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ANALYTIC FUNCTION MODELS OF NOISE AND MODULATION

by

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SUMMARY

This paper is essentially a presentation of the theory of complex analytic time functions applied to a representation of stochastic and deterministic time processes. By using analytic notation, noise and modulated signals are represented in the same manner. An intuitively satisfying, yet mathematically rigorous, method of transition from real to analytic processes is given. By use of the theory of distributions, the rotating vector, or phasor, method is generalized to arbitrary functions and stochastic processes. The advantages of analytic time function notation are exploited to derive correlation and spectral functions for both signal and noise processes. The polar and quadrature forms for band-pass processes are derived.

INTRODUCTION

The purpose of this paper is to present, in a comprehensible, yet reasonably rigorous manner, the fundamentals of a method of analysis whereby stochastic processes (noise) and signals are treated with the use of a common notation; that is, a notation is presented which allows the writing down of signal and noise expressions in the same form. This procedure eases the examination of those common characteristics of signals and noise which are of interest, such as correlation and spectral functions. The notational concept presented is that of expressing real signal and noise processes as analytic time functions or processes. Here, the word analytic is used in the sense of the theory of complex variables.

This analytic concept has existed for approximately 20 years. However, documentation is fragmentary and scattered. Also, a method of transition from real to analytic processes which is intuitionally satisfying and mathematically rigorous is thought, by the authors, to be lacking. This paper is written to remedy both of these difficulties.

The paper starts with a generalization of the rotating-vector method for steady-state analysis of linear time-invariant systems. Next, the analytic form of a real deterministic time function along with the Hilbert transform relations is developed. Then, the transition from deterministic functions to stochastic processes is made. A linear transform of the convolution type is introduced as a method of generating analytic functions or processes from their real parts. With the use of this transform, the correlation and spectral relations between real and analytic counterparts are developed.

The second part of the paper emphasizes band-pass processes; band-pass noise and modulated carriers are special cases of these processes. The quadrature form of a band-pass process is presented and its analytic counterpart is derived. Stationarity conditions are examined in detail. The text of the paper concludes with the development of the polar form of a band-pass process, which is particularly suited for representation of modulated carriers.

Three appendixes are given which serve to derive rigorously certain results needed in the text. Appendix A gives insight into the relationship between the Hilbert transform and analytic time functions. Appendix B covering correlation properties of complex stochastic processes is given for completeness. Appendix C contains some original results from distribution theory.

SYMBOLS AND NOTATION

This section defines the symbols and mathematical notation used in the body of the paper. Symbols for the appendixes are not listed here. Since the appendixes are short and self-contained, each new symbol is defined within each appendix.

Subscripting, as applied to functions, is standard. Thus, the symbol $Q_{xy}(\ )$ means the function $Q(\ )$ which is related to functions $x(\ )$ and $y(\ )$. General mathematical notation and functions are standard except as detailed below.

Mathematical notation:

$E\{\ \}$	statistical expectation operator
$\longleftrightarrow$	Fourier transform correspondence
$\triangleq$	equality by definition
$\wedge$	denotes Hilbert transform
$*$	midline asterisk denotes convolution; superscript asterisk denotes complex conjugate

$j = \sqrt{-1}$

Re( ) real part of variable

Real numbers:

A amplitude constant, dimensionless

t, τ independent time variables, seconds

α constant, dimensionless

ϕ phase constant, radians

σ dummy variable of integration

ω, ω_c frequency variables, radians per second

Deterministic functions and stochastic processes:

h( ) complex deterministic network impulse response function

m( ) complex low-pass modulation function (deterministic or stochastic, depending on context)

n( ) real band-pass stochastic noise process

n_1( ) real low-pass stochastic noise process

s( ) real band-pass signal function (deterministic or stochastic, depending on context)

x( ),y( ) real low-pass noise functions (deterministic or stochastic, depending on context)

z( ) complex low-pass noise function (deterministic or stochastic, depending on context)

A( ) real, nonnegative, low-pass amplitude function (deterministic or stochastic, depending on context)

H( ) Fourier transform of h( )

R( ) correlation function

S( ) Fourier transform of R( )

$\delta(\ )$ Dirac delta distribution

$\phi(\ )$ real low-pass signal or noise function (deterministic or stochastic, depending on context)

$\nu(\ )$ complex band-pass signal or noise function (deterministic or stochastic, depending on context)

ANALYTIC TIME FUNCTIONS AND STOCHASTIC PROCESSES

Generalization of the Phasor

Complex time functions have been used many years in the analysis of linear electrical systems. Theoretical treatments of lumped electrical networks (refs. 1 and 2), modulation techniques (ref. 3), and electromagnetic fields (ref. 4) have been simplified by the phasor or rotating vector concept. It is fundamental to the analysis of linear electrical systems that cause and effect in such systems may be related through linear nonhomogeneous integrodifferential equations. Where the electrical structure of such systems is not time varying, the integrodifferential equations have constant coefficients. Where the forcing function is sinusoidal and the solution is bounded (stable system), a steady-state solution (particular integral) which is also sinusoidal exists. By making a substitution of the form

$$A \cos(\omega t + \phi) = \mathrm{Re}\left[A \exp(j\phi) \exp(j\omega t)\right] \tag{1}$$

for each sinusoid in the integrodifferential equation, the solution, in the steady-state case, is reduced to solution of an algebraic equation involving phasors of the form $A \exp(j\phi)$.

The phasor convention is an aid in obtaining steady-state solutions of sinusoidally excited linear time-invariant systems. It has also found application as an aid to visualization for modulation techniques involving sinusoidal modulating functions. However, it is possible to generalize the phasor concept to treat nonsinusoidal time functions. By using results from the theory of analytic functions of a complex variable, it is possible to consider arbitrary real time functions as the real parts of complex analytic time functions. This consideration is particularly useful in the analysis of modulation techniques and treatment of noise (stochastic) processes. Although it might be possible to construct a means of analysis using complex functions which are not analytic, it will be seen that use of analytic time functions provides many advantages.

Deterministic Functions

Consider an arbitrary real time function $x(t)$ which is assumed, for the time being, to be deterministic; that is, it is assumed that although $x(t)$ is arbitrary, it is a member of the class of time functions whose members possess unique and explicit function descriptions. Examples of such functions are $\sin \omega t$, $\exp(-\alpha t)$, and so on. For the moment, functions which are members of the class of random functions (sample functions of stochastic processes) are excluded. Let the complex function related to $x(t)$ be defined as

$$z(t) = x(t) + jy(t) \tag{2}$$

The relationship between $y(t)$ and $x(t)$ must now be determined. Before considering $y(t)$, the question of what is meant by $z(t)$ being analytic must be answered. An analytic function, in the sense of the theory of complex variables, is a function which satisfies the Cauchy-Riemann conditions in some region of a complex plane. Thus to define analyticity, $z(t)$ must be defined in a complex plane the real axis of which is the t-axis.

If a function $z(t)$ which is analytic on the real t-axis of a complex plane is given, $x(t)$ and $y(t)$ are uniquely related by an integral relation known as the Hilbert transform:

$$y(t) \equiv \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} \, d\tau \triangleq \hat{x}(t) \tag{3}$$

where the integral is the Cauchy principal value. This relationship is derived in appendix A for a suitably restricted class of functions. The properties and behavior of the Hilbert transform are interesting and well known. (See refs. 5 and 6.)

