0 |k|^{d_1} |P_k|, \tag{2.4}$$

in which c_0, d_1 $(d_1 \geqslant d)$ are certain positive constants, independent of $\underline{k}$, is valid for Y_k.

In order to obtain the estimates of (2.4), we shall represent the matrix A in the form

$$A = UJU^{-1}. \tag{2.5}$$

where J is the matrix in the Jordanian form:

$$J = \{J_{Q_1}(\lambda_1), \ldots, J_{Q_l}(\lambda_l)\} \tag{2.6}$$

where

$$J_{Q_j}(\lambda_j) = \lambda_j E_{Q_j} + Z_{Q_j}, \tag{2.7}$$

E_{ϱ_l} is a ϱ_l-dimensional unitary matrix, Z_{ϱ_l} is a ϱ_l-dimensional matrix, whose subdiagonal consists of unities, while the other elements are zero:

$$E_{\varrho_l} = \delta_{\alpha\beta}^{(l)} = \begin{cases} 0, & \alpha \neq \beta, \\ 1, & \alpha = \beta \end{cases} \qquad (\alpha, \beta = 1, \ldots, \varrho_l); \tag{2.8}$$

$$Z_{\varrho_l} = \delta_{\alpha-1,\beta}^{(l)} = \begin{cases} 0, & \alpha - 1 \neq \beta \\ 1, & \beta = \alpha - 1 \end{cases} \qquad \begin{matrix} (\alpha, \beta = 1, \ldots, \varrho_l), \\ (\alpha = 2, \ldots, \varrho_l). \end{matrix} \tag{2.9}$$

Substituting according to (2.5) the value of A into Eq.(2.2), we obtain:

$$i(k\omega)Y_k = UJU^{-1}Y_k - Y_kUJU^{-1} + P_k$$

or

$$i(k\omega)U^{-1}Y_kU + U^{-1}Y_kU \cdot J = J \cdot U^{-1}Y_kU + U^{-1}P_kU. \tag{2.10}$$

Let us denote

$$U^{-1}Y_kU = X, \qquad U^{-1}P_kU = Q; \tag{2.11}$$

then Eq.(2.10) will be written in the form

$$i(k\omega)X + XJ = JX + Q. \tag{2.12}$$

Let us now break down matrices X and Q in sets:

$$X = \{X_{\varrho_j\varrho_r}\}, \qquad Q = \{Q_{\varrho_j\varrho_r}\}, \tag{2.13}$$

where $X_{\varrho_j\varrho_r}$, $Q_{\varrho_j\varrho_r}$ $(j, r = 1, \ldots, l)$ are matrices having ϱ_j lines and ϱ_r columns.

Equation (2.12) disintegrates into a system of independent equations

$$i(k\omega)X_{\varrho_j\varrho_r} + X_{\varrho_j\varrho_r}J_{\varrho_r}(\lambda_r) = J_{\varrho_j}(\lambda_j)X_{\varrho_j\varrho_r} + Q_{\varrho_j\varrho_r} \qquad (j, r = 1, \ldots, l). \tag{2.14}$$

Let us now denote by $x_\alpha^{(jr)}$, $q_\alpha^{(jr)}$ $(\alpha = 1, \ldots, \varrho_j)$ the vectors

$$x_\alpha^{(jr)} = (x_{\alpha 1}^{(jr)}, \ldots, x_{\alpha\varrho_r}^{(jr)}), \qquad q_\alpha^{(jr)} = (q_{\alpha_1}^{(jr)}, \ldots, q_{\alpha_{\varrho_r}}^{(jr)}); \tag{2.15}$$

which are α-th lines of matrices $X_{\varrho_j\varrho_r}$, $Q_{\varrho_j\varrho_r}$.

Then, taking into account the denotation (2.7), system (2.14) may be represented in the form

$$x_\alpha^{(jr)}J_{\varrho_r}(\lambda_r - \lambda_j + i(k\omega)) = x_{\alpha-1}^{(jr)} + q_\alpha^{(jr)}, \tag{2.16}$$

where $x_0^{(jr)} = 0$, $\alpha = 1, \ldots, \varrho_j$, $j, r = 1, \ldots, l$.

According to condition (2.1), it is obvious that system (2.16) is resolvable, whereupon this solution has the form

$$x_\alpha^{(jr)} = \{x_{\alpha-1}^{(jr)} + q_\alpha^{(jr)}\}J_{\varrho_r}^{-(jr)}(\lambda_r - \lambda_j + i(k\omega \tag{2.17}$$

where

$$J_{\varrho_r}^{-(jr)}(\lambda_r - \lambda_j + i(k\omega)) = \frac{1}{\lambda_r - \lambda_j + i(k\omega)}\left\{E_{\varrho_r} - \frac{Z_{\varrho_r}}{\lambda_r - \lambda_j + i(k\omega)} + \right.$$

$$\left. + \ldots + (-1)^{\varrho_r - 1}\frac{Z_{\varrho_r}^{\varrho_r - 1}}{(\lambda_r - \lambda_j + i(k\omega))^{\varrho_r - 1}}\right\}. \tag{2.18}$$

Now we shall find the estimate for expressions (2.17).

According to (2.18) and (2.9), we have

$$|x_1^{(jr)}| \leq |q_1^{(jr)}|\left\{\frac{\varrho_r}{|\lambda_r - \lambda_j + i(k\omega)|} + \frac{\varrho_r - 1}{|\lambda_r - \lambda_j + i(k\omega)|^2} + \cdots + \frac{1}{[\lambda_r - \lambda_j + i(k\ )]^{\varrho_r}}\right.$$

or, taking into account condition (2.1)

$$|x_1^{(lr)}| \leqslant |q_1^{(lr)}|(\varepsilon^{-1}|k|^d \varrho_r + \varepsilon^{-2}|k|^{2d}(\varrho_r - 1) + \ldots + \varepsilon^{-\varrho_r}|k|^{\varrho_r d}),$$
$$|x_1^{(lr)}| \leqslant c_1 \frac{|k|^{\varrho_r d}}{\varepsilon^{\varrho_r}} \cdot |q_1^{(lr)}|, \tag{2.19}$$

where c_1 is a constant independent of $\underline{k}$ and ε.

Taking into account the inequality (2.19), we find from expression @2.17) for $\alpha = 2, 3, \ldots, Q_j$:

$$|x_\alpha^{(lr)}| \leqslant (|q_1^{(lr)}| + \ldots + |q_\alpha^{(lr)}|) c_1^\alpha \frac{|k|^{\alpha\varrho_r d}}{\varepsilon^{\alpha\varrho_r}}. \tag{2.20}$$

After that, taking into account (2.13), we obtain for X the following estimate:

$$|X| \leqslant c_2 \frac{|k|^{d_1}}{\varepsilon^{d_1/d}} |Q|, \tag{2.21}$$

where
$$d_1 = \max_{1 \leqslant l \leqslant L} \varrho_l \max_{1 \leqslant r \leqslant L} \varrho_r \cdot d,$$

c_2 being a positive constant, independent of both $\underline{k}$ and ε.

