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ANALYSIS OF THE PERTURBATIONS IN THE ANGULAR ORBITAL ELEMENTS FOR THE SATELLITE RELAY II

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GREENBELT, MARYLAND

ANALYSIS OF THE PERTURBATIONS IN THE ANGULAR ORBITAL ELEMENTS FOR THE SATELLITE RELAY II

A Dissertation
submitted to the Faculty of the
Graduate School of Georgetown University
in partial fulfillment of the requirements for the
degree of

Master of Arts

By

Victor T. Laczo

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June 1969

PREFACE

This work was undertaken because an analytical method used to predict lunar and solar perturbations for earth satellites failed to give suitable values for the perturbations in the orbital elements of the satellite Relay II. A study of this problem has yielded a feasible method by which one may accurately compute perturbations in the angular elements for Relay II. This study partially satisfies the requirement toward a Masters Degree at Georgetown University.

I wish to express my especial gratitude to Mr. David Fisher of Goddard Space Flight Center for the guidance he has given me throughout this work. In addition, I am indebted to Father Francis Heyden, S.J. of Georgetown University for the unfailing interest he has shown in this work and in all my studies at the University. I also wish to thank Dr. Thomas Margrave of Georgetown University and Mr. Theodore Felsentreger of Goddard Space Flight Center for their advice and encouragement.

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INTRODUCTION

Six orbital elements, coordinates, at any instant completely specify an orbit. These six elements or coordinates are usually the three Cartesian coordinates of position and the three of velocity or the six Keplerian elements. Since earth satellites can in the first approximation be assumed to move in elliptical orbits, the Keplerian elements form a suitable set of coordinates with which to describe a satellite's motion. In a simplified case, the two-body problem, there is a complete analytical solution. In other words, there is a solution given by a simple algebraic formula which yields the value of the six elements at any desired time. The actual orbit problem for a satellite, however, is much more complicated. The solution to the problem of describing an actual satellite's orbit is approached either numerically or by use of an infinite series technique. The use of such methods still leaves the Keplerian elements as suitable quantities with which to solve problems but their use can occasionally cause difficulties.

One such difficulty arises when one attempts to describe the motion of the artificial satellite Relay II. The period of an argument of the motion, the longitude of perigee, is about 570 years. This condition is referred to as resonance. Resonance complicates the computations of the Keplerian angular elements ω and

Ω for Relay II. The series expressions for ω (or g which is the argument of perigee) and Ω (or h which is the longitude of the ascending node) have some luni-solar terms which have the derivative of the longitude of perigee in the denominator. Since the longitude of perigee for Relay II has a value of almost zero, such luni-solar terms produce a small divisor problem. These near singular terms create sizable errors in the computed values for the orbital elements. This work shows the feasibility of an approach to treat the angular orbital elements ω and Ω for Relay II. The approach numerically integrates Lagrange's planetary equations and thereby bypasses forming terms which have the derivative of the longitude of perigee in the denominator. By using this method, one avoids all near singular terms which cause large errors. The approach has been checked by two computer programs and has been compared to the work of Theodoré Felsentrager. A determination of the J_3 zonal harmonic of the earth's gravitational potential was made as a check to the work.

PART I

DERIVATION OF THE EQUATIONS DESCRIBING THE MOTION OF THE SATELLITE RELAY II

CHAPTER I

FORMULATION OF THE TWO-BODY PROBLEM

Due to the rapid development of satellite theory, the problem of describing the motion of a satellite which has a near-resonant orbit condition seems rather removed from basic celestial mechanics. The equations which are finally arrived at to describe the behavior of the angular orbital elements for Relay II are actually simple developments from such fundamental laws as those of Newton. A detailed development starting from these basic laws will give physical insight to the problem and acquaint the reader with the notation.

Newton's first law states that if the resultant force acting on a particle is zero, the particle will move with constant velocity. Where this law holds, we have an inertial frame of reference. The first law is described by the equation

$$\bar{r} = \bar{r}_0 + \dot{\bar{r}}\, t \tag{1}$$

where $\dot{\bar{r}}$ is constant. The second law states that if the resultant force acting on a particle is not zero, the particle will suffer an acceleration proportional to the magnitude of the resultant force and in the direction of the resultant force. In vector notation

$$\bar{F} = \frac{d(m\dot{\bar{r}})}{dt} = m\,\ddot{\bar{r}}\,. \tag{2}$$

The third law states that when two bodies act on each other, the forces they exert will be of equal magnitude but opposite in direction. This is described by the equation

$$\bar{F}_{21} = -\bar{F}_{12}\,. \tag{3}$$

In addition to these three laws, there is Newton's law of gravitation which states that any two particles of mass m_1 and m_2 separated by a distance $\bar{r}_{12}$ mutually attract each other with a force directly proportional to the product of their masses and inversely proportional to the square of the distance. The force exerted on particle one by particle two can be written as

$$\bar{F}_{12} = \frac{k^2\, m_1\, m_2\, \bar{r}_{12}}{|\bar{r}_{12}|^3}\,. \tag{4}$$

By using the previous four equations, one can immediately investigate the simplest orbit problems. One such problem can be described as occurring when two bodies, each of whose mass is spherically distributed, orbit about a mutual center of gravity. Let these masses be denoted m_1 and m_2. The external gravity field for a body with a spherical mass distribution is the same as for a body having all its mass concentrated at its geometric point center.

The center of mass of the two-body system is defined as

$$\bar{r}_{CM} = \frac{\sum_{i=1}^{2} m_i\, \bar{r}_i}{\sum_{i=1}^{2} m_i}\,. \tag{5}$$

If the origin of the coordinate system is taken to be at the system's center of mass, then the motion of m_2 relative to m_1 can be obtained by setting equations (2) and (4) equal, i.e.,

$$m_1 \ddot{\bar{\rho}}_1 = \frac{k^2 m_1 m_2 (\bar{\rho}_2 - \bar{\rho}_1)}{|\bar{\rho}_2 - \bar{\rho}_1|^3} \tag{6}$$

and

$$m_2 \ddot{\bar{\rho}}_2 = \frac{-k^2 m_1 m_2 (\bar{\rho}_2 - \bar{\rho}_1)}{|\bar{\rho}_2 - \bar{\rho}_1|^3}. \tag{7}$$

See Figure 1. Let

$$\bar{r}_{12} = \bar{\rho}_2 - \bar{\rho}_1 .$$

Then

$$\ddot{\bar{r}}_{12} = \frac{-k^2 (m_1 + m_2) \bar{r}_{12}}{|\bar{r}_{12}|^3}. \tag{8}$$

If m_1 and m_2 can both be considered point masses, then equation (8) has a solution

$$r_{12} = \frac{a\left(1 - e^2\right)}{1 + e \cos f} . \tag{9}$$

This equation describes a conic section; see figure 2.[1] Only the elliptic orbit case will be considered here ($0 < e < 1$). In this case, three orbital elements can be immediately defined. The size of the ellipse is given by the value of its

[1]Forest Moulton, <u>An Introduction to Celestial Mechanics</u> (New York: Macmillan,Co., 1935), p. 321.

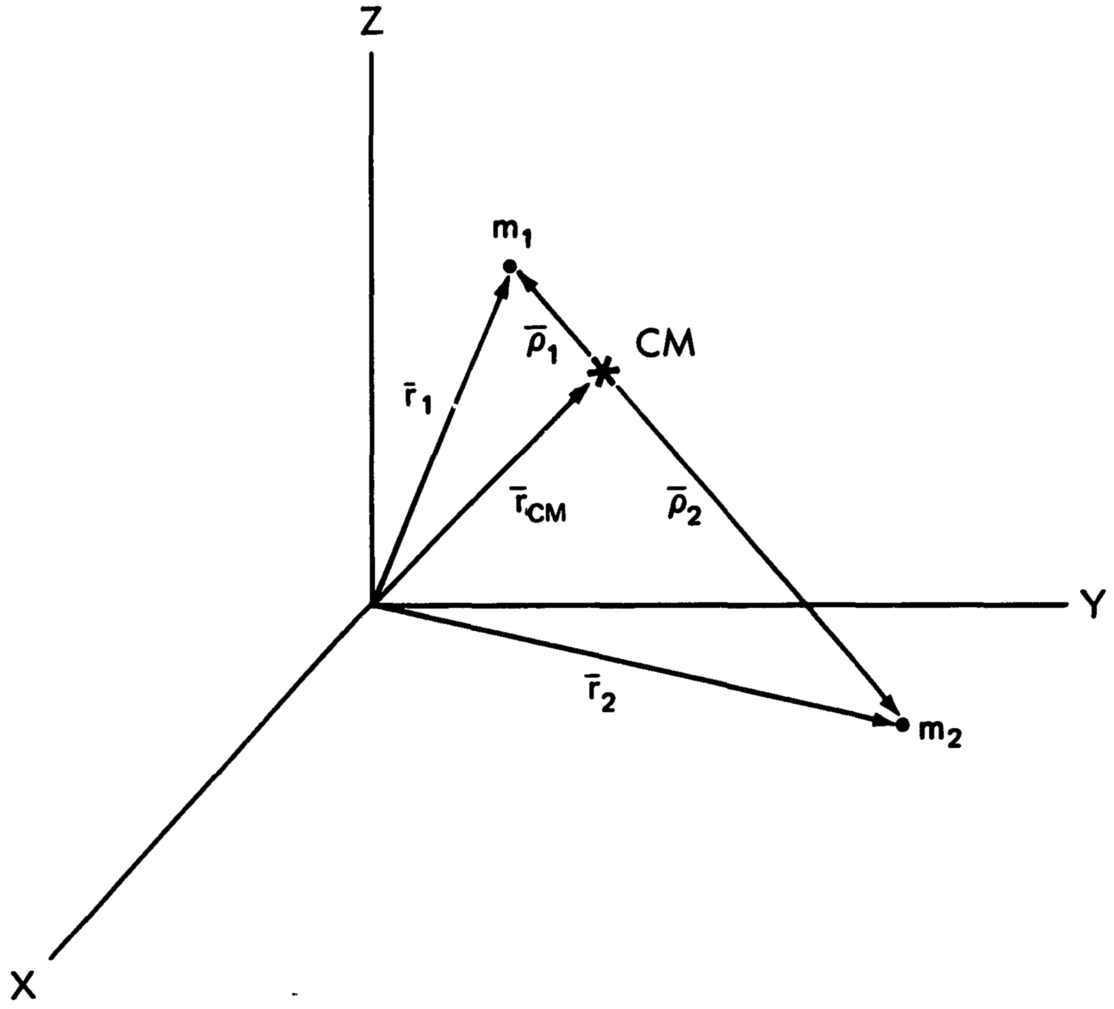

Figure 1.--The two-body problem as viewed in an inertial coordinate system with axes x, y, z.

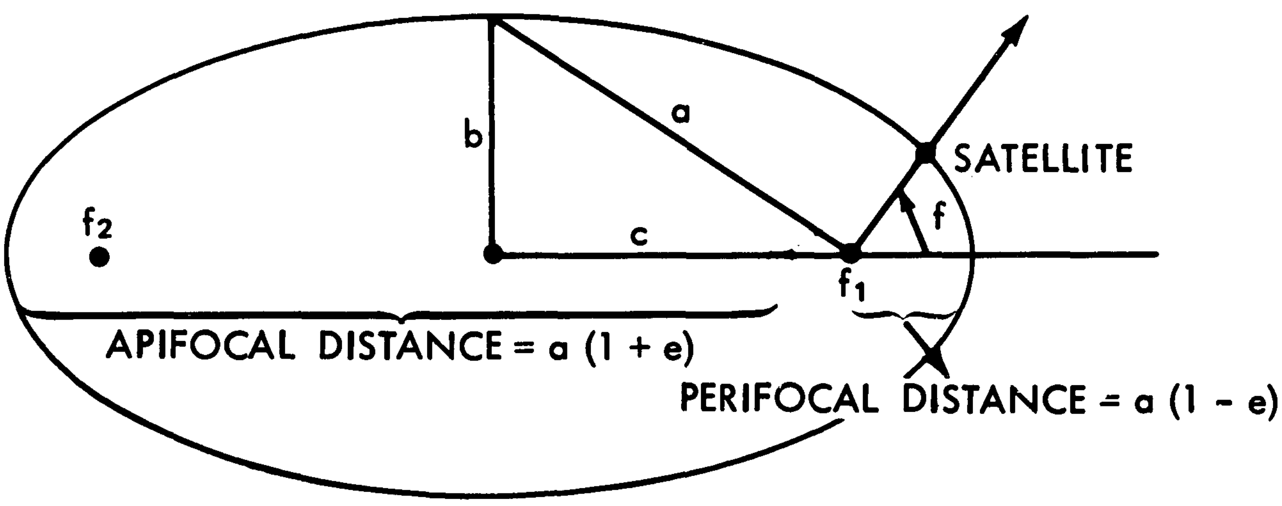

Figure 2.--Elliptical orbit of a satellite with respect to a primary body located at focus f_1.

semimajor axis, a . The semimajor axis is proportional to the total energy in the orbit. The shape of the ellipse is described by the value of the eccentricity, e. From figure 2

$$e = c/a = \sqrt{a^2 - b^2}/a \, .$$

The quantity f is termed the true anomaly. It is the angle formed at the focus f_1 between the radius vector of the satellite at any time t and that to the perifocal point at time T, which is called the time of perifocal passage. These definitions can be used to calculate the perigee and apogee heights of the satellite Relay II ($a = 1.7449 \ r_e$ and $e = .239$). The

$$\begin{aligned} \text{Apogee height} &= (1.7449 \ r_e \ (1 + .239) - r_e) \ (3963 \text{ mi.}/r_e) \\ &= 4728 \text{ miles} \end{aligned}$$

while the

$$\begin{aligned} \text{Perigee height} &= (1.7449 \ r_e \ (1 - .239) - r_e) \ (3963 \text{ mi.}/r_e) \\ &= 1299 \text{ miles.} \end{aligned}$$

For the two-body problem, a complete analytical solution can give the position of the satellite at any specific time in the past or present. It is not necessary to do a step by step time integration to obtain the coordinates at a desired time (which is one method of solving problems involving more than two bodies). For the two-body problem, any of several sets of six independent coordinates will suitably describe the orbit. For example, the six coordinates x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ at any one time would specify an orbit. The two-body problem however has a preferred set of elements, the Keplerian elements. They are convenient because the time derivatives of the Keplerian elements are then zero. In other words, when there are no perturbations, the Keplerian elements are constant. They thus describe two-body motion for all time.[1]

The Keplerian elements a and e which specify the size and shape of the ellipse have been mentioned. To define the others, refer to the figure 3 which depicts a celestial sphere. The projection of the earth's equator on the celestial sphere forms one fundamental great circle, the celestial equator. The projection of the sun's apparent path during the course of the year defines the second fundamental great circle, the ecliptic. The point where the sun moving from south to north along the ecliptic crosses the equator is called the vernal equinox, γ. A suitable coordinate system can be set up by pointing x toward γ and z toward the north celestial pole. The satellite orbit plane as seen from the center of the earth projected against the sky is also plotted. The spherical angle between it and the celestial equator defines another Keplerian element, the inclination, i. The angular distance measured westward along the equator between γ and the point where the satellite crosses the equator traveling in a northerly direction defines the element Ω, the longitude of the ascending node. The angle between the ascending node and the perigee point on the orbit is referred to as ω, the argument of perigee. Any of several elements can be taken to be the sixth Kepler element. If the mean motion of a satellite, n, is given then the sixth orbital element generally used is the mean anomaly, M. The mean motion,

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[1]Wolf Research and Development Corporation, "Orbital Dynamics," Satellite Geodesy, Theory and Applications (College Park, Maryland: Wolf Research and Development Corporation, 1968) VI, p. 1.

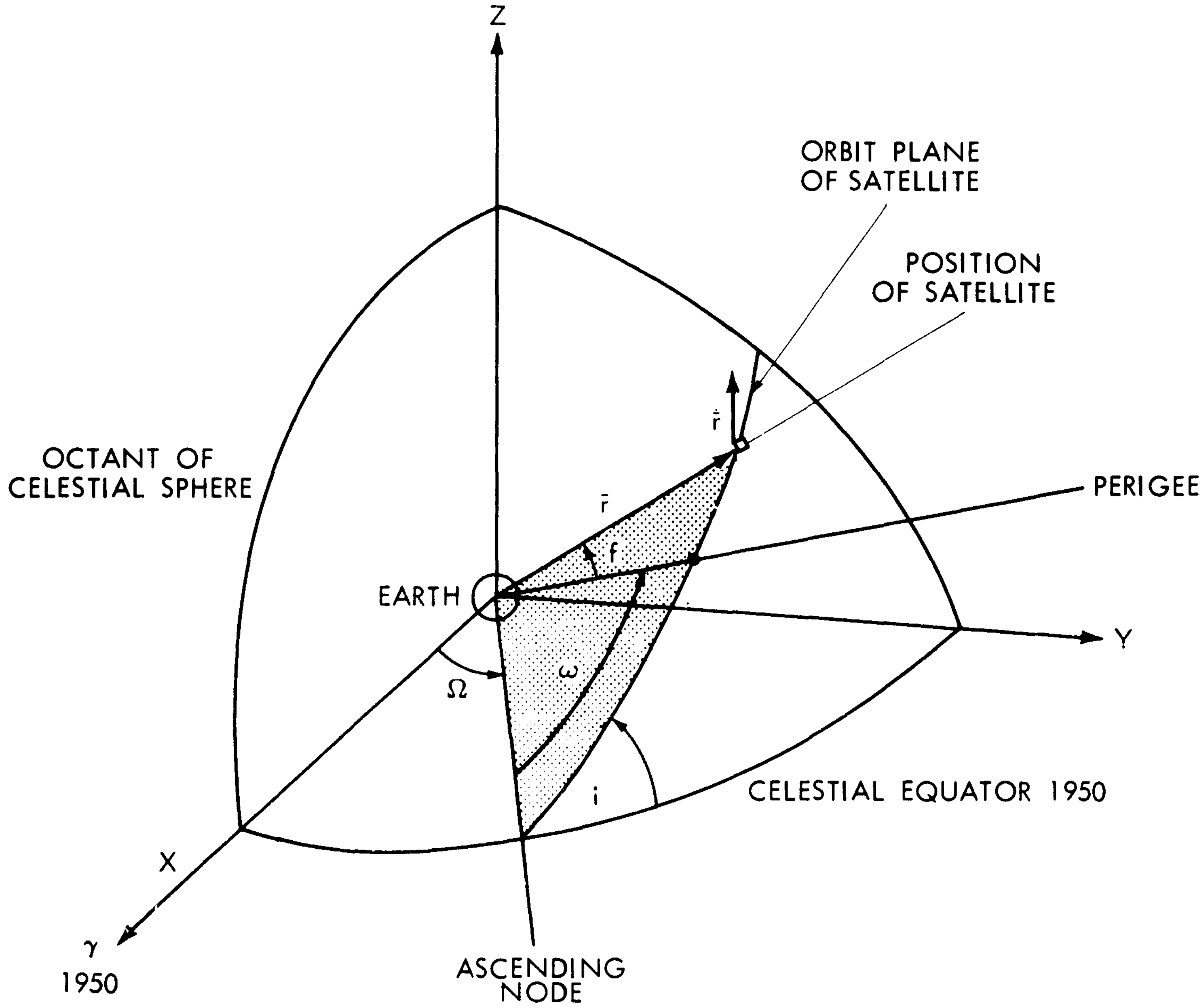

Figure 3.–Projection of the satellite's orbit upon the celestial sphere.

$$n = 2\pi / P \tag{10}$$

where P is the orbital period. The quantity M is defined as the product of the mean motion and the time elapsed since the perigee passage, $(t - T)$. A hybrid element $\tilde{\omega}$, the longitude of perigee, is defined as the sum of Ω and ω.

