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ON THE SHAPE PRESERVING EXPANSION AND CONTRACTION OF OBJECTS IN E^3

by P. Argentiero

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Greenbelt, Md.

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ABSTRACT

An S-map from a metric space X into a metric space Y is a map f such that there exists a K: $0 \leq K \leq 1$ and for all x and $y \in X$, $d[f(x), f(y)] = k[d(x, y)]$.

The set of S-maps, $S(X)$, on a compact subset of a metric space X under the compact-open topology and under composition is a topological semi-group. Several theorems are given relating the geometric structure of X to the fashion in which $S(X)$ is topologically and algebraically imbedded in $C(X)$.

An S-homotopy on X (a compact metric space) is a homotopy $H : X \otimes I \rightarrow X$ such that $H : X \otimes 0 \rightarrow X$ is the identity map on X and $H : X \otimes 1 \rightarrow X$ is a contractive S-map. Also for all $T \in I$, $H : X \otimes T \rightarrow X$ is an S-map. An S-homotopy on $X \in E^N$ is regular if there exists a coordinate set C and a $T_0 \in I$ such that if $T < T_0$, then $H : X \otimes T \rightarrow X$ has the canonical matrix representation relative to C of a rotation followed by a contraction. A subset of a metric space is a Regular Euler Set if it is compact, non-starlike, and supports a regular S-homotopy.

It is shown that an arc in E^N is a regular Euler set if, and only if, it is a generalized logarithmic spiral. This is a generalization of a classical result obtained by Euler. Also, a proof is given that a subset of E^3 is a regular Euler set if, and only if, it can be represented as the union of a family of logarithmic spirals.

CONTENTS

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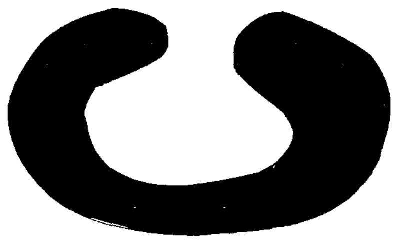
INTRODUCTION AND DISCUSSION

It is clear intuitively that certain subsets of E^2 can be continuously shrunk into themselves so that the "shape" of the set is altered at no time during the shrinking process. A disk, for instance, can be continuously shrunk into itself and yet remain a disk at all times. It is also easy to believe that some sets fail to have this property. Consider as an example the set below.



A continuous shrinking of such a set into itself would necessarily alter its metric ratios and destroy what is intuitively called its "shape." It is of interest to characterize sets which have this special shrinkability property. To provide such a characterization, intuition can be replaced by a mathematical description of a continuous shape preserving shrinking of a set into itself.

The idea of continuous shrinking of a set X into itself suggests a homotopy from the identity map X to some other map from X into X. The definition of the homotopy relation between two maps is the following:

Definition 1: Let f_1 and f_2 be continuous transformations from a topological space X into a topological space Y. Then f_1 and f_2 are HOMOTOPIC, if there exists a continuous transformation $H : X \otimes I \to Y$ (I is the closed unit interval) from $X \otimes I$ into Y such that $H : X \otimes 0 \to Y = f_1$ and $H : X \otimes 1 \to Y = f_2$. The map $H : X \otimes I \to Y$ is called a HOMOTOPY from f_1 to f_2. The homotopy relation is an equivalence relation (Reference 1).

A shape-preserving shrinking of a set X into itself will be defined as a homotopy $H : X \otimes I \to X$ such that $H : X \otimes 0 \to X$ is the identity map on X and for all $t \in I$; the map $H : X \otimes T \to X$ is shape preserving. The intuitive idea of the shape of a set X is intimately connected with the ratios of distances between points in X. Hence a rigorous treatment of shape must restrict itself to metric spaces. Also, any map between metric spaces which preserves properties of shape must preserve ratios of distances. In other words, a map f from a metric space X with metric d into a metric space Y with metric δ is called *shape preserving,* or an S-map, if given any three points, z_1, z_2, $z_3 \in X$, then

$$\frac{\delta\left[f(z_1), f(z_2)\right]}{\delta\left[f(z_2), f(z_3)\right]} = \frac{d\left[z_1, z_2\right]}{d\left[z_2, z_3\right]} .$$

It is not difficult to show that this property is identical with the one incorporated in the following definition:

Definition 2: A map f from a metric space X into a metric space Y is an S-map if there exists a K, $0 \leq K \leq 1$ such that for any $x \in X$ and $y \in X$, $d\left[f(x), f(y)\right] = Kd[x, y]$. The number K is called the *scale* of the S-map f. If $K < 1$, f is called a contractive S-map.

It will be shown that S-maps in E^n are actually similarity transformations, which are defined as maps that can be represented as a composition of a translation, an orthogonal transformation, and either a dilation or a contraction. Such maps have received considerable attention in E^n. Here, however, S-maps will be studied in the more general context provided in definition 2.

The concept of a shape preserving shrinking of a set into itself can now be formalized:

Definition 3: A subset X of a metric space is *S-shrinkable* if there exists a homotopy $H : X \otimes I \to X$ such that $H : X \otimes 0 \to X$ is the identity map on X, and $H : X \otimes 1 \to X$ is a contractive S-map. Furthermore, for every $t \in I$, the map $H : X \otimes T \to X$ is an S-map.

Definition 4: A subset X of E^n is *starlike* if there exists a point $p \in X$ such that if $x \in X$, the line segment $\langle p, x \rangle$ is contained in X.

The starlike property, which later is generalized to arbitrary metric spaces, is closely related to the S-shrinkable property as demonstrated in theorem 1.

Theorem 1: Let X be a starlike subset of E^n. Then X is S-shrinkable.

The proof of theorem 1 will be provided in the next section. It is of interest to note that the converse of theorem 1 is false; that is, an S-shrinkable subset of E^n is not necessarily a starlike set. A counterexample is provided by a subset of E^2; it is one referred to on other occasions. Let the points in the plane have the usual polar coordinate representation, (r, θ), and consider an arc X with parametric representation: $X = \{(r, \theta) \mid r = r_0 e^{-\ell}, \theta = \theta_0 + \alpha^{-1}\ell :\ 0 \leq \ell \leq \infty\}$. X contains the origin since we permit the parameter ℓ to assume the value of ∞. This well-known arc is called the logarithmic spiral. Intuitively it begins at a point (r_0, θ_0) and spirals down indefinitely into the origin. The positive number α is called the twisting coefficient of the logarithmic spiral and it governs the "tightness" of the spiraling. The logarithmic spiral is certainly nonstarlike and is now shown to be S-shrinkable. Define the required S-homotopy as follows: Let $H : X \otimes I \to X$ be such that for $t \in I$, $H : X \otimes t \to X$ maps the point $\left(r_0 e^{-\ell}, \theta_0 + \alpha^{-1}\ell\right)$ onto $\left[\left(r_0 e^{-(\ell+\alpha t)}, \theta_0 + \alpha^{-1}(\ell + \alpha t)\right)\right]$.

Clearly $H : X \otimes I \to X$ as defined is a homotopy. $H : X \otimes 0 \to X$ is the identity map; and for any t, $H : X \otimes t \to X$ is the composition of a rotation by angle t and a contraction by factor $e^{-\alpha t}$. Hence $H : X \otimes t \to X$ is an S-map and the logarithmic spiral X is S-shrinkable.

John Bernoulli in the middle of the eighteenth century was aware of these shrinkability properties of the logarithmic spiral. He noticed that if the logarithmic spiral is contracted by any factor, it becomes congruent to a subset of itself. This is just a rephrasing of the S-shrinkability property. He also observed that this shrinkability property provides the logarithmic spiral with its well-known hypnotic powers (Reference 2).*

It remained for Euler (Reference 3) to show that this special shrinkability property of the logarithmic spiral is in fact characteristic. That is, the logarithmic spiral is the only twice-differentiable curve in the plane that is both non-starlike and S-shrinkable. The differentiability assumption permitted Euler to use certain tools he had invented in order to establish his characterization of the logarithmic spiral. This information motivates the next definition.

Definition 5: A subset X of E^n is an *Euler Set* if it is compact, non-starlike, and S-shrinkable.

The logarithmic spiral is an example of an Euler set (it is compact because it contains the origin); and, as Euler showed, among twice-differentiable curves in the plane, it is the only Euler set. The aim here is to extend Euler's classical result by deleting differentiability assumptions and generalizing from the plane to E^n. The result will be a characterization of a general class of Euler arcs in E^n. Finally, a characterization is provided of Euler sets in E^3, and the significance of this characterization is briefly discussed with respect to the shape-preserving shrinking or expansion of objects in space. Before these objectives can be accomplished, a thorough study of the properties of S-maps is needed, the results of which will be useful in characterizing Euler arcs in E^n. These results also have some intrinsic interest as a further elucidation of the algebraic and topological properties of an important class of transformations.

PROPERTIES OF S-MAPS

There are some easily established but important facts about S-maps:

Theorem 2: If f is an S-map from M into M (a metric space) whose scale K is non-zero, f is topological. If M is complete and $K < 1$, f has a unique fixed point.

Proof: The first part of the theorem is an easy application of the definitions involved. The remainder of the theorem follows from the well-known result that a continuous contractive transformation has a unique fixed point on a complete metric space.

*Bernoulli called these properties of the spiral "reproductive properties," thinking that such properties had mystical significance. He wanted the logarithmic spiral inscribed on his tomb with the words,"Eadem mutata resurgo—though changed I rise unchanged."

It is next necessary to establish relationships between geometric properties of subsets of metric spaces (properties relating to the metric) and the fashion in which the set of S-maps on a set is embedded in a certain function space. In what follows, the symbol X will always represent a compact subset of a metric space. A common way to topologize the set of continuous functions from X into X is given in definition 6.

Definition 6: Let $C(X)$ be the set of continuous transformations from X into X. A metric on $C(X)$ is defined as follows: Let $f, g \in C(X)$. Then

$$d[f, g] \equiv \text{SUP}\left\{d\left[f(x), g(x)\right] \mid x \in X\right\} .$$

The topology induced by this metric will be called the *metric topology* on $C(X)$.

Another frequently used topology on $C(X)$ is given in the next definition.

Definition 7: Let K and U be subsets of X. Define the subset $W[K, U] \subset C(X)$ of $C(X)$ as follows: $W[K, U]$ is the set of all maps $f \in C(X)$ such that $f(K) \subset U$. The family of all sets $W[K, U]$ where K is compact and U is open forms a subbase for the *compact-open topology* on $C(X)$.

The metric topology on $C(X)$ is considerably simpler conceptually than the compact-open topology on $C(X)$. (It is routine to show that what is defined in definition 6 is a metric, provided X is compact.) However, the compact-open topology has several convenient properties that make its use almost unavoidable in some contexts. The property of the compact-open topology needed here is the one which permits us to identify homotopies on X with arcs in $C(X)$ and, conversely, to identify arcs in $C(X)$ in an obvious fashion with homotopies on X (Reference 1). Fortunately the metric topology on $C(X)$ is equivalent to the compact-open topology on $C(X)$ when X is compact (Reference 4). Hence the metric topology can be used on $C(X)$, while at the same time all the convenient properties of the compact-open topology are available.

Definition 8: Let $S(X)$ represent the set of all S-maps from X into itself. Clearly $S(X) \subset C(X)$ and hence $S(X)$ can inherit the topology of $C(X)$. $S(X)$ also has an algebraic structure, namely function composition. Certain relationships will be obtained between geometric properties of X and the topological and algebraic structure of $S(X)$.