Returning from X and Q respectively to Y_k and P_k, according to formulas (2.11), we obtain the inequality

$$|Y_k| \leqslant |U|^2 |U^{-1}|^2 c_2 \frac{|k|^{d_1}}{\varepsilon^{d_1/d}} |P_k|. \tag{2.22}$$

Denoting in (2.22)

$$c_0 = |U|^2 |U^{-1}|^2 \frac{c_2}{\varepsilon^{d_1/d}}, \tag{2.23}$$

we obtain the inequality (2.4) sought for.

From the solutions of Eqs.(2.2) let us now compose the series

$$Y(\varphi) = \sum_{|k| \neq 0} Y_k e^{i(k\varphi)}. \tag{2.24}$$

Differentiating it, we become convinced that $Y(\phi)$ is a formal solution of the equation

$$\frac{\partial Y}{\partial \varphi} \omega + YA = AY + P(\varphi) - \overline{P(\varphi)}, \tag{2.25}$$

where

$$\overline{P(\varphi)} = P_0 = \frac{1}{(2\pi)^m} \int_0^{2\pi} \ldots \int_0^{2\pi} P(\varphi)\, d\varphi. \tag{2.26}$$

3. When constructing the reducing matrix $\Phi(\phi)$, which, on the strength of transformation (1.4) transfers the system of Eqs.(1.2) to the integrable system (1.5), we shall make use of the iteration process with accelerated convergence. Let us describe here the s-th step of this process (s > 1). To that effect we shall denote by M_{s-1}, δ_{s-1}, ϱ_{s-1} the constants, with the help of which we shall characterize the (s − 1)-th iteration, and by M_s, δ_s, ϱ_s the constants with the aid of which we shall characterize the s-th iteration.

The transition from the (s − 1)-th to the s-th iteration may be characterized by the following assertion.

THEOREM 1. Assume that for the system of equations

$$\begin{aligned} \frac{dx}{dt} &= Ax + [P(\varphi, \xi) + \xi]x, \\ \frac{d\varphi}{dt} &= \omega \qquad (\omega = (\omega_1, \ldots, \omega_m)) \end{aligned} \tag{3.1}$$

A, ξ are n-dimensional matrices, $P(\phi, \xi)$ is the n-dimensional matrix, periodical with respect to $\phi = (\phi_1, \ldots, \phi_m)$ with period 2π, analytical by complex arguments ϕ, ξ in the region

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_{s-1}, \qquad |\xi| \leqslant M_{s-2} \tag{3.2}$$

ans satisfying the inequality

$$|P(\varphi, \xi)| \leqslant M_{s-1}. \tag{3.3}$$

Besides, eigenvalues of matrix A satisfy the inequality (2.1).

Then, for a sufficiently small M_0 and any s > 1, there exists the transformation

$$\begin{aligned} x &= [E + Y(\varphi, \xi_1)]x_1, \\ \xi &= \xi(\xi_1), \end{aligned} \tag{3.4}$$

periodical with respect to ϕ with period 2π and analytical with respect to ϕ, ξ in the region

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_s, \qquad |\xi_1| \leqslant M_{s-1}, \tag{3.5}$$

bringing the system of equations (3.1) to the form

$$\begin{aligned} \frac{dx_1}{dt} &= Ax_1 + [P_1(\varphi, \xi_1) + \xi_1]x_1, \\ \frac{d\varphi}{dt} &= \omega, \end{aligned} \tag{3.6}$$

where ξ_1 is a constant, and $P_1(\phi_1,\xi_1)$ is periodical with respect to ϕ with period 2π of the matrix, analytical with respect to ϕ, ξ in the region (3.5), and such that inequalities

$$|P_1(\varphi,\xi_1)| \leqslant M_s, \tag{3.7}$$

$$|Y(\varphi,\xi_1)| \leqslant \frac{M_{s-1}^{\varkappa-1}}{4n}, \qquad |\xi(\xi_1)-\xi_1| \leqslant M_{s-1}. \tag{3.8}$$

are valid in the region (3.5).

At the same time, constants M_s, δ_s, ϱ_s are linked with constants M_{s-1}, δ_{s-1}, ϱ_{s-1} by means of relations

$$M_s = M_{s-1}^{\varkappa}, \qquad \delta_s = \gamma\delta_{s-1}, \qquad \varrho_s = \varrho_{s-1} - 2\delta_{s-1} \qquad (s \geqslant 1), \tag{3.9}$$

where

$$1 < \varkappa < 2, \qquad \gamma = \frac{\varrho_0}{4+\varrho_0}, \qquad \delta_0 = \gamma, \qquad \varrho_0 = \varrho, \qquad M_{-1} = \sigma. \tag{3.10}$$

<u>DEMONSTRATION</u>. First of all let us bring system (3.1) to the form (3.6). To that effect let us choose the coordinate transformation in the form

$$x = [E + Y(\varphi,\xi)]\,x_1, \tag{3.11}$$

where $Y(\phi, \xi)$ is the solution of Eq.(2.25).

Substituting (3.11) into Eq.(3.1) and taking (2.25) into account, we obtain the following system of equations:

$$\frac{dx_1}{dt} = Ax_1 + [P_1^{(1)}(\varphi,\xi) + \xi + \overline{P(\varphi,\xi)}]\,x_1,$$
$$\frac{d\varphi}{dt} = \omega, \tag{3.12}$$

where the following denotation has been used

$$P_1^{(1)}(\varphi,\xi) = (E+Y)^{-1}[(P+\xi)Y - Y(\bar{P}+\xi)]. \tag{3.13}$$

Let us define the transformation $\xi = \xi(\xi_1)$ as the solution of the equation

$$\xi + \overline{P(\varphi,\xi)} = \xi_1. \tag{3.14}$$

Then, the substitution of variables (3.11), (3.14) will bring the initial equation (3.1) to the form required by us:

$$\frac{dx_1}{dt} = Ax_1 + [P_1(\varphi,\xi_1) + \xi_1]\,x_1,$$
$$\frac{d\varphi}{dt} = \omega,$$

where

$$P_1(\varphi,\xi_1) = P_1^{(1)}(\varphi,\xi(\xi_1)). \tag{3.15}$$

Let us examine, first of all, Eq.(3.14). According to theorem condition matrix $P(\phi, \xi)$ is an analytical function $\xi = (\xi_{\alpha\beta})$, $(\alpha, \beta = 1, \ldots, n)$, and satisfies in the region (3.2) the inequality