Any position vector $\bar{r}$ and velocity vector $\dot{\bar{r}}$ at time t has a corresponding unique set of Kepler elements a, e, i, M, ω and Ω. For any instant, there is a coordinate transform relating the two sets of coordinates. The problem can be viewed in either of the two sets of coordinates.

CHAPTER II

MODIFICATIONS TO THE TWO-BODY PROBLEM IN ORDER TO DESCRIBE THE MOTION OF A REAL SATELLITE

The two-body problem does not describe the actual motion of a satellite like Relay II very well. This is because the earth does not have a spherical mass distribution. Consequently the gravity field of the earth cannot be construed as acting from a point source located at the earth's center. Also the sun and the moon significantly effect the orbit of Relay II. For a close earth satellite, the simple two-body problem still remains a good first approximation to the actual physical solution. Since these two solutions are close for Relay II, the actual solution can be considered to be made up of a two-body motion and a deviation from the two-body motion, a perturbation. It is inconvenient to describe this motion analytically in terms of the x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ coordinates; they change too rapidly. The Keplerian elements a, e, i, M, ω, Ω on the other hand change only relatively slowly.[1] At any instant, there exists a set of Keplerian elements that would be followed if there were no forces other than the two-body force.

[1]Ibid., "Orbital Dynamics," VI, p. 1.

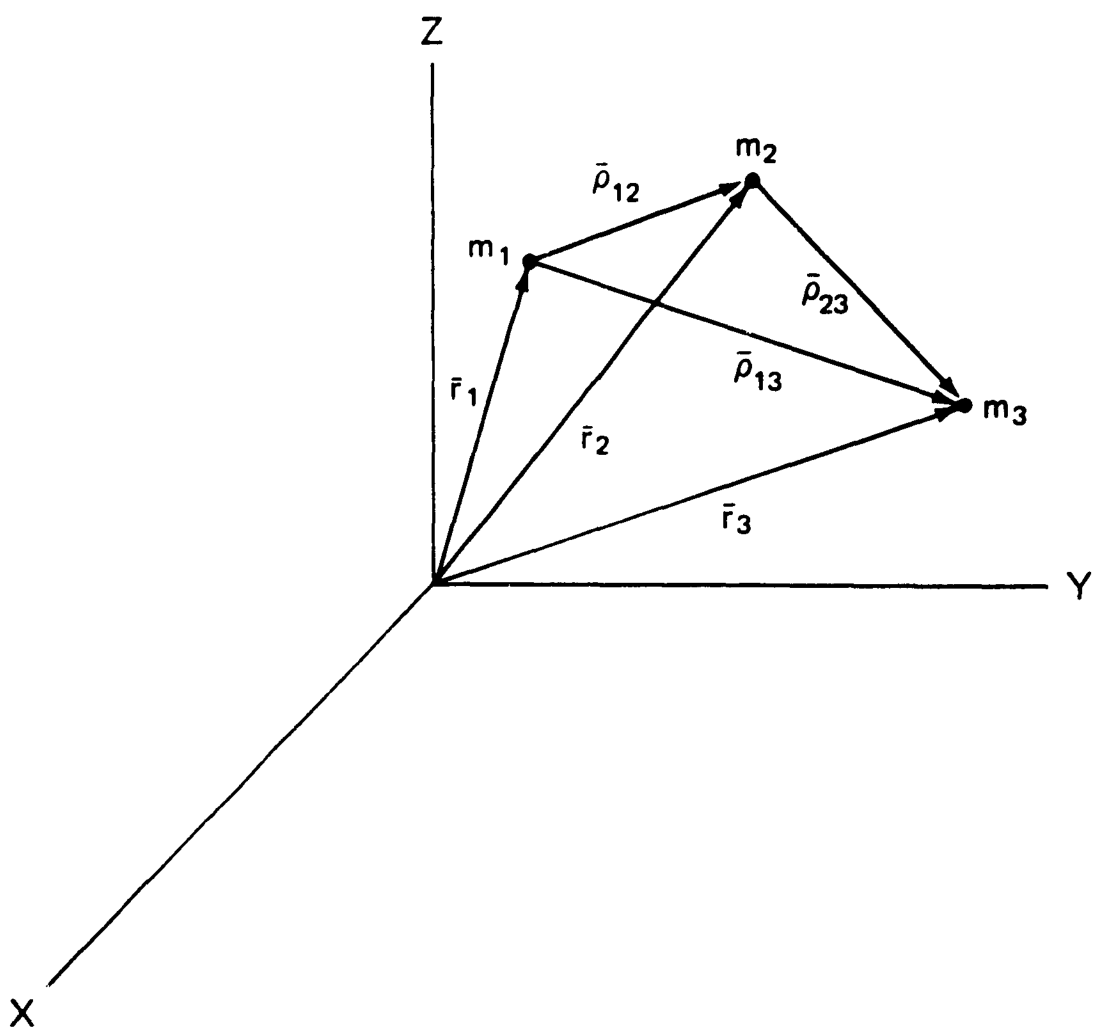

Figure 4. -- Position vectors of bodies m_1, m_2 and m_3 in the three-body problem.

These are called osculating elements. When disturbing forces do act, the coordinates in the actual orbit are changed from those which the body would have if there were no disturbing force. This change in position is referred to as a perturbation in the coordinate. A perturbation (as δg) must be added to the two-body value of the coordinate (as g) to obtain the actual value $g + \delta g$.

The analytical expression for the motion of Relay II can be obtained by assuming that the perturbations to the two-body motion come from three sources. The equations describing the perturbations due to two of these sources, the sun and the moon, can be derived by considering a simple form of the three-body problem. The perturbations due to the third source, the nonspherical earth, must be considered separately. In the simple version of the three body problem, we assume that there are two sizable masses, m_1 and m_3; each has a spherical mass distribution. Let a third body m_2, of negligible mass, orbit around body m_1. Let a remote body m_3 perturb the two-body motion of m_2 about m_1. To obtain the equation of motion for m_2, the forces given by Newton's second law and the law of gravitation are equated. The total force on body one is

$$\bar{F}_1 = m_1 \ddot{\bar{r}}_1 = \frac{k^2 m_1 m_2}{|\bar{\rho}_{12}|^3} \bar{\rho}_{12} + \frac{k^2 m_1 m_3}{|\bar{\rho}_{13}|^3} \bar{\rho}_{13} . \tag{11}$$

The total force on body two is

$$\bar{F}_2 = m_2 \ddot{\bar{r}}_2 = - \frac{k^2 m_1 m_2}{|\bar{\rho}_{12}|^3} \bar{\rho}_{12} + \frac{k^2 m_2 m_3}{|\bar{\rho}_{23}|^3} \bar{\rho}_{23} . \tag{12}$$

By transferring the origin of coordinates to m_1, one obtains the equation of motion of m_2 relative to m_1, i.e.,

$$\ddot{\bar{\rho}}_{12} = \ddot{\bar{r}}_2 - \ddot{\bar{r}}_1 = -k^2 (m_1 + m_2) \frac{\bar{\rho}_{12}}{|\bar{\rho}_{12}|^3} + k^2 m_3 \left(\frac{\bar{\rho}_{23}}{|\bar{\rho}_{23}|^3} - \frac{\bar{\rho}_{13}}{|\bar{\rho}_{13}|^3} \right) \tag{13}$$

where

$$k^2 (m_1 + m_2) = \mu .$$

Equation (13), describes the motion of m_2 as being composed of a two-body motion plus a perturbation.[1] Using the notation of figure 5, let

$$\bar{\rho}_{12} = \bar{r},$$

$$\bar{\rho}_{13} = \bar{r}'$$

and

$$\bar{\rho}_{23} = \bar{\Delta}.$$

In terms of these variables

$$\ddot{\bar{r}} = -k^2 (m_1 + m_2) \frac{\bar{r}}{|\bar{r}|^3} + k^2 m_3 \left(\frac{\bar{\Delta}}{|\bar{\Delta}|^3} - \frac{\bar{r}'}{|\bar{r}'|^3} \right) . \tag{14}$$

Let R denote the disturbing function and U denote the potential; then the above equation can be written as

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[1]Ibid., "Coordinate Reference Systems Used in Astrodynamics," II, p. 7.

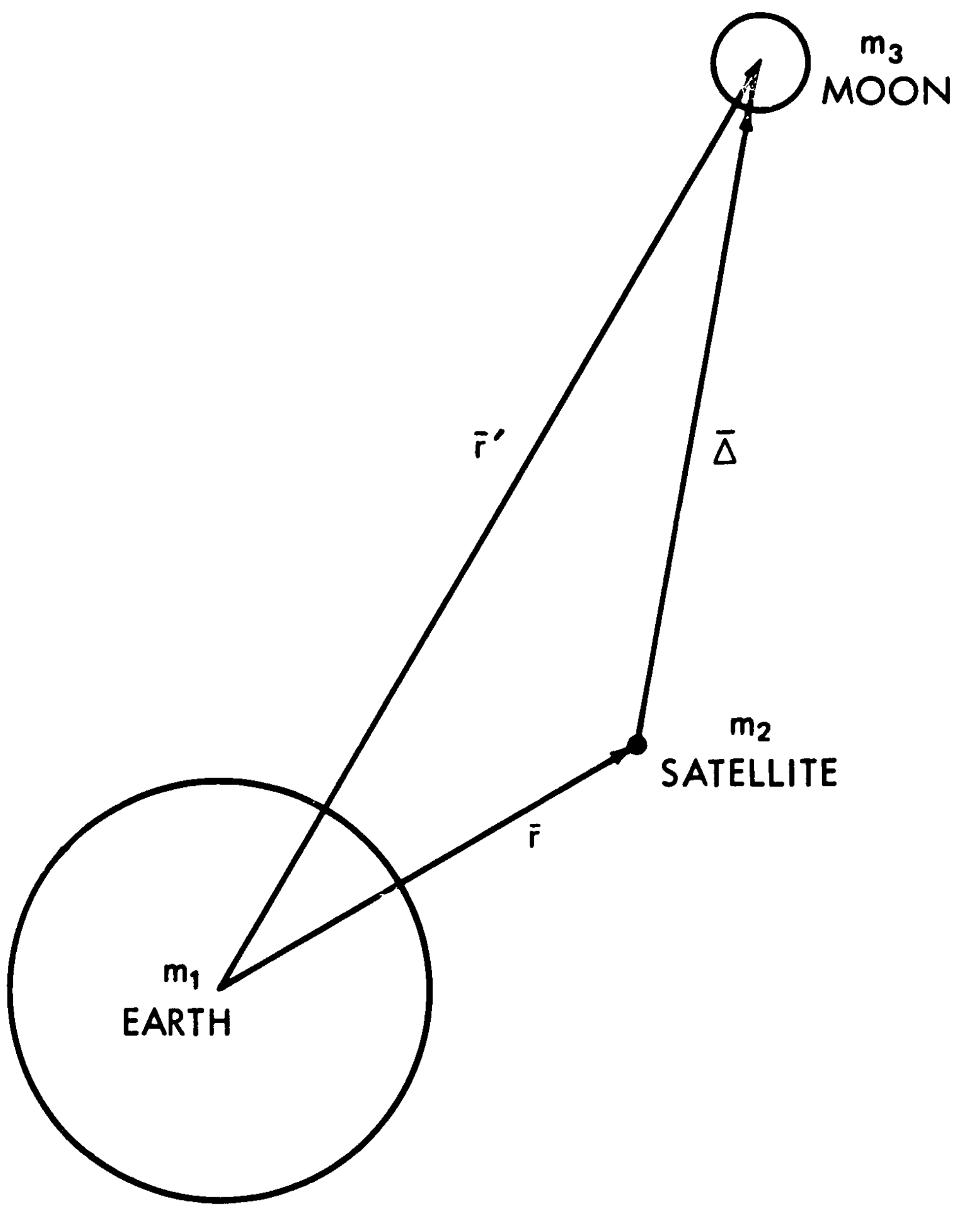

Figure 5.--The three-body problem consisting of the earth, the satellite and the moon.

$$\ddot{\bar{r}} = \bar{\nabla} U = \frac{-k^2(m_1 + m_2)\bar{r}}{|\bar{r}|^3} + \bar{\nabla} R . \tag{15}$$

Integrating equation (15) yields the forcing function (or the potential)

$$U = \frac{k^2(m_1 + m_2)}{|\bar{r}|} + R . \tag{16}$$

The disturbing function

$$R = k^2 m_3 \left(\frac{1}{|\bar{\Delta}|} - \frac{\bar{r}' \cdot \bar{r}}{|\bar{r}'|^3} \right) \tag{17}$$

where

$$\frac{\bar{r}' \cdot \bar{r}}{|\bar{r}'|} = |\bar{r}| \cos \widehat{rr'} .$$[1]

The term $(k^2 m_3/\bar{\Delta})$ is called the direct perturbation and the term $[k^2 m_3 (|\bar{r}|/|\bar{r}'|^2) \cos \widehat{rr'}]$ is called the indirect perturbation.

Henceforth the vector notation will be dropped so as not to complicate notation. To obtain an expression describing the effect of luni-solar forces on a satellite, one must visualize the following situation. A perturbing body, mass m_3, acts upon an earth satellite, mass m_2. The perturbing body m_3 can be either the sun, mass $m_{\odot}$, or the moon, mass $m_{☾}$. The direct perturbation on the satellite due to m_3 is

$$R = k^2 m_3/\Delta .$$

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[1] S. W. McCuskey, Introduction to Celestial Mechanics (Reading, Massachusetts: Addison-Wesley Co., 1963), p. 101.

By the law of cosines,

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos \widehat{rr'} ;$$

then

$$\frac{r'}{\Delta} = \frac{1}{\left[1 - 2\frac{r}{r'} \cos \widehat{rr'} + \left(\frac{r}{r'}\right)^2\right]^{1/2}} .$$ [1]

This equation can be expanded into an infinite series

$$\frac{r'}{\Delta} = 1 + \sum_{i=1}^{\infty} \left(\frac{r}{r'}\right)^i P_i(\cos \widehat{rr'}) . \tag{18}$$

The term P_i is the Legendre polynomial of the i^{th} order. The disturbing function then becomes

$$R = \frac{k^2 m_3}{r'} \left[1 + \sum_{i=1}^{\infty} \left(\frac{r}{r'}\right)^i P_i(\cos \widehat{rr'}) - \frac{r}{r'} \cos \widehat{rr'}\right] . \tag{19}$$ [2]

There is no simple analytical solution describing the motion of a body in a three or more body system. Numerical and infinite series approaches must then be used. The use of equation (19) for R is an infinite series approach. Kozai

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[1] D. Brouwer and G. Clemence, Methods in Celestial Mechanics (New York: Academic Press, 1959), p. 119.

[2] Wolf Research and Development Corporation, "Orbital Dynamics," X, p. 6.

used this disturbing function to develop a formula describing luni-solar effects to the order $i = 2$.[1] Murphy and Felsentreger's paper extend this to $i = 3$ but they consider only long-period terms (terms independent of the mean anomaly).[2] Among these terms are the two long-period luni-solar resonant terms for Relay II. These two terms are given in Felsentreger's paper as F_R.[3] They are eferred to in this work as $R_{R(SM)}$. The resonant part of the luni-solar disturbing function is

$$R_{R(SM)} = + \frac{15}{64} a^2 e^2 \left(1 + 2 \cos i + \cos^2 i\right) \Big\{ n_\odot^2 m_\odot \sin^2 i_\odot \cos 2\left(\Omega_\odot - g - h\right)$$

$$+ n_☾^2 m_☾ \Big[\sin^2 i_☾ \left(\cos^2 i'' - \frac{1}{2} \sin^2 i''\right) \cos 2(g + h)$$

$$- \sin i'' \cos i'' \sin i_☾ \left(1 - \cos i_☾\right) \cos (\Omega'' + 2g + 2h)$$

$$+ \sin i'' \cos i'' \sin i_☾ \left(1 + \cos i_☾\right) \cos (\Omega'' - 2g - 2h)$$

$$+ \frac{1}{4} \sin^2 i'' \left(1 - \cos i_☾\right)^2 \cos 2(\Omega'' + g + h)$$

$$+ \frac{1}{4} \sin^2 i'' \left(1 + \cos i_☾\right)^2 \cos 2(\Omega'' - g - h) \Big] \Big\} . \qquad (20)$$

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[1]Y. Kozai, "On the Effects of the Sun and the Moon Upon the Motion of a Close Earth Satellite," SAO Special Report, Number 22 (March, 1959).

[2]J. P. Murphy and T. L. Felsentreger, "An Analysis of Lunar and Solar Effects on the Motion of Close Earth Satellites," NASA TN-D-3559, (August, 1966).

[3]T. L. Felsentreger, "Long Period Lunar and Solar Effects on the Motion of Relay II," GSFC X-547-66-102, (March, 1966), 3.

To obtain simplified solutions in the previous cases, it was assumed that each mass was spherically symmetric. In such a case, the body would attract an external body as if all its mass were concentrated at its center. To represent reality more closely one must complicate such a simple mass model. An accurate mass model must take into account any mass distributions which cause the gravitation field of the earth to significantly deviate from a spherical potential. One model for the earth's mass separates the irregular mass distribution which deviates from a spherical distribution into mass components that are symmetric to the axis of rotation. The major component in this model is the equatorial bulge of the earth (the earth's equatorial diameter exceeds its polar dimaeter by some 27 miles). The earth also has a mass distribution component deviating from sphericity which is referred to as pear-shaped. This small effect only amounts to a 50-foot departure at maximum from a spherical earth. See figure 6. There are many other significant mass distribution components. Such axially symmetric mass distribution components, which account for the earth's irregular mass distribution, produce the various zonal harmonics in the earth's gravitational potential.[1]

The equation of motion for a satellite moving in a nonspherical gravitational field of potential U is

$$\ddot{r} = \nabla U. \tag{21}$$

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[1]C. A. Wagner, "The Drift of a 24-Hour Equatorial Satellite Due to an Earth Gravity Field Through 4th Order," NASA TN-D-2103, (February, 1964).

