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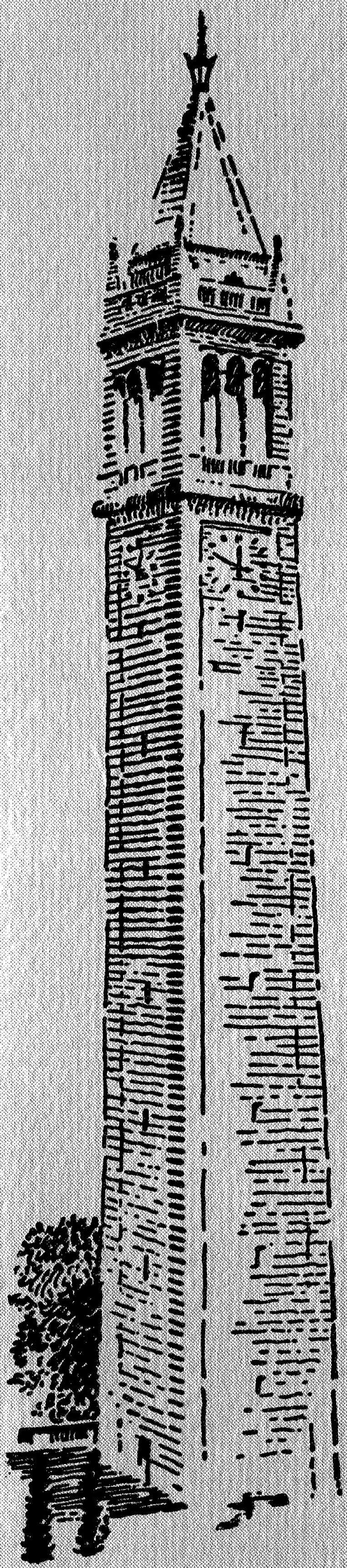

Testing Function Techniques in System Identification

by

Ronald Marshall Bass

12 May 1969

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ELECTRONICS RESEARCH LABORATORY

College of Engineering

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TESTING FUNCTION TECHNIQUES IN SYSTEM IDENTIFICATION

ABSTRACT

The following general problem was investigated: identify a linear time-invariant differential system from observations of input and output signals, over which the observer is assured to have no control. The observations of either the input or the output signals, but not both, may be corrupted by additive zero mean noise.

Moments of the observed signals with derivatives of functions called "testing functions" are taken to obtain linear algebraic equations in the unknown differential equation coefficients. These moments are calculated simply by multiplying the observed signal times a derivative of a testing function and integrating the product; thus the calculation is very stable numerically. If the observations are noise-free, the algebraic equations are solved in a straight-forward manner. If not, an estimation procedure called the instrumental variable method is used to obtain an estimate of the system coefficients which is asymptotically unbiased as the observation interval gets large. The question of what constitutes a good testing function is investigated.

This method may be applied to systems of the Hammerstein form, i.e., a memoryless nonlinearity followed by a linear plant. In this case the nonlinearity is represented as a linear combination of known functions, and the coefficients of this combination as well as the coefficients of the linear plant are identified.

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CHAPTER 1

INTRODUCTION

1.0 Introduction

The identification problem may be described informally as follows:[1]

> Given a system A and a class of systems C, to which A is known to belong. Find a system in C which is equivalent (or approximately equivalent) to A.

Usually this is assumed to be accomplished through observation, often in the presence of noise, of the response of A to "probe functions", i.e., inputs which may be specified at will according to the particular representation of the class of systems to which A belongs. For example, the desired representation of A may be an impulse response or a transfer function, in which cases the probe functions are a delta function, and a set of sinusoids of different frequencies, respectively.

There are, however, many practical situations in which we wish to model a system A, but have no control over the nature of the input function. In this case the identification algorithm which is applied must be such that whatever input that is available can be used. We will refer to such a situation as "in-service" identification.

In recent years a great deal of attention has been devoted to the development of identification algorithms for various classes of systems. The class of linear differential systems is of particular interest for two important reasons:

1. There is a large class of systems such that the behavior of each member of this class may be well approximated by the behavior of a linear differential system for a sufficiently small range of dynamic behavior and a sufficiently small time interval;
2. General solutions to the problems of optimal state estimation and optimal control of linear differential systems are well known.

We are concerned here with a particular linear differential system in-service identification algorithm which we shall call "the method of testing functions", developed in the noise free case by Loeb and Cahen.[2] In particular we obtain a general algorithm for in-service system identification, with observations taken in the presence of noise, and show how testing functions may be used in this algorithm to identify linear time invariant differential systems and some common types of nonlinear systems. We examine the properties of testing functions and show how to choose a "good" testing function. Finally, we develop a multiple input, multiple output model for linear time-invariant system identification.

Advantage of the Method of Testing Functions Over Other Known Methods

The method of testing functions will be shown to be superior to other known in-service linear differential systems identification algorithms for several reasons. First, other methods (the so-called error decomposition method[3] (Section 1.5) and the instrumental variable method using a sampled linear filter sequence generator[4] (Section 1.6)) both require the introduction of linear filters with which to process the observations of the system input and output. It is possible that these filters might cancel or nearly cancel a pole or zero of the unknown system, thus yielding an incomplete or inaccurate model of the system. Such filtering is not necessary with the method of testing functions.

Second, if the algorithm is implemented by digital computer, the introduction of these filters adds to the computational requirements either the numerical solution of a set of linear differential equations or the repeated evaluation of convolution integrals (an operation which is either ill posed or very time consuming) while the method of testing functions requires only evaluations of well posed integrals, which may be interpreted as linear filtering of the observations but can not be characterized by finite dimensional differential systems.

Third, the methods which use filters to process the data ignore transient response in the unknown system (which approximation, however, does not introduce much error) whereas no such approximation is made in the method of testing functions.

1.1 Statement of the Identification Problem

The following is a statement of the identification problem considered in this thesis:

Given

1. a system A;
2. observations of input and output functions of A (over which the observer has no control) possibly in the presence of noise, over an arbitrarily long time interval;
3. a class C of systems to which A belongs; find a member of C which is equivalent to A.

We will consider two differential systems to be equivalent if they can both be represented by the same differential equation.

Comment

We will assume A to be a single input, single output system. The multiple input, multiple output case is treated in Appendix III.

1.2 Outline of Thesis

By way of introduction we first discuss two linear time-invariant differential system in-service identification algorithms. One of these uses testing functions. In Chapter 2 we discuss the classes of systems and conditions on the noise processes for which we can devise identification algorithms which use testing functions. In Chapter 3 we discuss some of the properties of testing functions. In Chapter 4 we give examples of various types of testing functions, derive the general form of polynomial testing functions, and give a polynominal testing function which is well suited to the algorithms presented in the first two chapters.

1.3 Notation

The following notation will be used throughout the thesis. Additional notation will be introduced where needed.

$\triangleq$ means "is defined to be"

Unless otherwise indicated, all functions will be of the form

$f: R \to R$

where R is the real line.

A function will be denoted by a single letter (possibly subscripted). e.g., f,g,y,u,w,v, or with a generalized argument, e.g., x(•). Open, semi-open, and closed intervals will be denoted as follows:

$[a,b] \triangleq \{x|a \leq x \leq b\}$

$(a,b] \triangleq \{x|a < x \leq b\}$

$[a,b) \triangleq \{x|a \leq x < b\}$

$(a,b) \triangleq \{x|a < x < b\}$

$p \triangleq$ the differential operator d/dt

$p^k \triangleq$ the differential operator d^k/dt^k

$y^{(i)}(a) \triangleq p^i y(t)\Big|_{t=a}$, y is i times differentiable

$E(x) \triangleq$ expected value of the random variable x

The value of a random process x will be denoted by $x(\omega,t)$, where $\omega\varepsilon\Omega$, the probability space of the random variable $x(\cdot,t)$, and $t\varepsilon R$

$R_x \triangleq$ the autocorrelation function of the wide sense stationary random process x, where

$$R_x(\tau) = E[(x(\cdot,t)-m)(x(\cdot,t+\tau)-m)]$$

where $m = E[x(\cdot,\xi)]$

$$(x,y) \triangleq \sum_{i=1}^{N} x_i y_i \text{ where } x = (x_1,\ldots,x_N) \text{ and } y = (y_1,\ldots,y_N)$$

$\{x_i\} \triangleq$ is the class of all x_i, the range of the index i to be taken from context.

$\{x_i\}_m^n$ or $\{x_i\}_{i=m}^n \triangleq$ the set $x_m, x_{m+1}, \ldots, x_{n-1}, x_n$

$\delta(\cdot) \triangleq$ delta function

$\delta^{(k)}(\cdot) \triangleq k^{th}$ derivative of δ, $\delta^{(0)} \triangleq \delta$

$\delta^{(k)}$ is called a delta function of order k.

$$\text{sign }(x) \triangleq \begin{cases} -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$$

$$1(x-x_0) \triangleq \begin{cases} 0 \text{ if } x \leq x_0 \\ 1 \text{ if } x > x_0 \end{cases}$$

Definition-bounded function - A function f is said to be bounded if

$$\sup_{x \varepsilon R} |f(x)| < \infty$$

1.4 Two Identification Algorithms - Preliminaries

In this section we give a description of the system and some notation that will be used in discussing both algorithms. The configuration of the system under consideration is shown in Figure 1.

Notation

$u \triangleq$ input function to A

$y \triangleq$ output function of A due to excitation by u and to the initial state

N_1 and N_2 are zero mean, wide sense stationary, second order random processes statistically independent of u and y.

Comment

Throughout this thesis we will be dealing with only additive noise. Thus the assumption that a noise process is of zero mean is equivalent to the assumption that the mean is

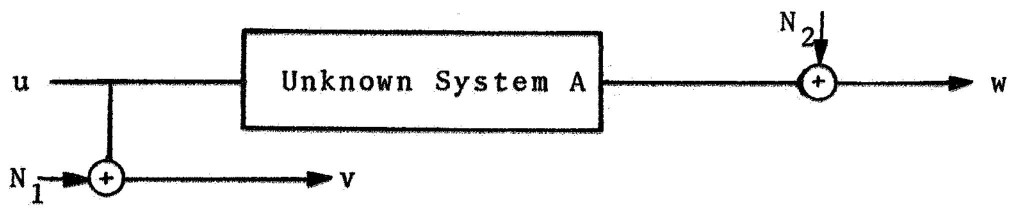


FIGURE 1

known, since the mean may be subtracted from the observations, yielding observations of signal plus zero mean noise.

v and w are the observed functions from which we will identify A. The functions u and y are assumed to satisfy the differential equation

$$L(p)y = M(p)u \tag{1}$$

where

$$L(p) = \sum_{i=0}^{M} a_i p^i \tag{2}$$

$$M(p) = \sum_{j=0}^{M} b_j p^j \tag{3}$$

We will assume that the system A is bounded-input, bounded-output stable and that the input is bounded in magnitude uniformly for all time.

Comment

In this case the identification problem is one of estimating the coefficients $\{a_i\}_{i=0}^{M}$ and $\{b_i\}_{i=0}^{M}$.

1.5 Error Decomposition Method[5]

In this algorithm, estimation of the unknown coefficient is accomplished by minimization of an "error function" which is in some sense a measure of the discrepancy between the model and the unknown system. The "error function" is decomposed into a sum of three functions, two of which depend upon the sample functions of N_1 and N_2, and one of which depends upon the input and output functions u and y. We give a brief summary of this algorithm in order to indicate its computational aspects.

We assume the following:

1. the noise processes N_1 and N_2 are uncorrelated, i.e.,

$$E(N_1(t)N_2(t+\tau)) = 0 \quad \text{for all } t \text{ and } \tau$$

2. R_{N_1} and R_{N_2} are known (Recall that $R_x(\tau)$ was defined in Section 1.3 as $R_x(\tau)=E[(x(\cdot,t)-E(x(\cdot,t)))(x(\cdot,t+\tau)-E(x(\cdot,t)))]$)

3. $b_0 = 1$

The error function is $(E(\mathcal{E}))^2$, where

$$\mathcal{E} = \frac{\hat{L}(p)}{Q(p)} w - \frac{\hat{M}(p)}{Q(p)} v \tag{4}$$

and

$$\hat{L}(p)y = \hat{M}(p)u \tag{5}$$

is the differential equation for best estimate of the unknown system, where

$$\hat{L}(p) = \sum_{i=0}^{M} \hat{a}_i p^i \qquad \hat{M}(p) = \sum_{j=0}^{M} \hat{b}_j p^j \tag{6}$$

where $\{\hat{a}_i\}$ and $\{\hat{b}_j\}$ are the best estimates of $\{a_i\}$ and $\{b_j\}$

respectively, and $Q(p)$ is a polynomial in p of degree M or greater which has roots only in the open half complex plane and which has no roots in common with $L(p)$ or $M(p)$.

Let

$$\alpha = (a_0,\ldots,a_M,\ b_1,\ldots,b_M) \tag{7}$$

$$\overline{\alpha} \text{ is any } 2M+1 \text{ vector} \tag{8}$$

$$\hat{\alpha} = (\hat{a}_0,\ldots,\hat{a}_M,\ \hat{b}_1,\ldots,\hat{b}_M), \quad \hat{\alpha} \text{ is the "current" estimate of } \alpha \tag{9}$$

Then

$$(E(\mathcal{E}))^2 = m_1(\alpha) + m_2(\alpha) + m_3(\alpha) \tag{10}$$

where

$$\begin{aligned} &m_1(\alpha) \text{ is a function of } \alpha,\ N_1,\ R_{N_1} \\ &m_2(\alpha) \text{ is a function of } \alpha,\ N_2,\ R_{N_2} \\ &m_3(\alpha) \text{ is a function of } \alpha,\ y,\ u \text{ and } E(\mathcal{E}) = m_3(\alpha) \end{aligned} \tag{11}$$

Under mild restrictions on u and y, it can be shown that $m_3(\overline{\alpha})$ is positive definite in u and y for any $\overline{\alpha}, \overline{\alpha} \neq \alpha$, and unimodal in $(\overline{\alpha}+\alpha)$ for any u and y, with

$$m_3(\alpha) = 0 \tag{12}$$

Thus we may find α by minimizing $m_3(\overline{\alpha})$ with respect to $\overline{\alpha}$. The gradient of $m_3(\alpha)$ with respect to $\overline{\alpha}$ cannot be obtained exactly; however,

$$z(t,\overline{\alpha}) = \text{grad } E(\mathcal{E}^2) - \text{grad } m_1(\overline{\alpha}) - \text{grad } m_2(\overline{\alpha}) \tag{13}$$

may be calculated from $\mathcal{E}, R_{N_1}$ and R_{N_2}. Further

$$E(z(t,\overline{\alpha})) = \text{grad } m_3(\overline{\alpha}) \tag{14}$$

Thus a stochastic approximation procedure may be used to solve

$$\text{grad } m_3(\overline{\alpha}) = 0 \tag{15}$$

for α.

<u>Comment</u>

We may calculate grad $E(\mathcal{E}^2)$, grad $m_1(\alpha)$, and grad $m_2(\alpha)$ using only the operations of filtering of observed signals through known linear filters $\hat{L}/Q(p)$ and $\hat{M}/Q(p)$, multiplication, and addition. Thus the algorithm may be implemented as a hybrid (analog-digital) computation, with

$$\frac{d\hat{\alpha}(t)}{dt} = -k(t)z(t,\hat{\alpha}(t)), k(t)\alpha \frac{1}{t} \tag{16}$$

or as a digital computation with

$$\hat{\alpha}_{n+1} = \hat{\alpha}_n - k(n) \; z(t,\hat{\alpha}_n), k(n)\alpha \; \frac{1}{n} \tag{17}$$

1.6 The Instrumental Variable Method[6]

Suppose that the unknown system is characterized by a vector x of N parameters. Further suppose that from noise-corrupted observations of the system input and output we are able to obtain some data which we arrange in a kNxN matrix S, which depends in part on the noise in the observations, and a kNx1 vector D, which does not depend upon the noise in the observations (we shall see that the entries in S and D are in fact simply integrals of the products of testing functions and their derivatives with observed signals). Suppose further that x satisfies the following equation:

$$E(S)x = D$$

where the expectation is taken over the sample space of the observation noise process. Let

$$B \triangleq E(S)$$

$$\eta \triangleq S - E(S).$$

Let S_k be the NxN matrix whose first row is 1/k (sum of the first k rows of S), whose second row is 1/k (sum of the second k rows of S), etc. Let D_k, B_k, and η_k be similarly defined. Then if the noise process is reasonably well behaved, we can expect by the central limit theorem that for large k, η_k will converge in probability to the NxN zero matrix, and S^{-1} D_k will converge to B_k^{-1} D_k, which equals x, if the inverses exist. This procedure of "adding together data to average out the noise" is the basic idea behind the instrumental variable method. We call the algorithm which generates S and D a "sequence generator". In essence, given a linear differential system with the differential equation

$$\sum_{\ell=0}^{M} a_\ell p^\ell y - \sum_{j=1}^{M} b_j p^j u = u$$

a sequence generator produces a sequence of random vectors $s_i \triangleq (s_{i_1},\ldots,s_{i_{2M}})$, $i = 1,2,\ldots$ and a sequence of deterministic scalars d_i, $i = 1,2,\ldots$ such that when s_i replaces $p^i y$, $s_{i_{M+j}}$ replaces $p^j u$ and d_i replaces u in the differential equation, we have

$$\sum_{\ell=0}^{M} a_\ell E(s_{i_\ell}) - \sum_{j=1}^{M} b_j E(s_{i_{M+j}}) = d_i,\ i = 1,2,\ldots$$

then

$$S = \begin{bmatrix} s_{1_1} & \cdots & s_{1_{2M}} \\ s_{2_1} & \cdots & s_{2_{2M}} \\ \vdots & & \end{bmatrix}$$

and

$$D = (d_1,d_2,\ldots)^T$$

In this case the vector of parameters x is $(a_0,\ldots,a_M,b_1,\ldots,b_M)^T$. In the actual instrumental variable algorithm the rows of S and D are not simply added together but are first weighted in a certain fashion. The matrix by which S and D are premultiplied to give this weighted sum is called the instrumental variable matrix.