Stochastic Processes

Consider the case where $x(t)$ represents a real stochastic process which has zero mean value and is weakly stationary. An immediate question is, "Can $z(t)$ be defined for the stochastic case, as in equation (2), where $y(t)$ is the Hilbert transform of $x(t)$?" The answer is yes. Under certain conditions (refs. 7 and 8), the integral (Hilbert transform of $x(t)$) converges in the mean-squared sense, for each value of t, to a random variable or to a stochastic process in t. However, it turns out that in the stochastic case, only the correlation and spectral properties of $x(t)$, $y(t)$, and $z(t)$ are of interest in this paper. Also, for $y(t)$ defined as the Hilbert transform of $x(t)$, it turns out that the auto-correlation function of $z(t)$ is an analytic function, as is seen later. Hence, for the purposes of this paper, it is sufficient to define $z(t)$ as "analytic" if its auto-correlation function is an analytic function.

An Analytic Distributional Convolution Transform

What is desired now is some relatively straightforward method for relating $x(t)$, $y(t)$, and $z(t)$ and their correlation functions. There are undoubtedly many approaches, each having an appeal depending on the individual reader's background. The following method is based on a linear network approach.

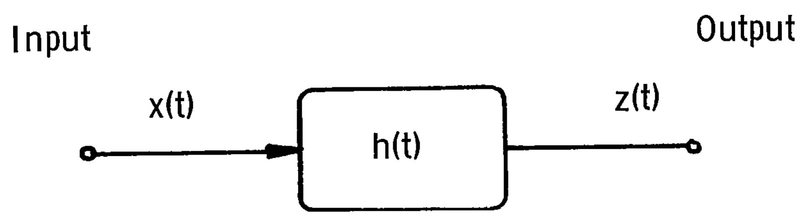

Figure 1.- Analytic function generating linear time-invariant network.

Consider a linear time-invariant network, as shown in figure 1 for which the driving function is $x(t)$ and the output is $z(t)$. The function $x(t)$ may represent either a deterministic time function or a zero-mean weakly stationary stochastic process. The impulse response (weighting) function of the network is $h(t)$. Thus, in the case where $x(t)$ is deterministic, $z(t)$ is represented as a linear transformation of the convolution type

$$z(t) = \int_{-\infty}^{\infty} x(\sigma)\, h(t - \sigma)\, d\sigma \triangleq x(t)*h(t) \tag{4}$$

which might be defined only as a distribution. In the case where $x(t)$ is stochastic, the auto-correlation function of $z(t)$ is (from appendix B)

$$R_{ZZ}(\tau) = R_{XX}(\tau)*h(\tau)*h^*(-\tau) \tag{5}$$

Now, $h(t)$ must be of such a form that both equations (4) and (5) are analytic.

It will be shown that equations (4) and (5) yield analytic results for $h(t)$ of the form

$$h(t) = \delta(t) + j\,\frac{1}{\pi t} \tag{6}$$

where $\delta(t)$ is the Dirac delta functional and is a singular distribution. The quantity $1/\pi t$ may also be viewed as a singular distribution. In fact, in the sense of distribution theory, $1/\pi t$ is the Hilbert transform of $\delta(t)$. (See ref. 9.) Thus, $h(t)$ may be said to be an analytic singular distribution.

Substituting equation (6) into equation (4) gives

$$\left.\begin{aligned} z(t) &= \int_{-\infty}^{\infty} x(\sigma)\left[\delta(t - \sigma) + j\,\frac{1}{\pi(t - \sigma)}\right]d\sigma \\ z(t) &= x(t) + j\,\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\sigma)}{t - \sigma}\,d\sigma \end{aligned}\right\} \tag{7}$$

which follows by the sifting property of the delta function. The imaginary part of $z(t)$ is, by definition, the Hilbert transform of $x(t)$. Thus $z(t)$ is analytic.

Evaluation of equation (5) for $h(t)$ given by equation (6) is carried out in two parts: First, it is shown that

$$h(\tau)*h^*(-\tau) = 2h(\tau) \tag{8}$$

Note that

$$h^*(-\tau) = \delta(-\tau) - j\,\frac{1}{-\pi\tau} = \delta(\tau) + j\,\frac{1}{\pi\tau} = h(\tau) \tag{9}$$

since $\delta(\tau)$ is even. Thus, formally

$$\left.\begin{aligned} h(\tau)*h^*(-\tau) &= \int_{-\infty}^{\infty}\left[\delta(\sigma) + j\,\frac{1}{\pi\sigma}\right]\left[\delta(\tau - \sigma) + j\,\frac{1}{\pi(\tau - \sigma)}\right]d\sigma \\ h(\tau)*h^*(-\tau) &= \int_{-\infty}^{\infty} \delta(\sigma)\;\delta(\tau - \sigma)\,d\sigma - \int_{-\infty}^{\infty} \frac{1}{\pi\sigma}\,\frac{1}{\pi(\tau - \sigma)}\,d\sigma + j\left[\int_{-\infty}^{\infty} \frac{1}{\pi\sigma}\,\delta(\tau - \sigma)d\sigma\right. \\ &\qquad \left. + \int_{-\infty}^{\infty} \delta(\sigma)\,\frac{1}{\pi(\tau - \sigma)}\,d\sigma\right] \\ h(\tau)*h^*(-\tau) &= \delta(\tau) + \delta(\tau) + j\left[\frac{1}{\pi\tau} + \frac{1}{\pi\tau}\right] \\ h(\tau)*h^*(-\tau) &= 2h(\tau) \end{aligned}\right\} \tag{10}$$

The convolution operations in equations (10) must be viewed as distributional convolutions. The convolutions exist in a distributional sense and this result is derived in appendix C. Additional reference material on distributional convolution may be found in references 10, 11, and 12.

Secondly, evaluation of equation (5) is simplified by the result of equations (10) as

$$R_{ZZ}(\tau) = R_{XX}(\tau)*2h(\tau) \tag{11}$$

Since $R_{xx}(\tau)$ is deterministic, it follows immediately, as in equation (7), that

$$R_{zz}(\tau) = 2\left[R_{xx}(\tau) + j\hat{R}_{xx}(\tau)\right] \tag{12}$$

when the Hilbert transform exists. Hence, for the $h(t)$ of equation (6), $R_{zz}(\tau)$ is analytic and $z(t)$ is an analytic stochastic process.

Correlation and Spectral Properties

Since a device in the network of figure 1 for generating analytic functions from their real parts has been conceived, the various correlation and spectral relations between $x(t)$, $y(t)$, and $z(t)$ may be developed.

From equation (B23), $R_{zx}(\tau)$ may be written as

$$R_{zx}(\tau) = R_{xx}(\tau)*h(\tau) = R_{xx}(\tau) + j\hat{R}_{xx}(\tau) \tag{13}$$

Also

$$R_{xz}(\tau) = R_{xx}(\tau)*h^{*}(-\tau) = R_{xx}(\tau)*h(\tau) = R_{zx}(\tau) \tag{14}$$

Thus, from equations (12), (13), and (14)

$$R_{zx}(\tau) = R_{xz}(\tau) = \frac{1}{2}R_{zz}(\tau) \tag{15}$$

From equations (B9)

$$\left.\begin{aligned} R_{zx}(\tau) &= R_{xx}(\tau) + jR_{yx}(\tau) \\ R_{xz}(\tau) &= R_{xx}(\tau) - jR_{xy}(\tau) \end{aligned}\right\} \tag{16}$$

From equations (13), (15), (16), and (B12), the fundamental result follows:

$$R_{yx}(\tau) = -R_{xy}(\tau) = -R_{yx}(-\tau) = \hat{R}_{xx}(\tau) \tag{17}$$

The result of equation (17) states that the cross-correlation function $R_{yx}(\tau)$ between the imaginary and real part of an analytic process is an odd function and is equal identically to the Hilbert transform of the auto-correlation function of the real part. This equation is a most useful result, as will be seen in the following sections.