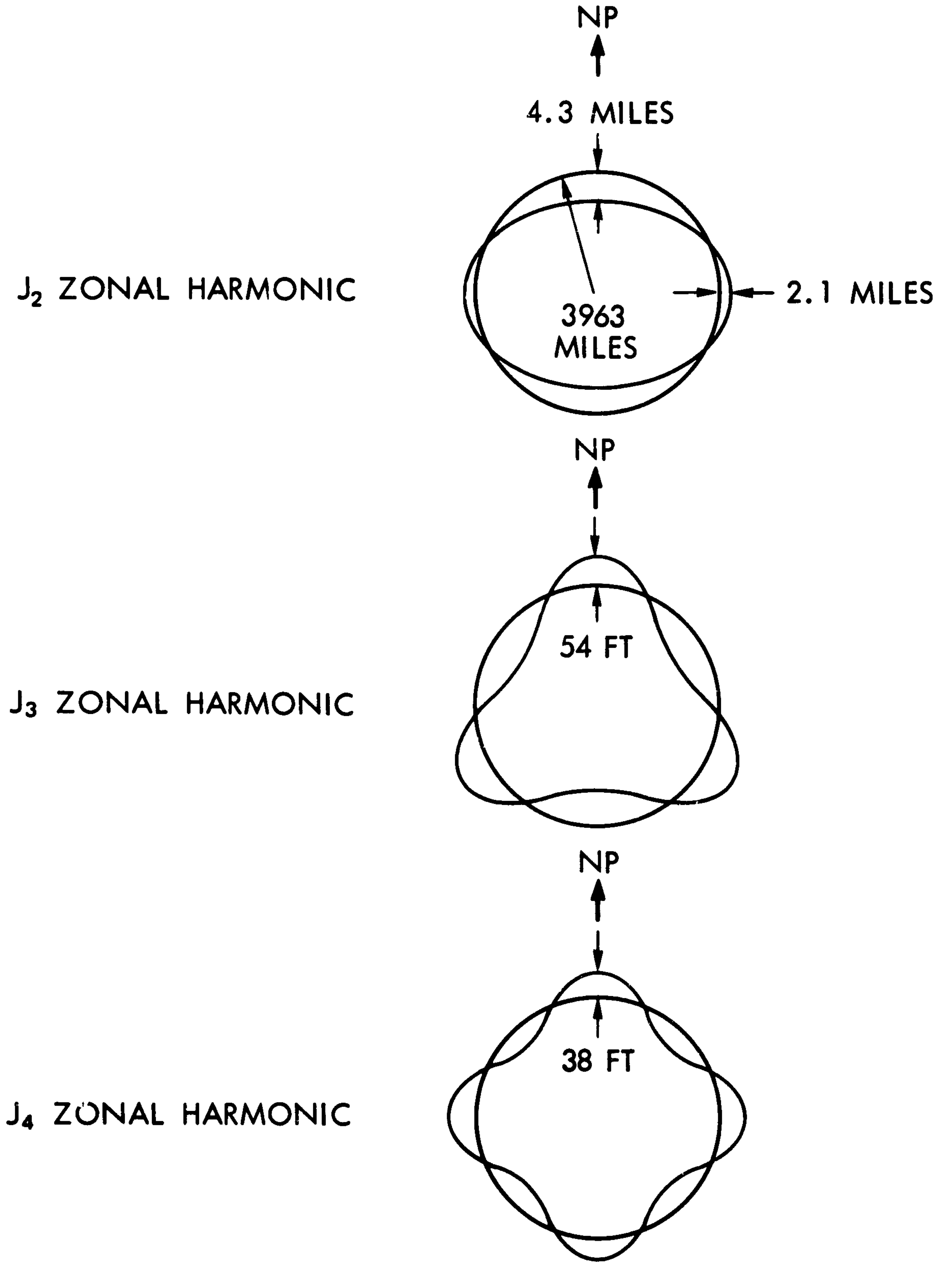

Figure 6.--The primary mass distribution components symmetric to the axis of rotation which depart from a spherical earth.

The nonspherical potential can be written as

$$U = \frac{\mu}{r} + R. \tag{22}$$

The term (μ/r) is the potential due to a spherical earth and R is the contribution to the potential arising from mass distributions that deviate from a spherical distribution.

To obtain an expression for the potential of an irregularly shaped body, one resorts to Laplace's equation. The potential must satisfy Laplace's equation outside the irregular body, the earth, i.e.,

$$\nabla^2 U = 0. \tag{23}$$

In spherical coordinates, equation (23) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial U}{\partial \theta}\right) = 0.$$

If one considers zonal harmonics only, then the mass distributions are symmetric to the axis of rotation and U is independent of ϕ. The general form of the potential U is then obtained by solving Laplace's equation assuming a solution of the form

$$U(r, \theta) = \tilde{R}(r)\, \Theta(\theta).$$

A convenient form of the final solution is

$$U = \frac{m_{earth}\, k^2}{r}\left[1 - \frac{r_e^2}{r^2} J_2 P_2(\cos\theta) - \frac{r_e^3}{r^3} J_3 P_3(\cos\theta) - \cdots\right] \tag{24}$$

where P_2 is the Legendre polynomial of the second order.[1] The quantity

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1).$$

The quantities J_2, J_3 and J_4 are the spherical harmonic coefficients of the earth's gravitational field; their magnitudes indicate the strength of the component of deviation from a spherical gravitational field. The first term in the brackets is due to a spherical earth; the second is due to the equatorial bulge and the third term, the J_3 term, is due to the pear-shaped earth.

Using this above expression of the potential, Brouwer derived expressions to suitably describe the motions of the Keplerian elements for earth satellites. Let

$$\theta = 90 - \delta$$

and

$$\mu = m_{earth}\, k^2 .$$

Then Brouwer's potential function takes the form

$$U = \frac{\mu}{r}\left[1 - \sum_{K=2}^{5} \frac{J_k\, r_e^K}{r^K} P_k(\sin\delta)\right] .^2 \qquad (25)$$

By spherical trigonometry,

$$\sin\delta = \sin i \sin(g + f);$$

[1]J. M. Danby, Fundamentals of Celestial Mechanics (New York: Macmillan Co., 1962), pp. 108-111.

[2]Brouwer and Clemence, Methods in Celestial Mechanics, p. 563.

then

$$U = \frac{\mu}{r} + \frac{\mu J_2 r_e^2}{a^3}\left[\left(-\frac{1}{4}+\frac{3}{4}\cos^2 i\right)\frac{a^3}{r^3}\right.$$

$$\left. + \left(\frac{3}{4}-\frac{3}{4}\cos^2 i\right)\frac{a^3}{r^3}\cos(2g+2f)\right] + \cdots \text{[1]} \quad (26)$$

In addition to this formula, Brouwer needed Hamilton's equations

$$\frac{dq_i}{dt} = -\frac{\partial \tilde{F}}{\partial p_i} \quad (27)$$

and

$$\frac{dp_i}{dt} = \frac{\partial \tilde{F}}{\partial q_i} . \quad (28)$$

These two equations determine the changes dq_i and dp_i in the coordinates q_i (x, y, z) and in the momenta p_i $(m\dot{x}, m\dot{y}, m\dot{z})$ that occur during an infinitesimal period of time dt. In these equations $\tilde{F}$ is the constructed Hamiltonian of the system. It is equal to the negative of the total energy of the system, i.e.,

$$\tilde{F}(p_i, q_i) = U - T_{KE} = \frac{\mu^2}{2L^2} + R . \quad (29)$$

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[1]D. Brouwer, "Solution of the Problem of Artificial Satellite Theory Without Drag," Astronomical Journal, LXIV, p. 378.

In the orbit problem, the coordinates p_i and q_i are often referred to as Delaunay canonical variables L , G, H and 1 , g , h. The Keplerian elements are related to the Delaunay variables in the following way

$$\left.\begin{array}{llllll} p_1 = L = (\mu a)^{1/2}, & q_1 = l = M, \\ p_2 = G = L\left(1 - e^2\right)^{1/2}, & q_2 = g = \omega, \\ p_3 = H = G\cos i, & q_3 = h = \Omega \end{array}\right\} .^1 \tag{30}$$

Brouwer solved Hamilton's equations using the disturbing function given in equation (26) by the method of canonical transformations. Among other things, he obtained the long-period and secular series expressions for the elements 1 , g and h as a function of the quantities J_2, J_4, J_2^2, a'', e'' and i'' etc.[2] The quantities a'', e'' and i'' are the Brouwer mean elements. The significant terms of the secular disturbing function due to the earth, as derived by Brouwer, can be found in Felsentreger's paper. He calls this function F_s.[3] This paper will call the secular part of the disturbing function due to the earth's nonspherical gravitational field $R_{s(E)}$. The quantity

$$R_{S(E)} = \frac{-J_2}{4L^6\left(1 - e^2\right)^{3/2}}\left(1 - 3\cos^2 i\right)$$

$$+ \frac{J_2^{\,2}}{L^{10}}\left[\frac{3}{128}\frac{1}{\left(1 - e^2\right)^{5/2}}\left(5 - 18\cos^2 i + 5\cos^4 i\right)\right.$$

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[1] W. M. Kaula, "Celestial Geodesy," NASA TN-D-1155, (March 1962), 8.

[2] Brouwer, "Artificial Satellite Theory," pp. 393-395.

[3] Felsentreger, "Motion of Relay II," p. 5.

$$+ \frac{3}{32} \frac{1}{\left(1 - e^2\right)^{6/2}} \left(1 - 6 \cos^2 i + 9 \cos^4 i\right)$$

$$\left. - \frac{15}{128} \frac{1}{\left(1 - e^2\right)^{7/2}} \left(1 - 2 \cos^2 i - 7 \cos^4 i\right) \right]$$

$$- \frac{3}{128} \frac{J_4}{L^{10}} \left[\left(\frac{5}{\left(1 - e^2\right)^{7/2}} - \frac{3}{\left(1 - e^2\right)^{5/2}} \right) \left(3 - 30 \cos^2 i + 35 \cos^4 i\right) \right] . \qquad (31)$$

Another set of formulas which are very useful to describe the motion of a satellite is Lagrange's planetary equations. These equations give the time derivative of an orbital element as a function of the values of the six orbital elements. For the angular variables g and h, these equations are

$$\frac{dg}{dt} = \frac{-\cos i}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial i} + \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial R}{\partial e} \qquad (32)$$

and

$$\frac{dh}{dt} = \frac{1}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial i} . \qquad (33)$$

These equations can be simplified by defining a convenient set of canonical units of time and distance. Such a consistent set of units also facilitates all computations. By Kepler's third law,

$$\mu = n^2 a^3 \qquad (34)$$

where

$$\mu = k^2 \left(m_{earth} + m_{satellite}\right) .$$

The quantity n is termed the mean angular motion of the satellite, where

$$n = 2\pi / \text{Period} .$$

If one uses the following canonical units for time and distance,

canonical time unit = 806.81242 seconds

and

canonical length unit = 6378.166 kilometers,

then

$$\mu = n^2 a^3 = 1 .$$

These units are currently used at Goddard Space Flight Center. With these units, the terms $n^2 a^4$ and na^2 in Lagrange's planetary equations reduce to a and $\sqrt{a}$ respectively.

PART II

DESCRIPTION OF THE MOTION IN THE ARGUMENT OF PERIGEE AND THE LONGITUDE OF THE ASCENDING NODE FOR THE EARTH SATELLITE RELAY II

CHAPTER I

NATURE OF THE PROBLEM OF RESONANCE IN THE LUNI-SOLAR DISTURBING FUNCTION FOR RELAY II

There is a computer program used at Goddard Space Flight Center which computes solar and lunar perturbations on a close earth satellite. The computer program is named SLOPE. It was formulated in a paper by Murphy and Felsentreger.[1] In general terms, the program employs the Lagrange planetary equations to form the derivatives of the orbital elements. The integration is then performed analytically to obtain the perturbation. For Relay II, the program predicts perturbations that are excessively large. Two terms in the luni-solar disturbing function can easily be identified as causing the problem. These terms are referred to as resonant. The resonant terms contain factors like $\cos 2(g + h)$. In the process of obtaining the argument of perigee, the factor is integrated and it yields the expression

$$\frac{\sin 2(g + h)}{2(\dot{g} + \dot{h})}.$$

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[1]Murphy and Felsentreger, "Lunar and Solar Effects," pp. 1-46.

The problem with this expression is that the time derivative of g and h have about the same magnitude but opposite sign. For Relay II,

$$\dot{h} = -1.1046373 \text{ degrees/day}$$

$$\dot{g} = +1.1063814 \text{ degrees/day.}^{1}$$

The term ($\dot{g}$ + $\dot{h}$) is equivalent to $\dot{\omega}$. In each analytical term where the time derivative of the longitude of perigee is the only term in the denominator, it produces a near singularity.

The period for the longitude of perigee is excessively long, i.e., the

$$\text{period of } \tilde{\omega} = 360/\dot{\tilde{\omega}} = 360/(\dot{\Omega} + \dot{\omega}) = 570 \text{ years} .$$

This means that the phenomenon of resonance occurs in the orbit of Relay II because of the derivative of the mean motion of an argument is equal to zero. When the derivative of the longitude of perigee

$$\dot{\tilde{\omega}} = \dot{\Omega} + \dot{\omega} \cong 0 , \tag{35}$$

then

$$\tilde{\omega} = \text{constant} .$$

This condition can be visualized by looking at figure 7. Physically, resonance occurs for Relay II (a close earth satellite, a = 1.7449 and a satellite of large eccentricity, e = .239) because it happens to be near a critical inclination. When the sun, moon and oblateness act simultaneously on such a satellite near a

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[1] Felsentreger, "Motion of Relay II," pp. 26-29.

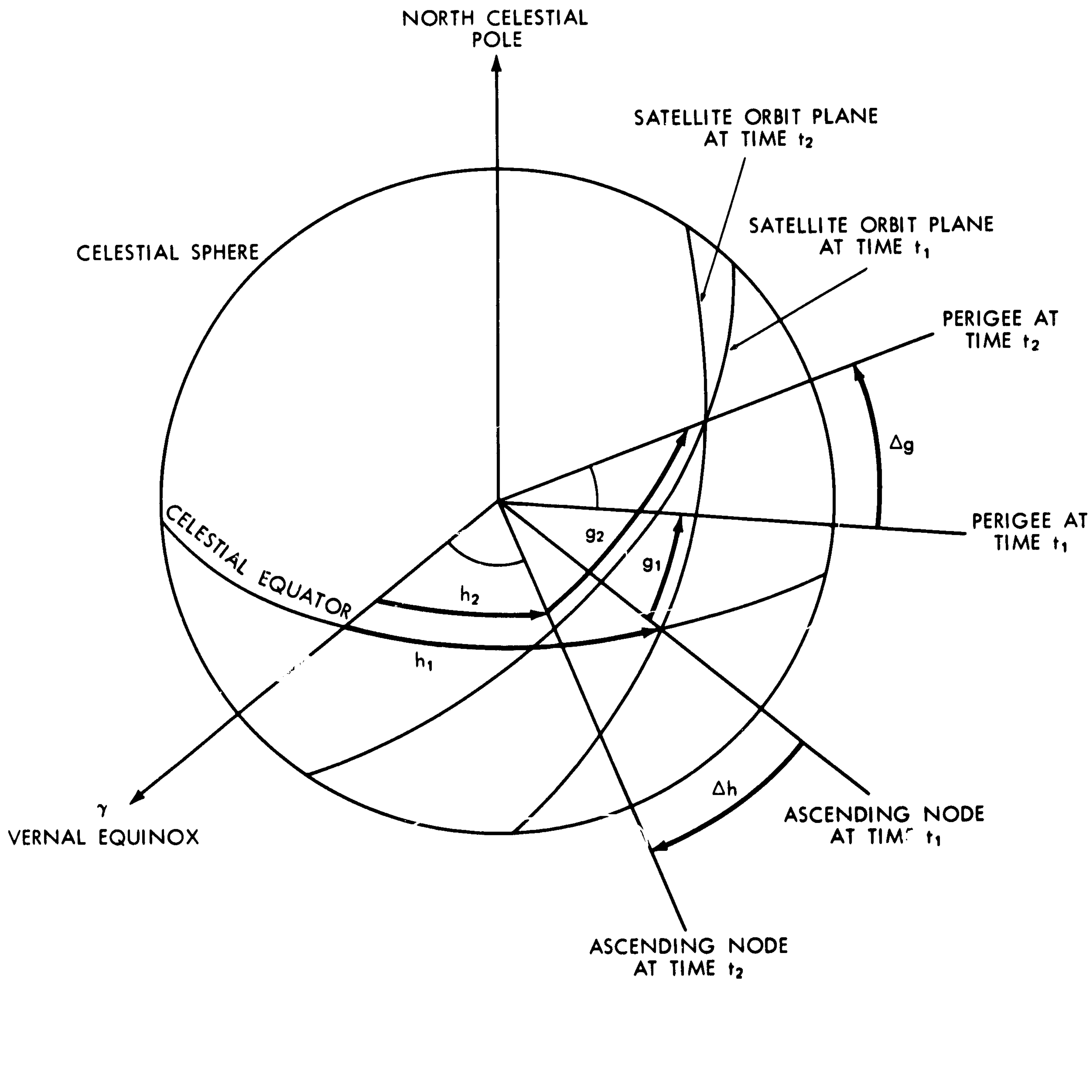

Figure 7.--The sum of the longitude of ascending node and argument of perigee is approximately constant for the satellite Relay II.

critical inclination, then resonance occurs. An appropriate value of the resonant inclination can be found by assuming that the main effects on g and h for Relay II are due to the oblateness effect, the J_2 or A_2 term. From Kozai's paper,

$$\dot{\Omega} = -\frac{A_2}{p^2} n \cos i \tag{36}$$

and

$$\dot{\omega} = \frac{A_2}{p^2} \frac{n\left(4 - 5 \sin^2 i\right)}{2} ; \tag{37}$$

then

$$\dot{\tilde{\omega}} = \dot{\Omega} + \dot{\omega} = \frac{A_2}{p^2} n\left(2 - \frac{5}{2} \sin^2 i - \cos i\right) = 0 .$$ [1]

Here p equals $\left[a\left(1 - e^2\right)\right]$. This reduces to the quadratic equation

$$5 \cos^2 i - 2 \cos i - 1 = 0$$

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[1] Y. Kozai, "The Earth's Gravitational Potential Derived from the Motion of Satellite 1958 Beta Two," SAO Special Report, Number 22 (March, 1959). 2.

which has the solution

$$\cos i = -.2896, \ .6896$$

or

$$i = 106^\circ\!.833, \ 46^\circ\!.400 .$$[1]

The inclination of Relay II averages 46.32 degrees which is close to the latter value.

Since the orbit of Relay II is near this critical inclination, problems result when one analyzes the motion of the Keplerian elements for Relay II. Felsentreger has investigated the elements e and i. This work will investigate the angular variables g and h.

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[1]Peter Musen, private communication, 1968.

CHAPTER II

INPUTS FOR THE SOLUTION OF THE PROBLEM

In order to obtain a solution describing the motion of g and h for the satellite Relay II, two inputs were required. One input was an appropriate analytical disturbing function R. The second input was a good stretch of observation data, which consisted of observed orbital elements of the satellite Relay II for a few hundred days. The disturbing function R is composed of several distinct components. Two components, $R_{R(SM)}$ and $R_{S(E)}$, have already been mentioned. For $R_{S(E)}$, only the terms with coefficients J_2, J_2^2 and J_4 will be considered. In addition to these components, there is a component $R_{S(SM)}$ which can be viewed as a secular term produced by the sun and moon.

The expression for $R_{S(SM)}$ in Felsentreger's paper is appropriate but a coordinate transform must be performed on the quantity $i_{\mathbb{C}}$.[1] The quantity $i_{\mathbb{C}}$ cannot be considered constant over a long data arc. The quantities i'' and $i_{\odot}$ are nearly constant and Ω'' is almost linear with time. These three variables are more appropriate for problems which have long data arcs. From figure 8, the relationship between these spherical angles is

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[1]Felsentreger, "Motion of Relay II," pp. 2-5.