There are two main parts to the instrumental variable method:

1. generate a sequence of scalars $\{d_i\}$ and a sequence of vectors $\{s_i\}$ from observations of v and w such that $(E(s_i),x) = d_i$ for all i, where x is a vector of parameters which specify the unknown system (in this case simply the coefficients of the differential equation);
2. estimate x from these sequences.

Both of these may be carried out on line.[7] The instrumental variable algorithms for linear and nonlinear systems differ only in the way in which the sequences are generated.

We present this algorithm in a framework more general than necessary for linear differential system identification. This will enable us to use the same algorithm with only slight modification for the identification of nonlinear systems as discussed in the next chapter.

We first give a general description of the properties that the sequence generator (1) must have in order that a valid estimate of x may be obtained. We then give a theorem which shows how the system parameters may be estimated from the generated sequences. Finally, we discuss the structure of the sequence generator in the case of linear differential system identification.

Notation

$x \triangleq$ Nx1 vector of parameters which uniquely specify the unknown system

$\{\beta_i\}$, $\{d_i\}$ are sequences of real scalars

$\{\alpha_i\}$, $\{s_i\}$ are sequences of real Nx1 vectors, where

$$\alpha_i \triangleq (a_{i1},\ldots,a_{iN})$$

$$s_i \triangleq (s_{i1},\ldots,s_{iN})$$

Sequence Generators-General Description

SG is an operator defined as follows: (See Figure 2)

SG operating on u and y gives $\{\alpha_i\}_{i=-\infty}^{\infty}$, $\{\beta_i\}_{i=-\infty}^{\infty}$;

SG operating on v and w or equivalently, on $u+N_1$ and $y+N_2$ yields $\{d_i\}_{i=-\infty}^{\infty}$, $\{s_i\}_{i=-\infty}^{\infty}$.

We require SG to be such that these sequences have the following properties:

1. $\{\alpha_i, x\} = \{\beta_i\}$, i.e., $\langle\alpha_i, x = \beta_i\rangle$ for all of i (1)

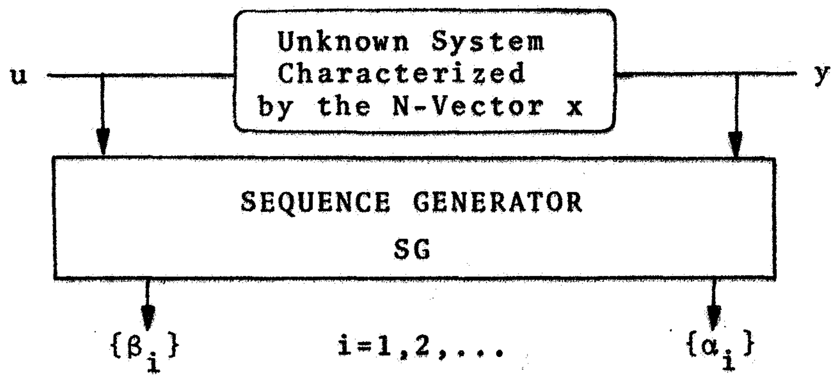
2. $E(s_{ij}) = \alpha_{ij}$ for all of i (2)
   j = 1,...,N where the expectation is taken over the sample spaces of the noise processes;

3. $|\alpha_{ij}|$, j = 1,...,N, and $|\beta_i|$ are uniformly bounded (3)
   with probability one for all i;

4. $\{s_i-\alpha_i\}$ is a sample function of a discrete, zero (4)
   mean wide sense stationary random process; and

$$\lim_{j\to\infty} |i-j|E[(s_i-\alpha_i)(s_j-\alpha_j)] = 0;$$

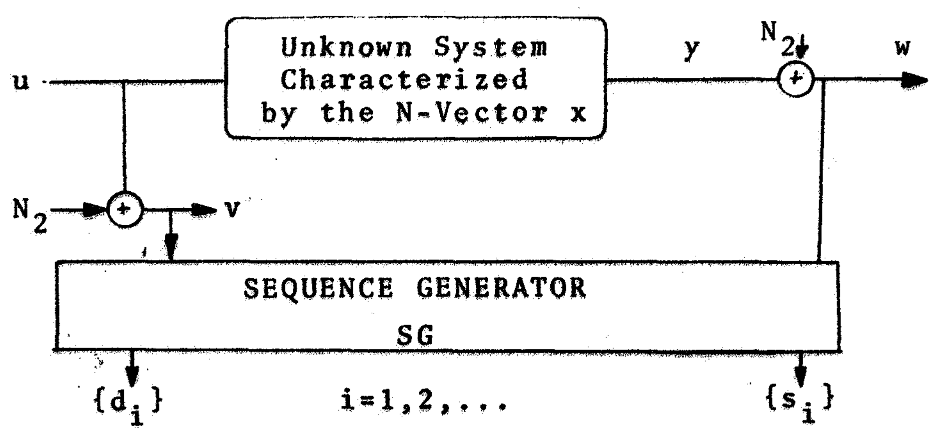
a) <u>No Noise Present</u>



β_i is a scalar $\qquad\qquad \alpha_i \triangleq (\alpha_{i1},\ldots,\alpha_{iN})$

$\langle\alpha_i,x\rangle = \beta_i$

b) <u>Noise Present</u>



d_i is a scalar $\qquad\qquad s_i \triangleq (s_{i1},\ldots,s_{iN})$

$E\left(s_i\right) = \alpha_i$

$d_i = \beta_i$

$\langle E\left(s_i\right),x\rangle = d_i$

Either N_1 or N_2 is zero.

FIGURE 2

5. $\{s_i\}$ and $\{d_i\}$ must be computable from available information. (5)

Without loss of generality, we let the estimation of x begin at i=1. We define an "array assembler" (Figure 3) with the sequences d_i and s_i as inputs and the arrays S_k and D_k as outputs, where

S_k is a kNxN matrix with i^{th} row s_i

D_k is a kNx1 vector with i^{th} row d_i (6)

With these definitions, we have

$$E(S_k)x = D_k \tag{7}$$

<u>Estimation of x from Generated Sequences</u>

The second major part of the algorithm, the estimation of x from the generated sequences, is based upon the following theorem:

<u>Theorem</u>[8] (8)

Let $v_1, v_2, \ldots$ be a sequence of vectors $\{v_i\}$, where $v_i \triangleq (v_{i1}, \ldots, v_{iN})^T$, $|v_{ij}|$ bounded uniformly for all i, j = 1,...,N.

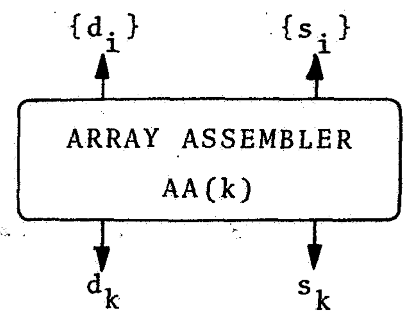
Let $V_k = (v_1 \vdots \ \ldots \ \vdots v_k)$.

Let $\lim_{k\to\infty} 1/k\ V_k S_k$ be nonsingular,

$\lim_{k\to\infty} 1/k\ V_k D_k \neq 0$.

Let $x_k \triangleq (V_k S_k)^{-1} V_k D_k$.

Then x_k is a consistent (and therefore asymptotically unbiased) estimate of x, i.e., x_k converges to x in probability



$$D_k \triangleq (d_1, \ldots, d_J)^T \quad S_k = \begin{bmatrix} s_i \\ \vdots \\ s_J \end{bmatrix}$$

$$J = kN \qquad (ES_k)\, x = D_k$$

FIGURE 3

as $k \to \infty$.

Comment

The sequence of matrices $\{v_k\}$ is called an instrumental variable sequence.

Thus, given SG, if we can find a sequence $\{v_i\}$ such that the hypotheses of Theorem (8) are satisfied, we can obtain, in the limit, an unbiased estimate of x.

Existence and Realizability of an Instrumental Variable Sequence

It is easily verified that such a sequence does exist. In fact the sequence $\{ES_k\}$ satisfies the requisite conditions if the input u is "sufficiently random". In practice, $\{ES_k\}$ cannot be obtained directly, but may be estimated recursively by using u and the best estimates of x and y available.[7] (See Appendix I). Unfortunately, there is no known method for estimating an instrumental variable sequence when both input and output noise are present. Thus we shall assume in the sequel that either $N_1 = 0$ or $N_2 = 0$. Note that $\{ES_k\}$ cannot be an instrumental variable sequence if (3) is not satisfied.

Existence of Sequence Generator

We show in the next chapter that for certain types of nonlinear systems and certain noise processes that sequence generators exist. In some cases we may require knowledge of moments of the noise processes in order to implement SG, but this is not always necessary, as we demonstrate with several examples in the next chapter.

Sequence generators exist for linear time invariant differential systems, and we present two of these below.

Two Sequence Generators

In this section we present two sequence generators for linear time invariant differential systems.

We use the following notation in discussing the sequence generators.

Notation

As specified in Section 1.4, the differential equation of the unknown system is

$$L(p)y = M(p)u$$

We set $b_0 = 1$ (This based upon the assumption that $b_0 \neq 0$. If $b_0 = 0$, we may set any non-zero coefficient b_j to 1 and proceed as indicated below.) Then

$$\sum_{j=0}^{M} a_i p^i y - \sum_{j=1}^{M} b_j p^j u = u \qquad (9)$$

Let $x = (a_0, \ldots, a_M, b_1, \ldots, b_M)^T$, where x is $(2M+1)x1$, i.e., $N = 2M+1$.

Without loss of generality, we begin estimation of x at $i = 1$, $t = 0$.

Comment

We will assume that $N_1 = 0$. If $N_2 = 0$, then the roles of y and u in the sequence generators are reversed, and whatever conditions were put on N_1 must be assumed to hold for N_2.

Sampled Linear Filter Sequence Generator[9]

To motivate the specification of this sequence generator, we show how linear differential filters can be used to obtain linear algebraic equations in the coefficients of the differential equation of the unknown system.

Let $Q(p)$ be a polynomial in p of degree M or greater such that

a) the roots of $Q(p)$ lie in the open left half half complex plane, (10)

b) $Q(p)$ has no roots in common with $L(p)$ or $M(p)$.

Then

$$\frac{L(p)}{Q(p)} y = \frac{M(p)}{Q(p)} u \tag{11}$$

and

$$\sum_{i=0}^{M} a_i \frac{p^i}{Q(p)} y - \sum_{j=1}^{M} b_j \frac{p^j}{Q(p)} u = \frac{1}{Q(p)} u \tag{12}$$

Thus we see that linear filters with transfer functions $p^i/Q(p)$, $i = 0,\ldots,M$ may be used to obtain linear equations in the unknown coefficients.

Notation

$h_i \triangleq$ zero state impulse response of the system with transfer function $p^i/Q(p)$.

<u>Specification of the Sequence Generator</u>

The sequence generator SG is given by the following relations:

Let $\Delta T > 0$, i_0 is "large".

$$\alpha_{ij} = \begin{cases} \displaystyle\int_0^{\Delta T(i+i_0)} h_{j-1}(t-\tau)y(\tau)\,d\tau & j=1,\ldots,M+1 \\ \displaystyle -\int_0^{\Delta T(i+i_0)} h_{j-(M+1)}(t-\tau)u(\tau)\,d\tau & j=M+2,\ldots,2M+1 \end{cases} \tag{13}$$

$$\beta_i = \int_0^{\Delta T(i+i_0)} h_o(t-\tau)u(\tau)\,d\tau \tag{14}$$

$$s_{ij} = \begin{cases} \displaystyle\int_0^{\Delta T(i+i_0)} h_{j-1}(t-\tau)w(\tau)\,d\tau & j=1,\ldots,M+1 \\ \displaystyle -\int_0^{\Delta T(i+i_0)} h_{j-(M+1)}(t-\tau)v(\tau)\,d\tau & j=M+2,\ldots,2M+1 \end{cases} \tag{15}$$

$$d_i = \int_0^{\Delta T(i+i_0)} h_0(t-\tau)v(\tau)\,d\tau = \beta_i \tag{16}$$

since N_1 is assumed to be zero.

Verification of 1 through 5

We now verify that conditions 1 through 5 are satisfied by the above specification.

We first note that for the system described by the differential equation

$$y = \frac{p^j}{Q(p)} u \tag{17}$$

if the state at $t = 0$ is zero, then

$$\left(\frac{p^j}{Q(p)} u\right)(t) = \int_0^t h_j(t-\tau)u(\tau)d\tau \tag{18}$$

and if the state is not zero at $t = 0$, then (18) is approximately true for large t since $p^j/Q(p)$ is stable by (10a). Thus 1 is satisfied to an arbitrary degree of accuracy for large i_0. We may write $h_j(t)$ as

$$h_j(t) = C_1\delta(t) + h_j'(t) \tag{19}$$

where $C_1 = 0$ if $j \neq M + 1$, and $|h_j'(t)|$ is uniformly bounded for all t by (10a). Thus

$$\left(\frac{p^j}{Q(p)} u\right)(t) = C_1 u(t) + \int_0^t h_j'(t-\tau)u(\tau)d\tau \tag{20}$$

With this relation it is clear that 2 is satisfied. Also, from (19) and (10a) (stability of the unknown system) we see that having u uniformly bounded with probability one ensures that 3 is satisfied.

To examine 4 we invoke the following well-known theorem.

<u>Theorem</u> [10] (21)

Let x be a zero mean wide sense stationary random process; let h be absolutely integrable on the positive real line; then

$$z(\cdot,t) \triangleq \int_0^t h(t-\tau)x(\cdot,\tau)d\tau$$

is a zero mean wide sense stationary random process and

$$\Phi_{zz} = |H|^2\Phi_{xx}$$

where

$\Phi_{zz} \triangleq$ the Fourier transform of R_z

$\Phi_{xx} \triangleq$ the Fourier transform of R_{xx}

$H \triangleq$ the Fourier transform of h.

Thus we may guarantee that 4 is satisfied if, for example, Φ_{N_2}, the spectrum N_2, is a proper ratio of polynomials, in which case $E[(s_{ij}-\alpha_{ij})(s_{kj}-\alpha_{kj})]$ goes to zero as $\exp(-a|i-k|)$ for some $a > 0$.

5 is clearly satisfied.

Thus conditions 1 through 5 are all satisfied, so that the relations (13) through (16) define a sequence generator.

Realizability of Sampled Linear Filter Sequence Generator

The sampled linear filter sequence generator may be realized either as a passive network[11] (with an appropriate gain factor applied to the integrals) or by analog or digital computer.

In fact this sequence generator is essentially a collection of filters with transfer functions $p^k/Q(p)$, $k = 0,\dots,M$ where $Q(p)$ is any M degree polynomial in p such that its roots have negative real parts and do not coincide with any of the poles or zeros of the unknown system.

Testing Functions--Definition and Elementary Property[12]

In order to present the second sequence generator, we first define testing functions and give an elementary property that makes them useful in system identification.