$$|\overline{P(\varphi, \xi)}| \leqslant M_{s-1}. \tag{3.16}$$

According to properties of analytical functions, it follows from (3.2) and (3.16) that in the region

$$|\xi| \leqslant \frac{M_{s-2}}{2} \tag{3.17}$$

the inequality

$$\sum_{\alpha,\beta=1}^{n} \left| \frac{\partial \overline{P(\varphi, \xi)}}{\partial \xi_{\alpha\beta}} \right| \leqslant \frac{2M_{s-1}n^2}{M_{s-2}} \leqslant 2n^2 M_{s-2}^{\varkappa-1} \leqslant \frac{1}{2}. \tag{3.18}$$

is valid, whence it follows that Eq.(3.15) is resolvable and $\xi = \xi(\xi_1)$ is an analytical function of ξ_1 in the region

$$|\xi_1| \leqslant M_{s-1},$$

whereupon according to (3.15) and (3.16) we have in this region the inequality

$$|\xi(\xi_1)| \leqslant 2M_{s-1} \leqslant \frac{M_{s-2}}{2},$$
$$|\xi(\xi_1) - \xi_1| \leqslant M_{s-1}. \tag{3.18'}$$

Let us derive still one more estimate of ξ. Differentiating the identity

$$\xi(\xi_1) + \overline{P(\varphi, \xi(\xi_1))} = \xi_1,$$

we have

$$\frac{\partial \xi_{kq}}{\partial \xi_{ij}^{(1)}} + \sum_{\alpha,\beta} \frac{\partial \overline{P(\varphi, \xi)}}{\partial \xi_{\alpha\beta}} \cdot \frac{\partial \xi_{\alpha\beta}}{\partial \xi_{ij}^{(1)}} = \frac{\partial \xi_{kq}^{(1)}}{\partial \xi_{ij}^{(1)}} = \begin{cases} 1, & k = i, \quad q = j, \\ 0, & k \neq i \quad q \neq j, \end{cases}$$

whence, taking into account (3.18), we find

$$\left| \frac{\partial \xi_{iq}}{\partial \xi_{ij}^{(1)}} \right| \leqslant \left| \frac{\partial \xi_{kq}^{(1)}}{\partial \xi_{ij}^{(1)}} \right| + 2n^2 M_{s-2}^{\varkappa-1} \max_{kq} \left| \frac{\partial \xi_{kq}}{\partial \xi_{ij}^{(1)}} \right|$$

or, summing up with repsect to i, j. we obtain

$$\max_{kq} \sum_{ij} \left| \frac{\partial \xi_{kq}}{\partial \xi_{ij}^{(1)}} \right| \leqslant \frac{1}{1 - 2n^2 M_{s-2}^{\varkappa-1}} \leqslant 1 + 4n^2 M_{s-2}^{\varkappa-1} \leqslant 2. \tag{3.19}$$

Let us now pause at the estimate of matrix $Y(\phi, \xi)$, entering in the formula for variable substitution (3,11).

According to the estimate (2.4) and denotations (2.24), we have the following inequality

$$|Y(\varphi, \xi)| \leqslant \sum_{|k|\neq 0} |Y_k||e^{i(k\varphi)}| \leqslant c_0 \sum_{|k|\neq 0} |k|^{d_1} |P_k(\xi)| e^{|\operatorname{Im}\varphi||k|} \tag{3.20}$$

for all ϕ, ξ from the region (3.5). But function $P(\phi, \xi)$ is analytical and bounded in the region (3.2); this is why for its Fourier coefficients $P_k(\xi)$ the estimate

$$|P_k(\xi)| \leqslant M_{s-1} e^{-\varrho_{s-1}|k|},$$

is valid; taking it into account, we obtain for $|Y(\phi, \xi)|$ the inequality

$$|Y(\varphi, \xi)| \leqslant c_0 M_{s-1} \sum_{|k|\neq 0} |k|^{d_1} e^{(|\operatorname{Im}\varphi| - \varrho_{s-1})|k|}. \tag{3.21}$$

For $|\mathrm{Im}| < \rho_s$, from (3,21) it follows

$$|Y(\varphi, \xi)| \leqslant c_0 M_{s-1} \sum_{|k|\neq 0} |k|^{d_1} e^{-2\delta_{s-1}|k|} \leqslant c_0 \left(\frac{d_1}{e}\right)^{d_1} \frac{(1+c)^m}{\delta_{s-1}^{d_1+m}} M_{s-1} =$$

$$= c_0 \left(\frac{d_1}{e}\right)^{d_1} (1+c)^m \frac{M_{s-1}^{2-\varkappa}}{\gamma^{s(d_1+m)}} M_{s-1}^{\varkappa-1} \leqslant \frac{M_{s-1}^{\varkappa-1}}{8n} < \frac{M_{s-1}^{\varkappa-1}}{4n}. \tag{3.22}$$

provided only M_0 is sufficiently small. Thus, all estimates (3.8) have been demonstrated.

The analyticity of function $Y(\phi, \xi(\xi_1))$ with respect to ξ_1 in the region $|\xi_1| < M_{s-1}$ follows from analyticity of function $P(\phi, \xi(\xi_1))$ on the strength of representation (2.24); the analyticity with respect to ϕ in the region $|\operatorname{Im}\varphi| \leqslant \varrho_s$ follows from the estimate

$$\left|\frac{\partial Y(\varphi, \xi)}{\partial\varphi}\right| \leqslant \sum_{|k|\neq 0} |Y_k||k| e^{\varrho_s|k|} \leqslant c_0 M_{s-1} \sum_{|k|\neq 0} |k|^{d_1+1} e^{-2\delta_{s-1}|k|} \leqslant$$

$$\leqslant c_0 \left(\frac{d_1+1}{e}\right)^{d_1+1} (1+c)^m \frac{M_{s-1}^{\varkappa}}{\gamma^{s(d_1+m+1)}} \leqslant c_1 M_{s-1}^{\varkappa-1}.$$

Let us now pass to the estimate of expression (3,15). We have

$$|P_1(\varphi, \xi_1)| = |P_1^{(1)}(\varphi, \xi(\xi_1))| \leqslant |(E + Y(\varphi, \xi))^{-1}| \times$$
$$\times [|P(\varphi, \xi) + \xi(\xi_1)| \cdot |Y(\varphi, \xi)| + |Y(\varphi, \xi)||\xi_1|]. \tag{3.23}$$

Therefore, taking into account the inequalities (3.3), (3.18') and (3.22), we find

$$|P_1(\varphi, \xi_1)| \leqslant \sum_{i=0}^{\infty} \left(\frac{M_{s-1}}{4n}\right)^i \cdot 4M_{s-1} \frac{M_{s-1}^{\varkappa-1}}{8n} < \frac{1}{n} M_{s-1}^{\varkappa} \leqslant M_s,$$

which precisely concludes the demonstration of theorem 1.