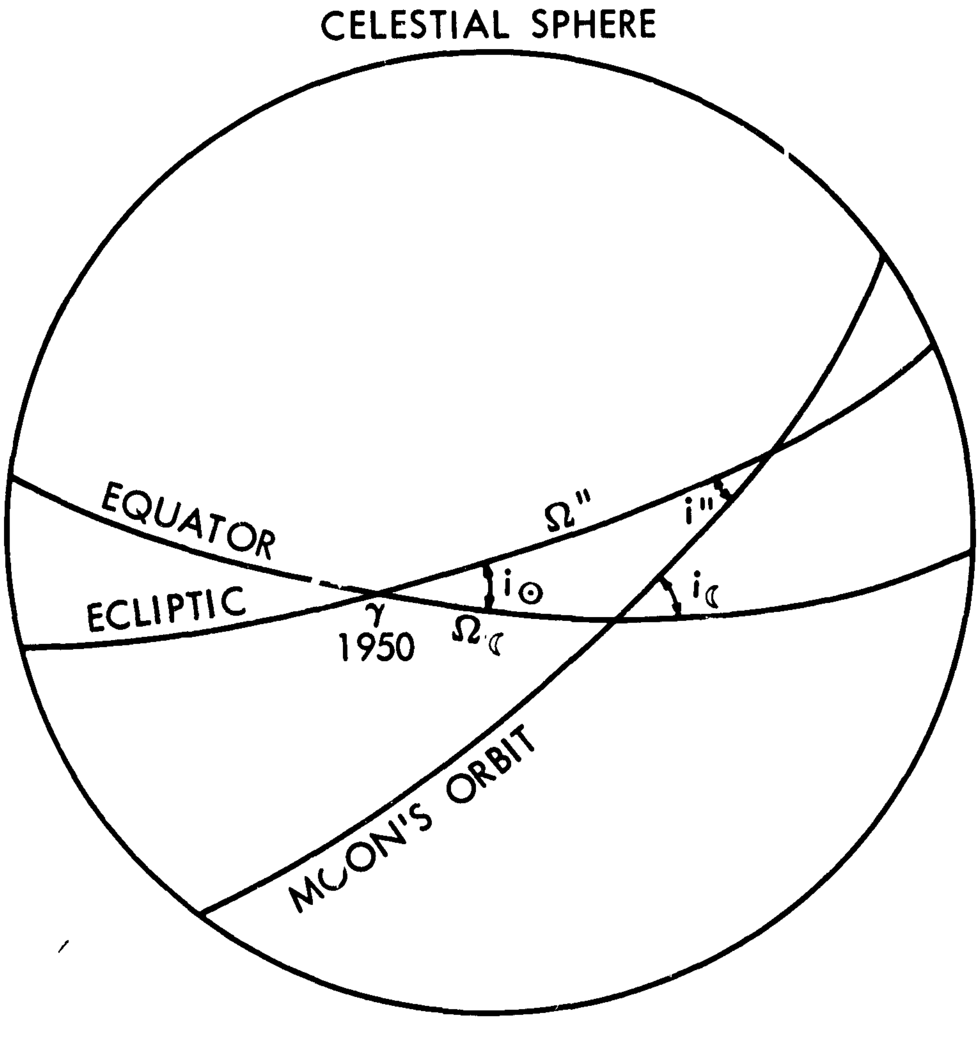

Figure 8.--Fundamental spherical angles relating the projection of the moon's orbit on the sky to the projection of the ecliptic and equator on the sky.

$$\cos\left(180 - i_c\right) = -\cos i'' \cos i_{\leftmoon} + \sin i'' \sin i_{\leftmoon} \cos \Omega'' . \tag{38}$$

Then

$$\begin{aligned}
\sin^2 i_c = \Big[1 - \cos^2 i'' \cos^2 i_{\leftmoon} & \\
& + 2 \cos i'' \cos i_{\leftmoon} \sin i'' \cos \Omega'' \sin i_{\leftmoon} \\
& - \sin^2 i'' \sin^2 i_{\leftmoon} \cos^2 \Omega'' \Big] .
\end{aligned}$$

The final expression for $R_{S(SM)}$ is

$$\begin{aligned}
R_{S(SM)} = -\frac{1}{32} \Big[n_{\odot}^2 m_{\odot} \left(2 - 3 \sin^2 i_{\odot}\right) & \\
& + n_c^2 m_c \left(-1 + 3 \cos^2 i'' \cos^2 i_{\leftmoon} - 6 \cos i'' \cos i_{\leftmoon} \sin i'' \sin i_{\leftmoon} \cos \Omega'' \right. \\
& \left. + 3 \sin^2 i'' \sin^2 i_{\leftmoon} \cos^2 \Omega'' \right) \Big] \Big[L^4 \left(2 + 3e^2\right) \left(1 - 3 \cos^2 i\right) \Big] .
\end{aligned} \tag{39}$$

The final form of the total disturbing function R for the system composed of Relay II, the earth, sun and the moon is

$$R = R_{R(SM)} + R_{S(E)} + R_{S(SM)} . \tag{40}$$

The disturbing function is composed of a series of terms containing the variables and parameters $g, h, a, e, i, \Omega, \Omega'', i''$, and $i_{\leftmoon}$. To obtain the necessary observed input values for a, e, i, g and h, elements were taken from Felsentreger's paper.[1] He obtained these by processing minitrack observation data with a Brouwer theory program. Such a program assumed the observation can be explained by a theory using the zonal harmonic terms J_2, J_3, J_4,

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[1] Ibid., pp. 18-29.

J_5 and J_2^2 in the earth's gravitational potential. By this method one obtains values of the orbital elements g and h which still have the effect of the sun and moon in them. Solar radiation and long-period luni-solar perturbations were then computed using the SLOPE program (not including the long-period resonant terms) and subtracted from the Brouwer elements. This yields the so-called corrected elements e_c, i_c, g_c and h_c. These corrected elements still contain the effects of secular and resonant terms due to the sun and the moon. The use of these corrected elements eliminates the effects of the long-period luni-solar terms in the elements. The long-period luni-solar terms in the disturbing function can then be eliminated. This makes it easy to spot periodic variation in the elements due to the odd zonal harmonics of the earth and to determine a value for J_3. The computation for J_3 will be made as a check to the work.

The corrected elements can still be viewed as observed orbital element data. These elements are given at roughly weekly intervals after the epoch date, January 21, 1964, 21 hours and 41 minutes U.T., for some 650 days. The parameters Ω'', $\Omega_\odot$, i'' and $i_\odot$ can be obtained for the desired date by polynomials given in Murphy and Felsentreger's paper.[1] With the observed orbital elements data and the disturbing function, $R(a, e_c, i_c, g_c, h_c, \Omega'', \Omega_\odot, i'', i_\odot)$, the problem of computing the perturbations in g and in h could be undertaken.

[1]Murphy and Felsentreger, "Lunar and Solar Effects," pp. 46-47.

CHAPTER III

NUMERICAL METHOD OF SOLUTION

Since the analytical series method as specified by the SLOPE computer program yielded excessively large perturbations for g and h , another approach was attempted. A better approach to computing the perturbations would be to avoid forming terms which had the derivative of the longitude of perigee in the denominator. This second approach to the problem again used Lagrange's planetary equations to obtain the expressions for dg/dt and dh/dt. Upon evaluation of these derivatives at each observation date, it was found that the scatter in the magnitude of dg /dt was roughly twice the scatter in the amplitude of dh /dt . For example, from 87 to 101 days past epoch, dg/dt changed by 1.74×10^{-4} degrees/day while dh/dt changed by $.42 \times 10^{-4}$ degrees/day. Refer to the data in tables 1 and 2. These points are plotted on graphs 1 and 3.

The integration of the derivatives dg/dt and dh /dt was done numerically using the trapezoidal rule for 84 observation dates. The formula used to obtain the perturbations in the argument of perigee was

$$\delta g_{computed} (t_n - t_0) = \sum_{i=0}^{n-1} \left[\frac{\dot{g}(t_{i+1}) + \dot{g}(t_i)}{2} \right] (t_{i+1} - t_i) \qquad (41)$$

where

$n = 1, 2, \ldots, 84$

t_0 = start time of integration was 1/24/64, 21 hours, 41 minutes U.T.,

and

$t_i = i^{th}$ observation date.

Upon integrating dg/dt and dh/dt, the perturbations $\delta g\ (t - t_0)$ and $\delta h (t - t_0)$ were obtained at roughly weekly intervals for 655 days.

The perturbations $\delta g (t - t_0)$ increased almost linearly from 7.7 degrees at ten days past epoch to 720 degrees on day 654. The perturbation $\delta h\ (t - t_0)$ decreased almost linearly by almost the same magnitude as δg. Refer to the data in tables 1 and 2. The computed values for the elements g and h were obtained by adding the computed perturbations δg and δh for any date past epoch to the value g_c and h_c on the date (t_0) at which the integration was begun. Then

$$g_{computed}\ (t_n) = g\ (t_0) + \delta g\ (t_n)$$

and

$$h_{computed}\ (t_n) = h\ (t_0) + \delta h\ (t_n)$$

where

$n = 1, 2, \ldots, 84.$

To investigate the accuracy of the computed g and h, their values were compared to the input g_c and h_c values. The latter are essentially observed elements. Residuals were formed. Let

$$g_{res}\ (t_n) = g_{computed}\ (t_n) - g_{corrected}\ (t_n) \tag{42}$$

and

$$h_{res}\ (t_n) = h_{computed}\ (t_n) - h_{corrected}\ (t_n). \tag{43}$$

The quantity g_{res} decreases almost linearly from zero to .34 degrees after 654 days. During the same period h_{res} increases almost linearly to .17 degrees. See tables 3 and 4 for the values of the compu..., corrected and r· idual values of the elements. Plotting the values of the residuals against time for 654 days gives a nearly straight line. For g_{res}, the line has noticeable arcs superimposed on it. For h_{res}, the line is very straight and has only slight arcs on it. The quantities g_{res} and h_{res} are plotted on graphs 5 and 6.

The main linear characteristic of the graphs for g_{res} and h_{res} can be accounted for. Such secular deviation can arise because of errors in J_2 and J_4. It also can arise because of omission of higher order even zonal harmonics such as J_6, J_8, etc. To better understand how the linear deviation of the computed from the observed elements arose, the residuals g_{res} and h_{res} were fitted to a least squares fit of the form

$$g_{res}(t_n) \approx A_0 + A_1(t_n)$$

and

$$h_{res}(t_n) \approx B_0 + B_1(t_n).$$

The resulting slope for g_{res} had almost twice the absolute magnitude of the slope for h_{res}. By adjusting the zonal harmonic J_4 from -2.123×10^{-6} to -1.672×10^{-6} the magnitude of the slope for g_{res} became almost the same as the slope for h_{res}. Three values of J_4 were tested: -1.672×10^{-6}, -1.840×10^{-6} and -2.123×10^{-6}. The least squares slope for g_{res} in degrees per day for these J_4 values was -2.748×10^{-4}, -3.644×10^{-4} and -5.163×10^{-4}. The corresponding slope for h_{res} in degrees per day was 2.919×10^{-4}, 2.781×10^{-4} and 2.489×10^{-4}. Such errors in J_4 are plausible. Higher order even harmonics presumably contribute to a lesser degree to the secular deviation.

To further investigate the residuals between the computed and observed elements, the straight-line least squares solution was subtracted from the residuals. This essentially left only the periodic components of g_{res} and h_{res}. Let these periodic components be denoted g_{pres} and h_{pres}. See the data in tables 5 and 7. The equations for g_{res} and h_{res} are

$$g_{pres}(t_n) = g_{res}(t_n) - [A_0 + A_1(t_n)] \tag{44}$$

and

$$h_{pres}(t_n) = h_{res}(t_n) - [B_0 + B_1(t_n)]. \tag{45}$$

Looking at the graphs of g_{pres} and h_{pres}, graphs 7 and 10, which consist of 84 points scattered from 10 to 654 days past epoch, one notices immediately that the graph for g_{pres} is definitely periodic. The amplitude is about 27 seconds of arc or .0075 degrees. The maxima occur at about 319 and 633 days past epoch and minima occur at about 158 and 458 days past epoch. This indicates the presence of a function composed predominantly of a term which has a frequency of $\dot{g}$ (or a period of 338 days).

The graph for h_{pres} appears to be quite different from the graph for g_{pres}. A long-period luni-solar term still seems to be present in h_{pres}. The luni-solar term has the same behavior as in Felsentreger's paper.[1] The amplitude of the

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[1]Felsentreger, "Motion of Relay II," p. 37.

term is about 8 seconds of arc. The existence of the term implies that the elements e_c, i_c, h_c, and g_c, which were input to the program, were not corrected for this term. Since the amplitude is so small, a similar luni-solar term would only slightly effect the graph for g. The graph for h_{pres} has an amplitude of only a few seconds of arc. It has many more arcs superimposed on it than g_{pres}. This means that terms with a frequency $2\dot{h}$ have a larger amplitude than terms with a frequency $\dot{h}$. Also a great deal of noise is evident in this small amplitude data. To investigate the components present in g_{pres} and h_{pres} will require different approaches due to the different characteristics of the graphs.

It is evident from the graph for g_{pres}, that there is a fair amount of noise present in the data. Visual inspection of the curve indicates that it must have an amplitude less than 43 seconds or arc or .012 degrees. Four of the 84 points for g_{pres} lie outside this limit. A typical maxima occurred around days 333 and 347. The amplitude of g_{pres} is .0079 and .0074 degrees on those dates. These values give a good indication of the actual amplitude of the periodic terms, g_{pres}. The amplitudes of the periodic components in g_{pres} had to be accurately determined to obtain a meaningful value for J_3. This implied that the effect of the noise in the data for g_{pres} had to be lessened. Either of two methods would be adequate: points which were fairly detached from the curve could be discarded or each pair of successive points could be averaged. The effect of both methods would be to lessen the maximum amplitude of g_{pres}. Since there was a great deal of noise, the first method was chosen. Four points which had an amplitude that was greater than .012 degrees and which would heavily bias a least squares solution were discarded. The remaining 80 points were fitted to an equation of the form

$$g_{pres}(t_n) \approx A_2 \sin \dot{g}(t_n) + A_3 \cos \dot{g}(t_n)$$

$$+ A_4 \sin 2\dot{g}(t_n) + A_5 \cos 2\dot{g}(t_n) . \qquad (46)$$

Then

$$A_2 \sin \dot{g}(t_n) + A_3 \cos \dot{g}(t_n) = \sqrt{A_2^2 + A_3^2} \sin \left(\dot{g}(t_n) + \tan^{-1} \frac{A_2}{A_3}\right)$$

$$= \sqrt{A_2^2 + A_3^2} \cos \left(\dot{g}(t_n) + \left(90 - \tan^{-1} \frac{A_2}{A_3}\right)\right) .$$

The size of $\sqrt{A_2^2 + A_3^2}$ indicates the magnitude of the $\cos \dot{g} t$ term which is produced by the earth's J_3 zonal harmonic. This term was deliberately not included in the constructed input disturbing function. The magnitude $\sqrt{A_4^2 + A_5^2}$ indicates the size of the $(J_4/J_2) \sin 2 \dot{g} t$ term. Performing the least squares fit yields the following values for the least squares coefficients:

$$A_2 = -.002649 \text{ degrees},$$

$$A_3 = +.005335 \text{ degrees},$$

$$A_4 = -.000717 \text{ degrees},$$

$$A_5 = -.000532 \text{ degrees},$$

$$\sqrt{A_2^2 + A_3^2} = .005956 \text{ degrees},$$

$$\sqrt{A_4^2 + A_5^2} = .000893 \text{ degrees}.$$

Refer to graph 8 and table 6 which were constructed using these coefficients.

From the least squares value for $\sqrt{A_2^2 + A_3^2}$ and a formula used in Kozai's paper, a value for J_3 can be determined.[1] The perturbation in g due to the J_3 harmonic by observation is

$$\delta g_{J_3} = \sqrt{A_2^2 + A_3^2}\cos g . \tag{47}$$

The perturbation according to Kozai is

$$\delta g_{J_3} = \frac{(-J_3\ J_2)(1 - \cos^2 i - e^2 \cos^2 i)}{2\, ea\, (1 - e^2) \sin i} \cos g . \tag{48}$$

The right-hand portions of each of the above equations are equated and solved for J_3 on each of the 80 observation dates. The average J_3 determined by using the angular variable g is -2.589×10^{-6}. To get an idea as to the error in this measurement, one can solve the case where no points were discarded, all 84 points were used. Then

$$\sqrt{A_2^2 + A_3^2} = .006508 \text{ degrees}$$

and

$$\sqrt{A_4^2 + A_5^2} = .001085 \text{ degrees} .$$

The quantity J_3 would then equal -2.826×10^{-6}. If the four g_{pres} points are discarded as stray maxima points (noise), then the value of J_3 determined from

[1]Y. Kozai, "Second Order Solution of Artificial Satellite Theory without Air Drag," Astronomical Journal, LXVII (September, 1962), 457.

the angular variable g agrees well with the J_3 determined from a study of the eccentricity by Felsentreger.[1] His value for J_3 was -2.503×10^{-6}.

Due to the presence of the conspicuous luni-solar term and of a great deal of noise, a determination of J_3 from h_{pres} posed difficulties. Also if one computed the predicted amplitude of the perturbation in h due to the J_3 harmonic term from the equation in Kozai's paper,

$$\delta h_{J_3} = \frac{(-J_3 \; J_2) \; e \cos i}{2a \; (1 - e^2) \sin i} , \tag{49}$$

then the amplitude of the perturbation was on the order of a half second of arc.[2] The magnitude of the scatter in the data for h_{pres} had about the same size. Thus a determination of J_3 from h_{pres} would have been meaningless.

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[1]Felsentreger, "Motion of Relay II," p. 9.

[2]Kozai, "Second Order Satellite Theory," p. 457.

CHAPTER IV

CONCLUDING REMARKS

The problem of resonance complicating the computation of the angular orbital elements for Relay II has been avoided. The problem had first been noticed when the computed values of the luni-solar perturbations for g and h using the SLOPE method, as given by Murphy and Felsentreger, were excessively large. In the SLOPE method, the derivative of the angular variable was obtained by using the Lagrange planetary equations. These derivatives were then integrated analytically. The perturbation in g or h is then in the form of a series of trigonometric terms. Several of these terms have the factor $(\dot{g} + \dot{h})$ in the denominator. Since $\dot{g}$ and $\dot{h}$ have about the same magnitude but opposite sign, terms which contain their sum in the denominator are nearly singular.

To avoid the problem encountered by the SLOPE method and to show the feasibility of computing the perturbations in the angular orbital elements accurately, another disturbing function of only a limited number of terms was formed. A coordinate transformation was done on $i_{\text{ℂ}}$ to eliminate it from the disturbing function. Lagrange's equations were again formed to yield the derivatives of the angular elements. No term in the analytical expression for $\dot{g}$ or

$\dot{h}$ contained the factor $(\dot{g} + \dot{h})$ in the denominator. The derivatives were evaluated and integrated numerically. The elements computed for g and h differed by approximately .3 degrees and .2 degrees from those observed at the end of a two-year period. The slope of the linear deviation in the computed and observed values for g and h can be added as a correction to any model attempting to explain the motion of g and h for Relay II. After accounting for the secular portion of the difference between the observed and computed differences in the angular variables, the remaining periodic difference for g was shown to be mainly due to the J_3 zonal harmonic of the earth. The remaining periodic difference for h was mostly noise superimposed upon a small amplitude long period luni-solar term. The accuracy of this numerical approach to calculate the perturbations in g and h was tested by making a computation for J_3 using g. The value of J_3 that was determined was consistent with the value obtained by Felsentreger, who studied the eccentricity of Relay II.

The primary motive of this work was to show that the perturbations in the argument of perigee and the longitude of the ascending node for Relay II could be obtained by numerical integration of Lagrange's planetary equations. Since this method yielded an acceptable value for J_3, the method was shown to be workable. The problem can now be reversed by solving only for the perturbations in g and h, rather than for the coefficient J_3. To evaluate the perturbations, the analytical expression for the J_3 component of the disturbing function should be added to the constructed disturbing function R. Lagrange's planetary equations can then be formed for the derivatives of the angular orbit elements. These derivatives should then be evaluated and integrated numerically. The resulting perturbations can then be used to generate an ephemeris for g and h.

APPENDIX A

SYMBOLS

a = semimajor axis of the satellite orbit

e = eccentricity of the satellite orbit

i = inclination of the satellite's orbital plane to the earth's equatorial plane

M = mean anomaly of the satellite

ω = argument of perigee of the satellite's orbit

Ω = longitude of ascending node of the satellite's orbit

f = true anomaly of the satellite

n = mean angular motion of the satellite

T = time of perigee passage

$\tilde{\omega}$ = longitude of perigee

$\dot{a}, \dot{e}, \dot{i}, \dot{\omega}, \dot{\Omega}, \dot{\tilde{\omega}}, \dot{\ell}$ = time derivatives of the orbital elements

$g(t_0), h(t_0)$ = value of g (or h) on January 24, 1964, 21 hours, 41 minutes U.T.