From equations (B9), $R_{zz}(\tau)$ may be written as

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) + j\left[R_{yx}(\tau) - R_{xy}(\tau)\right] \tag{18}$$

A second fundamental result follows from equations (12) and (18):

$$R_{xx}(\tau) = R_{yy}(\tau) \tag{19}$$

The results of equations (12) to (19) have been derived for stochastic $x(t)$. However, they obviously hold for deterministic $x(t)$ provided suitable correlation functions may be defined, for example, as time averages. Results identical to fundamental results (eqs. (17) and (19)) have been derived for deterministic or ergodic $x(t)$ by Dugundji (ref. 13) by using time average correlation functions. His results were extended to weakly stationary $x(t)$ by Zakai (ref. 14) by using a frequency domain analysis based on a defined Fourier operator for the Hilbert transformation.

By the Wiener-Khintchine relations, the power spectral densities of $x(t)$, $y(t)$, and $z(t)$ are Fourier transforms of their respective auto-correlation functions. By use of a standard notation employed by Papoulis (ref. 15) and others, the correspondence between correlation function and power spectral density is denoted as

$$R_{ZZ}(\tau) \longleftrightarrow S_{ZZ}(\omega) \tag{20}$$

It is understood that the Fourier transform may exist in a distributional sense only. For functions $R_{ZZ}(\tau)$ which possess a Fourier transform in the classical sense (ref. 16) and which are convolutions of functions as in equation (11), the transform of the convolution is equal to the product of the transforms of the functions being convolved. A similar result is also valid for distributional convolution (ref. 10). Thus the spectral density of $z(t)$ may be written as

$$S_{ZZ}(\omega) = 2S_{XX}(\omega)\ H(\omega) \tag{21}$$

where

$$\left.\begin{aligned} S_{XX}(\omega) &\longleftrightarrow R_{XX}(\tau) \\ H(\omega) &\longleftrightarrow h(\tau) \end{aligned}\right\} \tag{22}$$

The Fourier transform of $h(\tau)$ is the sum of the transforms of the two singular distributions $\delta(\tau)$ and $j\frac{1}{\pi\tau}$ and is given as

$$H(\omega) = 1 + \mathrm{sgn}(\omega) = \begin{Bmatrix} 2; & (\omega > 0) \\ 1; & (\omega = 0) \\ 0; & (\omega < 0) \end{Bmatrix} \tag{23}$$

Thus

$$S_{ZZ}(\omega) = \begin{Bmatrix} 4S_{XX}(\omega); & (\omega > 0) \\ 2S_{XX}(\omega); & (\omega = 0) \\ 0\ ; & (\omega < 0) \end{Bmatrix} \tag{24}$$

Equations (24) is a third fundamental result. It shows that the spectral density of the analytic counterpart of $x(t)$ is just four times the spectral density of $x(t)$ for positive frequency and is zero for negative frequency. This result, then, implies the value and produces the motivation for use of analytic time functions. Such functions give a tool for modeling processes or signals having "one-sided" spectra. It might be suspected that such a model is of importance in treating single-sideband transmission systems.

By similar development, it may be shown that convolving the conjugate analytic kernel $h^*(t)$ with a real function or process produces the conjugate analytic $z^*(t)$. The auto-correlation function for $z^*(t)$ is also conjugate analytic. The power spectral density is zero for positive frequencies.

BAND-PASS PROCESSES

A band-pass process is a process having a spectral density with finite frequency domain so that the spectral density does not extend to the origin and becomes zero for sufficiently large ω. Included in this class of processes are purely stochastic (random) processes and also deterministic functions. In communication theory, both signals and noise may be modeled as band-pass stochastic processes. Signals may also be modeled as band-pass deterministic functions. As in previous sections, the stochastic processes are considered to be weakly stationary and of zero mean value.

The Quadrature Form

It is usual in dealing with band-pass processes to use a notation which explicitly refers the process to some frequency in the neighborhood of the process spectrum. A notational form which has been used for some time expresses a band-pass process $n(t)$ in terms of two low-pass processes $x(t)$ and $y(t)$, referring to a frequency ω_c. (See refs. 17 and 18.) This notation

$$n(t) = x(t) \cos \omega_c t - y(t) \sin \omega_c t \tag{25}$$

is closely related to that of Rice (ref. 19). Equation (25) expresses an amplitude modulation by $x(t)$ and $y(t)$ on two carriers which are in phase quadrature; hence, the name "quadrature form."

Now, $n(t)$ is, by definition, weakly stationary. Thus, the statistical properties of $n(t)$, through the second order, are time-invariant; thus, the two terms in equation (25) must be jointly stationary in the wide sense. It is not necessary that they be individually stationary since, in fact, they cannot be. That this is true may be demonstrated by forming the auto-correlation function of the first component process $n_1(t)$:

$$n_1(t) = x(t) \cos \omega_c t \tag{26}$$

This auto-correlation function is

$$\left.\begin{aligned}
R_{n_1n_1}(t+\tau,t) &= E\left\{n_1(t+\tau)\ n_1(t)\right\} \\
R_{n_1n_1}(t+\tau,t) &= E\left\{x(t+\tau)\cos\omega_c(t+\tau)\ x(t)\cos\omega_c t\right\} \\
R_{n_1n_1}(t+\tau,t) &= \frac{1}{2} R_{xx}(\tau)\left\{\cos\omega_c\tau + \cos\omega_c(2t+\tau)\right\}
\end{aligned}\right\} \tag{27}$$

Since $R_{n_1n_1}(t+\tau,t)$ is not a function of τ alone, it is not time-invariant. Therefore, the $n_1(t)$ component of $n(t)$ is not weakly stationary. A similar argument holds for the other component.

To find necessary conditions on $n(t)$ so that it is weakly stationary, the auto-correlation function is formed:

$$\begin{aligned}
R_{nn}(t+\tau,t) = E\left\{n(t+\tau)\ n(t)\right\} = \frac{1}{2} E\Bigg\{&\cos\omega_c\tau\left[x(t+\tau)\ x(t) + y(t+\tau)\ y(t)\right] \\
&+ \sin\omega_c\tau\left[x(t+\tau)\ y(t) - y(t+\tau)\ x(t)\right] \\
&+ \cos 2\omega_c t\Big\{\cos\omega_c\tau\left[x(t+\tau)\ x(t) - y(t+\tau)\ y(t)\right] \\
&- \sin\omega_c\tau\left[y(t+\tau)\ x(t) + x(t+\tau)\ y(t)\right]\Big\} \\
&+ \sin 2\omega_c t\Big\{\sin\omega_c\tau\left[y(t+\tau)\ y(t) - x(t+\tau)\ x(t)\right] \\
&- \cos\omega_c\tau\left[y(t+\tau)\ x(t) + x(t+\tau)\ y(t)\right]\Big\}\Bigg\}
\end{aligned} \tag{28}$$

Inspection of equation (28) shows that time invariance of $R_{nn}(t+\tau,t)$ requires that $x(t)$ and $y(t)$ be weakly stationary and that

$$\left.\begin{aligned}
R_{yy}(\tau) &= R_{xx}(\tau) \\
R_{yx}(\tau) &= -R_{xy}(\tau) = -R_{yx}(-\tau)
\end{aligned}\right\} \tag{29}$$

Equations (29), the necessary conditions for the quadrature form of a stochastic process to be weakly stationary, are satisfied by equations (17) and (19), the fundamental results of $x(t) + jy(t)$ being analytic. It is now but a short step to the expression of $n(t)$ as the real part of an analytic process $\nu(t)$ where

$$\nu(t) = z(t)\exp\left(j\omega_c t\right) \tag{30}$$

Sufficient conditions for $n(t)$ to be weakly stationary are for $z(t)$ to be analytic. But is it necessary for $z(t)$ to be analytic? The answer is no. Observe that the necessary conditions of equations (29) are weaker than those of equations (17) and (19) in that equations (29) do not require that $R_{yx}(\tau)$ be the Hilbert transform of $R_{xx}(\tau)$. Hence analyticity is sufficient, but not necessary.