Theorem 3: $S(X)$ with the algebraic and topological structures defined above is a topological semigroup.

Proof: Because $S(X)$ is closed under composition, the result will follow if it is shown that $C(X)$ under composition is a topological semigroup. U_x^ϵ will represent an epsilon neighborhood of x in X. If f is a map on X, U_f^ϵ will represent an epsilon neighborhood of f in $C(X)$. A symbol $f(x)$ is an element of X, namely the image of x under f.

To prove that $C(X)$ is a topological semigroup it must be shown that composition in $C(X)$ is continuous. This is equivalent to showing that for any neighborhood U_{fg}^{ϵ}, $f, g \in C(X)$ (fg of course represents composition of f with g) of fg there exist neighborhoods $U_f^{\delta f}$ and $U_g^{\delta g}$ of f and g respectively such that if $f' \in U_f^{\delta f}$ and $g' \in U_g^{\delta g}$ then $f' g' \in U_{fg}^{\epsilon}$.

Let $\epsilon > 0$ be prescribed and let $x \in X$. There exists a neighborhood $U_{g(x)}^{\delta}$ of radius δ which can be chosen independently of x, because X is compact, so that $f\left[U_{g(x)}^{\delta}\right] \subset U_{fg(x)}^{\epsilon/2}$. Now if $f' \in U_f^{\epsilon/2}$ and $g' \in U_g^{\delta}$, then $g'(x) \in U_{g(x)}^{\delta}$; and also, because $d[f', f] < \epsilon/2$, then $f'\left[U_{g(x)}^{\delta}\right] \subset U_{fg(x)}^{\epsilon}$. Thus $f' g'(x) \in U_{fg(x)}^{\epsilon}$ and $d[f' g'(x), fg(x)] < \epsilon$. Because $x \in X$ was arbitrary and δ was chosen independently of x, then $f' g' \in U_{fg}^{\epsilon}$. Hence composition is continuous in $C(X)$, and by definition this makes $C(X)$ a topological semigroup under composition.

Theorem 4: Let $\{f_n\}$ be a convergent sequence of S-maps on a compact subset X of a metric space. Then, if K_n represents the scale of the n^{th} element in the convergent sequence, $\{f_N\}$, $\lim_{n\to\infty} K_n = K$ exists.

Theorem 5: $S(X)$ is a closed subset of $C(X)$.

Proof of theorems 4 and 5: Let $\{f_n\}$ be a sequence of S-maps converging to $f \in C(X)$. It will be shown that $f \in S(X)$. For any two points $x \in X$ and $y \in X$,

$$d\left[f(x), f(y)\right] = \lim_{n\to\infty} d\left[f_n(x), f_n(y)\right] = \lim K_n \, d[x, y] = d[x, y] \lim_{n\to\infty} K_n .$$

Hence

$$\lim_{n\to\infty} K_n = K$$

exists and thus

$$d\left[f(x), f(y)\right] = Kd[x, y] .$$

Since the points x and y were arbitrary, this proves theorems 4 and 5.

Theorem 6: $S(X)$ is complete.

Proof: Because X is compact, $C(X)$ is complete. Hence $S(X)$ as a closed subset of a complete space is complete.

It is now convenient to extend the definition of the starlike property given in definition 4 to arbitrary metric spaces.

Definition 9: A subset Y of a metric space is *starlike* if there exists a $p \in Y$ so that, for every $x \in Y$, there is an S-map f from the closed interval $[0, a]$ (a dependent on x) into Y such that $f(0) = p$ and $f(a) = x$.

There will be no need to distinguish the sense of the word "starlike," as used in definition 9, from the sense of the same word as used previously in definition 4 since it can be readily shown that the two definitions coincide in E^n. Let Y be a set in E^n which is starlike in the sense of definition 4. Then there exists a $p \in Y$ such that if $x \in Y$ the line segment

$$\langle p, x\rangle = \left[p(1 - t/a) + x\,t/a, \qquad a = d[x, p] : 0 \leq t \leq a\right]$$

is included in Y. Now it is easy to construct an S-map f (in fact an isometry) from the closed interval $[0, a]$, $a = d[p, x]$ into Y such that $f(0) = p$ and $f(a) = x$. For $0 \leq t \leq a$, define

$$f(t) = p(1 - t/a) + x\,t/a\,,$$

where the addition is, of course, vector addition. Clearly $f(0) = p$ and $f(a) = x$. It must also be shown that f is an S-map. Let t_1, $t_2 \in [0, a]$, $t_1 < t_2$. Then

$$d\left[f(t_2), f(t_1)\right] = \left\|p - p\frac{t_2}{a} + \frac{xt_2}{a} - p + p\frac{t_1}{a} - \frac{xt_1}{a}\right\| = \left\|(t_2 - t_1)\frac{p - x}{a}\right\| = t_2 - t_1 = d\left[t_2, t_1\right] .$$

Thus f is an isometry and hence an S-map. Next it is shown that if Y, a subset of E^n, is starlike in the sense of definition 9, it is also starlike in the sense of definition 4. If Y is starlike in the sense of definition 9, there exists a $p \in Y$ such that for any $x \in Y$, there is an S-map f from an interval $[0, a]$ into Y and $f(0) = p$ and $f(a) = x$. It can be shown that Y is starlike in the sense of definition 4 by showing that

$$f\left[(0, a)\right] = \langle p, x\rangle .$$

Let $t \in (0, a)$. Then since $(0, a)$ is an interval and f is an S-map,

$$d\left[f(0), f(t)\right] + d\left[f(t), f(a)\right] = d\left[f(0), f(a)\right] .$$

This implies that $f(t) \in \langle 0, p\rangle$ and hence $f\left[(0, a)\right] \subset \langle p, x\rangle$. But since $f(0) = p$ and $f(a) = x$, and since the image of $(0, a)$ under f must be connected, $f\left[(0, a)\right] = \langle p, x\rangle$; the proof is complete.

Notice that with the extension of the starlike property to arbitrary metric spaces, the definition of Euler sets provided in definition 5 can be extended. An Euler set in an arbitrary metric space is compact, S-shrinkable, and nonstarlike in the sense of definition 9. Several other properties usually defined for linear spaces can just as easily be extended to arbitrary metric spaces. For instance, a subset Y of an arbitrary metric space can be called convex if for any two points x, $y \in Y$, there exists an S-map, f, from the closed interval $[0, a]$ into Y such that $f(0) = x$ and $f(a) = y$. This can be shown to be a valid generalization of the convexity property as usually defined in a linear space. Some interesting properties of convex sets in metric spaces can then be proved. For instance if the metric space is complete, a subset is convex if and only if it has the midpoint property (Reference 5). A generalization of this result can be found in (Reference 6).

Theorem 7: If X is a starlike subset of E^n, $S(X)$ is arc-wise connected.

Proof: Since X is starlike there exists a point $p \in X$ such that if $x \in X$, the line segment $\langle p, x\rangle$ is wholly contained in X. It is shown that if f is an S-map on X, then there is a closed arc (a homeomorph of the interval) in $S(X)$ with f as one endpoint and the constant map on p as the other endpoint. This will be sufficient to show that $S(X)$ is arc-wise connected. This arc is constructed as follows. For $t \in [0, 1]$ define

$$g_t(x) = (1-t)x + tp,$$

where the addition is vector addition. Next, g_t is shown to be an S-map. Let $x, y \in X$. Then

$$d[g_t(x), g_t(y)] = \|(1-t)x + tp - (1-t)y - tp\| = \|(1-t)(x-y)\| = (1-t)d[x, y] .$$

Since x and y were arbitrary, g_t is an S-map, and hence $g_t \in S(X)$. Define $f_t = g_t f$ (f_t is the composition of g_t with f). Since f is an S-map, so is f_t. Now the map $\alpha[(0, 1) \to s(X)]$ defined by $\alpha(t) = f_t$ is continuous since

$$d\left[f_{t_1}, f_{t_2}\right] = \mathrm{SUP}\left[\|(1-t_1)f(x) - t_1 p - (1-t_2)f(x) + t_2 p\| : x \in X\right]$$

$$= \mathrm{SUP}\left[\|(t_2 - t_1)f(x) + (t_2 - t_1)p\| : x \in X\right],$$

and clearly this number can be made as small as desired by making t_2 sufficiently close to t_1. Also notice that $\alpha(0) = f$ and $\alpha(1) = \{$constant map on $p\}$. Hence the proof is complete.

In this proof it is interesting to notice the role played by the properties of the point $p \in X$. In defining $g_t(x)$ as

$$g_t(x) \equiv (1-t)x + tp ,$$

success in defining a map on X was tacitly assumed. In other words, it was assumed that for any $t \in (0, 1)$ and any $x \in X$, the point $(1-t)x + tp$ was a member of X. This would not necessarily be true if $p \in X$ were not the special point defined in the beginning of the proof.

With theorem 7 established, a proof of theorem 1 can now be provided. The properties of the metric topology on $S(X)$ had been used in theorem 7. But since X is compact this topology is equivalent to the compact-open topology. As mentioned before, in the compact-open topology arcs in the function space may be identified with homotopies on the underlying space X. Hence there exists an arc from the identity map on X to a contractive S-map and the arc lies in $S(X)$. But this arc obviously defines an S-homotopy. Hence X is S-shrinkable if X is a starlike subset of E^n. The following theorem incorporates an interesting fact about contractive, S-maps on a complete Riemannian space. A proof is found in Reference 7.

Theorem 8: A complete Riemannian space which supports an into, contractive, S-map is Euclidean.

The assumption of completeness cannot be deleted from the statement of theorem 8. It is possible to construct into, contractive, S-maps on noncomplete, non-Euclidean Riemannian spaces (Reference 7).

The construction of machinery necessary to establish an interesting mapping theorem on $C(X)$ when X is a starlike subset of E^n can now be initiated.

Lemma 1: Let f be an S-map on X and choose $\epsilon > 0$. Let g be another S-map on X such that $g \in U_f^{\epsilon} \subset S(X)$. Then

$$\left|K_f - K_g\right| < \frac{2\epsilon}{M} ,$$

where M is the diameter of X.

Proof: Let g be an S-map such that $g \in U_f^{\epsilon}$, and let x and y be arbitrary elements of X. Then $d\left[f(x), g(x)\right] < \epsilon$, and $d\left[f(y), g(y)\right] < \epsilon$. Writing

$$d\left[g(x), g(y)\right] \leq d\left[g(x), f(x)\right] + d\left[f(x), f(y)\right] + d\left[f(y), g(y)\right] \leq 2\epsilon + d\left[f(x), f(y)\right] ,$$

leads to

$$\left|d\left[g(x), g(y)\right] - d\left[f(x), f(y)\right]\right| < 2\epsilon .$$

Since f and g are S-maps, then

$$\left|K_g\, d[x, y] - K_f\, d[x, y]\right| < 2\epsilon .$$

This implies

$$|K_g - K_f| < \frac{2\epsilon}{d[x, y]} .$$

Let M be the diameter of X. Since X is compact there exist points x' and y' such that

$$d[x', y'] = M .$$

Since x and y were arbitrary points of X, let $x = x'$ and $y = y'$. In that case

$$|K_g - K_f| < \frac{2\epsilon}{M} ,$$

and the proof is complete.

Lemma 2: Define $S_K \subset S(X)$ to be the set of S-maps of scale K in $S(X)$. Then sets of the form

$$\bigcup_{K_1<K<K_2} S_K$$

are open in $S(X)$.