$p_1, p_2, p_3, q_1, q_2, q_3$ = are the canonical variables. The p_i are the canonical conjugates of the q_i.

P = orbital period

ℓ, g, h = Delaunay canonical variables which are equivalent to the Keplerian elements M, ω and Ω

$L = \sqrt{a}$

$G = L\sqrt{1 - e^2}$

$H = G \cos i$

$t_n - t_0$ = days since January 24, 1964, 21 hours, 41 minutes U.T.

e_c, i_c, h_c, g_c = element corrected for solar radiation pressure and long-period luni-solar effects, excluding near-resonant terms

γ = direction of the vernal equinox

$m_\odot, m_☾$ = mass of sun (moon)/mass of sun (moon) + mass of earth

$n_\odot, n_☾$ = mean motion of the sun (moon) relative to the earth

$i_\odot, i_☾$ = inclination of the sun's (moon's) orbital plane to the earth's equatorial plane

$\Omega_\odot, \Omega_☾$ = longitude of the mean ascending node of the sun's (moon's) orbit on the earth's equator, measured from the mean equinox of date

i'' = inclination of the moon's orbital plane to the ecliptic = 5.1453967 degrees

Ω'' = longitude of the mean ascending node of the lunar orbit on the ecliptic, measured from the mean equinox of date.

J_2, J_3, J_4 = zonal harmonic coefficients in the earth's gravitational potential

A_2 = Kozai's equivalent to (3/2) J_2

$\bar{i}, \bar{j}, \bar{k}$ = Cartesian unit vectors in the x, y, z directions respectively

$\bar{\nabla}$ = gradient vector operator. In Cartesian coordinates it is

$$\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$$

∇^2 = Laplacian scalar differential operator $\bar{\nabla} \cdot \bar{\nabla}$. In Cartesian coordinates it is

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$\partial R/\partial e$, $\partial R/\partial i$ = partial derivative of the disturbing function R with respect to e (or i), holding all the other elements constant

δg, δh = size of the perturbation in g (or h) over the time interval $t - t_0$

V = potential as used in physics ($r = \infty$, $V = 0$), ($r = 0$, $V = -\infty$)

U = potential as used in astronomy $U = -V$. U is also called the forcing function, ($r = \infty$, $U = 0$), ($r = 0$, $U = +\infty$)

k^2 = universal gravitational constant

$\hat{F}$ = Hamiltonian of a dynamical system

μ = $k^2 (m_1 + m_2)$. In canonical units $\mu = 1$

T_{KE} = kinetic energy of satellite

$\bar{F}$ = force exerted on a body as given by Newton's laws

R = disturbing function. The contribution to the forcing function of a body which makes the body deviate from simple two body motion

F_R = sum of the two near-resonant terms in the luni-solar disturbing function

F_S = secular disturbing function

$R_{R(SM)}$ = the two resonant terms in the luni-solar disturbing function

$R_{S(E)}$ = secular part of the disturbing function due to the earth's spherical harmonics

$R_{S(SM)}$ = secular part of the disturbing function due to the sun and moon

m_1, m_2 = mass of body one (two) as implied in Newton's second law

r_e = radius of the earth

r_0 = initial vector position

r = position vector

$\dot{\bar{r}}$ = first time derivative of the position vector

$\ddot{\bar{r}}$ = second time derivative of the position vector

$\bar{r}_1$ = position vector of body one with respect to the center of mass of the system

$\ddot{\bar{r}}_1$ = second derivative of the vector $\bar{r}_1$

x, y, z = Cartesian coordinate components of $\bar{r}$

$\dot{x}, \dot{y}, \dot{z}$ = first time derivatives of the Cartesian coordinates

$\bar{r}_{12}$ = position vector of body two with respect to body one

r = magnitude of the vector $\bar{r}$

$|\bar{\Delta}|$ = magnitude of the vector $\bar{\Delta}$

$\bar{r}'$ = position vector of the perturbing body, the sun or moon, with respect to the earth

θ = geocentric colatitude of the satellite

ϕ = geocentric longitude of the satellite

δ = declination or the angular distance above or below the celestial equator at which the satellite appears in the sky

$\tilde{R}(r)$ = function dependent on the coordinate r

APPENDIX B

TABLES

TABLE 1

COMPUTED DERIVATIVES AND PERTURBATIONS FOR THE ARGUMENT OF PERIGEE

Obs. No.	Number of Days Since Epoch Date	Derivative of the Arg. of Perigee (Degrees/Day)	Computed Pert. in the Arg. of Perigee (Degrees)
2	10.0000000	1.1056376	7.7393818
3	17.0000000	1.1055355	15.4787064
4	24.0000000	1.1056547	23.2180786
5	31.0000000	1.1055241	30.9574280
6	38.0000000	1.1056795	38.6968536
7	45.0000000	1.1055374	46.4363098
8	52.0000000	1.1056471	54.1756897
9	59.0000000	1.1055746	61.9151611
10	66.0000000	1.1055965	69.6544800
11	73.0000000	1.1055908	77.3938446
12	80.0000000	1.1057444	85.1337128
13	87.0000000	1.1057653	92.8741608
14	101.0000000	1.1055708	108.3539429
15	108.0000000	1.1056337	116.0933685
16	115.0000000	1.1057196	123.8332825
17	129.0000000	1.1056538	139.3133240
18	151.0000000	1.1057796	163.6397858
19	158.0000000	1.1057682	171.3804169
20	165.0000000	1.1059513	179.1215820
21	172.0000000	1.1058750	186.8631897
22	186.0000000	1.1058359	202.3455963
23	193.0000000	1.1058092	210.0865479
24	200.0000000	1.1058912	217.8277130
25	207.0000000	1.1057758	225.5687256
26	214.0000000	1.1058426	233.3095703
27	221.0000000	1.1057758	241.0504150
28	228.0000000	1.1058369	248.7912598
29	235.0000000	1.1057882	256.5319824
30	256.0000000	1.1058044	279.7543945
31	263.0000000	1.1058874	287.4953613
32	270.0000000	1.1058111	295.2365723
33	277.0000000	1.1059160	302.9777832
34	284.0000000	1.1058273	310.7192383
35	291.0000000	1.1058645	318.4602051
36	298.0000000	1.1057329	326.2011719
37	305.0000000	1.1057596	333.9414063
38	312.0000000	1.1058540	341.6823730
39	319.0000000	1.1057377	349.4230957
40	326.0000000	1.1058130	357.1638184
41	333.0000000	1.1056538	364.9040527
42	340.0000000	1.1058273	372.6445312

Epoch Date = 1/21/64, 21 hours, 41 minutes, U.T.

TABLE 1--Continued

Obs. No.	Number of Days Since Epoch Date	Derivative of the Arg. of Perigee (Degrees/Day)	Computed Pert. in the Arg. of Perigee (Degrees)
43	347.0000000	1.1057196	380.3850098
44	354.0000000	1.1057577	388.1254883
45	361.0000000	1.1056681	395.8657227
46	368.0000000	1.1057587	403.6059570
47	375.0000000	1.1057186	411.3461914
48	382.0000000	1.1057720	419.0866699
49	389.0000000	1.1057663	426.8271484
50	395.0964355	1.1057425	433.5683594
51	402.0964355	1.1056852	441.3085938
52	409.0964355	1.1057806	449.0488281
53	423.0964355	1.1057978	464.5302734
54	430.0964355	1.1059437	472.2717285
55	437.0964355	1.1059017	480.0134277
56	444.0964355	1.1060905	487.7553711
57	451.0964355	1.1061144	495.4982910
58	458.0964355	1.1060953	503.2412109
59	465.0964355	1.1060152	510.9838867
60	472.0964355	1.1059046	518.7258301
61	479.0964355	1.1059008	526.4672852
62	486.0964355	1.1059599	534.2089844
63	493.0964355	1.1060343	541.9511719
64	500.0964355	1.1059532	549.6933594
65	507.0964355	1.1060085	557.4353027
66	514.0964355	1.1059675	565.1774902
67	521.0964355	1.1060104	572.9196777
68	528.0964355	1.1060324	580.6621094
69	535.0964355	1.1060295	588.4045410
70	549.0964355	1.1060581	603.8894043
71	556.0964355	1.1061239	611.6323242
72	563.0964355	1.1060085	619.3750000
73	570.0964355	1.1060467	627.1174316
74	577.0964355	1.1060257	634.8598633
75	584.0964355	1.1060181	642.6022949
76	591.0964355	1.1061344	650.3449707
77	598.0964355	1.1060696	658.0878906
78	605.0964355	1.1059628	665.8300781
79	612.0964355	1.1060429	673.5720215
80	619.0964355	1.1060848	681.3146973
81	626.0964355	1.1060591	689.0576172
82	633.0964355	1.1061192	696.8005371
83	640.0964355	1.1060762	704.5434570
84	647.0964355	1.1059570	712.2851563
85	654.0964355	1.1059685	720.0273438

TABLE 2

COMPUTED DERIVATIVES AND PERTURBATIONS FOR
THE LONGITUDE OF THE ASCENDING NODE

Obs. No.	Number of Days Since Epoch Date	Derivative of the Long. of the Asc. Node (Degrees/Day)	Computed Pert. in the Long. of Asc. Node (Degrees)
2	10.0000000	-1.1040640	-7.7285700
3	17.0000000	-1.1040258	-15.4870990
4	24.0000000	-1.1040773	-23.1856689
5	31.0000000	-1.1040268	-30.9142609
6	38.0000000	-1.1040850	-38.6428833
7	45.0000000	-1.1040449	-46.3715210
8	52.0000000	-1.1040735	-54.1001740
9	59.0000000	-1.1040449	-61.8287811
10	66.0000000	-1.1040697	-69.5573730
11	73.0000000	-1.1040649	-77.2860413
12	80.0000000	-1.1041193	-85.0148621
13	87.0000000	-1.1041250	-92.7438965
14	101.0000000	-1.1040831	-108.2018280
15	108.0000000	-1.1041126	-115.9306946
16	115.0000000	-1.1041431	-123.6597900
17	129.0000000	-1.1041489	-139.1182556
18	151.0000000	-1.1042042	-163.4108429
19	158.0000000	-1.1042194	-171.1405334
20	165.0000000	-1.1042585	-178.8703461
21	172.0000000	-1.1042728	-186.6004181
22	186.0000000	-1.1042757	-202.0606842
23	193.0000000	-1.1042643	-209.7907715
24	200.0000000	-1.1043177	-217.5210114
25	207.0000000	-1.1042614	-225.2512054
26	214.0000000	-1.1043062	-232.9813995
27	221.0000000	-1.1042652	-240.7115784
28	228.0000000	-1.1043119	-248.4418335
29	235.0000000	-1.1042957	-256.1721191
30	256.0000000	-1.1043158	-279.3630371
31	263.0000000	-1.1043472	-287.0937500
32	270.0000000	-1.1043434	-294.8242188
33	277.0000000	-1.1043758	-30[illegible].[illegible]549316
34	284.0000000	-1.1043539	-310.2556445
35	291.0000000	-1.1043625	-318.0163574
36	298.0000000	-1.1043348	-325.7470703
37	305.0000000	-1.1043482	-333.4777832
38	312.0000000	-1.1043844	-341.2084961
39	319.0000000	-1.1043501	-348.9392090
40	326.0000000	-1.1043596	-356.6659219
41	333.0000000	-1.1043177	-364.4003906
42	340.0000000	-1.1043653	-372.1311035

TABLE 2--Continued

Obs. No.	Number of Days Since Epoch Date	Derivative of the Long. of the Asc. Node (Degrees/Day)	Computed Pert in the Long. of Asc. Node (Degrees)
43	347.0000000	-1.1043348	-379.8618164
44	354.0000000	-1.1043367	-387.5922852
45	361.0000000	-1.1043186	-395.3227539
46	368.0000000	-1.1043444	-403.0532227
47	375.0000000	-1.1043472	-410.7839355
48	382.0000000	-1.1043425	-418.5144043
49	389.0000000	-1.1043520	-426.2451172
50	395.0964355	-1.1043730	-432.9777832
51	402.0964355	-1.1043310	-440.7084961
52	409.0964355	-1.1043587	-448.4392090
53	423.0964355	-1.1043854	-463.9006348
54	430.0964355	-1.1044445	-471.6318359
55	437.0964355	-1.1044273	-479.3630371
56	444.0964355	-1.1045084	-487.0944824
57	451.0964355	-1.1045132	-494.8261719
58	458.0964355	-1.1045256	-502.5581055
59	465.0964355	-1.1045103	-510.2897949
60	472.0964355	-1.1044903	-518.0217285
61	479.0964355	-1.1044865	-525.7531738
62	486.0964355	-1.1045332	-533.4851074
63	493.0964355	-1.1045475	-541.2170410
64	500.0964355	-1.1045599	-548.9489746
65	507.0964355	-1.1045761	-556.6811523
66	514.0964355	-1.1045790	-564.4135742
67	521.0964355	-1.1045856	-572.1457520
68	528.0964355	-1.1046238	-579.8781738
69	535.0964355	-1.1046124	-587.6105957
70	549.0964355	-1.1046324	-603.0755277
71	556.0964355	-1.1046696	-610.8085938
72	563.0964355	-1.1046410	-618.5412598
73	570.0964355	-1.1046667	-626.2741699
74	577.0964355	-1.1046486	-634.0068359
75	584.0964355	-1.1046677	-641.7397461
76	591.0964355	-1.1046839	-649.4726563
77	598.0964355	-1.1046743	-657.2055664
78	605.0964355	-1.1046362	-664.9384766
79	612.0964355	-1.1046705	-672.6708984
80	619.0964355	-1.1046810	-680.4038086
81	626.0964355	-1.1046734	-688.1369629
82	633.0964355	-1.1046934	-695.8701172
83	640.0964355	-1.1046877	-703.6030273
84	647.0964355	-1.1046572	-711.3354492
85	654.0964355	-1.1046562	-719.0683594

TABLE 3

COMPUTED, OBSERVED AND RESIDUAL VALUES FOR THE ARGUMENT OF PERIGEE

Number of Days Since Epoch Date	Computed Argument of Perigee (Degrees)	Observed Argument of Perigee (Degrees)	Residual (Comp.--Obs.) (Degrees)
10.0000000	195.7567444	195.7681274	-0.0113831
17.0000000	203.4960632	203.5079803	-0.0119171
24.0000000	211.2354431	211.2566376	-0.0211945
31.0000000	218.9747925	218.9953461	-0.0205536
38.0000000	226.7142181	226.7438965	-0.0296783
45.0000000	234.4536743	234.4820099	-0.0283356
52.0000000	242.1930542	242.2241669	-0.0311127
59.0000000	249.9325256	249.9664917	-0.0339661
66.0000000	257.6716309	257.7160645	-0.0444336
73.0000000	265.4111328	265.4548340	-0.0437012
80.0000000	273.1508789	273.1989746	-0.0480957
87.0000000	280.8913574	280.9433594	-0.0520020
101.0000000	296.3710937	296.4265137	-0.0554199
108.0000000	304.1105957	304.1713867	-0.0607910
115.0000000	311.8505859	311.9147949	-0.0642090
129.0000000	327.3305664	327.4052734	-0.0747070
151.0000000	351.6569824	351.7507324	-0.0937500
158.0000000	359.3977051	359.4963379	-0.0986328
165.0000000	7.1389160	7.2409868	-0.10207[illegible]
172.0000000	14.8803711	14.9842863	-0.10391[illegible]
186.0000000	30.3627930	30.4728851	-0.1100922
193.0000000	38.1037598	38.2148438	-0.1110840
200.0000000	45.8449707	45.9591675	-0.1141968
207.0000000	53.5859375	53.6982880	-0.1123505
214.0000000	61.3269043	61.4508209	-0.1239166
221.0000000	69.0676270	69.1942291	-0.1266022
228.0000000	76.8085938	76.9362183	-0.1276245
235.0000000	84.5493164	84.6751556	-0.1258392
256.0000000	107.7717285	107.9191132	-0.1473846
263.0000000	115.5126953	115.6549835	-0.1422882
270.0000000	123.2539062	123.3988800	-0.1449738
277.0000000	130.9951172	131.1443176	-0.1492004
284.0000000	138.7365723	138.8918762	-0.1553040
291.0000000	146.4775391	146.6333771	-0.1558380
298.0000000	154.2185059	154.3786011	-0.1600952
305.0000000	161.9587402	162.1191101	-0.1603699
312.0000000	169.6997070	169.8631744	-0.1634674
319.0000000	177.4404297	177.6011353	-0.1607056
326.0000000	185.1811523	185.3511047	-0.1699524
333.0000000	192.9213867	193.0941315	-0.1727448
340.0000000	200.6618652	200.8407593	-0.1788940

TABLE 3--Continued

Number of Days Since Epoch Date	Computed Argument of Perigee (Degrees)	Observed Argument of Perigee (Degrees)	Residual (Comp.-Obs.) (Degrees)
347.0000000	208.4023437	208.5828094	-0.1804657
354.0000000	216.1428223	216.3295593	-0.1867371
361.0000000	223.8830566	224.0752106	-0.1921539
368.0000000	231.6232910	231.8216248	-0.1983337
375.0000000	239.3635254	239.5666809	-0.2031555
382.0000000	247.1040039	247.3149567	-0.2109528
389.0000000	254.8444824	255.0584717	-0.2139893
395.0964355	261.5856934	261.8024902	-0.2167969
402.0964355	269.3259277	269.5471191	-0.2211914
409.0964355	277.0661621	277.2915039	-0.2253418
423.0964355	292.5476074	292.7868652	-0.2392578
430.0964355	300.2890625	300.5310059	-0.2419434
437.0964355	308.0307617	308.2761230	-0.2453613
444.0964355	315.7727051	316.0175781	-0.2448730
451.0964355	323.5156250	323.7636719	-0.2480469
458.0964355	331.2585449	331.5146484	-0.2561035
465.0964355	339.0012207	339.2556152	-0.2543945
472.0964355	346.7431641	347.0026855	-0.2595215
479.0964355	354.4846191	354.7470703	-0.2624512
486.0964355	2.2263184	2.4898586	-0.2635403
493.0964355	9.9685059	10.2401361	-0.2716303
500.0964355	17.7106934	17.9889221	-0.2782288
507.0964355	25.4526367	25.7281647	-0.2755280
514.0964355	33.1948242	33.4798889	-0.2850647
521.0964355	40.9370117	41.2181702	-0.2811584
528.0964355	48.6794434	48.9686737	-0.2892303
535.0964355	56.4218750	56.7084198	-0.2865448
549.0964355	71.9067383	72.1979218	-0.2911835
556.0964355	79.6496582	79.9532013	-0.3035431
563.0964355	87.3923340	87.6879272	-0.2955933
570.0964355	95.1347656	95.4409027	-0.3061371
577.0964355	102.8771973	103.1825104	-0.3053131
584.0964355	110.6196289	110.9314423	-0.3118134
591.0964355	118.3623047	118.6692200	-0.3069153
598.0964355	126.1052246	126.4133453	-0.3081207
605.0964355	133.8474121	134.1591034	-0.3116913
612.0964355	141.5893555	141.9064484	-0.3170929
619.0964355	149.3320312	149.6548920	-0.3228607
626.0964355	157.0749512	157.3941040	-0.3191528
633.0964355	164.8178711	165.1446075	-0.3267365
640.0964355	172.5607910	172.8916931	-0.3309021
647.0964355	180.3024902	180.6322021	-0.3297119
654.0964355	188.0446777	188.3854523	-0.3407745