Definition - M-differentiable testing function (21)

Let ϕ satisfy the following conditions:

1. let $T > 0$, then $\phi^{(k)}$, $k = 0,\dots,M$ exists in the ordinary sense on $[0,T]$;
2. $\phi^{(k)}(t) = 0$ for $t \leq 0$, $t \geq T$, $k = 0,\dots,M$.

Then $\phi \triangleq$ M-differentiable testing function on $[0,T]$.

Definition - Infinitely differentiable testing function (22)

Let ϕ be an M-differentiable testing function for all integers $M > 0$. Then

$\phi \triangleq$ infinitely differentiable testing function on $[0,T]$.

Notation

In the sequel, unless otherwise specified, ϕ will denote an M-differentiable testing function on [0,T].

An infinitely differentiable testing function is a testing function as defined in the theory of generalized functions. M-differentiable testing functions share some properties with these, as seen below.

Remark (23)

Let $\hat{\phi}(t) = \phi(T-t)$. Then $\hat{\phi}$ is an M-differentiable testing function on [0,T].

Remark (24)

The union of the set of all M-differentiable (or infinitely differentiable) testing functions with the zero function is a linear vector space.

Remark (25)

$\phi^{(k)}$ is an M-k differentiable testing function on [0,T].

Remark (26)

Let f be M times differentiable on [0,T]. Then $f\phi$ (i.e., $f(\cdot)\phi(\cdot)$) is an M-differentiable testing function on [0,T].

Property (27)

Let y be M times differentiable on [0,T]. Then for all $k \leq M$

$$\int_0^T \phi(t)y^{(k)}(t)dt = (-1)^k \int_0^T \phi^{(k)}(t)y(t)dt$$

<u>proof</u> (induction)

Let $k = 1$

$$\int_0^T \phi(t)y^{(1)}(t)dt = \phi(t)y(t)\Big|_0^T - \int_0^T \phi^{(1)}(t)y(t)dt$$

and $\phi(0) = \phi(T) = 0$. Thus the property holds for $k = 1$.

Inductive hypothesis: the property holds for $k = n$, $n < M$.

Then

$$\int_0^T \phi(t)y^{(n)}(t)dt = (-1)^n \int_0^T \phi^{(n)}(t)y(t)dt$$

and

$$\int_0^T \phi(t)y^{(n+1)}(t)dt = \phi(t)y^{(n)}(t)\Big|_0^T - \int_0^T \phi^{(1)}(t)y^{(n)}(t)dt$$

but by remark (25), $\phi^{(1)}(t)$ is an M-1 differentiable testing function, so by inductive hypothesis

$$-\int_0^T \phi^{(1)}(t)y^{(n)}(t)dt = -(-1)^n \int_0^T \phi^{(n+1)}(t)y(t)dt$$

<u>end proof</u>

<u>Corollary</u> (28)

Let $L(p)$ be a polynomial in p of degree $\leq M$. Then

$$\int_0^T \phi(t)L(p)y(t)dt = \int_0^T y(t)L(-p)\phi(t)dt$$

<u>Definition</u> (29)

A bounded function f is said to be finitely differentiable in the distribution sense if it is piecewise continuous and has piecewise continuous derivatives of all finite orders on every finite interval.

<u>Corollary</u> (30)

Let y be bounded and finitely differentiable in the distribution sense. Then Property (27) holds.

<u>Note</u>

When a delta function of any order appears in the integrand of an integral with upper and lower limits a and b respectively, we will interpret those limits as a^+ and b^-.

<u>proof</u>

We can write $y^{(k)}$ as

$$y^{(k)} = y_b^{(k)} + y_\delta^{(k)}$$

where $y_b^{(k)}$ is a bounded piecewise continuous function and $y_\delta^{(k)}$ is a sum of delta functions of order at most k-1. Then

$$y = y_b + y_\delta, y_b = y_b(0) + \int_0^t \cdots \int_0^{t_{k-1}} y_b^{(k)}(t_k)dt_k \ldots dt_1$$

$$y_\delta(t) = \int_0^T \cdots \int_0^{T_{k-1}} y_\delta^{(k)}(t_k)\,dt_k \ldots dt_1$$

Then

$$\int_0^T \phi(t) y^{(k)}(t)\,dt = \int_0^T \phi(t) y_b^{(k)}(t)\,dt + \int_0^T \phi(t) y_\delta^{(k)}(t)\,dt$$

By Property (27),

$$\int_0^T \phi(t) y_b^{(k)}(t)\,dt = (-1)^k \int_0^T \phi^{(k)}(t) y_b(t)\,dt$$

From the sifting theorem, we have

$$\int_0^T \phi(t)\delta^{(j)}(t-t_0)\,dt = (-1)^j \int_0^T \phi^{(j)}(t)\delta(t-t_0)\,dt$$

$$= (-1)^j \phi^{(j)}(t_0)$$

Suppose that

$$y_\delta^{(k)}(t) = \delta^{(j)}(t-t_0), \qquad j < k$$

then

$$y_\delta(t) = \left(\frac{(t-t_0)^{k-1-j} 1(t-t_0)}{(k-j-1)!}\right),$$

and by Property (27),

$$(-1)^k \int_0^T \phi^{(k)}(t)\ y_\delta(t)dt$$

$$= (-1)^j \int_0^T \phi^{(k-(k-1-j))}(t)\ 1(t-t_0)dt$$

$$= (-1)^j[\phi^{(j)}(t_0)-\phi^{(j)}(0)] = (-1)^j \phi^{(j)}(t_0)$$

Thus

$$\int_0^T \phi(t)\delta^{(j)}(t)dt = (-1)^k \int_0^T \phi^{(k)}(t)\delta^{(j-k)}(t)dt$$

which implies that

$$\int_0^T \phi(t)y_\delta^{(k)}(t)dt = (-1)^k \int_0^T \phi^{(k)}(t)y_\delta(t)dt$$

<u>end proof</u>

With these definitions and properties, we can now discuss the second sequence generator.

<u>Testing Function Sequence Generator</u>

To motivate the specification of this sequence generator, we first show how we can use testing functions to obtain a linear equation in the coefficients of the differential equation of the unknown system.

We repeat the system differential equations as given by (9).

$$\sum_{i=0}^{M} a_i p^i y(t) - \sum_{j=1}^{M} b_j p^j u(t) = u(t) \tag{31}$$

Multiplying (31) on both sides by $\phi(t-nT)$, where ϕ is an M-differentiable testing function on [0,T], and integrating from nT to (n+1)T, we have

$$\sum_{i=0}^{M} a_i \int_{nT}^{(n+1)T} \phi(t-nT) p^i y(t)dt$$

$$- \sum_{j=1}^{M} b_j \int_{nT}^{(n+1)T} \phi(t-nT) u(t)dt \tag{32}$$

$$= \int_{nT}^{(n+1)T} \phi(t-nT) u(t)dt$$

which by Property (27) becomes

$$\sum_{i=0}^{M} a_i (-1)^i \int_{nT}^{(n+1)T} \phi^{(i)}(t-nT) y(t)dt$$

$$- \sum_{j=1}^{M} b_j (-1)^j \int_{nT}^{(n+1)T} \phi^{(j)}(t-nT) u(t)dt = \int_{nT}^{(n+1)T} \phi(t-nT) u(t) \tag{33}$$

Thus, with knowledge of the testing function and its derivatives, u, and y, we can obtain a linear algebraic equation in the unknown coefficients.

<u>Specification of the Sequence Generator</u>

With this introduction we specify the sequence generator as follows:

$$\left.\begin{aligned}
\alpha_{ij} &= \begin{cases} (-1)^{j-1} \displaystyle\int_{iT}^{(i+1)T} \phi^{(j-1)}(t-iT)y(t)\,dt & j = 1,\ldots,M+1 \\[2ex] (-1)^{j-M} \displaystyle\int_{iT}^{(i+1)T} \phi^{(j-(M+1))}(t-iT)u(t)\,dt & j = M+2,\ldots,2M+1 \end{cases} \\
\beta_i &= \int_{iT}^{(i+1)T} \phi(t-iT)u(t)\,dt
\end{aligned}\right\} \quad (34)$$

$$\left.\begin{aligned}
s_{ij} &= \begin{cases} (-1)^{j-1} \displaystyle\int_{iT}^{(i+1)T} \phi^{(j-1)}(t-iT)w(t)\,dt & j = 1,\ldots,M+1 \\[2ex] (-1)^{j-M} \displaystyle\int_{iT}^{(i+1)T} \phi^{(j-(M+1))}(t-iT)\,v(t)\,dt & j = M+2,\ldots,2M+1 \end{cases} \\
d_i &= \int_{iT}^{(i+1)T} \phi(t-iT)v(t)\,dt = \beta_i \text{ since } N_1 \text{ is assumed to be zero}
\end{aligned}\right\} \quad (35)$$

Verification of 1 through 5

We now need to verify that conditions 1 through 5 are satisfied by the hypothesized sequence generator.

From (33), (34), and (35), 1 are and 2 are clearly satisfied. From (35) it is clear that 3 is satisfied since $|u|$ is bounded and the system is b.i.b.o. stable.

5 is clearly satisfied.

To verify that 4 is satisfied, we put the following restriction on the noise process N_2. Let

$$\lim_{\tau\to\infty} \tau R_{N_2}(\tau) = 0 \qquad (36)$$

Then, the following theorem guarantees that 4 is satisfied by the testing function sequence generator.

Theorem (37)

Let g be a continuous function.

Let x be a wide sense stationary second order random process such that

1. $E(x(\cdot,t)) = 0$,
2. $x(\cdot,t)$ is bounded with probability one uniformly for all t.

Let y be the random process defined by

$$y(\cdot,t_0) \triangleq \int_{t_0}^{t_0+\Delta} g(t-t_0)x(\cdot,t)dt, \qquad \Delta > 0$$

Let

$$\lim_{\tau\to\infty} \tau R_x(\tau) = 0.$$

Then

1. $E(y(\cdot,t_0)) = 0$ for all t_0,

2. y is wide sense stationary,

3. $\lim_{\tau\to\infty} \tau R_y(\tau) = 0$

<u>proof</u>

1. Is clearly true.

2.

$$I(\delta) \triangleq [y(t_0)y(t_0+\delta)]$$

$$= \int_{t_0}^{t_0+\Delta} \int_{t_0+\delta}^{t_0+\Delta+\delta} \phi(t-t_0)\phi(s-t_0+\delta)R_x(t-s)\,dt\,ds$$

let $z = t - t_0$ $\quad w = s - t_0 + \delta$

Then

$$I(\delta) = \int_0^{\Delta}\int_{\delta}^{\delta+\Delta} \phi(z)\phi(w-z)R_x(w-z)\,dz\,dw$$

is clearly a function of δ only for Δ fixed, which together with (1) implies that y is wide sense stationary.

3. Let

$$M_g = \max_{t \varepsilon [t_0, t_0+\Delta]} |g(t)|$$

let $\delta > 0$, since $R_y(-\tau) = R_y(\tau)$ for all τ.

Note that $I = R_y(\delta)$, so that

$$\lim_{\delta \to \infty} |\delta R_y(\delta)| = \lim_{\delta \to \infty} |\delta I(\delta)|$$

$$\leq M_g^2 \lim_{\delta \to \infty} \left| \int_0^{\Delta} \int_{\delta}^{\Delta+\delta} R_x(w-z)\, dw\, dz \right|$$

$$\leq M_g^2 \, \Delta^2 \lim_{\delta \to \infty} \max_{\tau \varepsilon [\delta-\Delta, \delta+\Delta]} |\delta R_x(\tau)|$$

$$\leq M_g^2 \Delta^2 \lim_{\tau \to \infty} [\, |\tau R_x(\tau)| + |K(\tau) R_x(\tau)| \,]$$

where $|K(\tau)| \leq \Delta$

$$= 0$$

<u>end proof</u>

Realizability of Testing Function Sequence Generator

Given a testing function and its derivatives, the testing function sequence generator may be realized by a digital computer. If the testing function is a polynomial, then the testing function sequence generator may be realized by an analog computer.

Comparison of Sequence Generators

We have seen that both of the sequence generators discussed above have essentially the same mathematical representation, namely that of a set of integral operators (linear filters). The testing function sequence generator has several advantages over the sampled linear filter sequence generator. First, the polynomial Q(p) need not be used in the former, thus eliminating the undesirable possibility of cancellation or near-cancellation by Q(p) of roots of L(p) and M(p). Second, if the sequence generator is implemented on a digital computer, the testing function sequence generator requires only repeated integrations over an interval of fixed length, while the sampled linear filter generator requires either integrations of the form

$$e^{a(i+1)T}\int_{iT}^{(i+1)T} e^{-a\tau}u(\tau)\,d\tau \qquad a > 0$$

or of the form

$$\int_0^{iT} e^{a(t-\tau)} u(\tau) d\tau$$

or the solution of a linear differential equation by some integration scheme such as Runge-Kutta or Adams method. In the first case we have to multiply the integral by e^{at}, a numerical operation which becomes rapidly unstable as t grows large. In the second case the integration time for each integral is proportional to the sample number i, so the total integration time grows as i^2. In the third case we have again a numerical operation which is relatively unstable compared to the integration used in the testing function sequence generator.

Comment

There are some extremely accurate numerical methods for solving differential equations. However, these methods use a variable integration step size in order to maintain consistently high accuracy and can at times be quite slow. As we shall see, we may actually "tailor" the testing function that we use in the testing function sequence generator so that we may use a reasonably large uniform integration step size in the testing function sequence generator and still maintain a high degree of accuracy.

In other words, the sampled linear filter sequence generator either takes more time or is less stable numerically than the testing function when digital computer implementation is used.

1.7 Comparison of Algorithms

The important points of comparison among the algorithms presented above are: method of implementation, computational complexity, and amount of information needed to implement the algorithm.

If analog implementation is used, the error decomposition method is the least computationally complex of the above algorithms. The instrumental variable algorithm must be at least partially implemented by digital computer. If the error decomposition method is implemented by digital computer, then the instrumental variable method using a testing function sequence generator is the least computationally complex and the most numerically stable of the above methods.

The error decomposition method can be applied when both input and output observations are corrupted by observation noise while implementation of the instrumental variable method depends upon the availability of noise free observations of either the input or output signals. However, knowledge of the autocorrelations of the noise processes is necessary in order to implement the error decomposition method. Further, the instrumental variable method can be modified so that it is applicable when both input and output observations are corrupted by noise requiring the added knowledge of the autocorrelations of the noise processes. Smith and Ornstein[13] have given an error decomposition algorithm (upon which that in

Section 1.5 is based) for the case when the input or output can be observed in the absence of noise and for which no knowledge of second order noise statistics is required. Although convergence for this method has not been proven, it has been found to work very well in practice.

The instrumental variable method has a similar shortcoming in that the practical procedure for estimation of an instrumental variable matrix given in Appendix I (and its more sophisticated version given by Wong and Polak[14]) has not been proven to converge, but works very well in practice.

The essential problem with these methods is that bias in the estimates or error functions cannot be removed (except perhaps asymptotically) without some knowledge of the second order statistics of the observation noise processes.

CHAPTER 2

APPLICATIONS TO NONLINEAR SYSTEMS

2.0 Introduction

In this chapter we discuss the types of systems to which the instrumental variable identification algorithm (See Section 1.6) is applicable. In other words, we give some systems for which we can find realizable sequence generators. Recall that the object of the instrumental method is to obtain many linear algebraic equations in the system parameters, the coefficients in these equations being obtained from the sequence generator, and then to add these equations together to "average out the noise". We discuss only sequence generators which use testing functions. However, for most of the systems discussed here, we can also find a sampled linear filter type of sequence generator. Further, some of these systems have discrete time counterparts for which a tapped delay line (multiple time sampler) is a sequence generator.

In devising sequence generators for linear systems, we used some information about the noise processes; namely the mean of the noise process, and the rate of convergence to zero of the autocorrelations function of the noise process. To devise sequence generators for nonlinear sytems, we must use additional information about the noise and signal processes. The amount and nature of such information depends upon the class of systems to which the unknown system is assumed to belong.

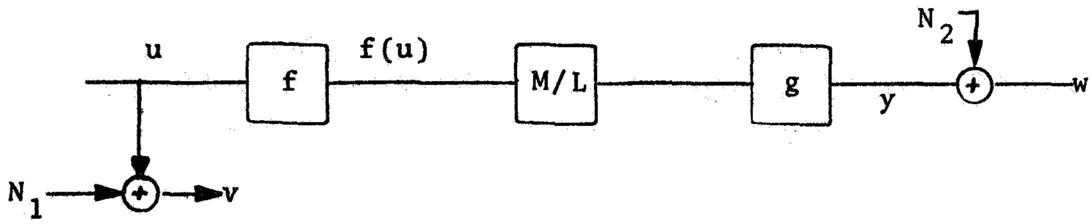
In developing sequence generators for nonlinear systems, we will use the following three step procedure:

1. Choose a representation of the class of systems to which the unknown system belongs such that the system differential equation is linear in a set of constants that completely specifies the system.
2. Specify a hypothetical sequenee generator for the unknown system.
3. Find sufficient conditions on any of all of the input, output, noise processes, and the system representation, such that the hypothesized sequence generator satisfies the conditions 1 through 5 (Section 1.6).