4. Let us now apply the preceding theorem for the construction of iteration process with accelerated convergence for system (1.2), which would subsequently increase the smallness of function $P(\phi, \xi)$ and yield an explicit expression for the reducing matrix $\Phi(\phi)$.

We shall assume in sequence:

$$x = [E + U^{(1)}(\varphi, \xi)]\, x^{(1)},$$
$$\xi = \xi(\xi^{(1)}), \tag{4.1}$$

where $U^{(1)}(\phi, \xi) = Y(\phi, \xi)$ is determined from Eq.(2.25) and satisfies the inequalities (3.8) at $s = 1$.

In this case $x^{(1)}$, ϕ will satisfy the equations

$$\frac{dx^{(1)}}{dt} = Ax^{(1)} + [P^{(1)}(\varphi, \xi^{(1)}) + \xi^{(1)}]\, x^{(1)},$$
$$\frac{d\varphi}{dt} = \omega \tag{4.2}$$

To Eqs.(4.2) we shall again apply the transformation corresponding to this new systen according to theorem 1. As a result, we shall obtain

$$x^{(1)} = [E + U^{(2)}(\varphi, \xi^{(1)})]\, x^{(2)},$$
$$\xi^{(1)} = \xi^{(1)}(\xi^{(2)}), \tag{4.3}$$

where $U^{(2)}(\phi, \xi^{(1)})$ are solutions of type-(2.5) equation, respectively composed for the system (4.2) and $x^{(2)}$, ϕ are the solutions of the system of equations:

$$\frac{dx^{(2)}}{dt} = Ax^{(2)} + [P^{(2)}(\varphi, \xi^{(2)}) + \xi^{(2)}]\, x^{(2)},$$
$$\frac{d\varphi}{dt} = \omega . \tag{4.4}$$

Pursuing the indicated process, we shall have on the s-th step:

$$x^{(s-1)} = [E + U^{(s)}(\varphi, \xi^{(s-1)})]\, x^{(s)},$$
$$\xi^{(s-1)} = \xi^{(s-1)}(\xi^{(s)}), \tag{4.5}$$

where $x^{(s)}$, ϕ satisfy the equations

$$\frac{dx^{(s)}}{dt} = Ax^{(s)} + [P^{(s)}(\varphi, \xi^{(s)}) + \xi^{(s)}]\, x^{(s)},$$
$$\frac{d\varphi}{dt} = \omega . \tag{4.6}$$

The durability of the iteration process for all $s \geqslant 1$ is assured by the ch.ice of parameters (3.9) and assertions of theorem 1. This is why we shall utilize the estimates of theorem 1 for the demonstration of the convergence of this process.

First of all, we shall express x, ξ by $x^{(s)}$ and $\xi^{(s)}$. According to formulas of variable substitution (4.1), (4.3) and (4.5), we have

$$x = [E + U^{(1)}(\varphi, \xi(\xi^{(1)}))] \ldots [E + U^{(s)}(\varphi, \xi^{(s-1)}(\xi^{(s)}))]\, x^{(s)}$$
$$\xi = \xi(\xi^{(1)} \quad (\xi^{(s)})), \tag{4.7}$$

whence, taking into account that $\xi^{(j)}(\xi^{(j+1)}) = \xi^{(j)}(\xi^{(j+1)} \ldots (\xi^{(s)}))$, we find the expression for x and ξ by $x^{(s)}$ and $\xi^{(s)}$:

$$x = \prod_{j=1}^{s} [E + U^{(j)}(\varphi, \xi^{(j-1)}(\ldots(\xi^{(s)})))]\, x^{(s)},$$
$$\xi = \xi(\xi^{(1)} \ldots (\xi^{(s)})), \tag{4.8}$$

whereupon in (4.8) the product of matrices ought to be taken in the order defined by formula (4.7).

Let us consider in thn first place the sequence

$$B^{(s)}(\xi^{(s)}) = \xi(\xi^{(1)} \ldots (\xi^{(s)})) \qquad (s = 1, 2, \ldots). \tag{4.9}$$

Since $\xi^{(j)}(\xi^{(j+1)})$,for $0 \leqslant j \leqslant s-1$, is the solution of the equation

$$\xi^{(j)}(\xi^{(j+1)}) + \overline{P^{(j)}(\varphi, \xi^{(j)}(\xi^{(j+1)}))} = \xi^{(j+1)}, \tag{4.10}$$

in which $|P^{(j)}(\varphi, \xi^{(j)})| \leqslant M_j$, by taking into account the estimate (3.19), we may write for $|\xi^{(j+1)}| \leqslant M_j$

$$\sum_{\alpha,\beta} \left| \frac{\partial \xi^{(j)}}{\partial \xi^{(j+1)}_{\alpha\beta}} \right| \leqslant 1 + 4n^2 M_{j-2}^{\varkappa-1} \tag{4.11}$$

But, from inequality $|\xi^{(s)}| \leqslant M_{s-1}$, and by virtue of estimates (3,18'), it follows

$$|\xi^{(s-1)}(\xi^{(s)})| \leqslant 2M_{s-1} \leqslant \frac{M_{s-2}}{2},$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$|\xi^{(j)}(\xi^{(j+1)} \ldots (\xi^{(s)}))| \leqslant 2M_j \leqslant \frac{M_{j-1}}{2},$$

whence we conclude that for $|\xi^{(s)}| < M_{s-1}$, the inequality (4.11) remains in force for

This is why, differentiating (4.9), we obtain the estimate

$$\sum_{\alpha,\beta}\left|\frac{\partial B^{(s)}(\xi^{(s)})}{\partial\xi^{(s)}_{\alpha\beta}}\right| \leqslant (1+4n^2\delta^{\varkappa-1})(1+4n^2M_0^{\varkappa-1})\ldots(1+4n^2M_{s-2}^{\varkappa-1}) \leqslant 2\prod_{j=0}^{\infty}(1+4n^2M_j^{\varkappa-1}) \leqslant c_1, \tag{4.12}$$

after which we may write

$$|B^{(s+1)}(0)-B^{(s)}(0)| = |B^{(s)}(\xi^{(s)}(0))-B^{(s)}(0)| \leqslant c_1|\xi^{(s)}(0)| \leqslant c_1 M_{s-1}. \tag{4.13}$$