Proof: Let

$$f \in \bigcup_{K_1<K<K_2} S_K .$$

It must be shown that there exists an $\epsilon > 0$ such that

$$f \in U_f^{\epsilon} \subset \bigcup_{K_1<K<K_2} S_K .$$

This will establish the result. Since

$$f \in \bigcup_{K_1<K<K_2} S_K ,$$

then $K_1 < K_f < K_2$ where K_f is the scale of f. Now choose an $\epsilon > 0$ such that

$$K_f + \frac{2\epsilon}{M} < K_2 ,$$

and

$$K_1 < K_f - \frac{2\epsilon}{M},$$

where again M is the diameter of X. Then, if $g \epsilon U_f^\epsilon$, from lemma 1

$$|K_g - K_f| < \frac{2\epsilon}{M}.$$

Hence

$$K_1 < K_f - \frac{2\epsilon}{M} < K_g < K_f + \frac{2\epsilon}{M} < K_2 .$$

Therefore

$$g \epsilon \bigcup_{K_1<K<K_2} S_K .$$

But g was an arbitrary element of U_f^ϵ. Thus

$$U_f^\epsilon \subset \bigcup_{K_1<K<K_2} S_K .$$

Lemma 3: If X is a starlike subset of E^n, then for any K such that $0 \leq K \leq 1$, there exists an $f \epsilon S(X)$ such that $K_f = K$ where K_f is the scale of f.

Proof: If X is starlike there exists a point $p \epsilon X$ such that for any $x \epsilon X$ the line segment

$$\langle p, x\rangle \equiv \left\{p(1-t)+x(t) \mid t \epsilon [0, 1]\right\},$$

where the addition is vector addition. Let $t_0 \epsilon [0, 1]$ and define on X the function:

$$\alpha_{t_0}(x) = p\left(1-t_0\right) + x\,t_0,\; x \epsilon X .$$

This is an into function on X because of the special properties of the point p. Also α_{t_0} is an S-map on X. To see this notice that

$$d\left[\alpha_{t_0}(x), \alpha_{t_0}(y)\right] = \|p\left(1-t_0\right) + x\,t_0 - p\left(1-t_0\right) - y\,t_0\| = \|t_0(x-y)\| = t_0\, d[x, y] .$$

Hence α_{t_0} is an S-map on X of scale t_0. Since t_0 is any point in the unit closed interval $[0, 1]$, the proof is complete.

Define S_K as the set of S-maps of scale K in S(X). Let $\{S_K\}_K$ be the collection of such subsets. By the natural transformation Υ from S(X) into $\{S_K\}_K$, is meant the transformation $\Upsilon(f) \equiv S_{K_f}$. $\{S_K\}_K$ is topologized by choosing the coarsest topology which makes the natural transformation continuous. This topology is the well known quotient topology induced on $\{S_K\}_K$ by the equivalence relation on S(X), $f \sim g$ if and only if f has the same scale as g. See Reference 8 for relevant properties of this topology. Also an algebraic structure is placed on $\{S_K\}_K$ by stating that

$$S_{K_1} \times S_{K_2} \equiv S_{K_1 K_2} .$$

Lemma 4: Let X be a starlike subset of E^n. Then with the above defined algebraic and topological structures, $\{S_K\}_K$ is a topological semigroup that is isomorphic to the closed unit interval under multiplication; that is, $\{S_K\}_K$ is a one parameter topological semigroup.

Proof: Let X be a starlike subset of E^n and define a function

$$f\left[\{S_K\}_K \rightarrow [0, 1]\right],$$

as $f(S_K) \equiv K$. The map is clearly a one to one homomorphism. The starlike quality of X and lemma 3 insures that f is onto. Lemma 2 insures that f is continuous since it shows that the inverse images of open sets under f are open. Since f is a one to one onto homomorphism, it is an algebraic isomorphism.

To prove that f is also topological notice that in view of lemma 2, sets of the form

$$\bigcup_{K_1 < K < K_2} S_K \subset \{S_K\}_K$$

are open in $\{S_K\}_K$. Furthermore, these sets form a basis for the topology on $\{S_K\}_K$. Hence f is a continuous transformation which maps basis elements of $\{S_K\}_K$ onto basis elements of $[0, 1]$ and thus is open. So f is topological, and the proof is complete.

Lemma 5: If X is a starlike subset of E^n, there exists a continuous transformation from S(X) onto $[0, 1]$.

Proof: The topology on $\{S_K\}_K$ is one which makes the natural transformation from S(X) onto $\{S_K\}_K$ continuous. In the proof of lemma 4, the existence of a topological transformation f was established from $\{S_K\}_K$ onto $[0, 1]$; thus the composition of the natural transformation with f is a continuous transformation from S(X) onto $[0, 1]$.

The following mapping theorem can now be stated.

Theorem 9: If X is a starlike subset of E^n, there exists a continuous transformation from $C(X)$ onto $[0, 1]$.

Proof: According to theorem 5, $S(X)$ is closed in $C(X)$. So apply Tietze's Extension Theorem (Reference 1) to the result of lemma 5.

Definition 10: Two maps f and g from X into X are *S-homotopic* if there exists a homotopy $H : X \otimes I \to X$ such that $H : X \otimes 0 \to X = f$ and $H : X \otimes 1 \to X = g$, and for all $T \in I$, $H : X \otimes T \to X$ is an S-map.

The next result is an application of theorem 3 and it has a geometric interpretation of some intuitive meaning.

Theorem 10: Let the identity map on X be S-homotopic to a contractive S-map on X. Then the identity map is also S-homotopic to a contractive S-map whose range is of arbitrarily small diameter.

Proof: According to theorem 3, $S(X)$ is a topological semigroup under composition. Also, since the topology used on $S(X)$ is the compact-open topology, the S-homotopy classes of S-maps are precisely the arc-wise connected components of $S(X)$ (Reference 1). Thus since composition is continuous, if f, $g \in S(X)$ are S-homotopic, so are f^2 and g^2. To see this notice that the map $\Upsilon[S(X) \to S(X)]$ defined as $\Upsilon(f) = f^2$ is continuous. Hence it preserves paths. Thus if the identity map I on X is S-homotopic to $f \in S(X)$, it is also S-homotopic to f^n where n is any positive integer.

Now let f be a contractive S-map on X. Let $p \in X$ be the fixed point of f (theorem 2). The point p is also the fixed point for f^n. For any $x \in X$,

$$d\left[f^n(x), f^n(p)\right] = d\left[f^n(x), p\right] = K_{f^n} d[x, p] = K_f^n d[x, p] .$$

So if M is the diameter of X, then $K_f^n M$ is a bound on the diameter of the range of f^N. But since f is contractive, $K_f < 1$ and it can be insured that the range of f^N has as small a diameter as desired by picking a sufficiently large n. Hence if the identity map on X is S-homotopic to a contractive S-map f, it is also S-homotopic to f^n whose range can be made arbitrarily small by choosing a sufficiently large n. This completes the proof.

Theorem 10 has the following intuitive meaning. It says that if a compact set in a metric space can be shrunk at all to a smaller subset of itself in a continuous fashion which preserves shape, then it can be continuously shrunk to as small a set as desired in a manner which preserves its shape. A necessary condition for S-shrinkability is now provided in arbitrary metric spaces.

Definition 11: X is *curled* if there exist two points $x, y \in X$ and a positive number $\epsilon > 0$, such that if $x' \in U_x^\epsilon$ and if $y' \in U_y^\epsilon$, then $d[x', y'] \geq d[x, y]$.

For example the set shown in the introduction is curled. So is the union of two closed disjoint line segments. In fact, every compact, disconnected subset of a metric space is curled.

Theorem 11: The property of being curled is preserved under S-maps.

Proof: The proof is a straightforward application of the definitions involved.

Theorem 12: Let X be S-shrinkable with S-homotopy $H : X \otimes I \to X$. There exists a $t_0 \in I$ such that $H : X \otimes t_0 \to X$ is an isometry, and if $t > t_0$, $H : X \otimes t \to X$ is a contractive S-map.

Proof: Let the map f_t represent $H : X \otimes t \to X$. Define the set $\beta \subset I$ as the set of all T such that f_t is an isometry. β is bounded and hence has a least upper bound t_0. Also $t_0 < 1$ since f_1 is by definition of an S-homotopy a contractive S-map. All that needs to be shown is that $t_0 \in \beta$. Assume that $t_0 \in \beta$. Then f_{t_0} is a contractive S-map. But since t_0 is the least upper bound of β and because of the continuity properties of the homotopy $H : X \otimes I \to X$, a sequence of isometries f_t can be constructed converging to f_{t_0}, a contractive S-map. ($H : X \otimes I \to X$ defines a continuous transformation from I into $C(X)$.) But a convergent sequence of isometries in $C(X)$ must converge to an isometry. This is a contradiction. Hence $t_0 \in \beta$, and the proof is complete.

In view of theorem 12, given an S-shrinkable set X (X compact), no generality is lost if it is assumed that there exists an S-homotopy, $H : X \otimes I \to X$, such that f_t for $0 < t$ is not an isometry. To show this, let $H : X \otimes I \to X$ be any S-homotopy on X. According to theorem 12, there exists a t_0 such that f_{t_0} is an isometry on X and if $t > t_0$, f_t is contractive. Consider the homotopy $H : X \otimes [t_0, 1] \to X$ obtained by restricting H to $X \otimes [t_0, 1]$. An isometry on a compact subset of a metric space is onto. Hence f_{t_0} has an inverse $f_{t_0}^{-1}$ which is an isometry on X. Define a new homotopy $G : X \otimes [t_0, 1] \to X$ as $G_t = f_t f_t^{-1}$, where $G_t = G : X \otimes t \to X$, $t \in [t_0, 1]$. Notice that G_{t_0} is the identity map on X and for $\in [t_0, 1]$, G_t is an S-map on X. A reparameterization of G such that G_{t_0} becomes G_0 will result in an S-homotopy on X such that the only isometry in the homotopy is the identity map G_0.

Theorem 13: If X is curled, it is not S-shrinkable.

Proof: Assume to, the contrary, that X is curled and S-shrinkable. Then there exist points $x \in X$ and $y \in X$ and a number $\epsilon > 0$ such that if $x' \in U_x^\epsilon$ and if $y' \in U_y^\epsilon$, then $d[x', y'] > d[x, y]$. Let $H : X \otimes I \to X$ be the S-homotopy on X. Again letting the symbol f_t represent $H : t \otimes t \to X$, $t \in I$, a $t_0 \in I$ can be named such that if $0 < t < t_0$, then

$$d[f_0, f_t] < \frac{\epsilon}{2},$$

where the metric in question is of course the one defined in definition 6. But f_0 is by definition the identity map i on X. Hence $f_t(y) \in U_y^\epsilon$ and $f_t(y) \in U_y^\epsilon$ for $0 < t < t_0$. But by theorem 12 we can assume that f_t is a contractive map of scale $K_t < 1$. Hence

$$d\left[f_t(x), f_t(y)\right] = K_t\, d[x, y] < d[x, y] .$$

But this is a contradiction since $f_t(x) \in U_x^{\epsilon}$ and $f_t(y) \in U_y^{\epsilon}$. Hence no set X can be both curled and S-shrinkable.

Corollary: Let V be an S-shrinkable, compact subset of a metric space. Then V is connected.