We discuss two classes of nonlinear systems and give a number of examples for which the knowledge of the noise and signal statistics necessary to implement the identification algorithm is minimal and for which the conditions 3. are reasonably weak and realistic.

2.1 Observation Configuration

The observation configuration is assumed to be the same as that for the linear system, namely that shown in Figure 1. As before, the noise processes are assumed to be second order wide sense stationary random processes. However, we do not



NONLINEAR SYSTEM I

Figure 4

assume that the means of the noise processes are known.

We have said (Section 1.6, (8)) that either N_1 or N_2 must be zero. In this chapter, we usually assume that N_1 is zero. If N_2 is zero and N_1 is not zero, then N_1 must satisfy whatever conditions that we may have put on N_2. Further, d_i in the sequence generator (9) and (10) must then be an integral involving the output y, thus we must interchange d_i with one of the s_{ij} which depends on y. The required modification will become clear in the course of the following discussion.

2.2 Nonlinear Systems I

In this section we discuss the class of nonlinear systems with the representation shown in Figure 4, i.e., a memoryless nonlinearity, followed by a linear differential plant (stable b.i.b.o.), followed by known invertible memoryless nonlinearity. Formally,

u is the system input function

y is the system output function

u is assumed to be an M times quadratic mean differentiable sample function of a second order wide sense stationary random process bounded with probability one.

$$x = f(u) \tag{1}$$

$$y = g(z) \tag{2}$$

$$L(p)z = M(p)x \tag{3}$$

where

$$L(p) = \sum_{i=0}^{M} a_i p^i \tag{4}$$

$$M(p) = \sum_{j=0}^{M} b_j p^j \tag{5}$$

and $M(p)/L(p)$ is proper and the roots of $L(p)$ all lie in the open left half complex plane.

f is finitely differentiable in the distribtuion sense (See Definition (24), Section 1.6) (6)

g is known and invertible, and g^{-1} is finitely differentiable in the distribution sense. (7)

Let $\gamma \triangleq g^{-1}$. Then we have the following system differential equation:

$$L(p)\gamma(y) = M(p)f(u) \tag{8}$$

We assume that the autocorrelations of the noise processes satisfy

$$\lim_{\tau\to\infty} \tau R_{N_1}(\tau) = 0, \qquad \lim_{\tau\to\infty} \tau R_{N_2}(\tau) = 0 \tag{9}$$

<u>System Differential Equation Expressed as a Linear Combination of Constant Coefficients</u>

In order to put the system differential equation into the desired form, we make the following assumption about the form of f;

$$f(u) = \sum_{k=0}^{K} d_k \psi_k(u) \tag{10}$$

where $\{\psi_k\}_0^K$ must satisfy the same conditions as f and must be a linearly independent set on R. Substituting this relation into the system differential equation (3), we have

$$L(p)\gamma(y) = M(p) \sum_{k=0}^{K} d_k \psi_k(u) \tag{11}$$

or

$$\sum_{i=0}^{M} a_i p^i \gamma(y) = \sum_{j=0}^{M} \sum_{k=0}^{K} b_j d_k p^j \psi_k(u) \tag{12}$$

Comment

With this representation of f, the system is completely specified by the sets of coefficients

$$\{a_i\}_0^M \ , \ \{b_j\}_0^M \ , \ \{d_k\}_0^M \tag{13}$$

This equation is clearly not linear in the system coefficients. However it is linear in the pairwise products coefficients

$$\left.\begin{aligned} & b_j d_k, \quad j = 0,\ldots,M; \quad k = 0,\ldots,K \\ & \text{and the coefficients} \\ & a_i,\ i = 0,\ldots,M \end{aligned}\right\} \tag{14}$$

Motivation for Specification of Sequence Generator

Assume that a_0 and b_0 are not zero. This it is clear that without loss of generality we may arbitrarily fix a_0 and b_0. We set

$$a_0 = 1, \quad b_0 = 1 \tag{15}$$

Now suppose we could find a sequence generator and instrumental variable sequence for the unknown system such that we could estimate the constants (14). Then knowledge of b_0 is sufficient to find $b_1,\ldots,b_m$ and $d_0,\ldots,d_K$ from the pairwise products $b_j d_k$ (if at least one of d_k, $k = 0,\ldots,K$ is not zero, i.e., if f is not identically zero).

Let us multiply both sides of (12) by $\phi(t-iT)$, where ϕ is an M-differentiable testing function on $[0,T]$, and integrate the resulting equation from iT to (i+1)T. Then we get an algebraic equation which is linear in the constants (14), namely,

$$\begin{aligned} & \sum_{m=0}^{M} a_m \int_{iT}^{(i+1)T} \phi(t-iT)p^m \gamma(y(t)) \\ & \quad = \sum_{j=0}^{M} \sum_{k=0}^{K} b_j d_k \int_{iT}^{(i+1)T} \phi(t-iT)p^j \psi_k(u(t))dt \end{aligned} \tag{16}$$

where

$$p^j\psi_k(u) = \psi_k^{(j)}(u)u^{(j)} \qquad j = 0,\ldots,M; \quad k = 0,\ldots,K \tag{17}$$

are well defined in the distribution sense since u is bounded and M-times differentiable. Thus, Corollary (30) (Section 1.6) is applicable and we have

$$\sum_{m=1}^{M} a_m(-1)^m \int_{iT}^{(i+1)T} \phi^{(m)}(t-iT)\gamma(y(t))dt$$

$$-\sum_{j=o}^{M}\sum_{k=0}^{K} b_j d_k(-1)^j \int_{iT}^{(i+1)T} \phi^{(j)}(t-iT)\psi_k(u(t))dt \tag{18}$$

$$= -\int_{iT}^{(i+1)T} \phi(t-iT)\gamma(y(t))dt$$

Clearly the integrals in (18) are computable from knowledge of the testing function and its derivatives and from (observations of) the input and output signals. This is exactly the procedure that we followed in the case of a linear differential equation. The only difference here is that we are dealing with known functions of the input and output rather than with the input and output themselves.

Comment

In the absence of noise, we can readily identify any system of the form described above. We need only observe u and y on

M + (M+1)(K+1) different time intervals, apply (18) and solve the resulting linear equations for

$$\{a_i\}_0^M \; , \; \{b_j d_k\}_{j=0}^M , \quad k = 0,\ldots,K$$

then using the fact that $b_0 = 1$, find

$$\{b_j\}_1^M \; , \; \{d_k\}_1^K \qquad \text{(See Section 4.4)}$$

Comment

For the rest of this section we shall assume that

$$f(u) = u$$

Since the generalization to the case where f is a nonlinear function is clear from the above discussion, and requires only an obvious modification of the sequence generator (See Figure 5).

Specification of the Sequence Generator

With this introduction, we specify the sequence generator as follows:

$$\alpha_{ij} = \begin{cases} (-1)^j \displaystyle\int_{iT}^{(i+1)T} \phi^{(j)}(t-iT)\gamma(y(t))dt, & j = 0,\ldots,M \\[2ex] (-1)^{\ell+1} \displaystyle\int_{iT}^{(i+1)T} \phi^{(\ell)}(t-iT)u(t)dt, & \ell = 1,\ldots,M \\ & \text{where } j = M + \ell + 1. \end{cases}$$

$$\beta_i = -\int_{iT}^{(i+1)T} \phi(t-iT)u(t)dt$$

$$s_{ij} = \begin{cases} (-1)^j \displaystyle\int_{iT}^{(i+1)T} \phi^{(j)}(t-iT)[\gamma(w(t))+E(\gamma(y(t))-\gamma(w(t)))]dt & j = 0,\ldots,M \\ (-1)^{\ell+1} \displaystyle\int_{iT}^{(i+1)T} \phi^{(\ell)}(t-iT)u(t)dt & \ell = 1,\ldots,M \\ & \text{where } j = M+\ell+1 \end{cases}$$

$$d_i = -\int_{iT}^{(i+1)T} \phi(t-iT)u(t)dt = \beta_i \quad \text{(since } N_1 \text{ is assumed to be zero).}$$

where the expectation is to be taken over the sample space of the noise processes.

<u>Verification of Conditions 1 through 5 (Section 1.6)</u>

We will show that we can verify that all conditions except 4 and 5 are always satisfied by the hypothesized sequence generator. Since these two conditions must be satisfied in order to implement the sequence generator, we therefore cannot identify every system of the form given above. However, there are a number of cases, which we discuss below, in which 4 and 5 may be verified. In particular we show (Theorems (23), (24) and Corollaries (25), (26), Section 1.6) that for a large class of observation noise processes 4 is satisfied.

<u>Verification of Conditions 1 through 3 (Section 1.6)</u>

We have shown that condition 1 holds in (18).

Condition 2 holds, since $N_1 = 0$ implies

$$E[v(t)] = u(t); \tag{21}$$

also

$$E\{\gamma(w(t)) + [\gamma(y(t))-\gamma(w(t))]\} = \gamma(y(t)) \tag{22}$$

Condition 3 holds because f and g^{-1} are finite for finite arguments, and u and y are bounded.

Having demonstrated that these conditions are satisfied, in each of the examples given below we need only show that conditions 4 and 5 are satisfied.

We now give two theorems which will enable us to verify that condition 4 is satisfied in each example below for certain types of noise.

The following theorem gives a bound on the autocorrelation function of a nonlinear time-varying zero mean function $g(\cdot,t)$ of a gaussian process x. Note that this g is not the system nonlinearity.

<u>Theorem</u> (23)

Let x be a zero mean wide sense stationary gaussian random process.

Let $\rho(\tau) \triangleq R_x(\tau)/R_x(0)$.

Let $\lim_{\tau\to\infty} \tau\rho(\tau) = 0$.

Let $\hat{g}:R^2\to R$ be a time-varying mapping with $g(t) \triangleq \hat{g}(x(\cdot,t),t)$, such that $E(g(t)) = 0$.

Let there exist $\varepsilon > 0$ such that

$$|E[g(t)g(t+\tau)]|$$

$$|E[g(t)x(\cdot,t)g(t+\tau)x(\cdot,t+\tau)]|$$

$$|E[g(t)x^2(\cdot,t)g(t+\tau)]|$$

exist and are bounded by $K_1 > 0$ for all t and for all τ such that $|\rho(\tau)| < \varepsilon$. Then

$$\lim_{\tau\to\infty} \tau R_g(\tau) = 0.$$

Comment

As we shall see in the next theorem and in Section 2.28 and 2.29, the function g in which we are interested is

$$g(N_2(\cdot,t),t) \triangleq \gamma(y(t)+N_2(\cdot,t)) - E[\gamma(y(t)+N_2(\cdot,t))].$$

The time dependence clearly arises from y(t). We must verify that Condition 4 is satisfied when this function appears in the sequence generator.

Notation

We will use x(t) as shorthand notation for $x(\cdot,t)$.

Proof

Let

$$\sigma^2 \triangleq E(x^2)$$

$$\rho(\tau) \triangleq E\left[\frac{x(t)x(t+\tau)}{\sigma^2}\right]$$

$p(x(t),x(t+\tau)) \triangleq$ joint probability density function

x(t) and $x(t+\tau)$,

$$p(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[\frac{-(x^2-2\rho xy+y^2)}{2\sigma^2(1-\rho^2)}\right]$$

and we have

$$\rho_g(t,\tau) \triangleq E[g(t)g(t+\tau)]$$

$$\rho_g(t,\tau)\Big|_{\rho=0} = E[g(t)]\ E[g(t+\tau)] = 0$$

Now

$$\left|\frac{\partial}{\partial\rho}\ \rho_g(t,\tau)\right|$$

$$= \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t)g(t+\tau)p(x(t),x(t+\tau)) \left[\frac{1}{(1-\rho^2)} + \frac{1}{(1-\rho^2)\sigma^2}\cdot\left[\frac{-\rho(x(t)^2+x(t+\tau)^2-2\rho x(t)x(t+\tau))}{(1-\rho^2)} + x(t)x(t+\tau)\right]\right] dx(t)dx(t+\tau)\right|$$

$$\leq \left|2\rho'(\tau) + \frac{-2\rho(\tau)+2\rho^2(\tau)+1}{\sigma^2}\right| \frac{k_1}{(1-\rho^2)} \quad \text{for } |\rho(\tau)| < \varepsilon$$

Let

$$\max_{|\hat{\varepsilon}|\leq\varepsilon} \left|2\hat{\varepsilon} + \frac{-2\hat{\varepsilon}+2\hat{\varepsilon}^2+1}{\sigma^2}\right| \frac{k_1}{(1-\hat{\varepsilon}^2)} = k_2$$

Then

$$\left|\frac{\partial}{\partial\rho}\,\rho_g(t,\tau)\right| < k_2$$

for $|\rho(\tau)|$ sufficiently small, i.e., for $|\tau|$ sufficiently large. Now let τ_0 to sufficiently large that

$$|\rho(\tau)| < \varepsilon, \quad |\tau| > \tau_0$$

Then for all $\tau \ni |\tau| > \tau_0$ and for all $t \in R$

$$|\rho_g(t,\tau)| \leq \int_0^{|\rho(\tau)|} \left|\frac{\partial}{\partial\rho(\tau)}\,\rho_g(t,\tau)\right|_{\rho(\tau)=\xi} d\xi + \left|\rho_g(t,\tau)\right|_{\rho(\tau)=0}$$

$$\leq k_2|\rho(\tau)|$$

Thus

$$\lim_{\tau\to\infty} \tau\rho_g(t,\tau) = \lim_{\tau\to\infty} \tau\rho(\tau) = 0$$

<u>end proof</u>

<u>Comment</u>

We next present Theorem (24), Corollaries (25) and (26). In Theorem (24) we show that given a bounded wide sense stationary process z such that the probability density function of z(•,t) is continuous, there exists a wide sense stationary gaussian random process x and a function g such that the random process g(x) has the same autocorrelation function as z and such that $g(x(\cdot,t))$ and $z(\cdot,t)$ have the same probability density functions.

In Corollary (25), using Theorems (23) and (24), we show that when the observation noise is of the exact form $g(x)$, where x is a gaussian process as described above, then condition 4 (Section 1.6) is satisfied.

In Corollary (26) and the succeeding discussion, we add some weak hypotheses ((*) and (**)) concerning the asymptotic distribution of the noise process, to the hypotheses on the process z in Theorem (24) and show that if this new set of hypotheses is satisfied by the observation noise then condition 4 is satisfied. A summary of these results follows the presentation.

Theorem (24)

Approximate Modelling of Bounded Noise Processes

Let x be a wide sense stationary gaussian random process. Let F be the probability distribution function of $x(\cdot,t)$. Let

$$E(x(\cdot,t)) = 0$$

$$E(x(\cdot,t)^2) = \sigma^2$$

$$\rho(\cdot) = R_x(\tau)/\sigma^2$$

Let z be a wide sense stationary random process such that $E(x(\cdot,t)) = 0$ and $M_2 \leq z \leq M_1$, $M_2 < M_1$.

Let H be the probability distribution function of $z(\cdot,t)$. Let $H^{(1)}$ exist and $0 < |H^{(1)}(\xi)| < \infty$, $M_2 < \xi < M_1$.

Then there exists a continuous monotone increasing mapping g such that

1. $G(\alpha) \triangleq P(g(x(\cdot,t))\leq\alpha)$, $G(\alpha) = H(\alpha)$ for all α in $[M_2,M_1]$.

2. If

   i. $g(-\xi) = g(\xi)$ or

   ii. $R_z(\tau) \geq E[g(-x(\cdot,t))g(x(\cdot,t)]$

   Then

   a. We can choose R_x such that for

   $$R_g(\tau) \triangleq E[g(x(\cdot,t))g(x(\cdot,t+\tau))]$$

   $$R_g(\tau) = R_z(\tau) \text{ for all } \tau$$

   b. $\lim_{\tau\to\infty} \tau R_g(\tau) = 0 \Rightarrow \lim_{\tau\to\infty} \tau R_x(\tau) = 0$

   for such a choice of R_x

3. For τ_0 sufficiently large, there exists a choice of R_x such that 2a and 2 b hold for all τ such that $|\tau| > \tau_0$.