Equations (29) may be compactly stated as

$$E\left\{z(t + \tau)\ z(t)\right\} = 0 \tag{31}$$

which is an auto-correlation type of average obtained without taking the conjugate of $z(t)$. Doob (ref. 20) shows that a complex zero-mean Gaussian process also satisfies equation (31). Thus, it is possible, in general, to write the analytic expression (eq. (30)) the real part of which has the quadrature form and is weakly stationary without $z(t)$ being analytic.

It is now necessary to determine the requirements that must be placed on $z(t)$ so that $\nu(t)$ will indeed be analytic. The auto-correlation function is developed as

$$\left.\begin{aligned} R_{\nu\nu}(t + \tau, t) &= E\left\{z(t + \tau)\ \exp\left[j\omega_c(t + \tau)\right] z^*(t)\ \exp\left(-j\omega_c t\right)\right\} \\ R_{\nu\nu}(t + \tau, t) &= E\left\{z(t + \tau)\ z^*(t)\ \exp\left(j\omega_c \tau\right)\right\} \\ R_{\nu\nu}(t + \tau, t) &= R_{zz}(\tau)\ \exp\left(j\omega_c \tau\right) \end{aligned}\right\} \tag{32}$$

The spectral density is

$$\left.\begin{aligned} S_{\nu\nu}(\omega) &= S_{zz}(\omega)*\delta(\omega - \omega_c) \\ S_{\nu\nu}(\omega) &= S_{zz}(\omega - \omega_c) \end{aligned}\right\} \tag{33}$$

since the distributional Fourier transform of the exponent function is the delta functional.

It is known from equation (24) that $\nu(t)$ being analytic requires that

$$S_{\nu\nu}(\omega) = 0; \quad (\omega < 0) \tag{34}$$

Obviously, then

$$S_{zz}(\omega) = 0; \quad (\omega < -\omega_c; \quad \nu(t) \quad \text{analytic}) \tag{35}$$

The result of equation (35) has been stated by Nuttall (ref. 21).

The Polar Form

A notational form which applies equally well to band-pass stochastic processes or deterministic functions is the polar form. This form is derived from the most general expression for a modulated carrier. Let s(t) be a real process or deterministic function of the form of a sinusoidal carrier which is simultaneously modulated in amplitude and in phase:

$$s(t) = A(t) \cos\left[\omega_c t + \phi(t)\right] \tag{36}$$

where A(t) is the amplitude function and $\phi(t)$ is the phase function.

The real s(t) may be written as the real part of an analytic $\nu(t)$ as in equation (30) where

$$z(t) = A(t) \exp\left[j\phi(t)\right] \tag{37}$$

In the polar form z(t) is usually termed "modulation function" and is sometimes denoted as m(t).

It is desired that when s(t) represents a stochastic process, it should be weakly stationary. The restrictions on the various functions due to stationarity of s(t) are now developed. The first-order results are obtained by taking the expected value of s(t):

$$E\left\{s(t)\right\} = E\left\{A(t) \cos \phi(t)\right\} \cos \omega_c t - E\left\{A(t) \sin \phi(t)\right\} \sin \omega_c t \tag{38}$$

For equation (38) to be independent of time requires the two expectations to be zero. These conditions are satisfied for

$$E\left\{A(t) \exp\left[j\phi(t)\right]\right\} = E\left\{z(t)\right\} = 0 \tag{39}$$

Thus the first-order requirement is that z(t) have a zero mean value.

The second-order results are obtained from the requirement that the autocorrelation function of s(t) should also be invariant with time. Thus,

$$\left.\begin{aligned}
E\left\{s(t+\tau)\, s(t)\right\} &= E\left\{A(t+\tau) \cos\left[\omega_c(t+\tau) + \phi(t+\tau)\right] A(t) \cos\left[\omega_c t + \phi(t)\right]\right\} \\
E\left\{s(t+\tau)\, s(t)\right\} &= \frac{1}{2} E\left\{A(t+\tau)\left\{\cos\left[2\omega_c t + \omega_c \tau + \phi(t+\tau) + \phi(t)\right]\right.\right. \\
&\qquad \left.\left. + \cos\left[\omega_c \tau + \phi(t+\tau) - \phi(t)\right]\right\}\right\} \\
E\left\{s(t+\tau)\, s(t)\right\} &= \frac{1}{2} E\left\{A(t+\tau)\, A(t) \cos\left[\omega_c \tau + \phi(t+\tau) - \phi(t)\right]\right\}
\end{aligned}\right\} \tag{40}$$

provided that

$$E\left\{A(t + \tau)\ A(t)\left\{\cos\left[2\omega_c t + \omega_c \tau + \phi(t + \tau) + \phi(t)\right]\right\}\right\} = 0 \tag{41}$$

It is easily verified that equation (41) is satisfied if

$$E\left\{A(t + \tau)\ A(t)\ \exp\left\{j\left[\phi(t + \tau) + \phi(t)\right]\right\}\right\} = E\left\{z(t + \tau)\ z(t)\right\} = 0 \tag{42}$$

Equation (42) is satisfied if $\phi(t)$ contains a uniformly distributed constant. Then, equations (40) is invariant with time if $z(t)$ is weakly stationary.

From equation (33), it is seen that when $s(t)$ represents a stochastic signal, its sideband structure is determined by the spectral density of $z(t)$. In particular, if $z(t)$ is analytic, $s(t)$ is a member of the class of signals having only an upper sideband. If $z(t)$ is conjugate, analytic $s(t)$ has only a lower sideband. If either $A(t)$ or $\phi(t)$ is constant, $s(t)$ is a classic amplitude or angle-modulated carrier.

When viewed as a signal, $\nu(t)$ in polar form is generally called an analytic signal (see refs. 22 to 24). The various choices for $z(t)$ and the resulting $A(t), \phi(t)$ to obtain hybrid modulations (single-sideband amplitude and angle modulation) have been treated by Bedrosian (ref. 24).

It is seen that by use of the polar form, signals may be treated in the same manner as noise; that is, by placing a signal (either deterministic or stochastic) in analytic form, all the results obtained for purely stochastic (noise) processes may be applied. Helstrom (ref. 25) has made a similar observation.