Taking advantage of (4.13), we find the estimate

$$|B^{(s+k)}(0)-B^{(s)}(0)| \leqslant c_1\sum_{j=0}^{k-1} M_{s+j-1} \leqslant c_2 M_{s-1}, \tag{4.14}$$

according to which the uniform convergence of matrices $B^{(s)}(0)$ is evident:

$$B^{(s)}(0)\to\xi^{(\infty)} \text{ as } s\to\infty, \tag{4.15}$$

whereupon

$$|\xi^{(\infty)}-B^{(s)}(0)| \leqslant c_2 M_{s-1} \leqslant M_{s-2}^{\varkappa-1}. \tag{4.16}$$

Let us now pass to the demonstration of uniform convergence of the sequence

$$\Phi^{(s)}(\varphi,\xi^{(s)}) = \prod_{j=1}^{s}[E+U^{(j)}(\varphi,\xi^{(j-1)}\ldots(\xi^{(s)}))]. \tag{4.17}$$

We shall show, first of all, that sequences $\xi^{(j)}(\ldots(\xi^{(s-1)}(0)))$ $(j=0,1,\ldots,s-1)$ converge uniformly as $s\to\infty$. By definition $\xi^{(j)}(\xi^{(j+1)})$ is a solution of the equation

$$\xi^{(j+1)} = \xi^{(j)} + \overline{P^{(j)}(\varphi,\xi^{(j)})} \qquad (\xi^{(0)} = \xi^{(\infty)}),$$

and this is why

$$\xi^{(j+1)} = \xi^{(j)}(\xi^{(j+1)}) + \overline{P^{(j)}(\varphi,\xi^{(j)}(\xi^{(j+1)}))}$$

is identical, whence we obtain

$$\xi^{(j+1)}(\ldots\xi^{(s-1)}(0)) = \xi^{(j)}(\xi^{(j+1)}\ldots(\xi^{(s-1)}(0))) + \overline{P^{(j)}(\varphi,\xi^{(j)}(\ldots\xi^{(s-1)}(0)))}. \tag{4.18}$$

Since by virtue of (4.15) $\xi(\xi^{(1)}\ldots(\xi^{(s-1)}(0))) \underset{s\to\infty}{\to} \xi^{(\infty)}$ by passing to limit in (4.18) we find

$$\begin{gathered}
\xi_0^{(1)} = \lim_{s\to\infty}\xi^{(1)}(\xi^{(2)}\ldots(\xi^{(s-1)}(0))) = \xi^{(\infty)} + \overline{P^{(1)}(\varphi,\xi^{(\infty)})}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\xi_0^{(j+1)} = \lim_{s\to\infty}\xi^{(j+1)}(\xi^{(j+2)}\ldots(\xi^{(s-1)}(0))) = \xi_0^{(j)} + \overline{P^{(j)}(\varphi,\xi_0^{(j)})}.
\end{gathered} \tag{4.19}$$

At the same time, the inequalities

$$|\xi_0^{(j)}| \leqslant 2M_j \leqslant \frac{M_{j-1}}{2} \tag{4.20}$$

are preserved for $\xi_0^{(j)}$, and identities

$$\xi^{(j)}(\xi^{(j+1)}(0)) = \xi_0^{(j)}. \tag{4.21}$$

do take place.

Assuming in (4.17) $\xi^{(s)} = \xi_0^{(s)}$, and taking into account (4.21), we find

$$\Phi^{(s)}(\varphi, \xi_0^{(s)}) = \prod_{j=1}^{s} [E + U^{(j)}(\varphi, \xi_0^{(j-1)})]. \tag{4.22}$$

Paying attention to the fact that

$$|U^{(j)}(\varphi, \xi_0^{(j-1)})| \leqslant \frac{M_{j-1}^{\varkappa-1}}{4n}, \tag{4.23}$$

we obtain

$$\begin{gathered} |\Phi^{(s+1)}(\varphi, \xi_0^{(s+1)}) - \Phi^{(s)}(\varphi, \xi_0^{(s)})| \leqslant |\Phi^{(s)}(\varphi, \xi_0^{(s)})| \, |U^{(s)}(\varphi, \xi_0^{(s-1)})| \leqslant \\ \leqslant \left| \prod_{j=1}^{s} \left(E + \frac{M_{j-1}^{\varkappa-1}}{4n} I \right) \right| \frac{M_{s-1}^{\varkappa-1}}{4n} \leqslant \frac{n}{4} \prod_{j=1}^{\infty} \left(1 + \frac{M_j^{\varkappa-1}}{4} \right) M_{s-1}^{\varkappa-1} \leqslant c_3 M_{s-1}^{\varkappa-1}, \end{gathered} \tag{4.24}$$

where I is the matrix, of which all elements equal the unity.

Making use of (4.24), we find the estimate

$$|\Phi^{(s+k)}(\varphi, \xi_0^{(s+k)}) - \Phi^{(s)}(\varphi, \xi_0^{(s)})| \leqslant c_3 \sum_{j=0}^{k-1} M_{s+j-1}^{\varkappa-1} \leqslant c_4 M_{s-1}^{\varkappa-1}, \tag{4.25}$$

according to which the uniform convergence of $\Phi^{(s)}(\phi, \xi_0^{(s)})$ is obvious:

$$\Phi^{(s)}(\varphi, \xi_0^{(s)}) \to \Phi(\varphi) \quad \text{as} \quad s \to \infty, \tag{4.26}$$

whereupon

$$|\Phi^{(s)}(\varphi, \xi_0^{(s)}) - \Phi(\varphi)| \leqslant c_4 M_{s-1}^{\varkappa-1}. \tag{4.27}$$

Functions $\Phi^{(s)}(\phi, \xi_0^{(s)})$ are analytical function of ϕ for $|\operatorname{Im}\varphi| \leqslant \varrho_s$, and consequently, $\Phi(\phi)$ is an analytical function of ϕ for

$$|\operatorname{Im}\varphi| \leqslant \varrho_0 - 2\sum_{s=0}^{\infty} \delta_s = \frac{\varrho_0}{2}.$$

We shall show that the matrix $\Phi(\phi)$ is nondegenerate. According to estinates (4.22), (4.23), the following inequality takes place:

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$$|\Phi^{(s)}(\varphi, \xi_0^{(s)}) - E| = \left|\prod_{j=1}^{s} (E + U^{(j)}(\varphi, \xi_0^{(j-1)})) - E\right| \leqslant$$

$$\leqslant \left|\left(\frac{M_0^{\varkappa-1}}{4n} + \ldots + \frac{M_{s-1}^{\varkappa-1}}{4n}\right) I + \left(\frac{M_0^{\varkappa-1}}{4n} \cdot \frac{M_1^{\varkappa-1}}{4n} n + \ldots + \frac{M_{s-2}^{\varkappa-1}}{4n} \cdot \frac{M_{s-1}^{\varkappa-1}}{4n} n\right) I +\right.$$

$$\left. + \ldots + \frac{M_0^{\varkappa-1}}{4n} \cdot \frac{M_1^{\varkappa-1}}{4n} \ldots \frac{M_{s-1}^{\varkappa-1}}{4n} \cdot n^{s-1} I\right| \leqslant \left[\prod_{j=1}^{s}\left(1 + \frac{M_{j-1}^{\varkappa-1}}{4}\right) - 1\right] |I| \leqslant c_5 < 1.$$

from which it follows that

$$|\Phi(\varphi) - E| \leqslant c_5 < 1. \tag{4.28}$$

At fulfillment of (4.28), the series $\sum_{l=0}^{\infty} (E - \Phi(\varphi))^l$ converges and defines the matrix, reciprocal (inverse) to $\Phi(\phi)$:

$$\sum_{l=0}^{\infty} (E - \Phi)^l = [E - (E - \Phi)]^{-1} = \Phi^{-1}; \tag{4.29}$$

the latter precisely meaning that $\Phi(\phi)$ is nondegenerate matrix.