Proof: Assume to the contrary that V, an S-shrinkable, compact subset of a metric space, is not connected. Let V_1 and V_2 be sets of a separation of V. The sets V_1 and V_2 must also be compact. Define the distance between V_1 and V_2 as

$$d\left[V_1, V_2\right] = \mathrm{INF}\left[d\left[x_1, x_2\right];\ x_1 \in V_1,\ x_2 \in V_2\right] .$$

Then

$$d\left[V_1, V_2\right] = \delta \neq 0 ,$$

and furthermore there exists $x_1' \in V_1$ and $x_2' \in V_2$ such that

$$d\left[x_1', x_2'\right] = \delta .$$

Now let $U_{x_1'}^{\delta/2}$ be an open neighborhood of x_1' of radius $\delta/2$. Notice that

$$U_{x_1'}^{\delta/2} \cap V \subset V_1 .$$

Let $U_{x_2'}^{\delta/2}$ be an open neighborhood of x_2' of radius $\delta/2$. And again we have

$$U_{x_2'}^{\delta/2} \cap V \subset V_2 .$$

Now let

$$x_1'' \in U_{x_1'}^{\delta/2} \cap V ,$$

and

$$x_2'' \in U_{x_2'}^{\delta/2} \cap V \ .$$

Then $x_1'' \in V_1$ and $x_2'' \in V_2$ and

$$d\left[x_1'', x_2''\right] \geq \delta = d\left[x_1', x_2'\right] \ .$$

This indicates according to definition 11 that V is curled. But by theorem 13 this implies that V cannot be S-shrinkable which is a contradiction. Hence V is connected.

S-MAPS IN E^n

It will be shown in this section that by using the linear structure of E^n a convenient representation of S-maps in E^n can be obtained. With the aid of this representation a characterization is possible of a general class of Euler sets in E^2, and enough information can be obtained on the structure of Euler sets in E^n to obtain a significant generalization of Euler's characterization of Euler arcs in E^2 to a similar characterization in E^n.

Let f be a contractive S-map on E^n with scale $K < 1$. Define the coordinate set C as an orthogonal coordinate set whose origin is the fixed point of f. Define the maps f_1 and f_2 as follows:

$$f_1(\bar{x}) = \frac{1}{K} f(\bar{x}) \ ,$$

and

$$f_2(\bar{x}) = K\bar{x} \ ,$$

where $\bar{x}$ is a vector with components given relative to C and the multiplication indicated is of course scalar multiplication of a vector. Notice that for vectors $\bar{x}$ and $\bar{y}$ in E^n,

$$d\left[f_1(\bar{x}), f_1(\bar{y})\right] = \frac{1}{K} d\left[f(\bar{x}), f(\bar{y})\right] = \frac{1}{K}\left[Kd(\bar{x}, \bar{y})\right] = d[\bar{x}, \bar{y}] \ .$$

Hence f_1 is an isometry which has a fixed point at the origin of the coordinate set C. This implies that relative to the coordinate set C, f_1 is an orthogonal transformation. The map f_2 is clearly a simple contraction with contractive coefficient K. Since $f = f_2 \cdot f_1$, we have proved the next theorem.

Theorem 14: Let f be a contractive S-map on E^n. Then with the proper choice of coordinate set, f can be represented as a composition of an orthogonal transformation and a contraction with contractive coefficient K, the scale of f.

Theorem 14 permits the following useful matrix representation of an S-map on E^n.

Theorem 15: Let f be an S-map on E^n with scale $K < 1$. Then there exists M numbers $\phi_1, \phi_2, \cdots \phi_M$, $M \leq n/2$ such that with the proper choice of coordinate set, the matrix of f has the representation

$$\begin{pmatrix} \pm K & & & & & & \bigcirc \\ & \pm K & & & & & \\ & & \ddots & & & & \\ & & & \pm K & & & \\ & & & & \sigma_1 & & \\ & & & & & \sigma_2 & \\ \bigcirc & & & & & & \ddots \;\sigma_M \end{pmatrix}$$

where

$$\sigma_i = \begin{pmatrix} K\cos\phi_i & K\sin\phi_i \\ -K\sin\phi_i & K\cos\phi_i \end{pmatrix}$$

for $i \leq M$, with zeroes in all other places. The numbers ϕ_i, satisfying $0 < \phi_i < 2\pi$ are called the rotational coefficients of f and are nonzero.

Proof: It is well known (Reference 9) that an orthogonal transformation on E^n has associated with it a set of M nonzero numbers $\phi_1, \phi_2, \cdots \phi_M$ such that with proper choice of coordinate set it has the matrix representation

$$\begin{pmatrix} \pm 1 & & & & & & \bigcirc \\ & \pm 1 & & & & & \\ & & \ddots & & & & \\ & & & \pm 1 & & & \\ & & & & \sigma_1 & & \\ & & & & & \sigma_2 & \\ \bigcirc & & & & & & \ddots \;\sigma_M \end{pmatrix}$$

where

$$\sigma_i = \begin{pmatrix} \cos\phi_i & \sin\phi_i \\ -\sin\phi_i & \cos\phi_i \end{pmatrix}$$

for $i \leq M$, with zeroes elsewhere. This result together with theorem 14 proves the theorem.

In characterizing Euler arcs in E^n, Euler considered only S-maps which had their fixed points at the origin of a fixed coordinate set C. In effect this implied that there exists a single coordinate set for E^2 in which all S-maps had the convenient representation of theorem 15. Hence Euler actually characterized a certain subclass of twice differentiable Euler arcs in E^2. An equivalent assumption in E^n would be to assume for each S-homotopy $H : X \otimes I \to X$ on $X \subset E^n$, the existence of a coordinate set C such that for $t \in I$, $f_t = H : X \otimes t \to X$ has the representation of theorem 15 relative to C. This restriction on S-homotopies is somewhat stronger than necessary. However, it is necessary to restrict S-homotopies in a somewhat weaker fashion.

Definition 12: An S-homotopy $H : X \otimes I \to X$ on $X \subset E^n$ is *regular* if there exists a coordinate set C and a $t_0 \in I$, $0 < t_0$ such that if $t < t_0$ then the S-map $f_t = H : X \otimes t \to X$ has the representation of theorem 15 relative to C. An Euler set $X \subset E^n$ is a *regular* Euler set if there exists a regular S-homotopy on X.

Intuitively speaking a regular S-homotopy on a set X represents a shrinking of X such that at least in the beginning of the shrinking process a specific point in X remains fixed (the origin of the coordinate set C), and the rotations which occur are restricted to take place on M specifically assigned two-dimensional subspaces. This condition rules out some pathological situations which are very difficult to consider. This condition also insures that in a sufficiently small neighborhood of the identity map in S(X), the S-maps in the S-homotopy have the same number of rotational coefficients. Another way of viewing this condition is to observe that it permits an algebraic characterization of the S-maps in the beginning of an S-homotopy in terms of their matrix representation relative to a given coordinate set, rather than simply a geometric characterization as provided in definition 2. Notice also that in E^2 this condition reduces to an insistence that the S-maps in the beginning of the S-homotopy have the same fixed point. Hence in E^2, the regularity assumption is somewhat weaker than the assumption Euler applied. At this point the following convention is adopted. It is assumed that if X is a subset of E^n, it is not contained in a subspace of dimension lower than n. Clearly no loss of generality is incurred by this convention, and it is a convenience in the proof of the next theorem.

In the case of a regular S-homotopy, the S-maps in the homotopy in a sufficiently small neighborhood of the identity map can be given a representation somewhat simpler than that of theorem 15.

Theorem 16: Let X be a regular Euler set in E^n with regular S-homotopy $H : X \otimes I \to X$. Then there exists a $t_0 \in I$ and a coordinate set C such that if $t < t_0$, then the S-map $f_t = H : X \otimes t \to X$ has the representation relative to C as

$$\begin{pmatrix} K_t & & & & & & \\ & K_t & & & & \bigcirc & \\ & & \ddots & & & & \\ & & & K_t & & & \\ & & & & e_1 & & \\ & \bigcirc & & & & e_2 & \\ & & & & & & \ddots \\ & & & & & & & e_M \end{pmatrix}.$$

where

$$e_i = \begin{pmatrix} K_t \cos \phi_{i,t} & K_t \sin \phi_{i,t} \\ -K_t \sin \phi_{i,t} & K_t \cos \phi_{i,t} \end{pmatrix},$$

$i \leq M$, and $0 < \phi_{i,t} < 2\pi$ are the rotational coefficients of f_t.

Proof: Let X be a regular Euler set with regular S-homotopy $H : X \otimes I \to X$. By theorem 15 and the definition of a regular S-homotopy, there exists a coordinate set C and a $t_0 \in I$ such that if $t < t_0$ then f_t is represented relative to C as

$$\begin{pmatrix} \pm K_t & & & & & & \\ & \pm K_t & & & & & \\ & & \ddots & & & & \\ & & & \pm K_t & & & \\ & & & & e_1 & & \\ & & & & & e_2 & \\ & & & & & & \ddots \\ & & & & & & & e_M \end{pmatrix}$$

where

$$e_i = \begin{pmatrix} K_t \cos \phi_{i,t} & K_t \sin \phi_{i,t} \\ -K_t \sin \phi_{i,t} & K_t \cos \phi_{i,t} \end{pmatrix},$$

$i \leq M$, and $0 < \phi_{i,t} < 2\pi$, $i \leq M$ are the nonzero rotational coefficients of f_t. The difference between this representation and the one in the statement of the theorem is the possibility of negative signs appearing in front of K_t in one of the first L diagonal elements; $L = n - 2M$. Assume that for every $t' \in I$ there exists a $t < t'$ such that f_t has a $-K_t$ in one of its first L diagonal elements. We search for a contradiction. Assume that the i^{th} diagonal element $i \leq L$ in the representation of f_t is $-K_t$. Let $\bar{x}$ be in X and let x_i be the i^{th} coordinate of $\bar{x}$. Then the i^{th} coordinate of $f_t(\bar{x})$ is $-K_t x_i$. Hence the distance between $\bar{x}$ and $f_t(\bar{x})$ is at least as great as $(1 + K_t)|x_i|$. Let $b_i = \max\left[|x_i| : x_i \text{ is the } i^{th} \text{ coordinate of } \bar{x} \in X\right]$. The number b_i is the maximum i^{th} coordinate over all points in X. Let $b = \min\left[b_i, i \leq L\right]$. Since it can be assumed that X is not contained in a subspace of E^n of dimension less than n, then $b > 0$. Let $\{t_j\}$ be a sequence of points such that $\lim_{j\to\infty} t_j = 0$ and such that for every j the map, f_{t_j} has at least one $-K_t$ is the first L diagonal places in its representation relative to C. Then the distance between f_{t_j} and i, the identity map on X in the metric on $S(X)$, is at least as great as $\left(1 + K_{t_j}\right)b$. As j approaches infinity, this term approaches 2b. But this is impossible since as j approaches infinity f_{t_j} by construction and by the continuity properties of a homotopy must approach the identity map i in the metric of $S(X)$ and hence the distance between f_{t_j} and i must approach zero as j approaches infinity. Hence it is concluded that there exists a t_0 such that if $t < t_0$, then f_t has the representation stated in the theorem.