CONCLUDING REMARKS

This paper has presented a set of mathematical tools with which modulated carriers and random noise processes may be modeled in essentially the same way. Furthermore, both deterministic and stochastic modulation processes are tractable with this notation. The notation and concepts presented form a compact basis for treatment of hybrid amplitude and angle modulation systems. In particular, these tools of analysis are well adapted for examination of generalized single-sideband signals in the presence of noise.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., April 30, 1969,
125-21-01-02-23.

APPENDIX A

COMPLEX TIME FUNCTIONS AND HILBERT TRANSFORMS

This appendix is given, for completeness, to show the relationship between Hilbert transform and analyticity. Similar treatments have appeared previously. (See ref. 23.)

Consider a complex time function $f(t)$ given as

$$f(t) = u(t) + jv(t) \tag{A1}$$

where $f(t)$ is a function of one independent variable, namely t. Then $f(t)$ may be described as analytic (on t) if it satisfies the Cauchy-Riemann conditions on the real t-axis of a complex plane; that is, if

$$z = t + j\sigma \tag{A2}$$

and

$$\left.\begin{aligned}\frac{\partial u(z)}{\partial t} &= \frac{\partial v(z)}{\partial \sigma} \qquad (-\infty < t < \infty; \quad \sigma = 0)\\ \frac{\partial u(z)}{\partial \sigma} &= -\frac{\partial v(z)}{\partial t} \qquad (-\infty < t < \infty; \quad \sigma = 0)\end{aligned}\right\} \tag{A3}$$

then $f(t)$ is analytic. When $f(t)$ is analytic, $u(t)$ and $v(t)$ are uniquely related by

$$\left.\begin{aligned}u(t) &= \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{v(\tau)}{\tau - t}\, d\tau\\ v(t) &= -\frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{u(\tau)}{\tau - t}\, d\tau\end{aligned}\right\} \tag{A4}$$

where the integral is the "Cauchy principal value" given by

$$⨍_{-\infty}^{\infty} \frac{u(\tau)}{\tau - t}\, d\tau = \lim_{\epsilon \to 0}\left\{\int_{-\infty}^{t-\epsilon} \frac{u(\tau)}{\tau - t}\, d\tau + \int_{t+\epsilon}^{\infty} \frac{u(\tau)}{\tau - t}\, d\tau\right\} \tag{A5}$$

The first and second integrals in equations (A4) are known as the inverse and direct Hilbert transforms, respectively.

To derive the Hilbert transform relationships rigorously, and yet in a manner which shows the implications of analyticity, requires entry to the complex plane. Consider a function $f(z)$ defined on a complex plane and analytic on the real axis and in the upper half-plane. By Cauchy's integral formula, $f(z)$ may be evaluated at any point $z = s$ by evaluating a line integral around a suitable contour enclosing s. Let the point $z = s$ lie on the real axis at t and the contour be as shown in figure 2. Then $f(t)$ is

$$f(t) = \frac{1}{2\pi j}\int_{\Gamma}\frac{f(z)}{z - t}\,dz \tag{A6}$$

where Γ is the closed contour composed of k_1, c_1, and c_2. The contour integral about Γ may now be broken into separate integrals along the paths c_2, c_1, and k_1. Let z be defined on c_2 and k_1, respectively, as

$$z = R\exp(j\phi); \quad (0 \leqq \phi \leqq \pi) \tag{A7a}$$

$$z = t + \rho\exp(j\theta); \quad (\pi \leqq \theta \leqq 2\pi) \tag{A7b}$$

In the limit as R becomes arbitrarily large and ρ approaches zero, Γ encloses the entire upper half-plane, and the integral along c_1 approaches the Cauchy principal value. By suitably restricting the class of functions $f(z)$, the evaluation of the contour integral may be simplified.

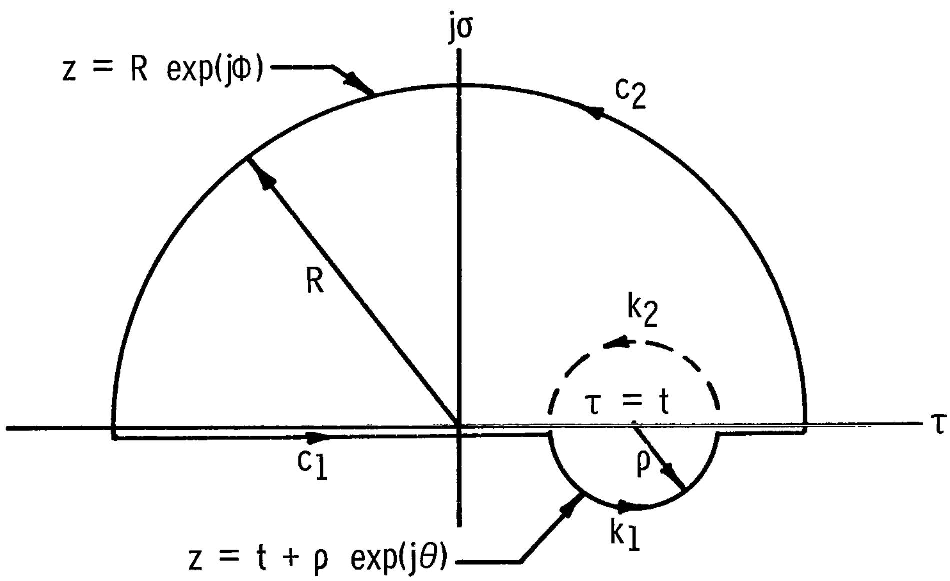

Figure 2.- Contour for evaluation of f(t).

The integral about Γ may be broken up as

$$f(t) = \frac{1}{2\pi j}\int_{\Gamma}\frac{f(z)}{z - t}\,dz \triangleq I_{c_1} + I_{c_2} + I_{k_1} \tag{A8}$$

where I_{c_1}, I_{c_2}, and I_{k_1} are the limiting case integrals along c_1, c_2, and k_1, respectively. Let $f(z)$ be restricted so that it is analytic on the τ-axis and in the entire upper half-plane. Then the integrand has only one simple pole at $z = t$. Let $f(z)$ be further restricted so that

$$\lim_{R\to\infty} I_{c_2} = 0 \tag{A9}$$

Thus $f(z)$ must be of the class for which $\frac{f(z)}{z - t}$ satisfies Jordan's Lemma. Actually, these restrictions may be weakened somewhat, but they suffice for the purpose of this appendix. The order of the class of functions for which the limit of I_{c_2} is zero may be determined as follows:

$$I_{c_2} = \frac{1}{2\pi j} \int_{c_2} \frac{f(z)}{z - t}\, dz \leqq \frac{1}{2\pi} \int_{c_2} \left|\frac{f(z)}{z - t}\right| |dz| \tag{A10}$$

by the Schwarz inequality. Also,

$$\frac{1}{2\pi} \int_{c_2} \left|\frac{f(z)}{z - t}\right| |dz| \leqq \frac{1}{2\pi} \int_{c_2} \frac{|f(z)|}{|z| - |t|} |dz| \tag{A11}$$

by the triangle inequalities. Applying the definition of z in equation (A7a) gives

$$I_{c_2} \leqq \frac{1}{2\pi} \int_{c_2} \left(\frac{R}{R - |t|}\right) |f(z)|\, d\phi \tag{A12}$$

Suppose now that

$$f(z) = kz^{\alpha} \qquad (k, \alpha \quad \text{constants}) \tag{A13}$$

Then

$$I_{c_2} \leqq \frac{1}{2\pi} \frac{kR^{\alpha+1}}{R - |t|} \int_0^{\pi} d\phi = \frac{k}{2} \frac{R^{\alpha+1}}{R - |t|} \tag{A14}$$

For I_{c_2} to approach 0 in the limit as R becomes arbitrarily large requires that α be negative

$$\lim_{R\to\infty} I_{c_2} \leqq \lim_{R\to\infty} \frac{k}{2} \frac{R^{\alpha+1}}{R - |t|} = 0; \quad (\alpha < 0) \tag{A15}$$

Thus I_{c_2} vanishes in the limit if $f(z)$ is of the order of z^{α} for α negative

$$\lim_{R\to\infty} I_{c_2} = 0 \qquad \left(f(z) = O\left(z^{\alpha}\right); \quad \alpha < 0\right) \tag{A16}$$

Next, the evaluation of the integrals along c_1, k_1 may be developed. It is desired that the integrals be evaluated in the limit as ρ approaches zero. It will be shown that in the limit, I_{k_1} equals I_{c_1}.