TABLE 4

COMPUTED, OBSERVED AND RESIDUAL VALUES FOR THE LONGITUDE OF THE ASCENDING NODE

Number of Days Since Epoch Date	Computed Longitude of Asc. Node (Degrees)	Observed Long. of Asc. Node (Degrees)	Residual (Comp.–Obs.) (Degrees)
10.0000000	212.5613098	212.5598907	0.0014191
17.0000000	204.8327789	204.8291626	0.0036163
24.0000000	197.1042175	197.0984192	0.0057983
31.0000000	189.3756256	189.3675842	0.0080414
38.0000000	181.6470032	181.6370544	0.0099487
45.0000000	173.9183655	173.9068146	0.0115509
52.0000000	166.1897125	166.1775513	0.0121613
59.0000000	158.4611053	158.4484558	0.0126495
66.0000000	150.7325134	150.7148895	0.0176239
73.0000000	143.0038452	142.9832153	0.0206299
80.0000000	135.2750244	135.2530365	0.0219879
87.0000000	127.5459900	127.5232849	0.0227051
101.0000000	112.0880585	112.0612640	0.0267944
108.0000000	104.3591919	104.3300323	0.0291598
115.0000000	96.6300964	96.6002808	0.0298157
129.0000000	81.1716309	81.1366272	0.0350037
151.0000000	56.8790436	56.8399200	0.0391235
158.0000000	49.1493530	49.1082153	0.0411377
165.0000000	41.4195404	41.3734589	0.0460815
172.0000000	33.6894684	33.6444244	0.0450439
186.0000000	18.2292023	18.1798401	0.0493622
193.0000000	10.4991179	10.4461279	0.0529900
200.0000000	2.7688780	2.7165384	0.0523396
207.0000000	355.0385742	354.9851074	0.0534668
214.0000000	347.3083496	347.2529297	0.0554199
221.0000000	339.5781250	339.5207520	0.0573730
228.0000000	331.8479004	331.7878418	0.0600586
235.0000000	324.1176758	324.0571289	0.0605469
256.0000000	300.9267578	300.8618164	0.0649414
263.0000000	293.1960449	293.1286621	0.0673828
270.0000000	285.4655762	285.3967285	0.0688477
277.0000000	277.7348633	277.6638184	0.0710449
284.0000000	270.0041504	269.9328613	0.0712891
291.0000000	262.2734375	262.1989746	0.0744629
298.0000000	254.5428314	254.4674225	0.0754089
305.0000000	246.8121185	246.7346039	0.0775146
312.0000000	239.0814056	239.0040894	0.0773163
319.0000000	231.3506927	231.2694702	0.0812225
326.0000000	223.6199799	223.5367126	0.0832672
333.0000000	215.8895111	215.8036346	0.0858765
340.0000000	208.1587982	208.0719604	0.0868378

TABLE 4--Continued

Number of Days Since Epoch Date	Computed Longitude of Asc. Node (Degrees)	Observed Long. of Asc. Node (Degrees)	Residual (Comp.–Obs.) (Degrees)
347.0000000	200.4280853	200.3385010	0.0895844
354.0000000	192.6976166	192.6063385	0.0912781
361.0000000	184.9671478	184.8748932	0.0922546
368.0000000	177.2366791	177.1418610	0.0948181
375.0000000	169.5059662	169.4085083	0.0974579
382.0000000	161.7754974	161.6759644	0.0995331
389.0000000	154.0447845	153.9466553	0.0981293
395.0964355	147.3121185	147.2118683	0.1002502
402.0964355	139.5814056	139.4793396	0.1020660
409.0964355	131.8506927	131.7459717	0.1047211
423.0964355	116.3892670	116.2817230	0.1075439
430.0964355	108.6580628	108.5491180	0.1089478
437.0964355	100.9270020	100.8151093	0.1118927
444.0964355	93.1955566	93.0823517	0.1132050
451.0964355	85.4638672	85.3509064	0.1129608
458.0964355	77.7319336	77.6152496	0.1166840
465.0964355	70.0002441	69.8845673	0.1156769
472.0964355	62.2683105	62.1520844	0.1162262
479.0964355	54.5368652	54.4187164	0.1181488
486.0964355	46.8049316	46.6855164	0.1194153
493.0964355	39.0729980	38.9510803	0.1219177
500.0964355	31.3410645	31.2163086	0.1247559
507.0964355	23.6088867	23.4806519	0.1282349
514.0964355	15.8764648	15.7472286	0.1292362
521.0964355	8.1442871	8.0136070	0.1306801
528.0964355	0.4118652	0.2791299	0.1327353
535.0964355	352.6794434	352.5441895	0.1352539
549.0964356	337.2141113	337.0761719	0.1379395
556.0964355	329.4814453	329.3403320	0.1411133
563.0964355	321.7487793	321.6081543	0.1406250
570.0964355	314.0158691	313.8742676	0.1416016
577.0964355	306.2832031	306.1394043	0.1437988
584.0964355	298.5502930	298.4033203	0.1469727
591.0964355	290.8173828	290.6701660	0.1472168
598.0964355	283.0844727	282.9362793	0.1481934
605.0964355	275.3515625	275.2001953	0.1513672
612.0964355	267.6191406	267.4675293	0.1516113
619.0964355	259.8862305	259.7302246	0.1560059
626.0964355	252.1530762	251.9956512	0.1574249
633.0964355	244.4199219	244.2615356	0.1583862
640.0964355	236.6870117	236.5265350	0.1604767
647.0964355	228.9545898	228.7889709	0.1656189
654.0964355	221.2216797	221.0537567	0.1679230

TABLE 5

RESIDUAL OF THE COMPUTED AND OBSERVED VALUE OF THE ARGUMENT OF PERIGEE AND ITS PERIODIC COMPONENT

Number of Days Past Epoch	Residual of the Comp. and Obs Value of the Arg. of Per. (Degrees)	Periodic Component of Original Res. (Degrees)
10.0000000	-0.0113831	0.0025341
17.0000000	-0.0119171	0.0056146
24.0000000	-0.0211945	-0.0000481
31.0000000	-0.0205536	0.0042074
38.0000000	-0.0296783	-0.0013028
45.0000000	-0.0283356	0.0036546
52.0000000	-0.0311127	0.0044921
59.0000000	-0.0339661	0.0052533
66.0000000	-0.0444336	-0.0015996
73.0000000	-0.0437012	0.0027474
80.0000000	-0.0480957	0.0019675
87.0000000	-0.0520020	0.0016759
101.0000000	-0.0554199	0.0054871
108.0000000	-0.0607910	0.0037306
115.0000000	-0.0642090	0.0039272
129.0000000	-0.0747070	0.0006584
151.0000000	-0.0937500	-0.0070244
158.0000000	-0.0986328	-0.0082926
165.0000000	-0.1020708	-0.0081160
172.0000000	-0.1039152	-0.0063458
186.0000000	-0.1100922	-0.0052935
193.0000000	-0.1110840	-0.0026708
200.0000000	-0.1141968	-0.0021690
207.0000000	-0.1123505	0.0032920
214.0000000	-0.1239166	-0.0046596
221.0000000	-0.1266022	-0.0037305
228.0000000	-0.1276245	-0.0011383
235.0000000	-0.1258392	0.0042616
256.0000000	-0.1473846	-0.0064399
263.0000000	-0.1422882	0.0022711
270.0000000	-0.1449738	0.0032002
277.0000000	-0.1492004	0.0025881
284.0000000	-0.1553040	0.0000992
291.0000000	-0.1558380	0.0031797
298.0000000	-0.1600952	0.0025371
305.0000000	-0.1603699	0.0058771
312.0000000	-0.1614674	0.0063941
319.0000000	-0.1607056	0.0127706
326.0000000	-0.1699524	0.0071384
333.0000000	-0.1727448	0.0079606
340.0000000	-0.1788940	0.0054259

TABLE 5--Continued

Number of Days Past Epoch	Residual of the Comp. and Obs. Value of the Arg. of Per. (Degrees)	Periodic Component of Original Res. (Degrees)
347.0000000	-0.1804657	0.0074689
354.0000000	-0.1867371	0.0048121
361.0000000	-0.1921539	0.0030099
368.0000000	-0.1983337	0.0004447
375.0000000	-0.2031555	-0.0007625
382.0000000	-0.2109528	-0.0049452
389.0000000	-0.2139893	-0.0043671
395.0964355	-0.2167969	-0.0040267
402.0964355	-0.2211914	-0.0048066
409.0964355	-0.2253418	-0.0053424
423.0964355	-0.2312578	-0.0120292
430.0964355	-0.2419434	-0.0111001
437.0964355	-0.2453613	-0.0109035
444.0964355	-0.2448730	-0.0068006
451.0964355	-0.2480469	-0.0063598
458.0964355	-0.2561035	-0.0108019
465.0964355	-0.2543945	-0.0054783
472.0964355	-0.2595215	-0.0069906
479.0964355	-0.2624512	-0.0063057
486.0964355	-0.2635403	-0.0037802
493.0964355	-0.2716303	-0.0082556
500.0964355	-0.2782288	-0.0112395
507.0964355	-0.2785280	-0.0049240
514.0964355	-0.2850647	-0.0108461
521.0964355	-0.2811584	-0.0033253
528.0964355	-0.2892303	-0.0077826
535.0964355	-0.2865448	-0.0014824
549.0964355	-0.2911835	0.0011081
556.0964355	-0.3035431	-0.0076369
563.0964355	-0.2985933	0.0039275
570.0964355	-0.3061371	-0.0030017
577.0964355	-0.3053131	0.0014369
584.0964355	-0.3118134	-0.0014488
591.0964355	-0.3089153	0.0070639
598.0964355	-0.3081207	0.0094731
605.0964355	-0.3116913	0.0095171
612.0964355	-0.3170929	0.0077301
619.0964355	-0.3228607	0.0055769
626.0964355	-0.3191528	0.0128994
633.0964355	-0.3267365	0.0089304
640.0964355	-0.3309021	0.0083793
647.0964355	-0.3297119	0.0131841
654.0964355	-0.3407745	0.0057361

TABLE 6

PERIODIC RESIDUAL, LEAST SQUARES APPROXIMATION AND REMAINING RESIDUAL, FOR (g)

Obs. No.	Number of Days Since Epoch Date	Periodic Residual (Degrees)	Least Squares Value (Degrees)	Remaining High Frequency Periodic Res. (Degrees)
2	10.0000000	0.0025341	0.0039983	-0.0014642
3	17.0000000	0.0056146	0.0033963	0.0022184
4	24.0000000	-0.0000481	0.0027798	-0.0028279
5	31.0000000	0.0042074	0.0021614	0.0020459
6	38.0000000	-0.0013028	0.0015504	-0.0028532
7	45.0000000	0.0036546	0.0009532	0.0027014
8	52.0000000	0.0044921	0.0003731	0.0041190
9	59.0000000	0.0052533	-0.0001890	0.0054423
10	66.0000000	-0.0015996	-0.0007340	-0.0008656
11	73.0000000	0.0027474	-0.0012643	0.0040117
12	80.0000000	0.0019675	-0.0017825	0.0037500
13	87.0000000	0.0016759	-0.0022909	0.0039667
14	101.0000000	0.0054871	-0.0032805	0.0087676
15	108.0000000	0.0037306	-0.0037578	0.0074884
16	115.0000000	0.0039272	-0.0042166	0.0081438
17	129.0000000	0.0006584	-0.0050419	0.0057003
18	151.0000000	-0.0070244	-0.0058780	-0.0011464
19	158.0000000	-0.0082926	-0.0059639	-0.0023287
20	165.0000000	-0.0081160	-0.0059393	-0.0021767
21	172.0000000	-0.0063458	-0.0057938	-0.0005521
22	186.0000000	-0.0052935	-0.0051154	-0.0001781
23	193.0000000	-0.0026708	-0.0045817	0.0019109
24	200.0000000	-0.0021690	-0.0039257	0.0017568
25	207.0000000	0.0032920	-0.0031597	0.0064516
26	214.0000000	-0.0046596	-0.0023002	-0.0023593
27	221.0000000	-0.0037305	-0.0013685	-0.0023620
28	228.0000000	-0.0011383	-0.0003891	-0.0007491
29	235.0000000	0.0042616	0.0006110	0.0036506
30	256.0000000	-0.0064399	0.0034535	-0.0098934
31	263.0000000	0.0022711	0.0042590	-0.0019879
32	270.0000000	0.0032002	0.0049554	-0.0017552
33	277.0000000	0.0025881	0.0055257	-0.0029376
34	284.0000000	0.0000992	0.0059580	-0.0058588
35	291.0000000	0.0031797	0.0062457	-0.0030659
36	298.0000000	0.0025371	0.0063875	-0.0038503
37	305.0000000	0.0058771	0.0063874	-0.0005103
38	312.0000000	0.0063941	0.0062541	0.0001400
39	326.0000000	0.0071384	0.0056415	0.0014969
40	333.0000000	0.0079606	0.0051955	0.0027651
41	340.0000000	0.0054259	0.0046809	0.0007451

TABLE 6--Continued

Obs. No.	Number of Days Since Epoch Date	Periodic Residual (Degrees)	Least Squares Value (Degrees)	Remaining High Frequency Periodic Res. (Degrees)
42	347.0000000	0.0074689	0.0041161	0.0033527
43	354.0000000	0.0048121	0.0035187	0.0012934
44	361.0000000	0.0030099	0.0029041	0.0001057
45	368.0000000	0.0004447	0.0022853	-0.0018406
46	375.0000000	-0.0007625	0.0016722	-0.0024347
47	382.0000000	-0.0049452	0.0010718	-0.0060169
48	389.0000000	-0.0043671	0.0004881	-0.0048551
49	395.0964355	-0.0040267	-0.0000055	-0.0040211
50	402.0964355	-0.0048066	-0.0005560	-0.0042506
51	409.0964355	-0.0053424	-0.0010908	-0.0042515
52	430.0964355	-0.0111001	-0.0026267	-0.0084734
53	437.0964355	-0.0109035	-0.0031201	-0.0077834
54	444.0964355	-0.0068006	-0.0036021	-0.0031985
55	451.0964355	-0.0063598	-0.0040678	-0.0022920
56	458.0964355	-0.0108019	-0.0045095	-0.0062923
57	465.0964355	-0.0054783	-0.0049171	-0.0005612
58	472.0964355	-0.0069906	-0.0052780	-0.0017126
59	479.0964355	-0.0063057	-0.0055781	-0.0007275
60	486.0964355	-0.0037802	-0.0058026	0.0020224
61	493.0964355	-0.0082556	-0.0059363	-0.0023193
62	500.0964355	-0.0112395	-0.0059655	-0.0052740
63	507.0964355	-0.0049240	-0.0058782	0.0009542
64	514.0964355	-0.0108461	-0.0056657	-0.0051804
65	521.0964355	-0.0033253	-0.0053230	0.0019977
66	528.0964355	-0.0077826	-0.0048497	-0.0029328
67	535.0964355	-0.0014824	-0.0042504	0.0027680
68	549.0964355	0.0011081	-0.0027169	0.0038250
69	556.0964355	-0.0076369	-0.0018165	-0.0058204
70	563.0964355	0.0039275	-0.0008564	0.0047839
71	570.0964355	-0.0030017	0.0001374	-0.0031391
72	577.0964355	0.0014369	0.0011371	0.0002998
73	584.0964355	-0.0014488	0.0021141	-0.0035629
74	591.0964355	0.0070639	0.0030406	0.0040233
75	598.0964355	0.0094731	0.0038906	0.0055825
76	605.0964355	0.0095171	0.0046410	0.0048761
77	612.0964355	0.0077301	0.0052727	0.0024574
78	619.0964355	0.0055769	0.0057714	-0.0001945
79	633.0964355	0.0089304	0.0063385	0.0025919
80	640.0964355	0.0083793	0.0064047	0.0019747
81	654.0964355	0.0057361	0.0061344	-0.0003983