The usefulness of theorem 16 can be demonstrated by switching from rectangular coordinates on E^n to generalized cylindrical coordinates. Let X be a regular Euler set in E^n and let $H : X \otimes I \to X$

be its regular S-homotopy. Then there exists a $t_0 \in I$ such that for $t < t_0$ there exists a coordinate set C such that f_t has the representation of theorem 16 relative to C. Hence on an L dimensional subspace of E^n, f_t is a simple contraction of contractive coefficient K_t. On each of M two-dimensional subspaces, f_t is represented as a rotation and a contraction. On the first two-dimensional subspace in question, f_t is represented as a rotation through angle $\phi_{1,t}$ composed with a contraction of contractive coefficient K_t. On the L dimensional subspace on which f_t is a contraction, ordinary rectangular coordinates are used to specify the projection of a point p into this space. In each of the M planes on which f_t is a rotation composed with a contraction, ordinary polar coordinates are used to specify the projection of a point p on the plane in question. What is achieved is a set of cylindrical coordinates of a general point $p \in E^n$,

$$p = (z_1, z_2, \cdots z_L, r_1, \theta_1, r_2, \theta_2, \cdots r_m, \theta_m) ,$$

which has the following property for $t < t_0$:

$$f_t(p) = (z_1 K_t, z_2 K_t, \cdots z_L K_t, r_1 K_t, \theta_1 + \phi_{1,t}, r_2 K_t, \theta_2 + \phi_{2,t}, \cdots r_m K_t, \theta_m + \phi_{m,t}) .$$

Hereafter, when speaking of a cylindrical coordinate set relative to C, it is this cylindrical coordinate set that is intended.

REGULAR EULER SETS IN E^n

The structure of regular Euler sets in E^n can now be studied. In what follows, X is a regular Euler set in E^n with regular S-homotopy $H : X \otimes I \to X$. There exists a coordinate set C and a $t_0 \in I$ such that if $t < t_0$, then f_t has the representation of theorem 16 relative to C. The nonzero numbers $\phi_{1,t}$, $\phi_{2,t}$ $\cdots$ $\phi_{m,t}$ are the rotational coefficients of f_t. Notice that C can always be choosen such that $\phi_{i,t}$, $i \leq m$, approaches zero as t approaches zero. The scale K_t of f_t approaches one as t approaches zero. Define the function

$$g_i(t) \equiv \frac{\ln K_t}{\phi_{i,t}} ,$$

$i \leq m$. Derivations of characterizations of various projections of an Euler set into orthogonal subspaces will be performed. With the aid of these characterizations the regular Euler arcs in E^n will be characterized and thus generalize Euler's result into E^n. This will be accomplished by characterizing the projections of X into various orthogonal subspaces in terms of the behavior of the functions $g_i(t)$, $i \leq m$, as t approaches zero.

Let p_0 be a point in X. Relative to p_0 and for every $t < t_0$ the following sets are defined: $\psi_t = \{f_t^n(p_0) : 0 \leq n \leq \infty\}$, where f_t^n represents the composition of f_t with itself n times; and with cylindrical coordinates relative to C, for every

$$t < t_0,\ \gamma_t = \Big\{(z_1, z_2, \cdots z_L, r_1, \theta_1, r_2, \theta_2, \cdots r_m, \theta_m) \mid z_1 = z_{1,0}\, e^{-\ell},\ z_2 = z_{2,0}\, e^{-\ell}, \cdots z_L = z_{L,0}\, e^{-\ell},\ r_1 = r_{1,0}\, e^{-\ell},\ \theta_1 = \theta_{1,0} - \frac{\phi_{1,t}}{\ell n\, K_t}\ell,\ r_2 = r_{2,0}\, e^{-\ell},\ \theta_2 = \theta_{2,0} - \frac{\phi_{2,t}}{\ell n\, K_t}\ell, \cdots r_m = r_{m,0}\, e^{-\ell},\ \theta_m = \theta_{m,0} - \frac{\phi_{m,t}}{\ell n\, K_t}\ell : 0 \le \ell \le \infty\Big\},$$

where

$$(z_{1,0}, z_{2,0}, \cdots z_{L,0}, r_{1,0}, \theta_{1,0}, r_{2,0}, \theta_{2,0}, \cdots r_{m,0}, \theta_{m,0})$$

are the cylindrical coordinates of p_0 relative to C. Notice that $\psi_t \subset X$ for $t \in I$.

Theorem 17: For every $t < t_0$, $\psi_t \subset \gamma_t$.

Proof: Let $p \in \psi_t$. Then there exists an n such that $p = f_t^n(p_0)$. The cylindrical coordinates of p relative to C are

$$p = (z_{1,0} K_t^n, z_{2,0} K_t^n, \cdots z_{L,0} K_t^n, r_{1,0} K_t^n, \theta_{1,0} + n\phi_{1,t}, r_{2,0} K_t^n, \theta_{2,0} + n\phi_{2,t}, \cdots r_{m,0} K_t^n, \theta_{m,0} + n\phi_{m,t}).$$

Now let

$$L' = -n\, \ell n\, K_t$$

(L' is positive since $K_t < 1$). Then p can be represented as

$$p = \Big(z_{1,0}\, e^{-L'}, z_{2,0}\, e^{-L'}, \cdots z_{L,0}\, e^{-L'}, r_{1,0}\, e^{-L'}, \theta_{1,0} - \frac{\phi_{1,t}}{\ell n\, K_t} L', r_{2,0}\, e^{-L'}, \theta_{2,0} - \frac{\phi_{2,t}}{\ell n\, K_t} L', \cdots r_{m,0}\, e^{-L'}, \theta_{m,0} - \frac{\phi_{m,t}}{\ell n\, K_t} L'\Big) .$$

This proves that $p \in \gamma_t$. But p was an arbitrary point of ψ_t. Hence $\psi_t \subset \gamma_t$.

Theorem 18: For any $\epsilon > 0$, there exists a $t' < t_0$ such that if $t < t'$ then for any point $p \in \gamma_t$ there exists a point $p' \in \psi_t$ such that $d[p, p'] < \epsilon$.

Proof: Define a map τ from $[0, \infty]$ to γ_t as

$$\tau(\ell) = \Big(z_{1,0}\, e^{-\ell},\ z_{2,0}\, e^{-\ell},\ \cdots\ z_{L,0}\, e^{-\ell},\ r_{1,0}\, e^{-\ell},\ \theta_{1,0} - \frac{\phi_{1,t}}{\ell n\, K_t}\, \ell,\ r_{2,0}\, e^{-\ell},\ \theta_{2,0} - \frac{\phi_{2,t}}{\ell n\, K_t}\, \ell,\ \cdots\ r_{m,0}\, e^{-\ell},\ \theta_{m,0} - \frac{\phi_{m,t}}{\ell n\, K_t}\, \ell\Big)\ .$$

This is in fact the parameterization map used to define the arc γ_t. Let p be any point on γ_t, $t < t_0$. Then there exists an L' such that $\tau(L') = p$. There exists an n such that

$$-n\, \ell n\, K_t \le L' \le -(n+1)\, \ell n\, K_t\ .$$

But

$$\tau\left(-n\, \ell n\, K_t\right) = p_1 \in \psi_t,\ \tau\left(-(n+1)\, \ell n\, K_t\right) = p_2 \in \psi_t\ .$$

In fact $p_1 = f_t^n\left(p_0\right)$ and $p_2 = f_t^{n+1}\left(p_0\right)$. It must be shown that, assuming t is sufficiently small, $d\left[p_1, p\right] \le d\left[p_1, p_2\right]$.

The distance between the projections of p_1 and p into the L dimensional subspace in which is a contraction, is seen to be

$$\sqrt{\left[z_{1,0}\left(e^{n\, \ell n\, K_t} - e^{-L'}\right)\right]^2 \cdots + \left[z_{L,0}\left(e^{n\, \ell n\, K_t} - e^{-L'}\right)\right]^2}$$

$$= \left|e^{n\, \ell n\, K_t} - e^{-L'}\right| \sqrt{\left(z_{1,0}\right)^2 + \left(z_{2,0}\right)^2 + \cdots \left(z_{L,0}\right)^2}\ .$$

By similar reasoning it is seen that the distance between the projections of p_1 and p_2 on this L dimensional space is

$$\left|e^{n\, \ell n\, K_t} - e^{(n+1)\ell n\, K_t}\right| \sqrt{\left(z_{1,0}\right)^2 + \left(z_{2,0}\right)^2 + \cdots \left(z_{L,0}\right)^2}\ .$$

Hence in this L dimensional subspace the distance between the projections of p_1 and p_2 is greater than the distance between the projections of p_1 and p in this space.

A similar result is proved for the m two-dimensional subspaces on which f_t is a rotation composed with a contraction. Consider the distance between the projections of p_1 and p on the i^{th}

such plane, $i \leq m$. The ordinary cosine law gives this distance as

$$\left[\left(r_{i,t}\, e^{-\ell}\right)^2 + \left(r_{i,t}\, e^{n \ln K_t}\right)^2 - r_{i,t}^2\, e^{-L' + n \ln K_t} \cos\left(\frac{\phi_{i,t}}{\ln K_t}\,(L' - n \ln K_t)\right)\right]^{1/2} ,$$

and the distance between the projections of p_1 and p_2 on this two-dimensional subspace is

$$\left[\left(r_{i,t}\, e^{(n+1)\ln K_t}\right)^2 + \left(r_{i,t}\, e^{n \ln K_t}\right)^2 - r_{i,t}^2\, e^{(2n+1)\ln K_t} \cos \phi_{i,t}\right]^{1/2} .$$

Since

$$-n \ln K_t \leq L' \leq -(n+1) \ln K_t ,$$

it is seen that the expression for the distance between the projections of p_1 and p_2 on this subspace is greater than the corresponding expression for the distance between the projections of p_1 and p provided $\phi_{i,t}$ is such that $0 \leq \phi_{i,\ t} \leq \pi/2$. This follows because

$$\phi_{i,t} \geq \phi_{i,t}\, (L' - n \ln K_t)/\ln K_t ,$$

and because the cosine function is monotone decreasing in the interval $[0, \pi/2]$. Let $t_1 < t_0$ be such that if $t < t_1$, then $0 \leq \phi_{i,t} \leq \pi/2$. If $t < t_1$ then for any $p \in \gamma_t$ there exists an L' such that $p = \tau(L')$ and there exists an n such that

$$-n \ln K_t \leq L' \leq -(n+1) \ln K_t .$$

Defining

$$p_1 = \tau - (n \ln K_t),$$

and

$$p_2 = \tau(-(n+1) \ln K_t) ,$$

it has been proven that the distance between the projections of p_1 and p_2 on the L dimensional subspace on which f_t is a contraction is greater than the distance between the projections of p_1 and p on the same subspace. Moreover, the same thing is true on each of the m two-dimensional subspaces on which f_t is a rotation followed by a contraction. Since the orthogonal sum of these subspaces is E^n this proves that

$$d[p_1, p_2] \geq d[p_1, p] \ .$$

But

$$d[p_1, p_2] = d\left[f_t^n(p_0), f_t^{n+1}(p_0)\right] = d\left\{f_t^n(p_0), f\left[f_t^n(p_0)\right]\right\}.$$

Given $\epsilon > 0$, choose t_2 such that if $t_2 < t_1$, and if $t < t_2$, then f_t is within an ϵ-neighborhood of the identity map I in $S(X)$. Then

$$d\left\{f_t^n(p_0), f_t\left[f_t^n(p_0)\right]\right\} < \epsilon \ .$$

This implies that $d[p_1, p] < \epsilon$. But $p_1 \epsilon \psi_t \subset X$. This completes the proof.

Now consider a sequence $\{t_j\}$ such that $\lim_{j\to\infty} t_j = 0$. We focus attention on the m sequences $\{g_i(t_j)\}$ where $g_i(t) = \ell n\, K_t/\phi_{i,t}$. Let $i = 1$. Then we distinguish three possibilities:

1. The sequence $\{g_1(t_j)\}$ is an unbounded set of points.
2. The sequence $\{g_1(t_j)\}$ is a bounded set of points with at least one nonzero limit point.
3. The sequence $\{g_1(t_j)\}$ is bounded and has just one limit point at zero.