<u>Proof</u>

<u>Notation</u>

We will use x as shorthand notation for $x(\cdot,t)$

1. Let $g(\cdot) = H^{-1}(F(\cdot))$

   (F is invertible since it is the (bounded) integral of a strictly positive function)

Then

$$G(\alpha) = P(g(x)\leq\alpha) = P(H^{-1}(F(x))\leq\alpha)$$
$$= P(x\leq F^{-1}(H(\alpha)))$$
$$= F(F^{-1}(H(\alpha))) = H(\alpha)$$

Thus g is in fact the mapping that satisfies (1).

<u>Lemma</u>

Let $p(\cdot,\cdot)$ be the density function of the jointly gaussian random variables x_1 and x_2.

$$p(x_1,x_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}\, e^{-\frac{x_1{}^2-2\rho x_1x_2+x_2{}^2}{2\sigma^2(1-\rho^2)}}$$

where

$$E(x_1{}^2) = E(x_2{}^2)$$
$$\sigma^2 \triangleq E(x_2{}^2)$$
$$\rho \triangleq E\left[\frac{x_1x_2}{\sigma^2}\right]$$

(We note that $E[g(x_1)g(x_2)]$ exists $\forall\, \rho\varepsilon[-1,1]$, since g is bounded.) Then

i. Sign $(p(x_1,x^2) - p(-x_1,x_2)) = \text{sign } \rho\, \forall\, x_1,x_2 > 0$

ii. If g is odd,

$$E[g(x_1)g(x_2)] = 2\int_0^\infty\int_0^\infty g(x_1)g(x_2)\,[p(x_1,x_2)-p(-x_1,x_2)]\; dx_1dx_2$$

iii. $E(g(x_1)g(x_2))$ is a continuous function ρ.

iv. Let sign $[g(x)-g(0)]$ = (sign x or 0), then

$$E\left[g(-x)g(x)\right] < 0.$$

v. Let $E(xg(x)) \neq 0$ then $E[g(x_1)g(x_2)]$ takes on all values ξ such that

$$E[g(-x)g(x)] \leq \xi \leq E[g^2(x)]$$

as ρ varies over $[-1,1]$, and there exists a function f such that

$$f[E(g(x_1)g(x_2))] = \sigma^2\rho \qquad \forall\, \rho\varepsilon[-1,1]$$

(Note that $E(x_1x_2) = \sigma^2\rho$).

and such that for ε sufficiently small, $\exists k_1, k_2 > 0 \ni$

$$\sigma^2 k_1|\rho| \leq |E[g(x_1)g(x_2)]| < k_2\sigma^2|\rho| \qquad \forall\, |\rho| < \varepsilon$$

Proof of Lemma

i. $p(x_1,x_2) - p(-x_1,x_2)$

$$= \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}\, e^{\frac{-(x_1{}^2+x_2{}^2)}{2\sigma^2(1-\rho^2)}} \left[e^{\frac{\rho x_1x_2}{\sigma^2(1-\rho^2)}} - e^{\frac{-\rho x_1x_2}{\sigma^2(1-\rho^2)}}\right]$$

for $x_1 > 0$ and $x_2 > 0$.

If $\rho \geq 0, \ \dfrac{\rho x_1 x_2}{\sigma^2(1-\rho^2)} \geq \dfrac{-\rho x_1 x_2}{\sigma^2(1-\rho^2)}$

and $p(x_1,x_2) - p(-x_1,x_2) \geq 0$

If $\rho \geq 0, \ \dfrac{-\rho x_1 x_2}{\sigma^2(1-\rho^2)} - \dfrac{\rho x_1 x_2}{\sigma^2(1-\rho^2)}$

and $p(x_1,x_2) - p(-x_1,x_2) \leq 0$.

ii.

$$E[g(x_1)g(x_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)g(x_2)p(x_1,x_2)dx_1dx_2$$

$$= 2\int_{0}^{\infty}\int_{0}^{\infty} g(x_1)g(x_2)p(x_1,x_2)dx_1dx_2$$

$$+ 2\int_{-\infty}^{0}\int_{0}^{\infty} g(x_1)g(x_2)p(x_1,x_2)dx_1dx_2$$

since g is odd and $p(-x_1,-x_2) = p(x_1,x_2)$. Finally,

$$2\int_{-\infty}^{0}\int_{0}^{\infty} g(x_1)g(x_2)dx_1dx_2 = -2\int_{0}^{\infty}\int_{0}^{\infty} g(x_1)g(x_2)p(-x_1,x_2)dx_1dx_2$$

iii. let x_1, x_2 be jointly gaussian, let

$$z_1 = g(x_1)$$
$$z_2 = g(x_2)$$

where

$$g(\cdot) = H^{-1}(F(\cdot))$$

Let

$$\mathcal{H}(\alpha,\beta) \triangleq P(z_1 \leq \alpha, z_2 \leq \beta) = P(x_1 \leq g^{-1}(\alpha), x_2 \leq g^{-1}(\beta))$$

$$= \int_{-\infty}^{g^{-1}(\alpha)} \frac{e^{-(x_1^2/2\sigma^2)}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{g^{-1}(\beta)} \frac{e^{-\frac{(x_2-\rho x_1)^2}{2\sigma^2(1-\rho^2)}}}{\sqrt{2\pi\sigma^2}} \, dx_1 dx_2$$

$$= \int_{-\infty}^{g^{-1}(\alpha)} \frac{e^{-(x_1^2/2\sigma^2)}}{\sqrt{2\pi\sigma^2}} \sqrt{1-\rho^2} \int_{-\infty}^{\frac{g^{-1}(\beta)-\rho x_1}{1-\rho^2}} \frac{e^{-x_2^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \, dx_2 dx$$

$$= \int_{-\infty}^{g^{-1}(\alpha)} \frac{e^{-x_1^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \sqrt{1-\rho^2} \, F\left[\frac{g^{-1}(\beta)-\rho x_1}{1-\rho^2}\right] dx_1 \quad (*)$$

where F is the distribution function of a zero mean gaussian random variable with variance σ^2. Now

$$E(z_1 z_2) = \int_{M_2}^{M_1} z_2 E(z_1/z_2)\,dz_2$$

where

$$E(z_1/z_2) = \int_{M_2}^{M_1} z_2 \frac{\partial \mathcal{H}}{\partial z_1}(z_1, z_2)\,dz_1$$

Then it is sufficient to show that

$$\frac{\partial \mathcal{H}}{\partial z_1}(z_1, z_2)$$

is continuous in $\rho\ \forall\, z_1, z_2\ \varepsilon\ [\,M_2, M_1]$, for then we have

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \ni \forall\, |\rho_1 - \rho_2| < \delta$$

$$\left| E(z_1/z_2)_{\rho=\rho_0} - E(z_1/z_2)_{\rho=\rho_2} \right| < E$$

and

$$\left| \int_{M_2}^{M_1} z_2 E(z_1/z_2)_{\rho=\rho_1} dz_2 - \int_{M_2}^{M_1} z_2 E(z_1/z_2)_{\rho=\rho_2} dz_2 \right|$$

$$< 2[\,|M_1| + |M_2|\,]\varepsilon$$

i.e., $E(z_1 z_2)$ is a continuous function of ρ . Thus we show that $E(z_1/z_2)$ is a continuous function of $\rho, z_1/z_2$ $\forall\ z_1 z_2 \epsilon [M_2, M_1]$ $\forall \rho \epsilon [-1,1]$. From (*) we have

$$\frac{\partial \mathcal{H}}{\partial \alpha}(\alpha,\beta) = \frac{e^{-(g^{-1}(\alpha))^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \left[\sqrt{1-\rho^2}\ F\left[\frac{g^{-1}(\beta)-\rho g^{-1}(\alpha))}{1-\rho^2}\right] g^{-1^{(1)}}(\alpha)\right]$$

which is continuous in ρ, α, β $\forall \alpha, \beta \epsilon [M_2, M1]$ $\forall \rho \epsilon [-1,1]$ if

a) $g^{-1^{(1)}}(\alpha)\ \dfrac{e^{-(g^{-1}(\alpha))^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$

is bounded for all α, and

b) $\left(1-\rho^2\right)\ F\left[\dfrac{g^{-1}(\beta)-\rho g^{-1}(\alpha)}{1-\rho^2}\right]$

is continuous in ρ, α and β

$\forall \rho \epsilon [-1,1]$ $\forall \alpha, \beta \epsilon [M_2, M_1]$

To prove a), we first compute $g^{-1^{(1)}}(\alpha)$

$$g(x) = H^{-1}(F(x))$$

then

$$g^{-1}(x) = F^{-1}(H(x))$$

and

$$d/dx\ g^{-1}(x) = g^{-1^{(1)}}(x) = F^{-1^{(1)}}(H(x))H^{(1)}(x)$$

Now $H(x)$ and $H^{(1)}(x)$ are found by hypothesis and we may find ${F^{-1}}^{(1)}(x)$ by differentiating

$$F(F^{-1}(x)) = x$$

then

$$F^{(1)}(F^{-1}(x)) \cdot {F^{-1}}^{(1)}(x) = 1$$

and

$${F^{-1}}^{(1)}(x) = \frac{1}{F^{(1)}(F^{-1}(x))}$$

so that

$${g^{-1}}^{(1)}(\alpha)e^{-(g^{-1}(\alpha)^{2}/2 \;\; 2} = \frac{1}{F^{(1)}(F^{-1}(H(x))} \quad F^{(1)}(g(x))$$

$$= \frac{F^{(1)}(g(x))}{F^{(1)}(g(x))} = 1$$

which is clearly bounded for all x. Thus a) is true.

To prove b), we note that F and g^{-1} are continuous functions, thus b) is certainly satisfied for $\alpha,\beta \varepsilon [M_2, M_1]$, $\rho \varepsilon (-1,1)$. Further

$$\lim_{\rho \to -1} \sqrt{1-\rho^2}\; F\left(\frac{g^{-1}(\beta) - g^{-1}(\alpha)}{(1-\rho^2)}\right)$$

$$= \lim_{\rho \to 1} \sqrt{1-\rho^2}\; F\left(\frac{g^{-1}(\beta) - g^{-1}(\alpha)}{1-\rho^2}\right) = 0 \quad \forall\, \alpha,\beta \varepsilon [M_2, M_1]$$

thus $\frac{\partial \mathcal{H}}{\partial \alpha}(\alpha,\beta)$ is continuous at $\rho = \pm 1$ $\forall \alpha\ \beta \epsilon [M_2, M_1]$ and therefore is continuous in α, β, ρ $\forall \alpha, \beta \epsilon [M_2, M_1]$ $\forall \rho \epsilon [-1, 1]$.

iv. $Eg(x) = 0$

Thus

$$E[g(-x)g(x)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(-x)g(x)p(x)dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(g(-x)-g(0))(g(x)-g(0))-g^2(0)]p(x)dx$$

Further, g is monotone increasing, so

$$(g(-x)-g(0))(g(x)-g(0)) \leq 0$$

and is zero only for $x = 0$, thus $E[(g(-x)g(x)] < 0$ since $p(x) > 0$ for all x.

v. The bounds are attained for $\rho = \pm 1$ (by definition of ρ) and from iii. we have $E[g(x_1)g(x_2)]$ is a continuous function of ρ. Now

$$\frac{\partial}{\partial \rho} E(g(x_1)g(x_2))\Big|_{\rho=0} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)g(x_2)p(x_1,x_2)x_1x_2dx_1dx_2$$

$$= (E[xg(x)])^2 \neq 0$$

by hypothesis. Thus $\exists\, \epsilon > 0$, ϵ sufficiently small, such that

$\forall |\rho| < \varepsilon, \quad \exists \hat{f}, \ \hat{f}$ is 1:1 such that

$$\hat{f}(E(g(x_1)g(x_2))) = E(x_1x_2).$$

Let

$$\hat{f}^{-1}(-\varepsilon) = \varepsilon_1, \quad \hat{f}^{-1}(\varepsilon) = \varepsilon_2$$

then let

$$f(\xi) = \begin{cases} \sigma^2 \max\limits_{\rho} \{\rho | E(g(x_1)g(x_2)) = \xi\} & \xi \leq \varepsilon_1, \quad \xi \geq \varepsilon_2 \\ \\ \hat{f}(\xi) \text{ otherwise} \end{cases}$$

Clearly f exists for all ξ such that

$$E[g(-x)g(x)] \leq \xi \leq E[g^2(x)]$$

and for

$$|Eg(x_1)g(x_2)| \leq \min(|\varepsilon_1|, |\varepsilon_2|), \quad \exists k_1, k_2 > 0$$

$$k_2|E(x_1x_2)| \leq |E[g(x_1)g(x_2)]| \leq k_1|E(x_1x_2)|$$

2a) from (v) of the lemma, we can choose R_x so that

$$R_g(\tau) = R_z(\tau)$$

if

$$E[g(-x)g(x)] \leq R_z(\tau) \leq E[g^2(x)] \tag{*}$$

(Note that $R_g(0) = E[g^2(x)] = E[z^2]$)

Now $R_z(\tau)$ can take on any value ξ such that

$$E[-z^2] \leq \xi \leq E[z^2]$$

We also have

$$\frac{dF(-x)}{dx} = -\frac{dF(x)}{dx}$$

Thus

$$E[g(-x)] = E[g^2(x)] \qquad (**)$$

In order that we may choose R_x so that $R_g(\tau) = R_z(\tau)$, it is sufficient that

1. $E[R_z(\tau)] \geq E[g(-x)g(x)]$

or

2. $E[g(-x)g(x)] \leq -E[g^2(x)]$

Note that 2. $\Rightarrow$ 1. but 1. $\not\Rightarrow$ 2.

We show that 2. holds iff $g(-x) = g(x)$ for all $x \in R$.

We have from (*) and (**)

3. $E[g(-x)g(x)] + E[g^2(x)] \leq 0$

4. $E[g(-x)g(x)] + E[g^2(-x)] \leq 0$

Adding 3. and 4., we get

$$E[(g(-x)+g(x))^2] \leq 0$$

which is true iff $g(-x) = g(x)$ for all $x \varepsilon R$. Further, from v. of the Lemma, there exists a function f such that

$$f(R_g(\tau)) = R_x(\tau) = \sigma^2 \rho(\tau)$$

where

$$\rho(\tau) \triangleq \frac{E[x(t)x(t+\tau)]}{\sigma^2}$$

which complets the proof.

2b. From 2a. $\exists k_1, k_2 > 0 \ni$ a function f such that

$$f(R_g(\tau)) = R_x(\tau)$$

and

$$k_1 |f(R_g(\tau))| \leq |R_g(\tau)| \leq k_2 |f(R_g(\tau)|$$

for $|R_g(\tau)|$ sufficiently small, i.e., for $|\tau|$ sufficiently large. But $|R_g(\tau)| \to 0$ as $\tau \to \infty$

$$\therefore \lim_{\tau\to\infty} |\tau R_x(\tau)| \leq \lim_{\tau\to\infty} \frac{1}{k_1} |\tau R_g(\tau)| = 0$$

3.

$$\lim_{\tau\to\infty} |R_z(\tau)| = 0$$

Thus

$$\forall \varepsilon > 0 \qquad \exists \tau_0 \quad \text{such that } |\tau| > \tau_0$$

$$|R_z(\tau)| < \varepsilon$$

Let

$$\varepsilon = E[g(-x)g(x)]$$

then 2.ii of the theorem is satisfied for all τ such that $|\tau| > \tau_0$.

Corollary (25)

Let x be a zero mean wide sense stationary gaussian random process. Let $g:[-\infty,\infty]\to[M_2,M_1]$ be a bounded invertible function such that

a. g is differentiable with bounded non-zero derivative.

b. $E(g(x)) = 0$

Let

$$z(\cdot,t) \triangleq g(x(\cdot,t))$$

Let $h:R\to R$ be a bounded time variable function of $z(\cdot,t)$.

Let

$$\hat{h}(t) \triangleq h(g(x(\cdot,t),t) = h(z(\cdot,t),t)$$

such that

$$E(\hat{h}(t)) = 0 \text{ for all } t$$

Let

$$\lim_{\tau\to\infty} \tau R_x(\tau) = 0,$$

then

$$\lim_{\tau\to\infty} \tau E(\hat{h}(t)\hat{h}(t+\tau)) = 0$$

Proof

Follows immediately from Theorems (23) and (24).

Comment

Let

$$h(g(x(\cdot,t),t)) = \psi_k(u(t)+N_1(\cdot,t)) - E[\psi_k(u(t)+N_1(t))]$$

Then this corollary shows that 1.6.4 is satisfied when N_1 is of the form $g(x)$ were x is a gaussian process.