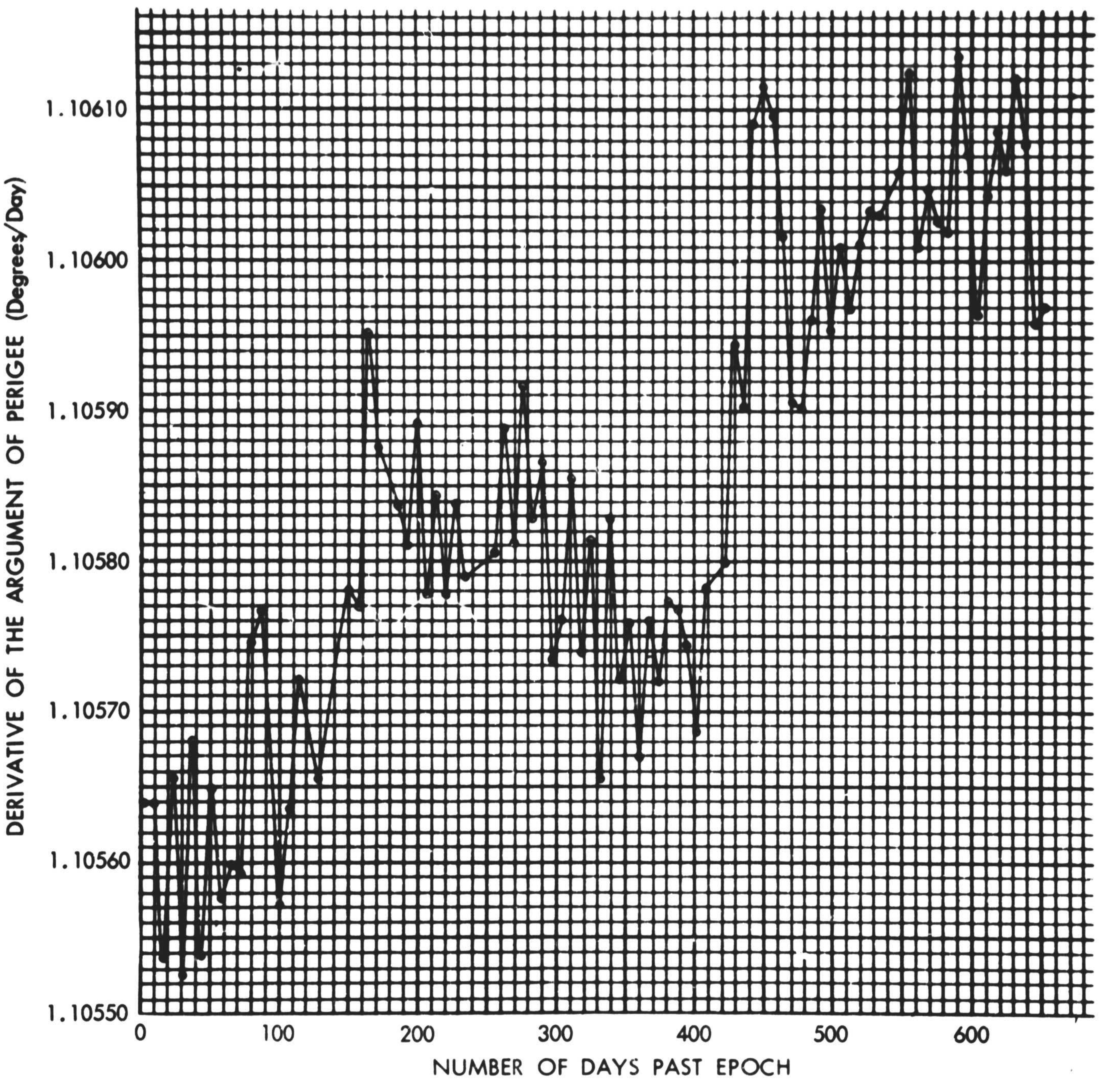
TABLE 7

RESIDUAL OF THE COMPUTED AND OBSERVED VALUE OF THE LONGITUDE OF THE ASCENDING NODE AND ITS PERIODIC COMPONENT

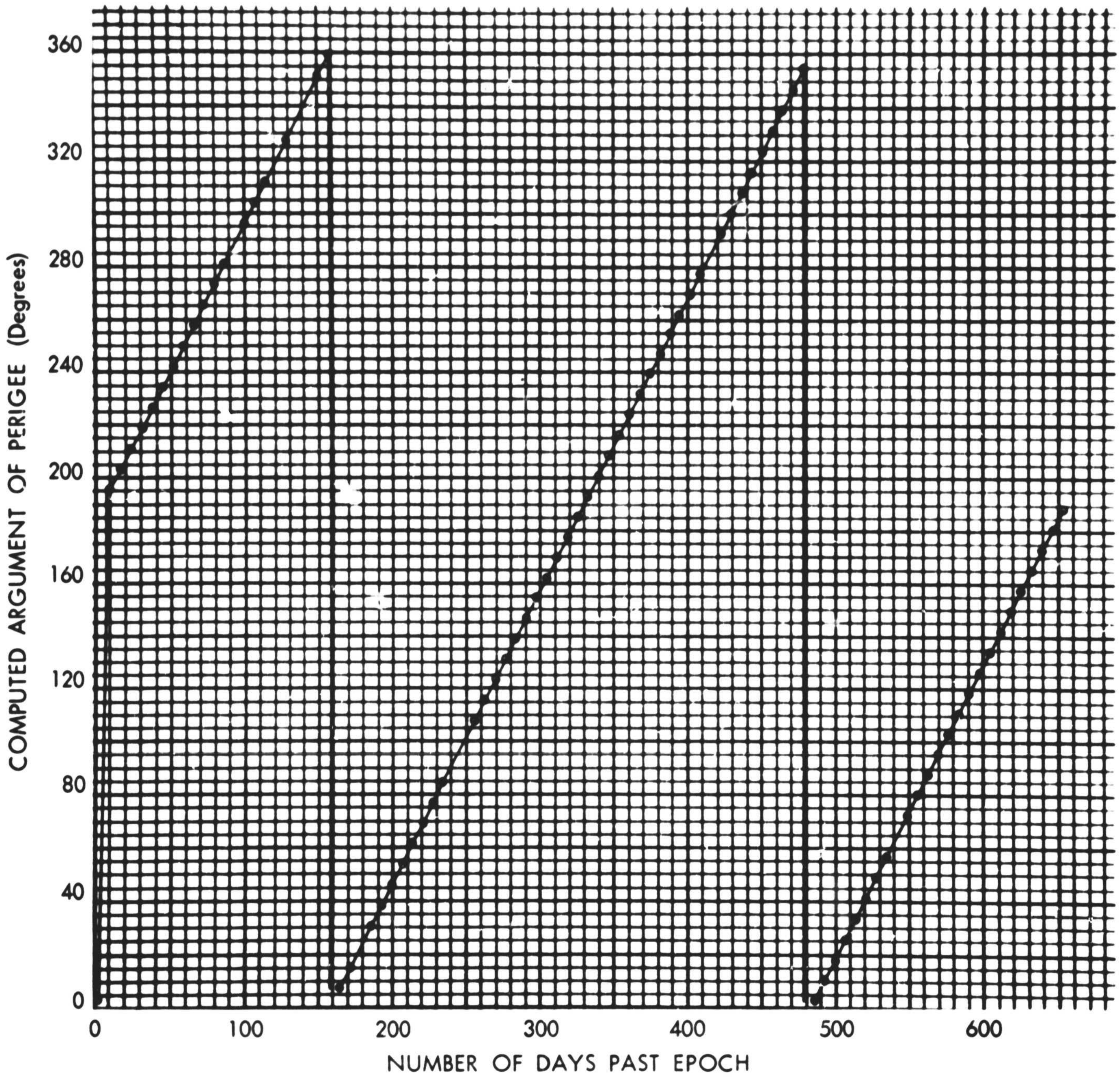
Number of Days Past Epoch	Residual of the Comp. and Obs. Value of the Long. of Asc. N. (Degrees)	Periodic Component of Original Res. (Degrees)
10.0000000	0.0014191	-0.0025982
17.0000000	0.0036163	-0.0021439
24.0000000	0.0057983	-0.0017049
31.0000000	0.0080414	-0.0012048
38.0000000	0.0099487	-0.0010404
45.0000000	0.0115509	-0.0011811
52.0000000	0.0121613	-0.0023137
59.0000000	0.0126495	-0.0035684
66.0000000	0.0176239	-0.0003370
73.0000000	0.0206299	0.0009260
80.0000000	0.0219879	0.0005411
87.0000000	0.0227051	-0.0004647
101.0000000	0.0267944	0.0001188
108.0000000	0.0291595	0.0007409
115.0000000	0.0298157	-0.0003459
129.0000000	0.0350037	0.0013562
151.0000000	0.0391235	-0.0000018
158.0000000	0.0411377	0.0002695
165.0000000	0.0460815	0.0034704
172.0000000	0.0450439	0.0006898
186.0000000	0.0493622	0.0015221
193.0000000	0.0529900	0.0034070
200.0000000	0.0523396	0.0010136
207.0000000	0.0534668	0.0003979
214.0000000	0.0554199	0.0006081
221.0000000	0.0573730	0.0008183
228.0000000	0.0600586	0.0017609
235.0000000	0.0605469	0.0005062
256.0000000	0.0649414	-0.0003281
263.0000000	0.0673828	0.0003704
270.0000000	0.0688477	0.0000923
277.0000000	0.0710449	0.0005466
284.0000000	0.0712891	-0.0009522
291.0000000	0.0744629	0.0004786
298.0000000	0.0754089	-0.0003183
305.0000000	0.0775146	0.0000445
312.0000000	0.0773163	-0.0018968
319.0000000	0.0812225	0.0002665
326.0000000	0.0832672	0.0005682
333.0000000	0.0858765	0.0014345
340.0000000	0.0868378	0.0006528

TABLE 7--Continued

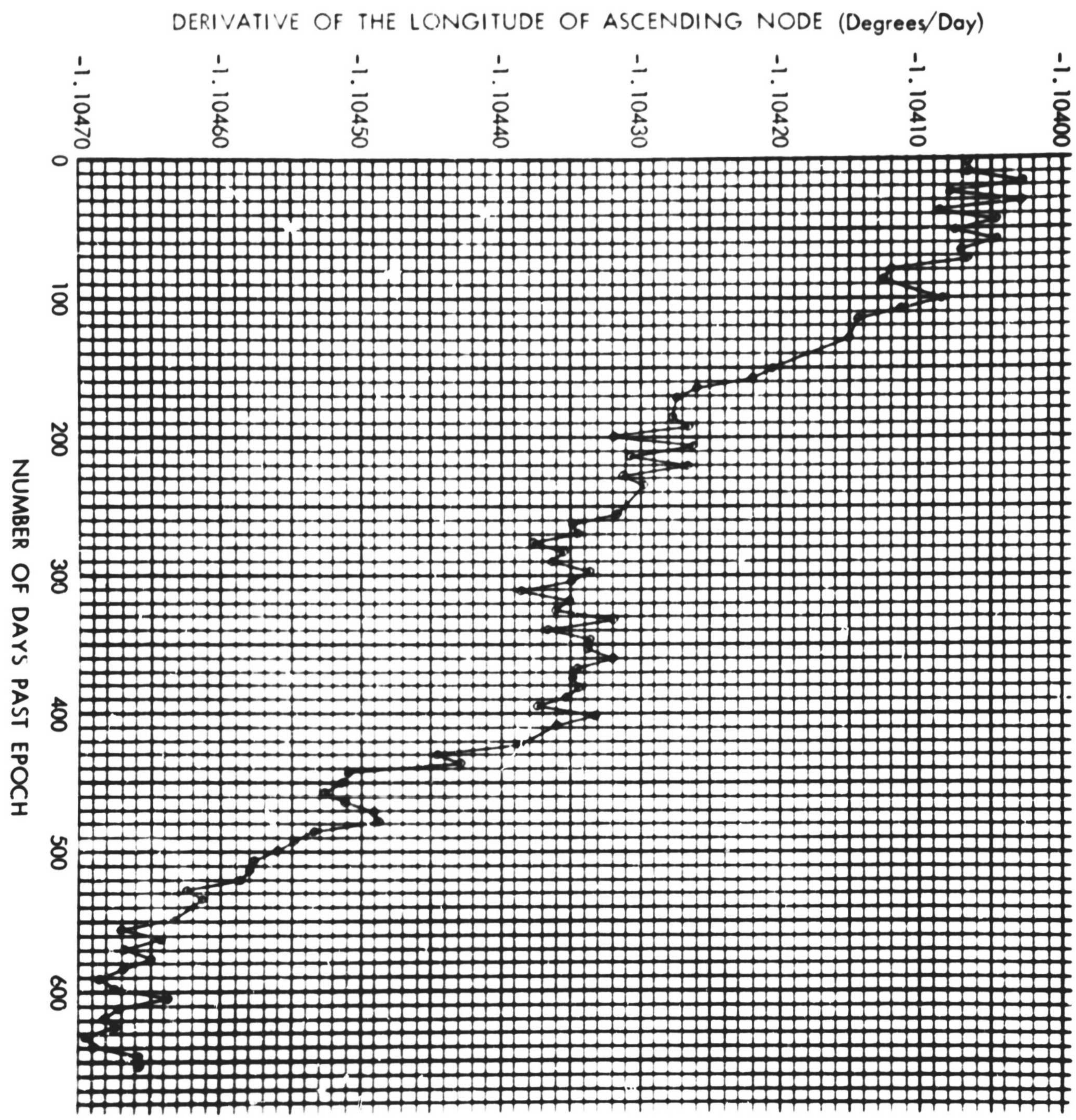
Number of Days Past Epoch	Residual of the Comp. and Obs. Value of the Long. of Asc. N. (Degrees)	Periodic Component of Original Res. (Degrees)
347.0000000	0.0895844	0.0016565
354.0000000	0.0912781	0.0016073
361.0000000	0.0922546	0.0008409
368.0000000	0.0948181	0.0016614
375.0000000	0.0974579	0.0025582
382.0000000	0.0995331	0.0028905
389.0000000	0.0981293	-0.0002563
395.0964355	0.1002502	0.0003467
402.0964355	0.1020660	0.0004196
409.0964355	0.1047211	0.0013316
423.0964355	0.1075439	0.0006686
430.0964355	0.1089478	0.0003295
437.0964355	0.1118927	0.0015315
444.0964355	0.1132050	0.0011008
451.0964355	0.1129608	-0.0008863
458.0964355	0.1166840	0.0010939
465.0964355	0.1156769	-0.0016561
472.0964355	0.1162262	-0.0028498
479.0964355	0.1181488	-0.0026701
486.0964355	0.1194153	-0.0031466
493.0964355	0.1219177	-0.0023871
500.0964355	0.1247559	-0.0012919
507.0964355	0.1282349	0.0004442
514.0964355	0.1292362	-0.0002974
521.0964355	0.1306801	-0.0005965
528.0964355	0.1327353	-0.0002843
535.0964355	0.1352539	0.0004914
549.0964355	0.1379395	-0.0003089
556.0964355	0.1411133	0.0011219
563.0964355	0.1406250	-0.0011093
570.0964355	0.1416016	-0.0018757
577.0964355	0.1437988	-0.0014214
584.0964355	0.1469727	0.0000095
591.0964355	0.1472168	-0.0014893
598.0964355	0.1481934	-0.0022557
605.0964355	0.1513672	-0.0008248
612.0964355	0.1516113	-0.0023236
619.0964355	0.1560059	0.0003279
626.0964355	0.1574249	0.0000041
633.0964355	0.1583862	-0.0007776
640.0964355	0.1604767	-0.0004300
647.0964355	0.1656189	0.0029692
654.0964355	0.1679230	0.0035303



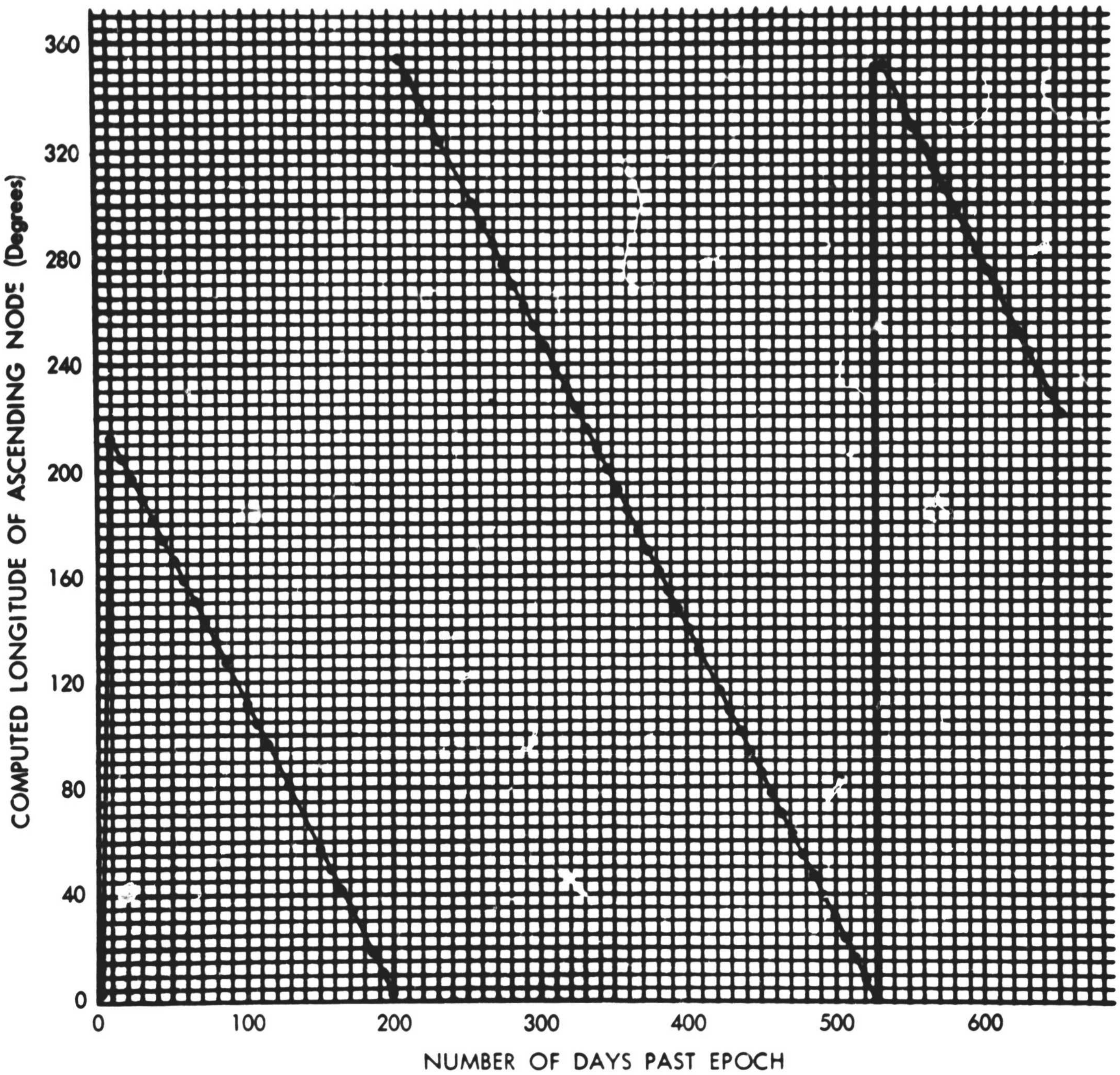
Graph 1.--Derivative of the argument of perigee as a function of the number of days past epoch.



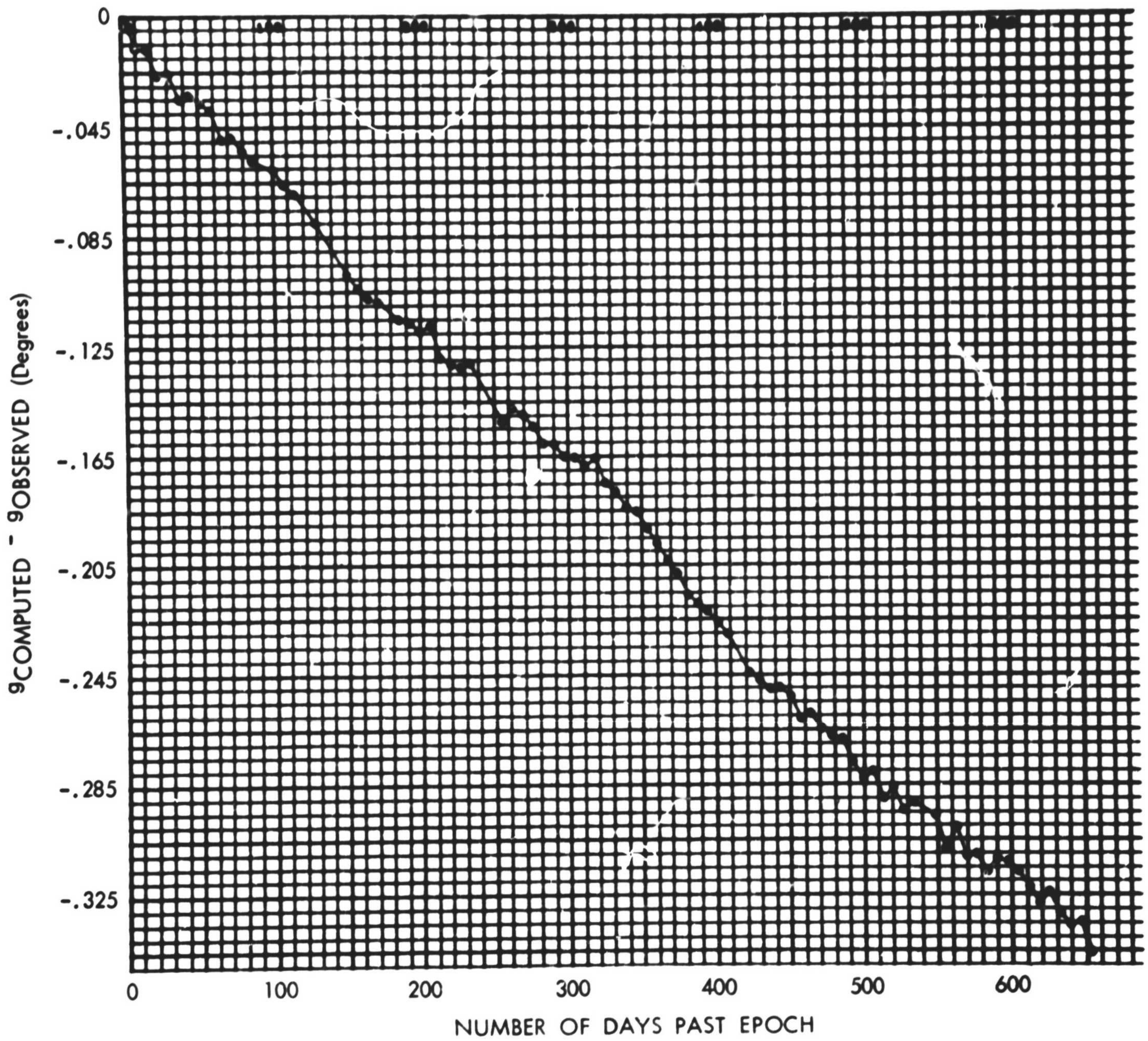
Graph 2.--Computed argument of perigee as a function of the number of days past epoch.