If the first case holds, a subsequence $\{t_j'\}$ can be defined such that

$$\lim_{j\to\infty} g_1(t_j') = -\infty \ .$$

If the second case holds $\{t_j'\}$ can be defined as a subsequence of $\{t_j\}$ which has the property that

$$\lim_{j\to\infty} g_1(t_j') = \alpha_1 \neq 0 \ .$$

If the third case holds $\{t_j'\}$ be simply defined as $\{t_j\} = \{t_j'\}$, and thus

$$\lim_{j\to\infty} g_1(t_j') = 0 \ .$$

In the same fashion, the sequence $\left\{g_2\left(t_j'\right)\right\}$ can be considered and a subsequence, $\left\{t_j''\right\}$ of $\left\{t_j'\right\}$, can be derived which would have the property that either

$$\lim_{j\to\infty} g_2\left(t_j''\right) = -\infty ,$$

$$\lim_{j\to\infty} g_2\left(t_j''\right) = \alpha_2 \neq 0 ,$$

or

$$\lim_{j\to\infty} g_2\left(t_j''\right) = 0 .$$

Proceeding in this fashion until the m functions $g_i(t)$, $i \leq m$, have been exhausted, a sequence is obtained which again can be called $\left\{t_j\right\}$ and about which can be stated the following theorem.

Theorem 19: There exists a sequence $\left\{t_j\right\}$, $\lim_{j\to\infty} t_j = 0$ and integers m', m'', $0 \leq m' \leq m'' \leq m$ such that with the proper reindexing of functions $g_i(t)$, $i \leq m$:

$$\lim_{j\to\infty} g_i\left(t_j\right) = \alpha_i \neq 0$$

for $i \leq m'$;

$$\lim_{j\to\infty} g_i\left(t_j\right) = -\infty$$

for i such that $m' < i \leq m''$; and

$$\lim_{j\to\infty} g_i\left(t_j\right) = 0$$

for i such that $m'' < i \leq m$.

Now let the integer m'' be defined as in theorem 19. The following theorem can be proved.

Theorem 20: There exists a set of $m - m''$ orthogonal two-dimensional subspaces on each of which the projection of X is a disk.

Proof: Let the m functions $g_i(t)$ be reindexed as in theorem 19. Let i be such that $m'' < i \leq m$. Consider the two-dimensional subspace E_i^2 on which the rotational coefficient $\phi_{i,t}$ is defined. For

$t < t_0$, the projection of the arc γ_t on this space is the arc parameterized as

$$\left\{(r, \theta) \mid r = r_{i,0}\, e^{-\ell},\ \theta = \theta_{i,0} - \frac{\phi_{i,t}}{\ln K_t}\ell : 0 \leq \ell \leq \infty\right\}.$$

We can reparameterize this arc and express it in terms of the function

$$g_i(t) = \frac{\ln K_t}{\phi_{i,t}},$$

as

$$\left\{(r, \theta) \mid r = r_{i,0}\, e^{g_i(t)S},\ \theta = \theta_{i,0} + S : 0 \leq S \leq \infty\right\}.$$

The reparameterization is accomplished by setting

$$S = -g_i^{-1}(t)\ell .$$

On the same two-dimensional space E_i^2, define the set

$$\left\{(r, \theta) \mid r = r_{i,0},\ \theta = \theta_{i,0} + S : 0 \leq S \leq \infty\right\}.$$

What has been defined is a circle centered at the origin and passing through the projection of p_0, the arbitrary point in X, onto this space. For any S' and any ϵ, there exists a j' such that if $t = t_j$, $j > j'$, then the points

$$\left(r_{i,0},\ \theta_{i,0} + S'\right),$$

and

$$\left(r_{i,0}\, e^{g_i(t)S'},\ \theta_{i,0} + S'\right),$$

are within a distance $\epsilon/2$ of each other. This follows because

$$\lim_{j\to\infty} g_i\left(t_j\right) = 0,$$

$m'' < i$. Let p_1 be a point on γ_t whose projection on the two-dimensional space E_i^2 in question is

$$\left(r_{i,0}\, e^{g_i(t)S'},\ \theta_{i,0} + S'\right),$$

By theorem 18, there exists a $j'' > j'$ such that if $t = t_j$, $j > j''$, then there exists a $p_2 \in \psi_t$ such that $d[p_1, p_2] < \epsilon/2$. But $\psi_t \subset X$; hence $p_2 \in X$. Also the distance between the projections of p_1 and p_2 must be less than $\epsilon/2$ provided $t = t_j$, $j > j''$. But the point

$$\left(r_{i,0},\ \theta_{i,0} + S'\right)$$

is within a distance of $\epsilon/2$ of the projection of p_1. So it can be concluded that the point

$$\left(r_{i,0},\ \theta_{i,0} + S'\right)$$

on the assigned two-dimensional subspace is within a distance ϵ of the projection onto this space of $p_2 \in \psi_t \subset X$ provided $t = t_j$, $j > j''$. The set X is compact. This implies that all its projections are compact and hence closed. So the point

$$\left(r_{i,0},\ \theta_{i,0} + S'\right)$$

is in the projection of X into the subspace in question. Since S' was arbitrary it has been demonstrated that on the two-dimensional subspace on which $\phi_{i,t}$ is defined, the projection of X contains a circle centered at the origin of C and passing through the projection of p_0 onto this space. By the corollary to theorem 13, X is connected, which implies that its projection sets are also connected. These facts, together with the facts that the origin of C is in X and hence is in all projection sets of X, and the p_0 was an arbitrary point of X, imply that the projection of X onto the two-dimensional space on which ϕ_i is defined, $m'' < i < m$, is a disk.

Next, the projection of X into the complement is characterized with respect to E^n of the orthogonal sum of the $m - m''$ two-dimensional subspaces on which the projections of X are disks. Let n' be the dimension of this space, where $n' = n - 2(m - m'')$. Call the space $E^{n'}$. For $t < t_0$ the projection of γ_t into this space can be characterized as:

$$\gamma_t' = \Big\{\left(z_1, z_2, \cdots z_L, r_1, \theta_1, r_2, \theta_2, \cdots r_{m'}, \theta_{m'}, \cdots r_{m''}, \theta_{m''}\right)$$

$$\mid z_1 = z_{1,0}\, e^{-\ell},\ z_2 = z_{2,0}\, e^{-\ell},\ \cdots z_L = z_{L,0}\, e^{-\ell},\ r_1 = r_{1,0}\, e^{-\ell}, \theta_1$$

$$= \theta_{1,0} - g_1^{-1}(t)\ell,\ r_2 = r_{2,0}\, e^{-\ell},\ \theta_2 = \theta_{2,0} - g_2^{-1}(t)\ell,\ \cdots r_{m'},$$

$$= r_{m',0}e^{-\ell}, \theta_{m'} = \theta_{m',0} - g_{m'}^{-1}(t)\ell, r_{m+1} = r_{m'+1}, e^{-\ell}, \theta_{m+1} = \theta_{m'+1,0} - g_{m'+1}^{-1}(t)\ell, \cdots r_{m''}$$

$$= r_{m'',0}e^{-\ell}, \theta_{m''} = \theta_{m'',0} - g_{m''}^{-1}(t)\ell : 0 \leq \ell \leq \infty \Big\} .$$

Of course, if $m'' = m$ or, in other words, if there exists no two-dimensional subspaces on which the projection of X is a disk then $\gamma_t' = \gamma_t$ and $E^{n'} = E^n$. In this case the characterization of the projection of X into $E^{n'}$ is actually the characterization of X itself in E^n. An arc is defined in $E^{n'}$ as follows:

$$\lambda = \Big\{ (z_1, z_2, \cdots z_L, r_1, \theta_1, r_2, \theta_2, \cdots r_{m'',m''}) \mid z_1$$

$$= z_{1,0}e^{-\ell}, z_2 = z_{2,0}e^{-\ell}, \cdots z_L = z_{L,0}e^{-\ell}, r_1 = r_{1,0}e^{-\ell}, \theta_1 = \theta_{1,0} - \alpha_1^{-1}\ell, r_2$$

$$= r_{2,0}e^{-\ell}, \theta_2 = \theta_{2,0} - \alpha_1^{-1}\ell, \cdots r_{m'} = r_{m',0}e^{-\ell}, \theta_{m'} = \theta_{m',0} - \alpha_{m'}^{-1}\ell, r_{m'+1}$$

$$= r_{m'+1,0}e^{-\ell}, \theta_{m'+1} = \theta_{m'+1,0}, \cdots r_{m''} = r_{m'',0}e^{-\ell}, \theta_{m''} = \theta_{m'',0} : 0 \leq \ell \leq \infty \Big\} .$$

Theorem 21: The arc λ is contained in the projection of X into $E^{n'}$.

Proof: For any point $p \in \lambda$ it is shown that for an arbitrary $\epsilon > 0$ there exists a point $p' \in X$ such that the projection of p' into $E^{n'}$ is within ϵ of p. Let $\{t_j\}$ be the sequence with the properties defined in theorem 19. By theorem 18, there exists a j' such that if $t = t_j$, $j > j'$, then for every point p_t on γ_t there exists a point in X within a distance $\epsilon/2$ of p_t.

Let p be a point in λ. Then there exists an L' such that

$$p = \big(z_{1,0}e^{-L'}, z_{2,0}e^{-L'}, \cdots z_{L,0}e^{-L'}, r_{1,0}e^{-L'}, \theta_{1,0} - \alpha_1^{-1}\ell, r_{2,0}e^{-L'}, \theta_{2,0}$$

$$- \alpha_2^{-1}L', \cdots r_{m',0}e^{-L'}, \theta_{m',0} - \alpha_{m'}^{-1}L', r_{m'+1,0}e^{-L'}, \theta_{m'+1,0}, \cdots r_{m''}e^{-L'}, \theta_{m''}\big) .$$

Define the point

$$p_t^0 \in \gamma_t', p_t^0 = \big(z_{1,0}e^{-L'}, z_{2,0}e^{-L'}, \cdots z_{L,0}e^{-\ell}, r_{1,0}e^{-L'}, \theta_{1,0}$$

$$- g^{-1}(t)L', r_{2,0}e^{-L'}, \theta_{2,0} - g_2^{-1}(t)L', \cdots r_{m'',0}e^{-L'}, \theta_{m',0} - g_m^{-1}(t)L'\big) .$$

The points p and p_t^0 differ only on their m'' angular coordinates. For $i \leq m'$, the difference in their i^{th} θ coordinates is

$$\left|\theta_{i,0} - \alpha_i^{-1} L' - \theta_{i,0} + g_i^{-1}(t) L'\right| = \left|L' \left[g_i^{-1}(t) - \alpha_i^{-1}\right]\right| ;$$

for i such that $m' < i \leq m''$, the difference in the i^{th} θ coordinate is

$$\left|\theta_{i,0} - \theta_i + g_i^{-1}(t) \ell'\right| = \left|\ell' g_i^{-1}(t)\right| .$$

But by theorem 19, the sequence $\{t_j\}$ has the property that for $i \leq m'$,

$$\lim_{j \to \infty} g_i(t_j) = \alpha_i ,$$

and for $m' < i \leq m''$ we have

$$\lim_{j \to \infty} g_i(t_j) = -\infty .$$

Hence for a sufficiently large j' we can insure that if $t = t_j$, $j > j'$ then the difference in the corresponding coordinates of p and p_t^0 can be made arbitrarily small. This proves that there exists a $j'' > j'$ such that if $t = t_j$, $j > j''$, then the distance between p and p_t^0 is less that $\epsilon/2$. But p_t^0 is in γ_t which is the projection of γ_t into $E^{n'}$. Hence there exists a $p_t \in \gamma_t$ such that the projection of p_t into $E^{n'}$ is p_t^0. Since $t = t_j$, $j > j'' > j'$, there exists a $p' \in X$ such that $d[p', p_t] < \epsilon/2$. Hence the distance between the projection of p' in $E^{n'}$ and the projection of p_t into $E^{n'}$, p_t^0 is also less than $\epsilon/2$. Hence there exists a point $p' \in X$, and the projection of p' into $E^{n'}$ is within a distance ϵ of $p \in \lambda$. But $\epsilon > 0$ was arbitrary and p was an arbitrary point in λ. Hence since the projection of X into $E^{n'}$ is closed this proves the theorem.