Corollary (26)

Let z be a zero mean wide sense stationary random process such that $M_2 \leq z \leq M_1$, where $M_2 < 0 < M_1$. Let the probability distribution function of $z\,(\cdot,t)$ be monotone increasing on $[M_2,M_1]$ and differentiable everywhere. Let $f(\cdot)$ = probability density function of $z(\cdot,t)$. Let $f(\cdot,\cdot)$ = joint probability density function of $z(\cdot,t)$ and $z(\cdot,t+\tau)$. (*) Let $z(\cdot,t)$ and $z(\cdot,t+\tau)$ be statistically independent in the limit as $\tau\to\infty$, and (**) $\tau|\hat{f}[z(\cdot,t),z(\cdot,t+\tau)] - f[z(\cdot,t)]f[z(\cdot,t+\ )]| \to 0$ uniformly for all z,t as $\tau\to\infty$. Let $h(\cdot,t)$ be a bounded time varying function such that $E[h(z(\cdot,t),t)] = 0$. Then

$$\lim_{\tau\to\infty} \tau E[h(z(\cdot,t),t)h(z(\cdot,t),t+\tau)] = 0$$

Proof

From Corollary (25), for z a wide sense stationary zero mean gaussian random process, there exists a monotone increasing function g and an autocorrelation R_x such that the random variable $g(x(\cdot,t))$ has the saome probability density as $z(\cdot,t)$ and the random process $g(x)$ has the same autocorrelation function as z. Let $M = \max(|M_1|, |M_2|)$. Then

$$\left| \lim_{\tau\to\infty} \tau E[h(z(\cdot,t),t)h(z(\cdot,t+\tau),t+\tau)] \right|$$

$$= \left| \lim_{\tau\to\infty} \tau \int_{-M}^{M}\int_{-M}^{M} h[z(\cdot,t),t]h[z(\cdot,t+\tau),t+\tau]\hat{f}[z(\cdot,t),z(\cdot,t+\tau)] \, dz(\cdot,t)dz(\cdot,t+\tau) \right|$$

$$= \left| \lim_{\tau\to\infty} \tau \int_{-M}^{M}\int_{-M}^{M} h(z_1,t)h(z_2,t+\tau)[(f(z_1)f(z_2)) + (\hat{f}(z_1,z_2)-f(z_1)f(z_2))] \, dz_1dz_2 \right|$$

$$\leq \left| \lim_{\tau\to\infty} \tau \int_{-M}^{M}\int_{-M}^{M} h(z_1,t)h(z_2,t+\tau)f(z_1)f(z_2)dz_1dz_2 \right|$$

$$+ \left| \lim_{\tau\to\infty} \tau \int_{-M}^{M}\int_{-M}^{M} h(z_1,t)h(z_2,t+\tau)[\hat{f}(z_1,z_2)-f(z_1)f(z_2)]dz_1dz_2 \right|$$

The first term is zero by Corollary (25). The second term is bounded by

$$\lim_{\tau\to\infty} \tau \; 4M^2 \Big| \max_{\substack{t\varepsilon R \\ z\varepsilon[M_2,M_1]}} (h(z,t)) \Big|^2 |\hat{f}(z_1,z_2)-f(z_1)f(z_2)|$$

$$= 0 \quad \text{by hypothesis}$$

<u>end proof</u>

<u>Comment</u>

The hypotheses in Corollary (26) that do not appear in

Corollary (25), (*) and (**), may be expected to hold in many realistic situations. Thus we may assume that in many applications, that when the hypotheses on z of Corollary (25) are satisfied by the observation noise, then (4) (Section 1.6) is satisfied even though the conditions (*) and (**) are not directly verifiable.

Verification of (4) (Section 1.6)

We now verify that for the noise processes treated in the above theorems, the sequence generator (13) through (16) satisfies condition (4). Recall that we have

$$\lim_{\tau\to\infty} \tau\, R_{N_2}(\tau) = 0 \tag{27}$$

N_2 is a Wide Sense Stationary Gaussian Random Process

Let N_2 be a zero mean wide sense stationary gaussian random process. Let us define (See Theorem (23))

$$g(t) \triangleq \gamma[y(t)+N_2(\cdot,t)] - E[\gamma[y(t)+N_2(\cdot,t)]] \tag{28a}$$

Then clearly

$$E(g(t)) = 0 \quad \text{for all } t \tag{28b}$$

Let us assume as is the case in each example below, that $|g(t)|$ is bounded by a polynomial in $|N_2|$ with coefficients depending upon $y(t)$. Since y is assumed bounded with probability one, the polynomial coefficients are bounded. Now all conditional finite moments of jointly gaussian random variables, conditioned by restricting the random variables to positive values, exist. Thus, the polynomial bound on $|g|$ together with (27) ensures that all the hypotheses of Theorem (23) are satisfied, so that

$$\lim_{\tau\to\infty} \tau E[g(t)g(t+\tau)] = 0 \tag{28c}$$

Then condition 4 (Section 1.6) is satisfied.

<u>N_1 is a Bounded Transformation of a Zero Mean Wide Sense Stationary Bounded Gaussian Random Process</u>

Let N_1 satisfy the hypotheses on z in Corollary (26). Let

$$h(t) \triangleq \gamma[y(t)+N_2(\cdot,t)] - E[\gamma[y(t)+N_2(\cdot,t)]] \tag{29}$$

Then by Corollary (26) (Section 1.6) is clearly satisfied.

<u>Summary of Conditions Under Which Condition 4 (Section 1.6) is Satisfied.</u>

We have shown that the above defined sequence generator satisifes Condition 4 if

a) N_2 is a wide sense stationary zero mean gaussian random process such that $\lim_{\tau\to\infty} \tau R_{N_2}(\tau) = 0$, and $|\gamma(\xi)|$ is bounded by $a + |\xi|^b$ for some a, b (30)

or

b) N_2 is a zero mean wide sense stationary random process bounded with probability one and satisfies the hypotheses on z and on the joint probability density of $z(\cdot,t)$ and $z(\cdot,t+\tau)$ in Corollary (26). (30)

EXAMPLES

In all the examples given below we will assume that the observation noise is one of the above two types of noise processes. With this assumption, we need only verify that Condition 5 (Section 1.6) is satisfied for each example given below. We recall that in order to verify Condition 5, we must show that we can compute

$$E[\gamma(y)-\gamma(y+N_2)]$$

Recall that we are assuming the differential equation of the unknown system to be of the form

$$L(p)g^{-1}(y) = M(p)f(u)$$

where

$$f(u) = u$$

$$\gamma \triangleq g^{-1}$$

(See Figure 5). Each example is concerned with a different type of nonlinearity for γ. If $f(u) \neq u$ then if N_1 is zero we simply add more terms to the sequence generator. If we assume $f(u) \neq u$, $N_1 \neq 0$, $N_2 \equiv 0$, then the algorithm is clearly applicable, with the obvious modifications in the sequence

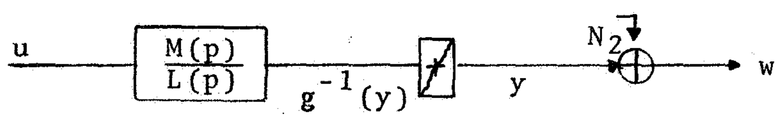


FIGURE 5

generator, if f is of the form of one of the nonlinearities discussed in the examples below.

Example (31)

N_2 is zero (and N_1 is zero)

Verification of Condition 5 (Section 1.6)

Condition 5 is trivially satisfied in this case since

$$E[\gamma(y)-\gamma(y+N_2)] = \gamma(y) - \gamma(y) = 0 \tag{32}$$

Comment

This is the case of identification in the absence of noise, which we have said may be accomplished simply by observing as many samples from the sequence generator as there are unknown coefficients and solving the resulting linear algebraic equations. The instrumental variable sequence in this case is simply one identity matrix. (See Section 4.4)

Example (33)

γ is "smooth", N_2 is zero mean and $|N_2|$ is bounded by a "small" number.

Let $|N_2| \leq \varepsilon$ with probability one for some $\varepsilon > 0$. We assume that γ is sufficiently smooth and ε is sufficiently small that for some $\alpha(y) \neq 0$,

$$\gamma(y+N_2) \cong \gamma(y) + \alpha(y)N_2 \tag{34}$$

Verification of Condition 5 (Section 1.6)

$$E[\gamma(y)-\gamma(y+N_2)] \cong E[\gamma(y)-\gamma(y)-\alpha(y)N_2] = 0$$

Example (36)

γ is a polynomial, noise moments of order $\leq$ degree γ are known

We let γ be a polynomial of degree k, and we assume that all noise moments

$$E[N^k], \quad k = 0,\ldots,K \tag{37}$$

are known. Specifically

$$\gamma(y) = \sum_{j=0}^{K} d_j \, y^j \tag{38}$$

Modification of Sequence Generator

Suppose we have a random variable ξ, where ξ is a function of v and $E[N^k]$, $k = 0,\ldots,K$ and

$$E[\xi] = E[\gamma(y)-\gamma(w)] \tag{39}$$

Then if we use ξ instead of $E[\gamma(y)-\gamma(w)]$ in the sequence generator, it is clear that conditions 1-4 (Section 1.6) are satisfied, with the possible exception of 4. However, we shall see that such a random variable ξ exists when γ is a polynomial and that $\xi - \gamma(u) + \gamma(w)$ is a zero mean polynomial in N_2 with coefficients depending upon y and moments of N_2; so that condition 4 is indeed satisfied (See (29)).

Existence and Specification of ξ, and Verification of Condition 5

Clearly, $E(\gamma(y)-\gamma(w))$ is a sum of terms of the form

$$y^j - E(w^j) \qquad j = 0, \ldots, K$$

Now

$$E(w^m) = E[(y+N_2)^m] = \sum_{T=0}^{M} \binom{m}{i} E(N_2{}^{m-i}) \, y^i \tag{40}$$

and

$$E(w^m) - y^m = \sum_{i=0}^{m-1} \binom{m}{i} E(N_2{}^{m-i})((y^i - E(w^i)) + E(w^i)) \tag{41}$$

Let

$$-\xi_m \triangleq \sum_{i=0}^{m-1} \binom{m}{i} E(N_1{}^{m-i})(-\xi_i + v^i), \qquad \xi_0 = 0 \tag{42}$$

<u>Assertion</u> (43)

$$E(-\xi_m) = E(w^m) - y^m \quad \text{i.e.,} \quad E(\xi_m) = y^m - E(w^m)$$

<u>Proof</u> (induction)

Let $m = 1$, then

$$E(w) - y = E(N_2) \text{ and } \xi_1 = (1)(E(N_4))(0+w^0) = E(N_2)$$

Now assume that

$$E(-\xi_i) = E(w^i) - y^i, \qquad i = 1, \ldots, m-1$$

Then

$$E(-\xi_m) = \sum_{i=0}^{m-1} \binom{m}{i} E(N_1^{m-i})[E(w^m)-y^m+E(w^i)] = E(w^m)-y^m$$

<u>end proof</u>.

Thus we may use $\xi \triangleq \xi_m$ in place of $y^m - E(w^m)$ in the sequence generator. We obtain ξ from (42). Thus condition 5 (Section 1.6) is satisfied, since ξ is clearly computable from observed signals and noise moments which are assumed to be known.

<u>N_2 is a Gaussian Process</u>

Suppose N_2 is a wide sense stationary gaussian random process. Then all moments are polynomials in the mean and the variance. Thus, to use the above sequence generator when N_2 is gaussian, we need only estimate the mean and variance of N_2.

<u>N_2 is a Zero Mean Wide Sense Stationary Random Process with an Even Probability Density Function</u>

Let $p(\cdot)$ be the probability density of $N_2(\cdot,t)$. Let

$$p(-x) = p(x) \tag{44}$$

Then if all higher order moments exist, the odd moments are all zero, i.e.,

$$E(N_2^{2k+1}) = 0, \quad k \geq 1 \tag{45}$$

so that in order to implement the sequence generator, we need only estimate the even moments of N_2.

2.3 Nonlinear Systems II

In this section, we consider systems described by differential equations of the form

$$L(p)y + f(y)y^{(1)} + g(y) = M(p)u + h(u)u^{(1)}x + w(u) \quad (1)$$

We shall discuss only the elements of the sequence generator involving the terms $f(y)y^{(1)}$ and $g(y)$, since we treat $h(u)u^{(1)}$ and $w(u)$ the same as these, and all other terms have been treated in the linear system case. We note that

$$\int_0^T \phi(t)f(y(t))y^{(1)}(t)dt = \int_0^T -\phi^{(1)}(t)F(y(t))dt \quad (2)$$

where

$$F(y(t)) \triangleq \left[\int f(\xi)d\xi\right](t) \quad (3)$$

is the indefinite integral of f. The elements of the sequence generator involving F and g are defined to be

$$-\int_0^T \phi^{(1)}(t)[F(w(t)) + E(F(y(t)-F(w(t)))]dt \quad (4)$$

and

$$\int_0^T \phi(t)[g(w(t))+E(g(y(t))-g(w(t)))]dt \quad (5)$$

respectively. Other components of the sequence generator are as those given for the linear differential system (35) (Section 1.6). We not need only note that these integrals are of the same form as the elements (20) (Section 2.2) of the sequence generator discussed in the last section and thus satisfy conditions 1 through 5 under the same conditions that the integrals (20) satisfy conditions 1 through 5 (Section 1.6).

Example (6)

Van der Pol equation, noise-free observations

Let the unknown system differential equation be a Van der Pol equation, i.e.,

$$x^{(2)} + a_1(1-x^2)x^{(1)} + a_0 x = 0 \qquad (7)$$

or

$$a_1(1-x^2)x^{(1)} + a_0 x = -x^{(2)} \qquad (8)$$

where x is observed on some time interval. Then, multiplying by a testing function and integrating, we get

$$a_1\int_0^T -\phi^{(1)}(t)[x-x^3/3]dt + a_0\int_0^T \phi(t)x(t)dt$$

$$= -\int_0^T \phi^{(2)}(t)x(t)dt \qquad (9)$$

since

$$\int(1-x^2)dx = x-x^3/3 + (\text{constant}) \qquad (10)$$

Then, assuming that the observation noise N is zero, we need only obtain two such equations (See Section 4.4) and solve for a_1 and a_0.

2.4 Summary

We have shown that for nonlinear systems with the following types of differential equations, with certain restrictions on knowledge of the noise statistics, (See Section 2.3 (30a), (30b), and Examples (31), (33) and (36)) we can devise identification algorithms. Let

$$L(p) = \sum_{i=0}^{M} a_i p^i$$

$$M(p) = \sum_{j=0}^{M} b_j p^j$$

1. $L(p)g^{-1}(y) = M(p)f(u)$ (See Figure 4)

   $f(u) = \sum_{i=0}^{I} d_i \psi_i(u)$, $\{\psi_i\}$ are known and linearly independent (See Examples (31), (33) and (36))

2. $L(p)y + f_1(y)y^{(1)} + f_2(y) = M(p)u + f_3(u)u^{(1)} + f_4(u)$

$f_i(x) = \sum_{j=0}^{J_i} d_{ji} \psi_{ji}(x), \left\{ \psi_{ji} \right\}_{j=0}^{J_i}$ are known

and linearly independent for each i, $i = 1,\ldots,4$

(See Section 2.3).

CHAPTER 3

PROPERTIES OF TESTING FUNCTIONS

3.0 Introduction

<u>Notation</u>

As before, unless otherwise noted, ϕ will denote an M-differentiable testing function on [0,T].

In this chapter we give several properties of testing functions. These properties give bounds on the number of zeros, number of sign changes, and rate of growth of the magnitude of the extrema of $\phi^{(k)}$ as k increases. These properties will be used in the next chapter in discussing the choice of a testing function to be used in the sequence generators given in the first two chapters.

<u>Lemma</u> (1) (1)

Lower bound on the absolute value of the derivative of function on a closed interval. Let

$$F = \{f \mid f:R\to R,\ f \text{ is differentiable on } [a,b],\ f(a) = A,\ f(b) = B\}$$

1. Let $D = \inf_{f\varepsilon F} \max_{t\varepsilon[a,b]} |f^{(1)}(t)|$ then $D = \left|\frac{B-A}{b-a}\right|$

2. Let $\hat{F} = \{f \mid f\varepsilon F,\ f^{(1)}(a) = f^{(1)}(b) = 0\}$ then

   for all f in $\hat{F}$, $\max_{t\varepsilon[a,b]} |f^{(1)}(t)| > \left|\frac{B-A}{b-a}\right|$

<u>Proof of Lemma</u>

We assume $B \geq A$ since

$$|f^{(1)}(t)| = |(-f(t))^{(1)}|$$

<u>Proof of 1.</u>

By the mean value theorem, for all f in F $\exists\, t_o$, $t_0 \in [a,b]$ such that

$$|f'(t_0)| = \left|\frac{B-A}{b-a}\right| \triangleq \hat{D}$$

thus $D \geq \hat{D}$.