Since the only singularity in the integrand is the simple pole at $z = t$, the closed line integral about c_1, k_1, and c_2 equals the closed-line integral about k_1,k_2 where k_2 is shown dashed in figure 2. This result follows from the residue theorem. Thus, abbreviating the integrals yields

$$\int_{\Gamma} = \int_{c_1} + \int_{c_2} + \int_{k_1} = \int_{k_1} + \int_{k_2} \tag{A17}$$

In the limit for R arbitrarily large and ρ arbitrarily small,

$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \int_{\Gamma} = \lim_{\rho\to 0}\left(\int_{c_1} + \int_{k_1}\right) = \lim_{\rho\to 0}\left(\int_{k_1} + \int_{k_2}\right) \tag{A18}$$

$$\left.\begin{aligned}
\lim_{\rho\to 0}\left(\int_{k_2}\right) &= \lim_{\rho\to 0} \frac{1}{2\pi j}\int_{k_2} \frac{f(z)}{z - t}\, dz \\
\lim_{\rho\to 0}\left(\int_{k_2}\right) &= \lim_{\rho\to 0} \frac{1}{2\pi j}\int_{0}^{\pi} \frac{f(z)j\rho \exp(j\theta)}{t + \rho \exp(j\theta) - t}\, d\theta \\
\lim_{\rho\to 0}\left(\int_{k_2}\right) &= \lim_{\rho\to 0} \frac{1}{2\pi}\int_{0}^{\pi} f(z)d\theta
\end{aligned}\right\} \tag{A19}$$

by substitution from equation (A7). Because $f(z)$ is continuous and analytic in the neighborhood of $z = t$, the integral in equations (A19) converges uniformly. Thus the limit and integral operations may be exchanged.

$$\lim_{\rho\to 0}\left(\int_{k_2}\right) = \frac{1}{2\pi}\int_{0}^{\pi} \lim_{\rho\to 0}\left[f(z)\right]d\theta \tag{A20}$$

But

$$\lim_{\rho\to 0}\left[f(z)\right] = f(t) \tag{A21}$$

which is independent of θ. Hence,

$$\lim_{\rho\to 0}\left(\int_{k_2}\right) = \frac{f(t)}{2} \tag{A22}$$

Likewise it may be shown that

$$\lim_{\rho\to 0}\left(\int_{k_1}\right) = \frac{f(t)}{2} = \lim_{\rho\to 0}\left(\int_{k_2}\right) \tag{A23}$$

Thus

$$\lim_{\substack{R\to\infty\\ \rho\to 0}}\left(\int_{\Gamma}\right) = \lim_{\rho\to 0}\left(\int_{k_1}\right) + \lim_{\rho\to 0}\left(\int_{k_2}\right) = f(t) \tag{A24}$$

which agrees with equations (A8) and (A17). From equations (A18), (A22), and (A23) it is obvious that

$$I_{k_1} = I_{c_1} \tag{A25}$$

as stated, and that

$$\left.\begin{aligned} f(t) &= \frac{1}{2\pi j}\int_{\Gamma}\frac{f(z)}{z - t}\,dz = 2I_{c_1} \\ f(t) &= \frac{1}{\pi j}\oint_{-\infty}^{\infty}\frac{f(\tau)}{\tau - t}\,d\tau \end{aligned}\right\} \tag{A26}$$

Substituting equation (A1) into equations (A26) yields

$$\left.\begin{aligned} u(t) + jv(t) &= \frac{1}{\pi j}\oint_{-\infty}^{\infty}\frac{u(\tau) + jv(\tau)}{\tau - t}\,d\tau \\ u(t) + jv(t) &= \frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{v(\tau)}{\tau - t}\,d\tau - j\,\frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{u(\tau)}{\tau - t}\,d\tau \end{aligned}\right\} \tag{A27}$$

Equating the real and imaginary parts of equations (A27) gives the final result

$$\left.\begin{aligned} u(t) &= \frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{v(\tau)}{\tau - t}\,d\tau \\ v(t) &= -\frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{u(\tau)}{\tau - t}\,d\tau \end{aligned}\right\} \tag{A28}$$

APPENDIX B

CORRELATION PROPERTIES OF COMPLEX STOCHASTIC PROCESSES

The purpose of this appendix is to derive the properties of, and establish the notation for, complex stochastic processes. A stochastic process is defined as a class of nondeterministic, or random, time functions. This statement means that an explicit defining equation cannot be written for a member function of such a process. Rather, such functions and processes can only be described in terms of their statistics.

Fundamentals

The notation and outlook of this appendix is essentially that of Papoulis (ref. 15). After his procedure, a dual definition shall be employed here so that the equation

$$z(t) = x(t) + jy(t) \tag{B1}$$

shall be used to define both a stochastic process and a representative member function (sample function) of the process. In general, the result of an operation on a process $z(t)$ is stated as some function of $z(t)$. However, the operation itself is defined on a member function $z(t)$. Such a duality may appear to be confusing at first, but will, in the end, be a most natural way of dealing with stochastic processes.

In equation (B1) $z(t)$ is complex, whereas $x(t)$ and $y(t)$ are both real. By employing equation (B1) as a definition of complex processes, the operations on $z(t)$ may be reduced to operations on real processes. Thus, the large body of knowledge which applies to real stochastic processes may also be applied to complex processes. However, certain results for real processes are more easily derived by considering complex processes. This appendix assumes knowledge of probability theory for real stochastic processes. What is presented here is the extension to complex processes.

The properties of interest, to be derived here, are the statistical mean and statistical cross-correlation functions. These properties are average properties, as might be expected, since it is generally not possible to discuss instantaneous properties of random processes. Initial consideration may be limited to these two average properties. From these two, others may be derived.

The mean, or expected, value of a complex stochastic process $z(t)$ denoted $\eta_z(t)$ is defined as

$$\eta_z(t) = E\left\{z(t)\right\} \tag{B2}$$

where $E\{\}$ denotes a statistical averaging process known as the "expectation." The expectation or expected value of a random variable x is defined as

$$E\left\{x\right\} = \int_{-\infty}^{\infty} x p_x(x)\ dx \tag{B3}$$

where $p_x(x)$ is the probability density function of x, the derivative of the distribution function $P_x(x)$. For continuous $P_x(x)$, equation (B3) may be defined as a Riemann integral. Where the random variable x is of the discrete type, $p_x(x)$ may be defined by using delta functions (Dirac) and equation (B3) defined as a distributional integral. (See appendix C.)