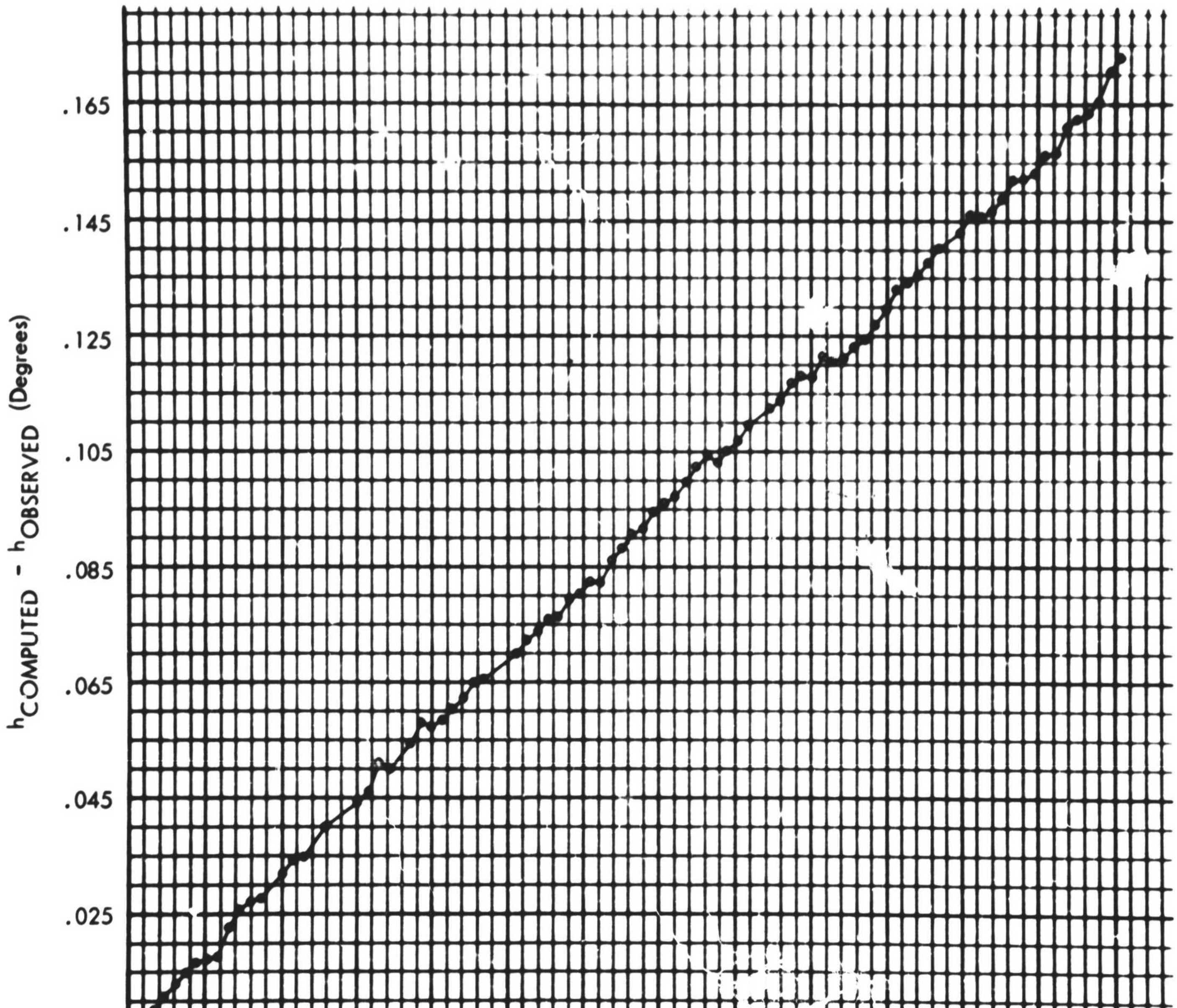
Graph 3.--Derivative of the longitude of ascending node as a function of the number of days past epoch.



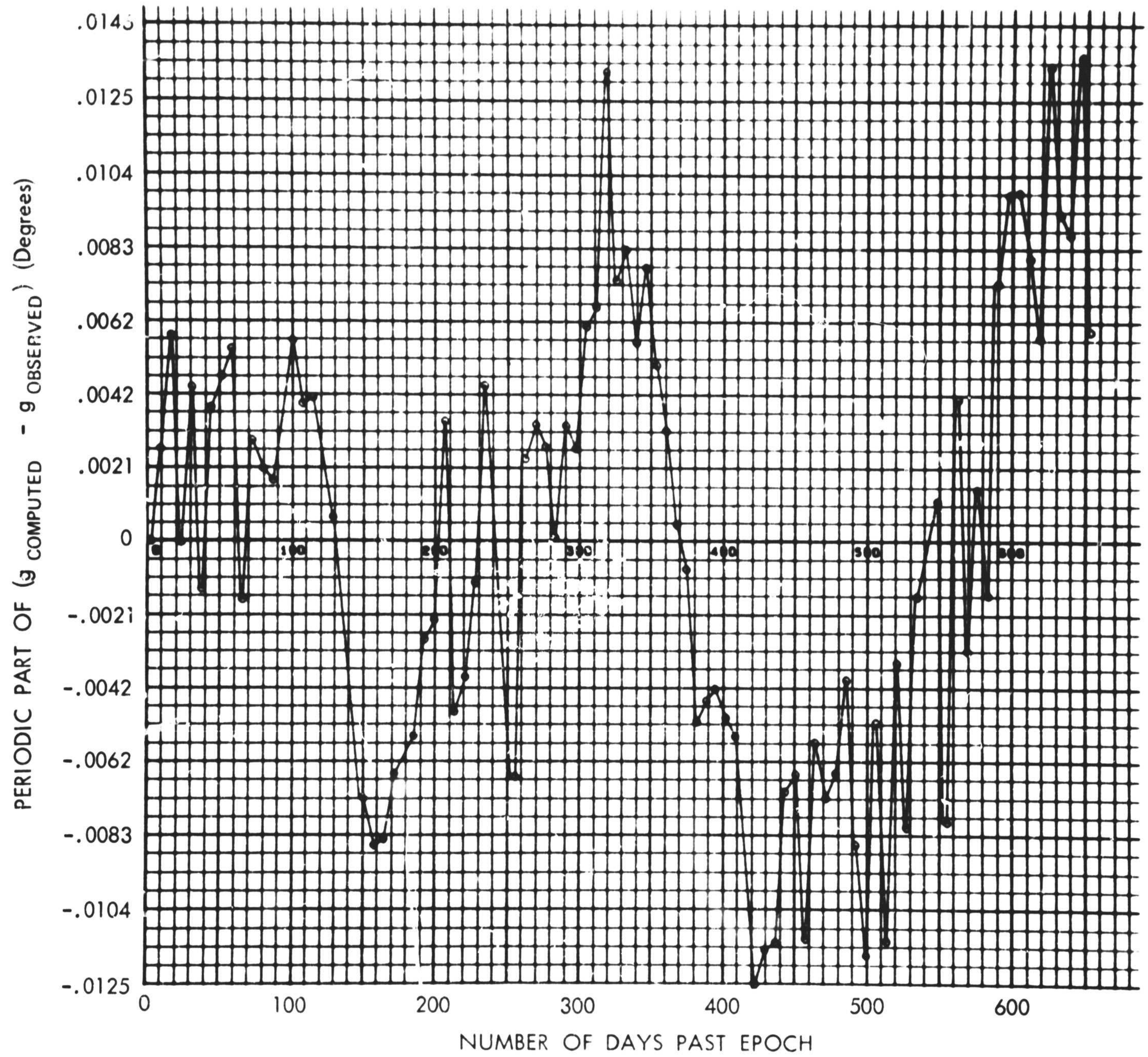
Graph 4.--Computed longitude of ascending node as a function of the number of days past epoch.



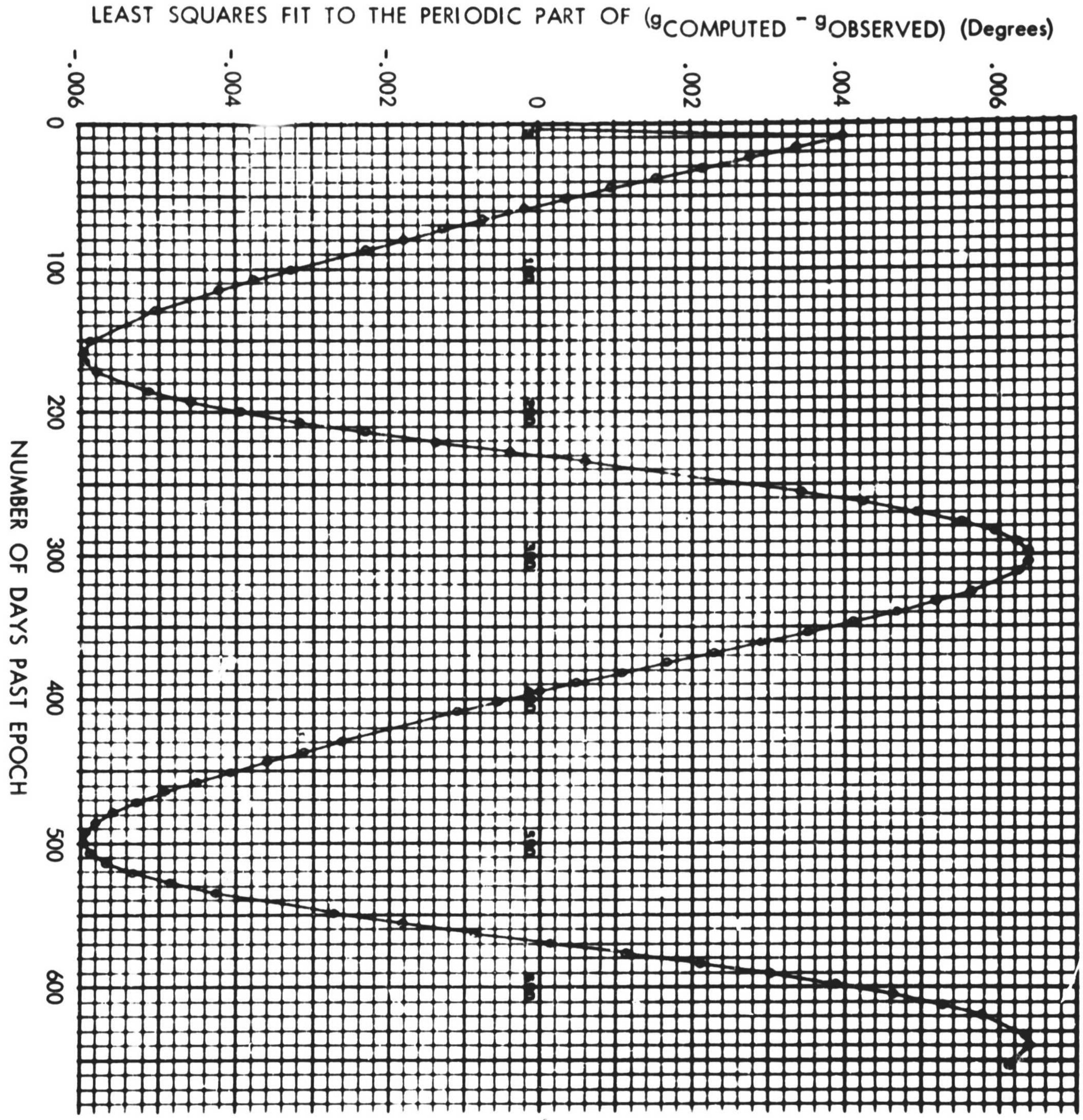
Graph 5.--Difference between the computed and observed values of the argument of perigee as a function of the number of days past epoch.



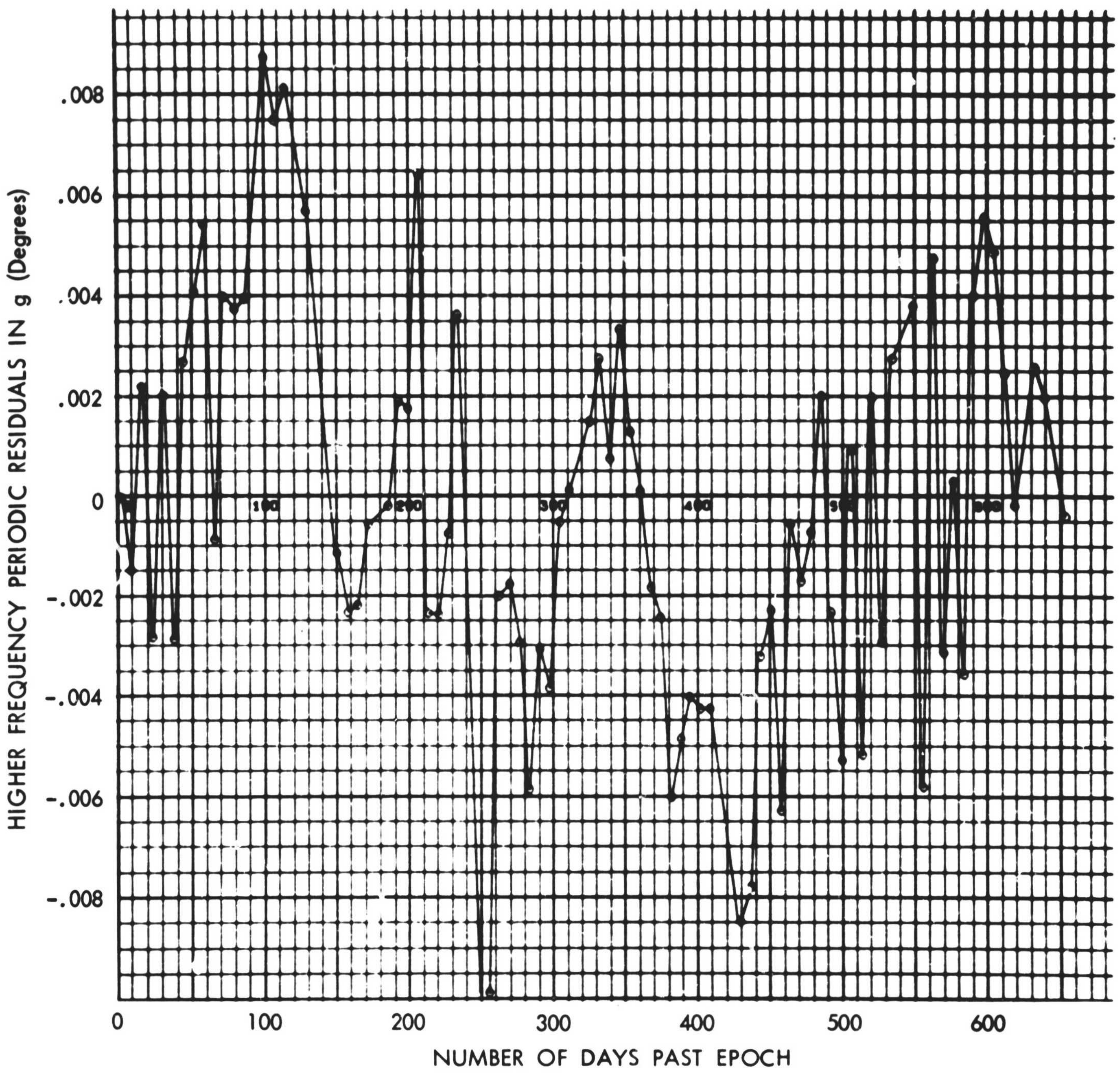
Graph 6.--Difference between the computed and observed values of the longitude of ascending node as a function of the number of days past epoch.



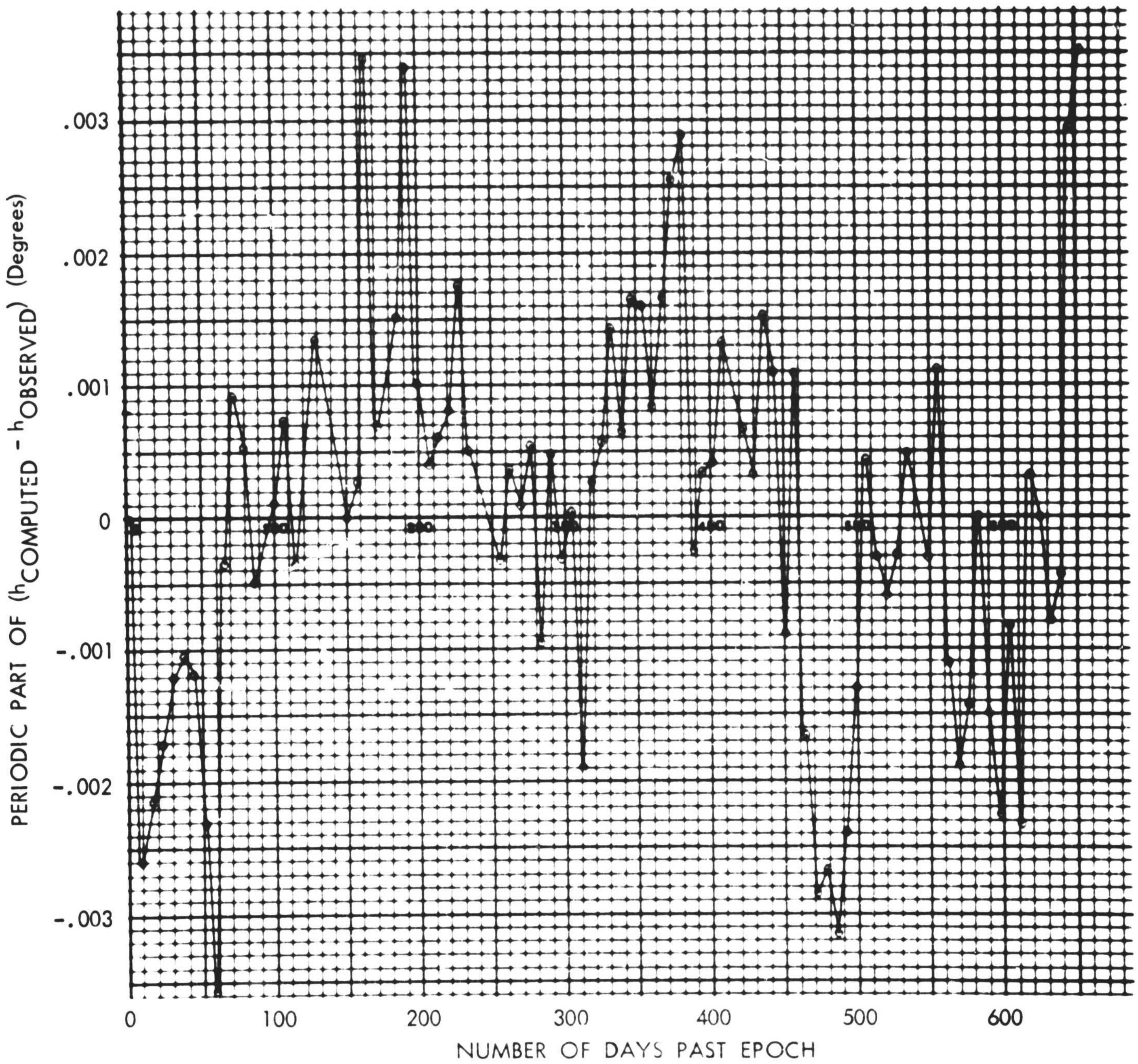
Graph 7.--Periodic part of the difference between the computed and observed values for the argument of perigee as a function of time.



Graph 8.--Least squares fit to the periodic part of the difference between the computed and observed values for the argument of perigee as a function of time.



Graph 9.--Periodic residuals with frequencies higher than $2\dot{g}$ in the curve depicting the difference between the computed and observed values for the argument of perigee as a function of time.



Graph 10.--Periodic part of the difference between the computed and observed values for the longitude of ascending node as a function of time.

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