Further, there exists some f, $f \in F$ such that

$$\max_{t\in[a,b]} |f'(t)| = \left|\frac{B-A}{b-a}\right| = \hat{D}$$

namely

$$f(t) = A + \left|\frac{B-A}{b-a}\right| (t-a)$$

thus $\hat{D} \geq D \geq \hat{D}$ or $D = \hat{D}$.

<u>Proof of 2.</u>

Suppose 2. is not true, then for some $f \in \hat{F}$ $\exists\, t_o$ such that $b > t_0 > a$ such that

$$\max_{t\in[a,b]} |f^{(1)}(t)| = |f^{(1)}(t_0)| \leq D$$

Now

$$f(t_0) < A + \frac{B-A}{b-a}(t_0-a),$$

else by the mean value theorem there exists $t_1 \varepsilon (0,t_0)$ such that

$$|f^{(1)}(t_1)| < \frac{A + \left|\frac{B-A}{b-a}\right|(t_0-a)-A}{t_0-a} = \left|\frac{B-A}{b-a}\right|,$$

which cannot be if 2. is false. Then

$$f(b) = f(t_0) + f(b) - f(t_0)$$

$$< A + \frac{B-A}{b-a}(t_0-a) + f(b) - f(t_0)$$

$$< A + \frac{B-A}{b-a}(t_0-a) + B - (A + \frac{B-A}{b-a}(t_0-a))$$

or $f(b) < B$, a contradiction.

<u>end proof Lemma 1</u>

<u>Property</u> (2) (2)

(Lower bound on number of distinct zeros of $\phi^{(k)}$.) There exist z_i^k, $i = 0,\ldots,k+1$, $z_i^k < z_{i+1}^k$, $k_0^k = 0$, $z_{k+1}^k = T$ for $k \leq M$, such that

1. $\phi^{(k)}(z_i^k) = 0$,

2. $\phi^{(k)}(t)$ is not identically zero for all t such that $z_i^k < t < z_{i+1}^k$

<u>Proof (induction)</u>

$\phi(0) = \phi(T) = 0$, $\phi(t)$ is not identically zero, $\therefore$ by Rolle's theorem, there exists t_0, $0 < t_0 < T$, such that

1. $\phi^{(1)}(t_0) = 0$, and
2. $\phi(t)$ is not identically zero for all $t \in (0,t_0)$ or for all $t \in (t_0,T)$

(If 2. were not true, then $\phi(t)$ would be identically zero). Thus the property holds for $k = 1$.

Inductive hypothesis: Property (2) holds for

$k = \ell$, $\ell < M$

thus letting $z_0^{\ell+1} = 0$, $z_{\ell+2}^{\ell+1} = T$, and applying Rolle's theorem to every two adjacent zeros z_i^ℓ and z_{i+1}^ℓ, $i = 0,\ldots,\ell$, of $\phi^{(\ell)}$ between which $\phi^{(\ell)}$ is not identically zero, we get: there exists

$$z_{i+1}^{\ell+1}$$

such that

$$z_i^\ell < z_{i+1}^{\ell+1} < z_{i+1}^\ell \,, \qquad i = 0,\ldots,\ell+1$$

$$\phi^{(\ell+1)}(z_{i+1}^{\ell+1}) = 0$$

and by the mean value theorem

$$\phi^{(\ell+1)}(z_{i+1}^{\ell+1}) \neq 0$$

$\phi^{(\ell)}(t) \equiv 0$, $z_i^\ell < t < z_{i\_1}^\ell$). Thus Property (2) holds for $k = \ell+1$.

<u>end proof</u>

Property (3) (3)

(Lower bound on number of zeros of $\phi^{(k+1)}$ given the number of zeros of $\phi^{(k)}$.)

Let $\ell < M$, $r \triangleq \ell+2$

Let $\phi^{(\ell)}$ have r distinct zeros z_j^ℓ, $j = 1,\ldots,r$ in $[0,T]$, with $z_j^\ell < z_{j+1}^\ell$ such that $\phi^{(\ell)}$ is not identically zero between any adjacent pair of these. Then $\phi^{(\ell+1)}$ has at least r+1 distinct zeros in $[0,T]$ between adjacent pairs of which $\phi^{(\ell+1)}$ is not identically zero.

Proof

Applying Rolle's theorem to each adjacent pair of the zeros z_j^ℓ, z_{j+1}^ℓ, we find that $\phi^{(\ell+1)}$ has at least one zero $z_j^{\ell+1}$ between these such that $\phi^{(\ell)}(z_j^{\ell+1}) \neq 0$ (else by the mean value theorem $\phi^{(\ell)}(t) \equiv 0$, $z_j^\ell \leq t \leq z_j^{\ell+1}$). Then also by the mean value theorem there exists

$$t_j^\ell,\ z_j^\ell \leq t_j^\ell \leq z_j^{\ell+1}$$

such that

$$\phi^{(\ell+1)}(t_j^\ell) \neq 0.$$

Now, there are at least r-1 adjacent pairs of zeros $\left(z_j^\ell, z_{j+1}^\ell\right)$ yielding r-1 zeros $z_j^{\ell+1}$, $j = 1,\ldots, r-1$, such that

$$0 < z_j^{\ell+1} < T$$

and $\phi^{(\ell+1)}$ is not identically zero between any adjacent pair of these. Further

$$\phi^{(\ell+1)}(0) = \phi^{(\ell+1)}(T) = 0$$

and $\phi^{(\ell+1)}$ is not identically zero for

$$t \in (0, z_1^{\ell+1})$$

or

$$t \in (z_{r-1}^{\ell+1}, T)$$

or else by the mean value theorem,

$$\phi^{(\ell)}(z_1^{\ell+1}) = \phi^{(\ell)}(z_{r-1}^{\ell+1}) = 0$$

which was not allowed. Thus there are at least r + 1 zeros of $\phi^{(\ell+1)}$ between adjacent pairs of which $\phi^{(\ell+1)}$ is not identically zero.

<u>end proof</u>

<u>Property</u> (4) (4)

(Lower bound on the number of sign changes of $\phi^{(k)}$ in $[0,T]$.) There exist t_i, $i = 1,\ldots,k+1$, $0 < t_i < t_{i+1} < T$, such that $\text{sign}\left(\phi^{(k)}(t_i)\right) = -\,\text{sign}\left(\phi^{(k)}(t_{i+1})\right)$ i.e., $\phi^{(k)}$ has at least k sign changes in $[0,T]$.

<u>Proof</u>

Let $\{z_j^i\}$ be the zeros defined in Property (2).

Let $1 \leq k \leq M$. Choose t_j, $j = 1,\ldots,k$, such that

1. $z_{j-1}^{k-1} < t_j < z_j^{k-1}$

2. $\phi^{(k-1)}(t_j) \neq 0$ (we can do this by property (2))

Then by the mean value theorem, there exists τ_1, τ_2

$$z_{j-1}^{k-1} < \tau_1 < t_j < \tau_2 < z_j^{k-1}$$

such that

$$\frac{\phi^{(k-1)}(t_j) - \phi^{(k-1)}(z_{j-1}^{k-1})}{t_j - z_{j-1}^{k-1}} = \phi^{(k)}(\tau_1) \qquad (*)$$

and

$$\frac{\phi^{(k-1)}(z_j^{k-1}) - \phi^{(k-1)}(t_j)}{z_j^{k-1} - t_j} = \phi^{(k)}(\tau_2) \qquad (**)$$

but

$$\phi^{(k-1)}(z_{j-1}^{k-1}) = \phi^{(k-1)}(z_j^{k-1}) = 0$$

and both denominators are positive. Thus,

$$\text{sign } [\phi^{(k)}(\tau_1)] = \text{sign } [\phi^{(k-1)}(t_j)]$$

$$= -\text{sign } [-\phi^{(k-1)}(t_j)] = -\text{ sign } \phi^{(k)}(\tau_2),$$

$$j = 1, \ldots, k$$

<ins>end proof</ins>

<u>Property</u> (5) (5)

(Linear independence of ϕ and its derivatives on $(0,T)$.)

The set

$$\Phi = \{\phi, \phi^{(1)}, \ldots, \phi^{(M)}\}$$

is a linearly independent set on $(0,T)$.

<u>Proof</u>

Suppose Φ is not a linearly independent set on $(0,T)$. Then there exist $a_0, a_1, \ldots, a_M$, not all zero, such that

$$\sum_{i=0}^{M} a_i \phi^{(i)}(t) \equiv 0 \quad \text{for all } t \in [0,T] \tag{*}$$

but $\phi(0) = \phi^{(1)}(0) = \ldots = \phi^{(M)}(0)$ which implies that the solution to (*) is

$$\phi^{(k)}(t) \equiv 0, \qquad k = \min_i \{i \mid a_i \neq 0\}$$

If $k = 0$, we have $\phi(t) \equiv 0$ which contradicts the definition of testing function. If $k > 0$, then $\phi^{(k-1)}(t) = \text{(constant)}$ for all $t \geq 0$ which also contradicts the definition of testing function.

<u>end proof</u>

<u>Property</u> (6) (6)

$$\int_0^T \phi^{(i)}(t)\phi^{(k)}(t)\,dt \quad \begin{cases} = 0 \text{ if } i-k \text{ is odd} \\ \neq 0 \text{ and has sign } (-1)^{|(i-k)/2|} \\ \quad \text{if } i-k \text{ is even} \end{cases}$$

for $0 \leq i,k \leq M$.

<u>Proof</u>

Let $M > i > k \geq 0$

Let $i - k$ be odd

Let $j \triangleq i - (\frac{i-k-1}{2}) = \frac{i+k+1}{2}$

Then

$$j - 1 = k + (\frac{i-k-1}{2}) = \frac{k+i-1}{2}$$

and

$$I_{ik} \triangleq \int_0^T \phi^{(i)}(t)\phi^{(k)}(t)\,dt = (-1)^{(\frac{i-k-1}{2})}\int_0^T \phi^{(j)}(t)\phi^{(j-1)}(t)\,dt$$

Let $x(t) = \phi^{(j)}(t)$, then $x(0) = x(T) = 0$ and

$$I_{ik} = \int_0^0 x\,dx = 0$$

Now let $i \geq k$

Let $i-k$ be even

Let $j \triangleq i-(\frac{i-k}{2}) = \frac{i+k}{2}$

Then

$$k + \left(\frac{i-k}{2}\right) = \frac{i+k}{2} = j$$

so that

$$\int_0^T \phi^{(i)}(t)\phi^{(k)}(t)dt = (-1)^{(i-k)/2}\int_0^T [\phi^{(j)}(t)]^2 dt \neq 0$$

which has sign $(-1)^{|(i-k)/2|}$.

<u>end proof</u>

<u>Property</u> (7) (7)

Let

$$\|\phi\| \triangleq \sqrt{\frac{1}{T}\int_0^T \phi^2(t)dt}$$

Then

$$\|\phi^{(j)}\| \geq \left(\frac{\|\phi^{(1)}\|}{\|\phi\|}\right)^{j-1} \|\phi^{(1)}\|$$

<u>Proof</u> (induction)

$$\|\phi^{(1)}\| = \frac{1}{T}\int_0^T (\phi^{(1)}(t))^2 dt = -\frac{1}{T}\int_0^T \phi^{(2)}(t)\phi(t)dt$$

$$= \| \phi^{(2)}, \phi \| < \|\phi^{(2)}\| \quad \|\phi\| \tag{1}$$

by the Schwarz inequality, where strict inequality follows from the linear independence of $\phi^{(2)}$ and ϕ (Property (5)).

Thus the property holds for j = 2.

Inductive hypothesis: assume Property (7) holds for $j = n$, $n \leq M$, i.e., assume

$$\|\phi^{(n)}\| > \left(\frac{\|\phi^{(1)}\|}{\|\phi\|}\right)^{n-1} \|\phi^{(1)}\| \tag{2}$$

Now since ϕ^{n+1} is an n-differentiable testing function, we apply (2) to it, and get

$$\|\phi^{(n+1)}\| > \left(\frac{\|\phi^{(2)}\|}{\|\phi^{(1)}\|}\right)^{n} \|\phi^{(2)}\| \tag{3}$$

and substituting (1) into (3) yields

$$\|\phi^{(n+1)}\| > \left(\frac{\|\phi^{(1)}\|^2}{\|\phi\| \; \|\phi^{(1)}\|}\right)^{n} \frac{\|\phi^{(1)}\|^2}{\|\phi\|}$$

or

$$\|\phi^{(n+1)}\| > \left(\frac{\|\phi^{(1)}\|}{\|\phi\|}\right)^{n+1} \|\phi^{(1)}\|$$

<u>end proof</u>

<u>Property</u> (8) (8)

(Lower bound on the rate of growth of absolute value of extrema of $\phi^{(j)}$ as j increases, given that $\phi^{(\ell+1)}$ has a zero in [0,1) or (T-1,T] and $j>\ell$.) Let there exist t_0, t_0 in [0,1) or (T-1,T], such that for $\ell + 1 < M$, $\phi^{(\ell)}(t_0) \neq 0$

and $\phi^{(\ell+1)}(t_0) = 0$. Then there exist t_j, $j = 0,\ldots,M-\ell-1$ in $(0,1)$ with $t_j > t_{j+1}$ such that for $j = 1,\ldots,M-\ell-1$

$$\sup_{t\varepsilon(0,T]} |\phi^{(\ell+j)}(t)| > \left|\frac{1}{t_0}\right|^j |\phi^{(\ell)}(t_0)| \text{ if } t_0\varepsilon[0,1)$$

$$> \left|\frac{1}{t_0-T}\right|^j |\phi^{(\ell)}(t_0)| \text{ if } t_0\varepsilon(T-1,T]$$

<u>Proof</u>

Let t_0 $[0,1)$. By the mean value theorem and Lemma (1) there exists t_1, $t_1\varepsilon(0,t_0)$, such that

1. $\phi^{(\ell+2)}(t_1) = 0$
2. $|\phi^{(\ell+1)}(t_1)| > \dfrac{|\phi^{(\ell)}(t_0)|}{t_0} > 0$

Inductive hypothesis: Property (8) holds for $j = n$, $n < M-\ell-1$, i.e., there exists t_n, $t_n\varepsilon(0,1)$ such that

$$\phi^{(\ell+n+1)}(t_n) = 0, \quad \phi^{(\ell+n)}(t_n) \neq 0$$

and

$$\sup_{t\varepsilon(0,1)} |\phi^{(\ell+n)}(t_n)| \geq (\frac{1}{t_0})^n |\phi^{(\ell)}(t_0)|$$

then, by Lemma (1)

$$\sup_{t\varepsilon(0,1)} |\phi^{(\ell+n+1)}(t)| \geq |\phi^{(\ell+n+1)}(t_n)| > \frac{1}{t_0}|\phi^{(\ell+n)}(t_n)|$$
$$> \left(\frac{1}{t_0}\right)^{n+1} |\phi^{(\ell)}(t_0)|$$

which completes the proof for $t_0 \varepsilon [0,1)$. Now let $t_0 \varepsilon (T-1,T)$. Let $\hat{\phi}(t) = \phi(T-t)$. Then $\hat{\phi}$ clearly satisfies the hypothesis of Property (8), so that

$$\sup_{t \varepsilon [0,T]} |\hat{\phi}^{(k)}(t)| > \left|\frac{1}{t_0 - T}\right|^k |\hat{\phi}(T-t_0)|$$

but

$$\sup_{t \varepsilon [0,T]} |\hat{\phi}^{(k)}(t)| = \sup_{t \varepsilon [0,T]} |\phi^{(k)}(t)|$$

and

$$\hat{\phi}(T-t_0) = \phi(t_0)$$

thus

$$\sup_{t \varepsilon [0,T]} |\phi^{(k)}(t)| > \left|\frac{1}{t_0 - T}\right|^k |\phi(t_0)|$$

<u>end proof</u>

<u>Corollary</u> (9)

Let $T < 2$, then

$$\max_{t \varepsilon [0,T]} |\phi^{(\gamma)}(t)| > \max_{t \varepsilon [0,T]} \left|\frac{2}{T}\right|^j |\phi(t)|, \quad j = 1,\ldots,M+1$$

<u>Property</u> (10) (10)

(Lower bound on the maximum absolute value of $\phi^{(k)}$ when ϕ is an infinitly differentiable testing function.) Let ϕ be an infinitely differentiable testing function on $[0,T]$. Then for any $\delta > 1$, there exist $A(\delta)$ and $N(\delta)$, both positive, such that for all n such that $n \geq N(\delta)$

$$\max_{t\varepsilon[0,T]} |\phi^{(n)}(t)| > A(\delta)\delta^{n-N(\delta)}$$

Proof

Lemma

There exist t_i^k, $i = 1,\ldots,k$ $k = 0,\ldots,M$; with

$$t_i^k < t_{i+1}^k$$

such that

1. $\phi^{(k+1)}(t_i^{k+1})$ is a local maximum or minimum value of $\phi^{(k+1)}$

2. $\phi^{(k+1)}(t_i^{k+1})$ is of the opposite sign of $\phi^{(k+1)}(t_{i+1}^{k+1})$

3. $$|\phi^{(k+1)}(t_i^{k+1})| \cdot |t_i^k - t_i^{k+1}| > |\phi^{(k)}(t_i^k)| + |\phi^{(k)}(t_{i-1}^k)|$$

4. $t_i^k < t_{i+1}^{k+1} < t_{i+1}^k$

Proof of Lemma

1., 2. and 4. follow immediately from properties (2), (3) and (4). 3. follows from Lemma (1), 1. and 2.

end proof of lemma

Let $\delta > 1$.