Drawing the balance sheet of the above obtained results, we arrive at the following theorem.

<u>THEOREM 2</u>. Let the matrix $P(\phi, \xi)$ be periodical with respect to

$$\phi = (\phi_1, \ldots, \phi_m)$$

with period 2π, and analytical with respect to $\varphi, \xi = \{\xi_{\alpha\beta}\} (\alpha, \beta = 1,$ in the region

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_0, \qquad |\xi| = \sum_{\alpha,\beta} |\xi_{\alpha\beta}| \leqslant \sigma. \tag{4.30}$$

Besides, $\lambda_1, \ldots, \lambda_n$ are eigenvalues of matrix A.

Assume that for certain $\varepsilon > 0$, $d > 0$ the inequality

$$|\lambda_j - \lambda_l + i(k\omega)| \geqslant \varepsilon |k|^{-d} \tag{4.31}$$

is fulfilled for all integral values of vectors $k = (k_1, \ldots, k_m)$, where

$$j, l = 1, \ldots, n, \qquad (k\omega) = k_1\omega_1 + \ldots + k_m\omega_m, \qquad |k| = \sum_{i=1}^{m} |k_i|.$$

Then we may point to such a sufficiently small positive constant $M_0 = M_0(\varepsilon, d)$ and such matrix $\xi, |\xi| < 2M_0^{\varkappa-1}$ $(1 < \varkappa < 2)$, that for

$$|P(\varphi, \xi)| \leqslant M_0 \tag{4.32}$$

the system of equations

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$$\frac{dx}{dt} = Ax + [P(\varphi, \xi) + \zeta]x,$$
$$\frac{d\varphi}{dt} = \omega \tag{4.33}$$

by way of substitution of variables

$$x = \Phi(\phi)y \tag{4.34}$$

by matrix $\Phi(\phi)$, periodical with respect to ϕ, of period 2π, analyiical and analytically convergent in the region

$$|\operatorname{Im}\varphi| \leqslant \frac{\varrho}{2} \tag{4.35}$$

is brought to the form

$$\frac{dy}{dt} = Ay, \qquad \frac{d\varphi}{dt} = \omega. \tag{4.36}$$

At the same time, matrix $\Phi(\phi)$ is representable in the form of the product

$$\Phi(\varphi) = \prod_{j=1}^{\infty} (E + U^{(j)}(\varphi)), \tag{4.37}$$

where

$$U^{(j)}(\varphi) = \sum_{|k| \neq 0} Y_k^{(j)} e^{i(k\varphi)}. \tag{4.38}$$

$Y_k^{(j)}$ is a solution of the equation

$$i(k\omega) Y_k^{(j)} = AY_k^{(j)} - Y_k^{(j)}A + P_k^{(j-1)}, \tag{4.39}$$

in which

$$P_k^{(j)} = (2\pi)^{-m} \int_0^{2\pi} \dots \int_0^{2\pi} P^{(j)}(\varphi) e^{-i(k\varphi)} d\varphi_1 \dots d\varphi_m$$

are Fourier coefficients of function $P^{(j)}(\phi)$, determined by relations

$$P^{(j)} = [E + U^{(j)}]^{-1} [P^{(j-1)}U^{(j)} - U^{(j)}\overline{P^{(j-1)}}],$$
$$P^{(0)} = P(\varphi, \xi) + \xi, \qquad \overline{P^{(j-1)}} = (2\pi)^{-m} \int_0^{2\pi} \dots \int_0^{2\pi} P^{(j-1)}(\varphi)\, d\varphi_1 \dots d\varphi_m, \tag{4.40}$$

and which satisfies the inequality

$$\left| \Phi(\varphi) - \prod_{j=1}^{s} [E + U^{(j)}(\varphi)] \right| \leqslant 2M_0^{(\varkappa-1)\varkappa^{s-1}}. \tag{4.41}$$

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5. Let us turn back to the inequality

$$|\lambda_j - \lambda_l + i(k\omega)| \geqslant \varepsilon |k|^{-d}. \tag{5.1}$$

Following is the relation required for its fulfillment:

$$|(k\omega)| \geqslant \varepsilon |k|^{-d}. \tag{5.2}$$

For matrices A with real eigenvalues, condition (5.2) is insufficient. For systems with such matrices the result of theorem 2 may be amplified. Namely, the following theorem takes place.

THEOREM 3. Let $P(\phi)$ be a matrix, periodical with respect to $\phi = (\phi_1, \dots \phi_m)$ with period 2π, and analytical in the region

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_0 \tag{5.3}$$

Assume that, besides, eigenvalues of matrix A are real and different.

We shall postulate that for certain $\varepsilon > 0$, $d > 0$ the inequality

$$|(k\omega)| \geqslant \varepsilon |k|^{-d} \tag{5.4}$$

is fulfilled for all integral values of vectors $k = (k_1, \dots, k_m)$.