Since for each $i > m'$ the i^{th} θ coordinate of points on λ is independent of the parameter ℓ, it is seen that the projection of λ on E_i^2 is a straight line. A simpler representation of λ can then be presented by converting r_i and θ_i into rectangular coordinates z_j, z_{j+1} by means of the equations $z_j = r_i \cos\theta_i$ and $z_{j+1} = r_i \sin\theta_i$. In that case, for any ℓ

$$z_j = z_{j,0} e^{-L}, \quad z_{j+1} = z_{j+1,0} e^{-\ell},$$

$$z_{j,0} = r_{i,0} \cos\theta_i , \quad z_{j+1,0} = r_{i,0} \sin\theta_i .$$

The parameterization of λ then takes the form

$$\lambda = \{(z_1, z_2, \cdots z_{L'}, r_1, \theta_1, r_2, \theta_2, \cdots r_{m'}, \theta_{m'}) \mid z_1 = z_{1,0}e^{-\ell}, z_2 = z_{2,0}e^{-\ell}, \cdots z_{L'} = z_{L',0}e^{-\ell}, r_1 = r_{1,0}e^{-\ell}, \theta_1 = \theta_{1,0} - \alpha_1^{-1}\ell, r_2 = r_{2,0}e^{-\ell}, \theta_2 = \theta_{2,0} - \alpha_2^{-1}\ell, \cdots r_{m'} = r_{m',0}e^{-\ell}, \theta_{m'} = \theta_{m',0} - \alpha_{m'}^{-1}\ell : 0 \le \ell \le \infty\}, L' = L + 2(m'' - m') .$$

Theorem 20 states that on certain two-dimensional subspaces E_i^2, $m' < i \le m''$, the projection of X is a disk. Theorem 21 together with the arbitrariness of $p_0 \in X$ shows that on the complement space $E^{n'}$ with respect to E^n of the orthogonal sum of these two-dimensional subspaces, the projection of X is the union of arcs with parameterization similar to λ.

Theorem 22: Let X be a regular Euler set in E^n. Then with the proper choice of coordinate set there exists an orthogonal set of D' two-dimensional subspaces, $0 \le D' \le n/2$ on each of which the projection of X is a disk. Furthermore, there exist D nonzero numbers, α_i, $i \le D$, $\alpha_i \neq 0$, such that on the complement space of $E^{n'}$ of the orthogonal sum of the D' subspaces on which the projection of X is a disk, called $E^{n'}$, the projection of X can be represented as the union of arcs that, with properly chosen cylindrical coordinates, can be parameterized as

$$\lambda = \{(z_1, z_2, \cdots z_L, r_1, \theta_1, r_2\,\theta_2, \cdots r_D, \theta_D) \mid z_1 = z_{1,0}e^{-\ell}, z_2 = z_{2,0}e^{-\ell}, \cdots z_L = z_{L,0}e^{-\ell}, r_1 = r_{1,0}e^{-\ell}, \theta_1 = \theta_{1,0} - \alpha_1^{-1}\ell, r_2 = r_{2,0}e^{-\ell}, \theta_2 = \theta_{2,0} - \alpha_1^{-1}\ell, \cdots r_D = r_{D,0}e^{-\ell}, \theta_D = \theta_{D,0} - \alpha_D^{-1}\ell : 0 \le L \le \infty\} ,$$

where

$$(z_{1,0}\, z_{2,0}, \cdots z_{L,0}, r_{1,0}, \theta_{1,0}, r_{2,0}, \theta_{2,0}, \cdots r_{D,0}, \theta_{D,0})$$

are the coordinates of a point in $E^{n'}$ which is the projection of a point $p_0 \in X$.

This theorem has some important implications.

Corollary 1: Let X be a regular Euler set in E^2. Then there exists an $\alpha \neq 0$ and a suitable coordinate set such that X can be represented as the disjoint union of arcs of the form

$$\lambda = \{(r, \theta) \mid r = r_0 e^{-\ell}, \theta = \theta_0 - \alpha^{-1}\ell : 0 \le \ell \le \infty\} ,$$

where

$$\left(r_0, \theta_0\right) \in X .$$

Proof: If X is a regular Euler set in E^2, then the integer D' associated with X as defined in theorem 22 must be zero. This follows because otherwise X would be a disk and hence starlike. Theorem 22 then implies that either X is represented as the union of rays

$$\left\{\left(z_1, z_2\right) \mid z_1 = z_{1,0}\, e^{-\ell},\ z_2 = z_{2,0}\, e^{-\ell} : 0 \leq \ell \leq \infty\right\},$$

or as the union of sets of the form

$$\left\{(r, \theta) \mid r = r_0\, e^{-\ell},\ \theta = \theta_0 - \alpha^{-1} \ell : 0 \leq \ell \leq \infty\right\} .$$

The first possibility can be ignored since this would also make X starlike. This completes the proof.

Corollary 1 constitutes a characterization of Euler sets in E^2 since any set representable in the fashion indicated in the statement of the corollary can readily be shown to be an Euler set. The following theorem is the major result of this section.

Theorem 23: Let X be a regular Euler arc in E^n. There exist D nonzero numbers $\alpha_1, \alpha_2, \cdots \alpha_D$, and a coordinate set C such that relative to C, X can be given the cylindrical coordinate parametric representation

$$\begin{aligned} X = \Big\{\left(z_1, z_2, \cdots z_L, r_1, \theta_1, r_2, \theta_2, \cdots r_D, \theta_D\right) \mid\ & z_1 = z_{1,0}\, e^{-\ell},\ z_2 = z_{2,0}\, e^{-\ell}, \cdots z_L \\ = z_{L,0}\, e^{-\ell},\ r_1 = r_{1,0}\, e^{-\ell},\ \theta_1 = \theta_{1,0} - \alpha_1^{-1} \ell,\ & r_2 = r_{2,0}\, e^{-\ell},\ \theta_2 = \theta_{2,0} - \alpha_2^{-1} \ell, \cdots r_D \\ & = r_{D,0}\, e^{-\ell},\ \theta_D = \theta_{D,0} - \alpha_D^{-1} \ell : 0 \leq \ell \leq \infty\Big\}, \end{aligned}$$

where

$$\left(z_{1,0}\, z_{2,0}, \cdots z_{L,0}, r_{1,0}, \theta_{1,0}, r_{2,0}, \theta_{2,0} \cdots r_{D,0}, \theta_{D,0}\right)$$

are the cylindrical coordinates of a point in X relative to C.

Proof: Let X be a regular Euler arc in E^n. Then there exists a regular S-homotopy $H : X \otimes I \to X$ on X. There exists a coordinate set C and a $t_0 \in I$ such that if $t < t_0$, then with respect to C, $f_t = H : X \otimes t \to X$ has the canonical representation of theorem 15. There exist D two-dimensional subspaces relative to C such that f_t, $t < t_0$, on each of these subspaces is the composition of a rotation and a contraction. According to theorem 20, the projection of X onto some of these two-dimensional subspaces may be a disk. But this is not possible. Let $i \le D$ and consider the i^{th} two-dimensional subspace E_i^2 on which f_t, $t < t_0$, is a contraction composed with a rotation. It can be shown that the projection of X on this subspace is not a disk.

Since X is an arc let $\tau[I \to E^n]$ be a parameterization function. It is assumed that the function τ is one to one. The map f_t is a transformation from X into X. Hence the composition map $e_t = \tau^{-1} f_t \tau$ is a well defined continuous transformation from I into I. Let the point $L' \in I$ be such that $\tau(L')$ is the origin of the coordinate set C. If $t < t_0$, $e_t(L') = L'$ since for $t < t_0$, $\tau(L')$ is the unique fixed point of f_t. Also if $\ell \neq L'$, $e_t(\ell) \neq L$. Also notice that if t is sufficiently small, K_t is nonzero and hence e_t is one to one. These results imply that by choosing a sufficiently small t in order to insure that f_t is sufficiently close to the identity map in the function space S(X), it is certain that the following conditions are satisfied:

1. e_t maps the interval $[0, L')$ into $[0, L')$ and for all $\ell \in [0, L')$, $e_t(\ell) > L$.

2. e_t maps the interval $(L', 1]$ into $(L', 1]$ and for all $\ell \in (L', 1]$, $e_t(\ell) < \ell$.

Now consider the subarc $[\tau(\ell) : \ell \in [0, L')] = X$. Define $R_i(\ell)$ and $\theta_i(\ell)$ to be the i^{th} r and θ coordinates of $\tau(\ell)$. The projection of the point $\tau(\ell_1) \in X_1$, on E_i is $(R_i(\ell_1), \theta_i(\ell_1))$. It will be shown that the projection function on X_1 is one to one by showing that the function R_i is monotone decreasing. This will be done by demonstrating that the function R_i has the property that for all $\ell_1 \in [0, L')$ there exists an ℓ_2 such that $\ell_1 < \ell_2$, and if ℓ is such that $\ell_1 < \ell < \ell_2$, then $R_i(\ell_1) > R_i(\ell)$.

Let $t' \in I$ be such that if $t \le t'$, then e_t satisfies conditions one and two. Also, for every point $p \in X$, let O_p be the orbit of p under the S-homotopy $H : X \otimes I \to X$. According to the first condition, for all $\ell \in [0, L')$, $e_{t'}(\ell) > \ell$. Then for every $p \in X$, there exist ℓ_1, $\ell_2 \in [0, L')$, such that $\ell_1 < \ell_2$ and $\tau(\ell_1) = p$ and $\tau(\ell_2) = f_{t'}(p)$. If $\ell \in [0, L']$ is such that $\ell_1 < \ell < \ell_2$, then because the orbit of p, O_p, is connected, there exists a $t \in [0, t']$ and $\tau(\ell) = f_t(p)$. If the i^{th} R coordinate of $\tau(\ell_1)$ is R_i, then the i^{th} R coordinate of $\tau(\ell)$ is $K_t R_i(\ell)$ and we have $R_i(\ell_1) > R_i(\ell)$. This implies that the function R_i is monotone decreasing and hence the projection of X_1 on E_i^2 is a one to one map. But a one to one map cannot raise dimension. Thus it is shown that the projection of X_1 on E_i^2 is one dimensional.

Define the subarc X_2 as

$$X_2 = [\tau(\ell) : \ell \in (L', 1]] .$$

By the same reasoning we can show that the projection of X_2 on E_i^2 is one dimensional. But the projection of the arc X_2 onto E_i^2 is the union of the projections of X_1, X_2, and the origin, $\tau(L')$. Hence the projection of X onto E_i^2 is one dimensional and cannot be a disk.