The cross-correlation function, defined for pairs of complex processes $w(t)$ and $z(t)$, denoted $R_{wz}(t + \tau,t)$ is

$$R_{wz}(t + \tau,t) = E\left\{w(t + \tau)\ z^*(t)\right\} \tag{B4}$$

where the superscript asterisk denotes a complex conjugate. The symbol τ denotes a displacement in time between the two functions whose product is being averaged. In general, the auto-correlation function is dependent on both t and τ.

It is shown by Papoulis (ref. 15) and others that the expectation $E\left\{\ \right\}$ is a linear transformation. That is,

$$E\left\{aX + bY\right\} = aE\left\{X\right\} + bE\left\{Y\right\} \tag{B5}$$

where a and b are scalars and X and Y are random variables. The domain of $E\left\{\ \right\}$ is a space of random variables. In general, the range is a space of functions of one or more independent variables.

The cross-covariance function $C_{wz}(t + \tau,t)$ is defined in terms of equations (B2) and (B4) as

$$C_{wz}(t + \tau,t) = R_{wz}(t + \tau,t) - \eta_w(t + \tau)\ \eta_z^*(t) \tag{B6}$$

For the case where $w(t)$ and $z(t)$ are identical, equations (B4) and (B6) specialize to the following auto-correlation and auto-covariance functions, respectively:

$$\left.\begin{aligned} R_{zz}(t + \tau,t) &= E\left\{z(t + \tau)\ z^*(t)\right\} \\ C_{zz}(t + \tau,t) &= R_{zz}(t + \tau,t) - \eta_z(t + \tau)\ \eta_z^*(t) \end{aligned}\right\} \tag{B7}$$

Suppose $w(t)$ is a complex stochastic process given by

$$w(t) = u(t) + jv(t) \tag{B8}$$

Expanding equation (B4) into its components in regular and reverse order, where $u(t)$ and $v(t)$ are real stochastic processes, gives

$$\left.\begin{aligned}
R_{wz}(t+\tau,t) &= R_{ux}(t+\tau,t) + R_{vy}(t+\tau,t) + j\left[R_{vx}(t+\tau,t) - R_{uy}(t+\tau,t)\right] \\
R_{wz}(t,\ t+\tau) &= R_{xu}(t+\tau,t) + R_{yv}(t+\tau,t) + j\left[R_{xv}(t+\tau,t) - R_{yu}(t+\tau,t)\right] \\
R_{zw}(t+\tau,t) &= R_{xu}(t+\tau,t) + R_{yv}(t+\tau,t) - j\left[R_{xv}(t+\tau,t) - R_{yu}(t+\tau,t)\right] \\
R_{zw}(t,\ t+\tau) &= R_{ux}(t+\tau,t) + R_{vy}(t+\tau,t) - j\left[R_{vx}(t+\tau,t) - R_{uy}(t+\tau,t)\right]
\end{aligned}\right\} \tag{B9}$$

By a suitable change of coordinates in the defining equations, it may be determined that the following general relation exists:

$$C_{wz}(t+\tau,t) = C^*_{zw}(t,\ t+\tau) = C^*_{zw}(t-\tau,t) \tag{B10}$$

Equation (B10) is a very general relation from which several specializations may be made. It follows that for complex $z(t)$ and $w(t)$,

$$\left.\begin{aligned}
R_{wz}(t+\tau,t) &= R^*_{zw}(t-\tau,t) \\
R_{zz}(t+\tau,t) &= R^*_{zz}(t-\tau,t)
\end{aligned}\right\} \tag{B11}$$

For real processes, for example, $p(t)$ and $q(t)$,

$$\left.\begin{aligned}
R_{pq}(t+\tau,t) &= R_{qp}(t-\tau,t) \\
R_{pp}(t+\tau,t) &= R_{pp}(t-\tau,t)
\end{aligned}\right\} \tag{B12}$$

A stochastic process is defined as stationary in the strict sense, if all its statistical properties are invariant with time. Thus, for a strict sense stationary process

$$R_{wz}(t+\tau,t) = R_{wz}(\tau) \tag{B13}$$

Independence of time t of the mean and auto-correlation function are not sufficient to insure strict sense stationarity. A process with this more restricted form of stationarity is defined as weakly stationary, or stationary in the wide sense. It should be noted in passing that when a process is composed of the sum of two component processes, it may be possible that the sum is weakly stationary but neither component is weakly stationary. For example, see equation (27).

Linearly Transformed Processes

The relationships for the case where $z(t)$ is a linear time-invariant transformation of $w(t)$ are now considered; that is

$$z(t) = f\left[w(t)\right] \tag{B14}$$

where

$$f\left[a_1 w_1(t) + a_2 w_2(t)\right] = a_1 f\left[w_1(t)\right] + a_2 f\left[w_2(t)\right] \tag{B15}$$

and

$$z(t + \tau) = f\left[w(t + \tau)\right] \tag{B16}$$

Equation (B15) expresses the linear property of the transformation f, whereas equation (B16) expresses the time invariance.

Consideration is limited to transformations of the convolution type where $z(t)$ may be expressed in the form

$$z(t) \triangleq w(t)*h(t) = \int_{-\infty}^{\infty} w(t - \sigma)\, h(\sigma)\, d\sigma \tag{B17}$$

where $h(t)$ is some weighting function. Papoulis (ref. 15) and others have shown that if $w(t)$ is weakly stationary, then $z(t)$ is also. Furthermore, the expectation transformation $E\{\ \}$ may be manipulated so that the mean of $z(t)$ and the various correlation functions are easily obtained.

For $w(t)$ weakly stationary, the mean is given as

$$\eta_z = E\left\{z(t)\right\} = E\left\{\int_{-\infty}^{\infty} w(t - \sigma)\, h(\sigma)\, d\sigma\right\} \tag{B18}$$

By a theorem of Papoulis (ref. 15), the expectation and integration operations may be taken in reverse order. Then,

$$\eta_z = \int_{-\infty}^{\infty} E\left\{w(t - \sigma)\right\} h(\sigma)\, d\sigma = \eta_w * h(t) \tag{B19}$$

If $w(t)$ (and, hence, $z(t)$) is weakly stationary, the various correlation functions are obtained as follows. First, the cross-correlation function $R_{zw}(\tau)$ is obtained:

$$\left.\begin{aligned}
R_{zw}(\tau) &= E\left\{z(t + \tau)\, w^*(t)\right\} = E\left\{z(t)\, w^*(t - \tau)\right\} \\
R_{zw}(\tau) &= E\left\{w^*(t - \tau) \int_{-\infty}^{\infty} w(t - \sigma)\, h(\sigma)\, d\sigma\right\} \\
R_{zw}(\tau) &= E\left\{\int_{-\infty}^{\infty} w(t - \sigma)\, w^*(t - \tau)\, h(\sigma)\, d\sigma\right\}
\end{aligned}\right\} \tag{B20}$$

Exchanging order of integration and expectation, as was done for the mean, gives

$$R_{zw}(\tau) = \int_{-\infty}^{\infty} E\left\{w(t - \sigma)\, w^*(t - \tau)\right\} h(\sigma)\, d\sigma \tag{B21}$$

But

$$E\left\{w(t - \sigma)\, w^*(t - \tau)\right\} = R_{ww}(\tau - \sigma) \tag{B22}$$

Thus

$$R_{zw}(\tau) = \int_{-\infty}^{\infty} R_{ww}(\tau - \sigma)\, h(\sigma)\, d\sigma = R_{ww}(\tau) * h(\tau) \tag{B23}$$