We then may indicate such a sufficiently small positive constant M, that for

$$|P(\varphi)| \leqslant M_0 \tag{5.5}$$

the system of equations

$$\frac{dx}{dt} = Ax + P(\varphi)x, \tag{5.6}$$

$$\frac{d\varphi}{dt} = \omega$$

by substitution of variables

$$x = \Phi(\varphi)y \tag{5.7}$$

with period with respect to ϕ of period 2π, analytical and analytically reversible matrix $\Phi(\phi)$ in the region

$$|\mathrm{Im}\,\varphi| \leqslant \frac{\varrho_0}{2} \tag{5.8}$$

is brought to the form

$$\frac{dy}{dt} = A^0 y, \qquad \frac{d\varphi}{dt} = \omega. \tag{5.9}$$

Moreover, matrices $\Phi(\phi)$ and A^0 are representable in the form

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$$\Phi(\varphi) = \prod_{j=1}^{\infty} (E + U^{(j)}(\varphi)), \tag{5.10}$$

where

$$A^0 = \sum_{j=1}^{\infty} \overline{P^{(j-1)}(\varphi)} + A,$$

$$U^{(j)}(\varphi) = \sum_{\|k\| \neq 0} Y_k^{(j)} e^{i(k,\varphi)}, \tag{5.11}$$

$Y_k^{(j)}$ is a solution of the equation

$$i(k\omega)\, Y_k^{(j)} = \left[A + \sum_{\alpha=0}^{j-1} \overline{P^{(\alpha)}(\varphi)}\right] Y_k^{(j)} - Y_k^{(j)} \left[A + \sum_{\alpha=0}^{j-1} \overline{P^{(\alpha)}(\varphi)}\right] + P_k^{(j-1)}, \tag{5.12}$$

in which

$$P_k^{(j)} = (2\pi)^{-m} \int_0^{2\pi} \dots \int_0^{2\pi} P^{(j)}(\varphi)\, e^{-i(k,\varphi)} d\varphi_1 \dots d\varphi_m, \quad \overline{P^{(j)}(\varphi)} = P_0^{(j)}$$

are Fourier coefficients of function $P^{(j)}(\phi)$, determined by the relations

$$P^{(j)} = (E + U^{(j)})^{-1} [P^{(j-1)} U^{(j)} - U^{(j)} \overline{P^{(j-1)}}], \qquad P^{(0)} = P(\varphi); \tag{5.13}$$

at the same time (*)

$$\left| \Phi(\varphi) - \prod_{j=1}^{s} (E + U^{(j)}(\varphi)) \right| \leq 2M_0^{(\varkappa-1)\varkappa^{s-1}}$$

$$\left| \sum_{j=s}^{\infty} \overline{P^{(j)}(\varphi)} \right| \leq 2M_0^{\varkappa^s} \tag{5.14}$$

DEMONSTRATION. First of all let us establish one property of matrices with real and different eigenvalues. In connection with this we shall demonstrate the following lemma.

L E M M A . Let eigenvalues $\lambda_1, \dots, \lambda_n$ of matrix A be real and different. We shall denote by $\mathfrak{A}_{r_0}$ the multitude of matrices A satisfying the inequality

$$|A - \overline{A}| = \sum_{i,j=1}^{n} |a_{ij} - \overline{a}_{ij}| \leq r. \tag{5.15}$$

We then may show such an $r = r_0(\lambda_1, \dots, \lambda_n) > 0$ that all matrices $\overline{A}$ from have real and different eigenvalues.

Indeed, let us include $\lambda_1, \dots, \lambda_n$ into non-intersecting segments $I_1, \dots, I_n$ with centers located respectively at points $\lambda_1, \dots, \lambda_n$.

Function $f(\lambda) = \det|A - \lambda E|$ continually reflects each of the segments I_j into the segment $f(I_j) = (\underline{m}_j, \overline{m}_j)$, where $\underline{m}_j = \min f(\lambda)$, $\overline{m}_j = \max f(\lambda)$; at the same time, since $\lambda_j \neq \lambda$, at $j \neq r$, we have

$$\underline{m}_j < 0, \qquad \overline{m}_j > 0 \tag{5.16}$$

Let us examine the function

$$f_r(\lambda) = \det|A + rA_1 - \lambda E| \tag{5.17}$$

where A_1 is any matrix, for which $|A_1| < 1$.

Function $f_r(\lambda)$ continually reflects each of the segments I_j in the segment $f_r(I_j) = [\underline{m}_j(r), \overline{m}_j(r)]$, where

$$\underline{m}_j(r) = \min f_r(\lambda), \qquad \overline{m}_j(r) = \max f_r(\lambda).$$

Moreover, function $f_r(\lambda)$ is a continuous function of $\underline{r}$. and this is why for $\lambda \in I_j$ relation

$$\lim_{r \to 0} f_r(\lambda) = f(\lambda), \tag{5.18}$$

takes place uniformly with respect to λ, from which, taking into account (5.16) follows the existence of such $r_0 > 0$ that for $r \leqslant r_0$

$$\underline{m}_j(r) < 0, \quad \overline{m}_j(r) > 0 \tag{5.19}$$

Inequalities (5.19) signify that over segments I_1, ... I_n there will found points

$$\overline{\lambda}_1 = \overline{\lambda}_1(r), \ldots, \overline{\lambda}_j = \overline{\lambda}_j(r), \ldots, \overline{\lambda}_n = \overline{\lambda}_n(r)$$

such that

$$f_r(\overline{\lambda}_j) = 0 \quad \text{for } j = 1, \ldots, n \tag{5.20}$$

The numbers $\overline{\lambda}_j$, $j = 1, \ldots, n$ are real and different and, besides, they constitute eigenvalues of matrix

$$\overline{A} = A + r_0 \frac{\overline{A} - A}{r_0} = A + rA_1,$$

which is arbitrary matrix from $\mathfrak{A}_{r_0}$. The last circumstance demonstrates the lemma.

Let us note also that, for any matrix $\overline{A}$ from $\mathfrak{A}_{r_0}$ one may always choose a matrix $\overline{U}$, leading $\overline{A}$ to Jordanian form $\overline{J}$:

$$\overline{A} = \overline{U}\overline{J}\overline{U}^{-1} \tag{5.21}$$

so that the inequality

$$|\overline{U}||\overline{U}^{-1}| \leq \overline{c}, \tag{5.22}$$

in which $\overline{c}$ is a constant, common for all matrices from $\mathfrak{A}_{r_0}$ be fulfilled.

Let us consider now the equation

$$i(k\omega)Y_k^{(j)} + Y_k^{(j)}A_{j-1} - A_{j-1}Y_k^{(j)} + P_k^{(j-1)} \quad (|k| \neq 0), \tag{5.23}$$

where A_{j-1} is an arbitrary matrix from $\mathfrak{A}_{r_0}$.

On the strength of assertions of section 2, Eq.(5.23) always has a solution, whereupon it is such that

$$\|Y_k^{(j)}\| \le \bar{c}_0 |k|^{d_1} \|P_k^{(j-1)}\|, \tag{5.24}$$

where $\bar{c}_0 = 2c_0 e^{-d_1} d_1^{d_1}$, c_0, d_1 are constants determined according to (2.23).