The result then follows from theorem 22. Since the number D' in the first part of the statement of theorem 22 must be zero, the second part of the statement of theorem 22 indicates that the arc X relative to C has the parametric representation

$$X = \Big\{\big(z_1, z_2, \cdots z_L, r_1, \theta_1, r_2, \theta_2, \cdots r_D, \theta_D\big) \mid z_1 = z_{1,0}\, e^{-\ell}, z_2$$

$$= z_{2,0}\, e^{-\ell}, \cdots z_L = z_{L,0}\, e^{-\ell}, r_1 = r_{1,0}\, e^{-\ell}, \theta_1 = \theta_{1,0} - \alpha_1^{-1}\ell, r_2$$

$$= r_{2,0}\, e^{-\ell}, \theta_2 = \theta_{2,0} - \alpha_2^{-1}\ell, \cdots r_D = r_{D,0}\, e^{-\ell}, \theta_D = \theta_{D,0} - \alpha_D^{-1}\ell : 0 \leq \ell \leq \infty\Big\},$$

where

$$\big(z_{1,0}, z_{2,0}, \cdots z_{L,0}, r_{1,0}, \theta_{1,0}, r_{2,0}, \theta_{2,0}, \cdots r_{D,0}, \theta_{D,0}\big)$$

are the coordinates of a point in X.

Theorem 23 provides the promised generalization of Euler's result concerning twice differentiable regular Euler arcs in E^2 to a characterization of all regular Euler arcs in E^n.

S-EXPANDABLE SETS

In definition 3, the concept of an S-shrinkable set was introduced. The purpose of this definition was to provide a mathematical model for the initiutive concept of a shape preserving shrinking of a set into itself. In a similar fashion, a mathematical model will be provided of the intuitive idea of a shape preserving expansion of a set. It will then be proved that in a wide class of metric spaces, the class of sets expandable in a shape preserving fashion is identical to the set of sets which are shrinkable in a shape preserving fashion.

Let X be a subset of a metric space M. Before discussing the meaning of a shape preserving expansion of X a definition is needed.

Definition 13: Let f be a map from X into M. The map f is an *expansive S-map* if there exists a $K > 1$, such that for any $x \in X$, $y \in X$, then

$$d\big[f(x), f(y)\big] = K d[x, y] .$$

As in the case of a contractive S-map, an expansive S-map on a subset of a complete metric space has a unique fixed point provided its scale K is greater than one.

Definition 14: A subset X of a metric space M is *canonically S-expandable* if there exists a homotopy $H : X \otimes I \to m$ such that $H : X \otimes 0 \to m$ is the identity map on X and for all $T \in (0, 1]$, $H : x \otimes t \to m$ is an expandable S-map. Also if $t \in I$,

$$H[X \otimes t] \subset H[X \otimes 1] .$$

Definition 15: A subset Y of a metric space M is S-expandable if there exists an S-map between Y and a canonically-expandable subset X of M.

This seemingly convoluted manner of defining S-expandable sets was used for the following reason. Intuitively, the property of S-expandablity is related only to the shape of the set, not to metric properties which are not properties of "shape," i.e., the diameter of a set. This is the reason for the distinction between S-expandable sets and canonically S-expandable sets. For instance, if a subset X of a metric space M has the same finite diameter as M, it cannot be canonically S-expandable in spite of the fact that its points may exhibit the same set of metric ratios and hence have the same "shape" as a subset of M which is canonically S-expandable. The above definition of S-expandability clearly removes this difficulty.

Theorem 24: A subset X of a metric space M is S-expandable if and only if it is S-shrinkable.

Proof: It is shown that if a subset X of a metric space M is S-shrinkable, it is S-expandable. The demonstration that if X is S-expandable, it is S-shrinkable is quite similar and will be omitted.

If X is S-shrinkable, there exists an S-homotopy $H : X \otimes I \to X$. Again, for all $t \in [0, 1]$, define $f_t = H : x \otimes t \to x$. The proof is complete if we show that $f_t\,[x]$ is canonically S-expandable. We lose no generality in assuming that $f_1\,[x]$ is not a single point and furthermore, that for $t \in [0, 1)$, $K_t > K_1$. For $t \in [0, 1]$, define $g_t = f_{1-t}\, f_1^{-1}$. The map g_t has $f_1\,[x]$ as its domain and being the composition of two S-maps is itself an S-map. Also its scale is K_{1-t}/K_1, which is greater than one. Hence, g_t for $t \in [0, 1]$ is an expansive S-map. An S-homotopy on $f_1\,[x]$ is constructed by stating $g_t \equiv G : f_1\,[x] \otimes t \to m$. This defines a map G from $f_1\,[x] \otimes I$ into M. The map G is an S-homotopy on $f_1\,[x]$ in which the maps of the homotopy are expansive S-maps and clearly g_0 is the identity map on $f_1\,[x]$. Hence, according to definition 14, $f_1\,[x]$ is canonically S-expandable and X is S-expandable.

A CHARACTERIZATION OF REGULAR EULER SETS IN E^3

Theorem 22 represents a powerful result on the structure of regular Euler sets in E^n. With the aid of this theorem a complete characterization of regular Euler sets in E^2 can be readily obtained. With considerably more difficulty, theorem 22 can be used to characterize regular Euler arcs in E^n. These results represent substantial generalizations of Euler's classical results discussed previously. At this point it is convenient to discuss what meaning these abstract results may have in E^3. Specifically, there is the question of what shapes are permitted to an object in E^3 which can exhibit a shape preserving growth or a shape preserving shrinking. Theorem 24 indicates that this constitutes not two questions but one. The question is answered by providing a characterization of regular Euler sets in E^3.

Theorem 25: Let X be a regular Euler set in E^3. Then X can be represented as the union of arcs that, with a properly chosen cylinderical coordinate set, can be parameterized as

$$\lambda = \left\{(z, r, \theta) \mid z = z_0 e^{-\ell}, r = r_0 e^{-\ell}, \theta = \theta_0 - \alpha^{-1} \ell : 0 \leq \ell \leq \infty\right\},$$

where α is a positive number and (z_0, r_0, θ_0) are the coordinates of a point in X.

Proof: Since by hypothesis X is a regular Euler set in E^3, there exists a regular S-homotopy $H : X \otimes I \to X$ such that for $t < t_0$, where t_0 is a fixed point in I,

$$f_t = H[X \otimes t \to X]$$

has the canonical representation of theorem 16. This means that there exists a coordinate set c and a cylindrical coordinate representation (z, r, θ) of points in E^3 such that if $t < t_0$, then f_t can be represented as

$$f_t[(z, r, \theta)] = (ZK_t, rK_t, \theta + \phi_t),$$

where K_t is the scale and ϕ_t is the rotational coefficient of f_t. Define

$$g(t) = \ln \frac{K_t}{\phi_t},$$

$t < t_0$. It will be shown that

$$\lim_{t \to 0} g(t) \neq 0 .$$

This will imply that the symbol D′ in the statement of theorem 22 is zero and an invocation of theorem 22 concludes the proof.

Let P be an arbitrary point in X; let its cylinderical coordinates relative to c be (z_0, r_0, θ_0). We define a set β_p as follows;

$$\beta_p = \left\{(z, r, \theta) \mid 0 \leq z \leq z_0, \; r = \frac{r_0}{z_0} z, \; 0 \leq \theta \leq 2\pi\right\}.$$

Intuitively, β_p is a cone with apex at the origin of c and containing P. It is shown that if

$$\lim_{t \to 0} g(t) = 0 ,$$

then $\beta_p \subset X$. As in the section entitled "S-Expandable Sets," for every $t < t_0$ define an arc γ_t as

$$\gamma_t = \left\{(z, r, \theta) \mid z = z_0 e^{-\ell}, r = r_0 e^{-\ell}, \theta = \theta_0 - g^{-1}(t)\ell : 0 \le \ell \le \infty\right\} .$$

Notice that $\gamma_t \subset \beta_p$. Let $\tau_t(\ell)$ be the parameterization function of γ_t. Let P' be an arbitrary point in β_p with coordinates relative to c of (z', r', θ'). Define

$$\ell_n = (\theta' - \theta_0 + 2n\pi)\, g(t) ,$$

and let the cylinderical coordinates of $\tau_t(\ell_n)$ be represented as $(z_{n,t}, r_{n,t}, \theta_{n,t})$. It is shown that for any $\epsilon > 0$, there exists a $t' < t_0$ such that for any $t < t'$, there exists an integer n such that the distance between $\tau_t(\ell_n)$ and P' is less than ϵ. Clearly, for all n,

$$z_{n,t} = z_0 e^{-(\theta' - \theta_0 + 2n\pi) g(t)} , \quad r_{n,t} = r_0 e^{-(\theta' - \theta_0 + 2n\pi) g(t)} , \quad \theta_{n,t} = \theta' .$$

For every $t < t_0$, there exists a unique $n(t)$ such that

$$Z_{n(t),t} \ge z' \ge Z_{n(t)+1,t} ,$$

and

$$r_{n(t),t} \ge r' \ge r_{n(t)+1,t} .$$

For any n, the distance between $z_{n,t}$ and $z_{n+1,t}$ can be shown by simple manipulations to be

$$z_0 \left[e^{2\pi g(t)} - 1\right] e^{(\theta' - \theta_0 + 2n\pi) g(t)} .$$

By definition

$$g(t) = \ell n \frac{K_t}{\phi_t} < 0 .$$

This implies that for a given $t' < t_0$,

$$\max\left[\left|z_{n,t'} - z_{n+1,t'}\right|\right] = z_0\left[e^{2\pi g(t)} - 1\right] .$$

Also by similar reasoning

$$\max\left[\left|r_{n,t'} - r_{n+1,t'}\right|\right] = r_0\left[e^{2\pi g(t)} - 1\right] .$$

If it is assumed that

$$\lim_{t \to 0} g(t) = 0 ,$$

then it is clear that for given any $\epsilon > 0$, there exists a $t' < t_0$ such that for some n',

$$\left|r_{n',t'}, r'\right| < \epsilon ,$$

and

$$\left|z_{n',t'}, z'\right| < \epsilon ,$$

and

$$\left|\theta_{n',t} - \theta'\right| < \epsilon .$$

Since $\tau_{t'}\left(\ell_{n'}\right) \epsilon\ \gamma_{t'}$, by choosing a sufficiently small t, a point in γ_t is within a preassigned value ϵ of the arbitrary point P'. But according to theorem 18, for any value ϵ, there exists a t' such that if $t < t'$, then every point of γ_t is approximated within ϵ by a point in X. Hence it is clear that every point in β_p can be arbitrarily closely approximated by a point in X. Since X is closed, this implies $\beta_p \subset X$. Because the point $P \epsilon X$ was arbitrary, X is starlike and thus contradicts the assumption that X is an Euler set. Hence

$$\lim_{t \to 0} g(t) \neq 0 ,$$

and the theorem is proved.

Theorem 25 shows that any object in E^3 that is shrinkable or expandable in the manner defined by definitions 3 or 15 is either starlike or has the representation given in the statement of theorem 25. The shape defined by theorem 25 is a common geometric form in nature and it is encountered in sciences as diverse as biology and astronomy. It is hoped that the results of this study will be of use in explaining the presence of this shape in some situations. For instance, it may be of some survival value for an organism to grow in a fashion that approximately preserves its original shape. If other considerations rule out a starlike shape for the organism, then the natural selection process could lead to a shape approximating the Euler set characterized for E^3 by theorem 25. The shells of several snails, for instance, form facsimiles of Euler sets. There are many other biological examples. Applications of the results of this paper to the explanation of the appearence of the Euler set geometric form in other disciplines such as astronomy are also possible.

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