Next the auto-correlation function $R_{zz}(\tau)$ is obtained:

$$\left.\begin{aligned}
R_{zz}(\tau) &= E\left\{z(t + \tau)\, z^*(t)\right\} \\
R_{zz}(\tau) &= E\left\{z(t + \tau) \int_{-\infty}^{\infty} w^*(t - \sigma)\, h^*(\sigma)\, d\sigma\right\} \\
R_{zz}(\tau) &= E\left\{\int_{-\infty}^{\infty} z(t + \tau)\, w^*(t - \sigma)\, h^*(\sigma)\, d\sigma\right\} \\
R_{zz}(\tau) &= \int_{-\infty}^{\infty} E\left\{z(t + \tau)\, w^*(t - \sigma)\right\} h^*(\sigma)\, d\sigma
\end{aligned}\right\} \tag{B24}$$

But

$$E\left\{z(t + \tau)\, w^*(t - \sigma)\right\} = R_{zw}(\sigma + \tau) \tag{B25}$$

Thus

$$\left.\begin{aligned}
R_{zz}(\tau) &= \int_{-\infty}^{\infty} R_{zw}(\sigma + \tau)\, h^*(\sigma)\, d\sigma = \int_{-\infty}^{\infty} R_{zw}(\tau - \sigma)\, h^*(-\sigma)\, d\sigma \\
R_{zz}(\tau) &= R_{zw}(\tau) * h^*(-\tau)
\end{aligned}\right\} \tag{B26}$$

Then, from equation (B23)

$$R_{zz}(\tau) = \left[R_{ww}(\tau) * h(\tau)\right] * h^*(-\tau) \tag{B27}$$

The brackets in equation (B27) may be dropped if the order of convolution is understood to be that of equations (B26) to give

$$R_{zz}(\tau) = R_{ww}(\tau) * h(\tau) * h^*(-\tau) \tag{B28}$$

Where the convolution of equation (B27) exists as a Riemann integral, the operation is both associative and commutative. Hence, the order is not important. However, for certain pathological functions (irregular distributions) where the convolution integral may exist only in a distributional sense, associativity may not apply and the order of convolution may become important.

APPENDIX C

RESULTS FROM DISTRIBUTION THEORY

The purpose of this appendix is to give a rigorous proof of equations (10). Such a proof must necessarily use techniques and results from the theory of distributions. (See refs. 10 to 12.) A working knowledge of the theory is assumed. Only the notation and two special lemmas are given here, prior to proof.

The reader should note that this proof contains within it the justification for the statement that "In the sense of distribution theory, $1/\pi t$ is the Hilbert transform of the Dirac delta distribution $\delta(t)$."

Definition 1: Let D denote the space of testing functions $\phi(t)$ (ref. 12, p. 2).

Definition 2: Let D' denote the space of (distributions) continuous linear functionals $f(t)$ over D (ref. 12, p. 7).

Definition 3: Let S denote the space of testing functions $\phi(t)$ of rapid descent (ref. 12, p. 100).

Definition 4: Let S' denote the space of (distributions of slow growth) continuous linear functionals $f(t)$ over S (ref. 12, p. 102).

Definition 5: Let $\langle f(t), \phi(t) \rangle$ denote the scalar which a distribution $f(t)$ associates with a testing function $\phi(t)$ (ref. 12, p. 7).

Definition 6: Let $f(t)*g(t)$ denote the distributional convolution of the distributions $f(t), g(t)$ (ref. 12, p. 124).

Definition 7: Let $f(t) \in S', \phi(t) \in S$. Let $\Phi(\omega)$ denote the Fourier transform of $\phi(t)$. Then, the distributional Fourier transform $F(\omega)$ of $f(t)$ is defined by the relation (ref. 12, p. 184; ref. 10, p. 64):

$$\langle F(\omega), \Phi(\omega) \rangle = \langle f(t), 2\pi\phi(-t) \rangle$$

Definition 8: Let $f(t) \in S'$. For p a natural number, $f(t)$ is said to be a member of the space L_p' if and only if:

$$f(t) = \sum_{m=1}^{M} \frac{d^{r_m}}{dt^{r_m}}\left[f_m(t)\right]$$

for some natural numbers M and r_m; and, for each $f_m(t) \in L_p$,

$$\int_{-\infty}^{\infty} \left|f_m(t)\right|^p dt < \infty$$

(See ref. 10, p. 163.)

Lemma 1: For $1 \leqq p,q \leqq 2$, let $f_1(t) \in L_p'$ and $f_2(t) \in L_q'$. Let $F_1(\omega)$ and $F_2(\omega)$ be the distributional Fourier transforms of $f_1(t)$ and $f_2(t)$, respectively. Then, the distributional Fourier transform of $f_1(t)*f_2(t)$ is $F_1(\omega)\,F_2(\omega)$. (See ref. 10, p. 167.)

Lemma 2: For p a natural number such that $p > 1$, $\frac{1}{\pi t} \in L_p'$. (See ref. 10, p. 171.)

Theorem: Let h(t) denote the distribution given by

$$h(t) = \delta(t) + j\,\frac{1}{\pi t}$$

where $\delta(t)$ is the Dirac delta distribution. Then

$$h(t)*h^*(-t) = 2h(t)$$

Proof: It is seen that

$$h^*(-t) = \delta(-t) - j\,\frac{1}{\pi(-t)} = \delta(t) + j\,\frac{1}{\pi t} = h(t)$$

since $\delta(t)$ is even. Hence,

$$h(t)*h^*(-t) = \left[\delta(t) + j\,\frac{1}{\pi t}\right]*\left[\delta(t) + j\,\frac{1}{\pi t}\right]$$

$$h(t)*h^*(-t) = \delta(t) + 2j\,\frac{1}{\pi t} - \frac{1}{\pi t} * \frac{1}{\pi t}$$

by the sifting property of $\delta(t)$. For $p > 1$, $\frac{1}{\pi t} \in L_p'$, by lemma 2. Hence,

$$\left\langle -\frac{1}{\pi t} * \frac{1}{\pi t}, \phi(t)\right\rangle = \frac{1}{2\pi}\left\langle \frac{1}{\pi t} * \frac{-1}{\pi t}, 2\pi\phi(-t)\right\rangle$$

$$\left\langle -\frac{1}{\pi t} * \frac{1}{\pi t}, \phi(t)\right\rangle = \frac{1}{2\pi}\left\langle \mathrm{sgn}^2(\omega), \Phi(\omega)\right\rangle$$

$$\left\langle -\frac{1}{\pi t} * \frac{1}{\pi t}, \phi(t)\right\rangle = \frac{1}{2\pi}\left\langle 1, \Phi(\omega)\right\rangle$$

$$\left\langle -\frac{1}{\pi t} * \frac{1}{\pi t}, \phi(t)\right\rangle = \frac{1}{2\pi}\left\langle \delta(t), 2\pi\phi(-t)\right\rangle$$

$$\left\langle -\frac{1}{\pi t} * \frac{1}{\pi t}, \phi(t)\right\rangle = \left\langle \delta(t), \phi(t)\right\rangle$$

Thus,

$$h(t)*h^*(-t) = \delta(t) + 2j\,\frac{1}{\pi t} + \delta(t)$$

$$h(t)*h^*(-t) = 2\left[\delta(t) + j\,\frac{1}{\pi t}\right]$$

$$h(t)*h^*(-t) = 2h(t)$$

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