Let us assume that matrix $P^{(j)}(\phi)$ is periodical with respect to ϕ with period 2π, and analytical in the region

$$|\mathrm{Im}\,\varphi| \le \rho_j \quad (j = 0, 1, \ldots) \tag{5.25}$$

satisfying the inequality

$$|P^{(j)}(\phi)| < M_j, \tag{5.26}$$

where ρ_j, M_j are constants, linked with relations (3.9). Taking into account the inequality (5.24), wemay then write that

$$\|Y_k^{(j)}\| \le \bar{c}_0 |k|^{d_1} M_{j-1} e^{-\rho_{j-1}|k|}, \tag{5.27}$$

provided only that

$$P_k^{(j-1)} = (2\pi)^{-m} \int_0^{2\pi} \ldots \int_0^{2\pi} P^{(j-1)}(\varphi) e^{-i(k,\varphi)} d\varphi_1 \ldots d\varphi_m.$$

From (5.27) it follows that matrix

$$U^{(j)}(\varphi) = \sum_{|k| \neq 0} Y_k^{(j)} e^{i(k,\varphi)} \tag{5.28}$$

is analytical in the region

$$|\mathrm{Im}\,\varphi| \le \rho_j \tag{5.29}$$

by the solution of the equation

$$\frac{\partial U^{(j)}}{\partial \varphi}\omega + U^{(j)}A_{j-1} - A_{j-1}U^{(j)} + P^{(j-1)} = \overline{P^{(j-1)}}, \tag{5.30}$$

satisfying the inequality

$$|U^{(j)}(\varphi)| \le \frac{M_{j-1}^{\varkappa-1}}{4n}. \tag{5.31}$$

Let us now pass to the direct demonstration of Theorem 3. To that effect we shall make in system (5.6) the substitution of variables

$$x = [E + U^{(1)}(\phi)]x^{(1)}, \tag{5.32}$$

having postulated in (5,23) $j = 1$, $A_0 = A$, $P^{(0)} = P$.

The substitution (5,32) brings the system (5.6) to the form

$$\frac{dx^{(1)}}{dt} = A_1 x^{(1)} + P^{(1)}(\varphi)x^{(1)}, \quad \frac{d\varphi}{dt} = \omega, \tag{5.33}$$

where the following denotation is made:

$$A_1 = A + \overline{P(\varphi)}, \quad P^{(1)} = (E + U^{(1)})^{-1}[PU^{(1)} - U^{(1)}\overline{P}]. \tag{5.34}$$

Matrix $P^{(1)}(\phi)$ is analytical in the region

$$|\mathrm{Im}\,\varphi| \leqslant \varrho_1$$

and satisfies the inequality

$$|P^{(1)}(\varphi)| \leq \sum_{j=0}^{\infty} \left(\frac{M_0^{\varkappa-1}}{4\pi}\right)^j 2M_0 \cdot \frac{M_0^{\varkappa-1}}{4\pi} \leqslant M_0^{\varkappa} = M_1. \tag{5.35}$$

Let us take M_0 so small that

$$\sum_{j=0}^{\infty} M_j < r_0, \tag{5.36}$$

where r_0 is determined by the lemma. At fulfillment of (5.36), matrix A_1 belongs to the multitude $\mathfrak{A}_{r_0}$ since

$$|A_1 - A| = |\overline{P(\phi)}| < M_0 < r_0. \tag{5.37}$$

For that reason we may effect in system (5.33) the substitution of variables

$$x^{(1)} = [E + U^{(2)}(\phi)]\, x^{(2)} \tag{5.38}$$

and arrive at the system

$$\frac{dx^{(2)}}{dt} = A_2 x^{(2)} + P^{(2)}(\varphi)x^{(2)}, \quad \frac{d\varphi}{dt} = \omega, \tag{5.39}$$

when the following denotation is made:

$$A_2 = A_1 + \overline{P^{(1)}(\varphi)} = A + \overline{P(\varphi)} + \overline{P^{(1)}(\varphi)},$$
$$P^{(2)}(\varphi) = (E + U^{(2)})^{-1}(P^{(1)}U^{(2)} - U^{(2)}\overline{P^{(1)}}); \tag{5.40}$$

whereupon $P^{(2)}(\phi)$ is an analytical function of ϕ in the region

$$|\mathrm{Im}\,\varphi| \leqslant \rho_2, \tag{5.41}$$

satisfying the inequality

$$|P^{(2)}(\phi)| \leqslant M_2, \tag{5.42}$$

A_2 is the matrix from $\mathfrak{A}_{\rho_2}$.

Pursuing further the indicated process, we shall arrive at the assertion that the substitution of variables

$$x = \prod_{j=1}^{s} [E + U^{(j)}(\varphi)]\, x^{(s)} = \Phi^{(s)}(\varphi)\, x^{(s)} \tag{5.43}$$

results in initial system of equations (5.6) taking the form

$$\frac{dx^{(s)}}{dt} = A_s x^{(s)} + P^{(s)}(\varphi)\, x^{(s)}, \quad \frac{d\varphi}{dt} = \omega, \tag{5.44}$$

where

$$A_s = \sum_{j=0}^{s-1} \bar{P}^{(j)} + A_0,$$

$$P^{(s)} = (E + U^{(s)})^{-1} (P^{(s-1)} U^{(s)} - U^{(s)} P^{(s-1)});$$

at the same time $\Phi^{(s)}$ and $P^{(s)}$ are analytical functions of ϕ in the region

$$|\mathrm{Im}\,\varphi| \leqslant \rho_s \tag{5.46}$$

satisfying the inequalities

$$|\Phi^{(s)}(\varphi)| \leqslant \left| \prod_{j=1}^{s} \left(E + \frac{M_{j-1}^{\varkappa-1}}{4n} I \right) \right| \leqslant n^2 \prod_{j=1}^{s} \left(1 + \frac{M_{j-1}^{\varkappa-1}}{4} \right). \tag{5.47}$$

and A_s is the matrix from $\mathfrak{A}_{\rho_s}$.

It follows from (5.47) that

$$\phi^{(s)}(\phi) \to \Phi(\phi), \text{ as } s \to \infty \tag{5.48}$$

uniformly with respect to from the region

$$|\mathrm{Im}\,\varphi| \leqslant \frac{\rho_0}{2}, \tag{5.49}$$

This is why the substitution

$$x = \Phi(\varphi) y = \prod_{j=1}^{\infty} (E + U^{(j)}(\varphi)) y \tag{5.50}$$

leads the initial system of equations (5.6) to the form

$$\frac{dy}{dt} = A^0 y, \quad \frac{d\varphi}{dt} = \omega, \tag{5.51}$$

where

$$A^0 = A + \sum_{j=0}^{\infty} \overline{P^{(j)}(\varphi)}. \tag{5.52}$$

Estimates (5.14) easily follow from (5.47), which completes the demonstration of the theorem.

In conclusion, let us note, that for the case of periodical systems (-scalar), theorem 3 remains in force, provided the condition of reality and distinctions in eigenvalues of matrix A is replaced by the condition

$$\lambda_1 - \lambda_l + ik\omega \neq 0 \tag{5.53}$$

where j, l = 1, 2, ... , n; k = 1, 2, ...

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