Let w = greatest integer in T/δ . Then there exist at most w intervals of length greater than or equal to $1/\delta$ in $[0,T]$ which do not intersect pairwise at more than one point.

Let $I_i^k = [t_{i-1}^k, t_i^k]$, $i = 1,\ldots,k+1$. There are $k+1$ such intervals.

Let k be sufficiently large that

$k + 1 > w + 1$.

Then at most w of the intervals $\{I_i^k\}$ are of length greater than or equal to $1/\delta$, and at least $k + 1 - w$ of them are of length less that $1/\delta$, and

$1/\delta > 1$.

Now starting at zero, we divide $[0,T]$ into w adjacent intervals at length $1/\delta$, and one interval of length $T-w/\delta$; namely

$$[0,1/\delta],\ [1/\delta,\ 2/\delta],\ \ldots,\ [w/\delta,T]$$

and label these $L_1,L_2,\ldots,L_{w+1}$ respectively. Since $k+1 > w+1$, at least one interval L_j contains at least one interval I_i^k.

Now as j increases, one of the following holds

1. There exists $N_1 < \infty$ such that for all $j > N_1-k$, $t_1^{k+j} \varepsilon\ L_2$. In this case the proof is complete by Property (7) with $N(\delta) = N_1$, $A(\delta) = |\phi^{(N_1)}(t_1^{N_1})|$
2. There exists $N_1 < \infty$ such that for all $j > N_1-k$, $t_1^{k+j} \varepsilon\ L_{j_i}$ for some j_1 such that $2 \leq j_1 \leq w+1$

In this case it is clear that there exists $N_i < \infty$ such that for all $j > N_i - k, t_i^{k+j}\ \varepsilon\ L_{j_i}$ for $i = 1,\ldots,k+1$

with $2 \leq j_1 \leq j_2 \leq ,\ldots, \leq j_{w+2} \leq w+1$ with $j_i = j_{i+1}$ for at least one value of i, $i = 1,\ldots,k+1$. Let $N_I = \max_i N_i$ $i = 1,\ldots,k+1$, then there exists n,ℓ, $2 \leq n, j_\ell \leq w+1$ such that $I_n^{N_I+M} \varepsilon L_{j_\ell}$ for all m, and the proof is complete by Property (7) with

$$N(\delta) = N_I$$

$$A(\delta) = |\phi^{(N_I)}(t_n^{N_I})| + |\phi^{(N_I)}(t_{n-1}^{N_I})|$$

<u>end proof</u>

<u>Corollary</u> (11)

For every infinite differentiable testing function ϕ and every $\delta > 1$ there exists $A(\phi,\delta) > 0$ such that

$$\sup_{t \varepsilon [0,T]} |\phi^{(n)}(t)| \geq A(\phi,\delta)\ \delta^n$$

CHAPTER 4

GENERAL FORM AND CHOICE OF TESTING FUNCTIONS

4.0 Introduction

In this chapter, we first give the general form of polynomial testing functions, some examples of other types of M-differentiable testing functions, and some examples of infinitely differentiable testing functions. We then give a procedure for choosing a specific polynomial testing function which was found to work well in practice.

4.1 M-differentiable Testing Functions

General Form of Polynomial Testing Functions

Theorem (1)

Let ϕ be an M-differentiable testing function on [0,T].

$$\text{Let } \phi(t) = \begin{cases} \hat{\phi}(t), & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

Then if ϕ is a polynomial,

$$\hat{\phi}(t) = (t(t-T))^{M+1} \sum_{i=0}^{I} c_i t^i$$

where $0 \leq 1$ and at least one of $c_0, c_1, \ldots, c_I$ is not zero.

Proof

Let

$$\phi(t) = \sum_{\ell=0}^{L} a_\ell \frac{t^\ell}{\ell!}$$

where we let $L \geq 2M+2$ so that we may choose tha a_ℓ to satisfy the 2M+2 boundary conditions at $t = 0$ and $t = T$, namely

$$\phi^{(k)}(0) = \phi^{(k)}(T) = 0, \quad k = 0,\ldots,m$$

Then

$$\phi^{(k)}(t) = \sum_{\ell=k}^{L} a_\ell \frac{t^{\ell-k}}{(\ell-k)!}$$

We require that

$$\hat{\phi}^{(k)}(0) = 0, \quad k = 0,\ldots,M$$

but

$$\hat{\phi}^{(k)}(0) = a_k$$

Thus

$$a_k = 0, \quad k = 0,\ldots,M$$

and

$$\hat{\phi}(t) = \sum_{\ell=M+1}^{L} a_\ell \frac{t^\ell}{\ell!}$$

Then there exist finite constants b_l, $l = 0,\ldots,L-(M+1)$, not all zero, such that

$$\hat{\phi}(t) = \frac{t^{M+1}}{(M+1)} \sum_{\ell=0}^{K-M-1} b_\ell \frac{(t-T)^\ell}{\ell!}$$

<u>Lemma</u>

By definition $b_j = 0$, $j = 0,\ldots,M$

$$\hat{\phi}^{(k)}(T) = 0, \; k = 0,\ldots,M$$

and

$$\hat{\phi}^{(k)}(T) = \sum_{i=0}^{k} \binom{k}{i} \frac{t^{M+1-k+i}}{(M+1-k+i)!} \sum_{\ell=i}^{L-M-1} b_\ell \frac{(t-T)^{\ell-i}}{(\ell-i)!} \Bigg|_{t=T}$$

$$= \sum_{i=0}^{k} \binom{k}{i} \frac{T^{M+1-k}}{(M+1-k)!} b_i$$

Let $k = 0$, then

$$\hat{\phi}(T) = 0 \text{ iff } b_0 = 0$$

Inductive hypothesis; assume $b_i = 0$, $i = 0,\ldots,J$, $J < M$.

Then

$$\hat{\phi}^{(k)}(T) = \sum_{i=J+1}^{k} \binom{k}{i} \frac{T^{M+1-i}}{(M+1-i)!} b_i$$

and

$$\hat{\phi}^{(J+1)}(T) = \frac{T^{M+1-J-1}}{(M+1-J-1)!} b_{j+1}$$

Since

$$j + 1 \leq M,$$

Then we require

$$\hat{\phi}^{(j+1)}(T) = 0,$$

which is true iff $b_{j+1} = 0$

<u>end proof of lemma</u>

Thus there exist finite constants c_i, $i = 0,\ldots,I$, $I = L-2M-2$, not all zero, such that

$$\hat{\phi}(t) = (t(t-T))^{M+1} \sum_{i=0}^{I} c_i t^i$$

<u>end proof of theorem</u>

<u>Definition</u> (2)

<u>Symmetric Testing Function</u>

A testing function on [0,T] is said to be symmetric if it is even about T/2, i.e., if

$\phi(t-T/2) = \phi(t+T/2)$

Corollary (3)

General Form of Symmetric Polynomial Testing Function

Let ϕ be a M-differentiable polynomial testing function on [0,T]. If ϕ is symmetric,

$$\hat{\phi} = (t(t-T))^{M+1} \sum_{i=0}^{I} c_i (t-T/2)^{2i}$$

where at least one of $c_0, c_1, \ldots, c_I$ is not zero.

Proof

First, note that

$$(t(t-T))^{M+1} = (t^2-tT)^{M+1} = ((t-T/2)^2-T^2/4)^{M+1}$$

is clearly even about T/2. Further, the part of the summation

$$\sum_{i=0}^{I} c_i t^i$$

that is even about T/2 can clearly be written as

$$\sum_{j=o}^{J} a_j (t-T/2)^{2j}$$

where J is the biggest even integer in I/2 for some set of coefficients $\{a_j\}_{j=0}^{J}$, not all zero.

end proof

<u>Corollary</u> (4)

Let

$\hat{\phi}(t) = (t(t-T))^{M+1}$

Then

1. $\hat{\phi}^{(k)}$ has exactly k+2 distinct zeros in [0,T], $k \leq M$.
2. 0 and T are roots of $\phi^{(k)}$ of multiplicity M+1-k and all other roots of $\phi^{(k)}$ are real and simple and lie in (0,T).

<u>Proof</u>

1. Let $\phi(t) = \begin{cases} \hat{\phi}(t) & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$

Then by the theorem, ϕ is an M differentiable testing function. Therefore, by Property (2) $\hat{\phi}^{(k)}$ has at least k+2 distinct zeros in [0,T]. We show that $\hat{\phi}^{(k)}$ has at most k+2 distinct zeros in [0,T]. Now

$$\frac{\hat{\phi}^{(k)}(t)}{(M+1)!} = \sum_{i=0}^{k} \binom{k}{i} \frac{t^{M+1-i}}{(M+1-i)!} \frac{(t-T)^{M+1-k+i}}{(M+1-k+i)!} \tag{*}$$

$$= t^{M+1-k}(t-T)^{M+1-k}\binom{k}{i} \frac{t^{k-i}}{(M+1-i)!} \frac{(t-T)^{i}}{(M+1-k-i)!}$$

Now $t^k(t-T)^k$ is clearly not zero except at $t = 0$ and $t = T$. Further, the summation is a polynomial of degree k and thus has at most k real zeros. Therefore

$$\frac{\hat{\phi}^{(k)}}{((M+1)!)^2}$$

(and consequently $\phi^{(k)}$) has at most k+2 real distinct zeros in (0,T).

2. From (*) it is clear that 0 and T are roots of $\hat{\phi}^{(k)}$ of multiplicity M+1-k. Now $\hat{\phi}^{(k)}$ has k+2 distinct roots in [0,T], two of which are 0 and T. Further, $\hat{\phi}^{(k)}$ is a polynomial of degree 2(M+1)-k = 2M+2-k and thus has 2M+2-k roots. Then $\hat{\phi}^{(k)}$ has (2M+2-k) - (2(M+1-k) = k roots which are not 0 or T, but $\hat{\phi}^{(k)}$ has k distinct real roots in (0,T) which are not 0 or T. Thus the distinct roots are all simple roots, and all roots of $\hat{\phi}^{(k)}$ are real and lie in [0,T].

<u>end proof</u>

<u>M-differentiable Testing Functions which are not Polynomials</u>

The following theorem gives a method for constructing M-differentiable testing functions which are not necessarily polynomials and are not necessarily infinitely differentiable testing functions.

<u>Theorem</u> (5)

$$\text{Let } \phi(t) = \begin{cases} (f(t))^{M+1}, & 0 \leq t \leq T \\ 0 \text{ elsewhere} \end{cases}$$

where

$$f(0) - f(T) = 0$$

and f is M+1 times differentiable on [0,T]. Then ϕ is an M-differentiable testing function on $[0,T]$.

Proof

We need only verify that $\phi^{(i)}(0) = \phi^{(i)}(T) = 0$, $i=1,\ldots,M$.

Lemma

$\phi^{(i)}(t) = f^{M-i+1}(t)\, g_i(t)$, $i = 0,\ldots,M-1$, where g_i is an $M-i+1$ differentiable function on $[0,T]$.

Proof For $i = 1$, $\phi^{(1)}(t) = Mf^{M-1}(t)\; f^{(1)}(t)$. Thus the lemma holds for $i = 1$. Now assume that the lemma holds for $i = k$, $k < M$. Then

$$\phi^{(k+1)}(t) = f^{M-k}(t) g_k(t) + f^{M-k+1}(t)\; g_k^{(1)}(t)$$

$$= f^{M-k}(t)(g_k(t) + f(t) g_k^{(1)}(t))$$

end proof of lemma

Thus $\phi^{(k)}$ is zero at $t = 0$ and at $t = T$ if $k \leq M$, since g_k is differentiable (and therefore bounded) on $[0,T]$ for $k \leq M$.

Example (6)

Let $f(t) = \sin(\pi t/T)$. The testing function constructed from this function by application of the above theorem is neither an infinitely differentiable testing function nor a polynomial testing function.

Comment

We note that both polynomial testing functions and infinitely differentiable testing functions may be of the above form. In fact, if f is a polynomial or an infinitely differentiable testing function, then the testing fucntion constructed from f by application of the above theorem is a polynomial testing function or infinitely differentiable testing function, respectively.

4.2 Infinitely Differentiable Testing Functions

We give some examples of infinitely differentiable testing functions on [0,T].

1. $$\phi(t) = \begin{cases} \exp(-1/p(t)) & 0 \leq t \leq T \\ 0 \text{ elsewhere} \end{cases}$$

Where p(t) is a polynomial which is a positive on (0,T) and has zeros at t = 0 and at t = T.

<u>Example</u>

$p(t) = t^2 - tT$

2. Let f be infinitely differentiable on [0,T].

Let ϕ be an infinitely differentiable testing function on (0,T). Then $\phi(t)f(t)$ is an infinitely differentiable testing function on [0,T].

3. The union of the set of all infinitely differentiable testing functions on [0,T] with the zero function is a linear vector space over the field of real numbers. Thus, let ϕ_1 and ϕ_2 belong to this space. Let a and b be real scalars. Let

$$\phi_3 = a\phi_1 + b\phi_2$$

Then if ϕ_3 is not zero, ϕ_3 is an infinitely differentiable testing function.

Comment

The first three derivatives of the testing function given in Example (1) are plotted in Figure 6 for $T = 1$. Such a testing function is undesirable because integrals involving the second and third derivatives will tend to emphasize the part of the signals near the ends of the integration interval rather than giving a nearly uniform emphasis over the entire interval as does the polynomial testing function shown in Figure 7. We will restrict further investigation to the class of polynomial testing functions.

4.3 Choice of a Specific Polynomial Testing Function

We base the choice of a testing function primarily upon the criterion that the numerical computations in the sequence generator must be well posed. First, each integrand in the sequence generator must be "well behaved". Thus we must choose a testing function such that the testing function and its derivatives are "well behaved", i.e., they are "smooth" and the maximum magnitudes of their slopes (normalized to their maximum magnitudes) are as small as possible. Second, the integrands in the sequence generator should all vary over approximately the same range of values, else the resulting numerical integrations could be very ill posed.

We will consider polynomial testing functions of the general symmetric form

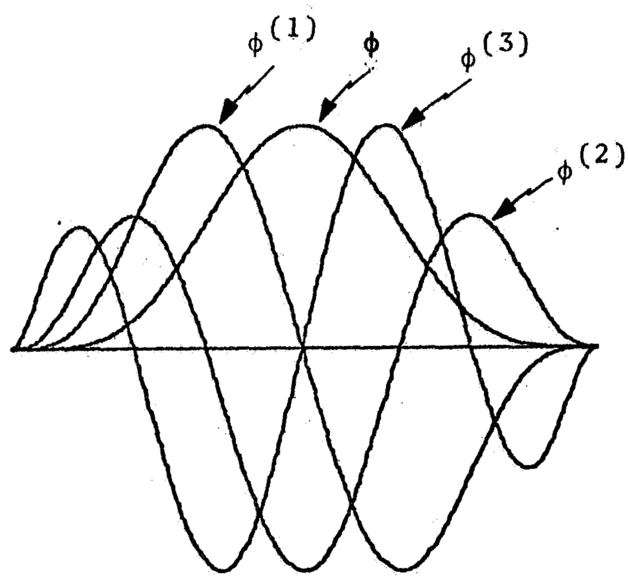


FIGURE 6

ϕ and the first three derivatives of ϕ on [0,2], $\phi = (t(2-t))^5$

$\max |\phi| = 1$

$\max |\phi^{(1)}| = 2.06$

$\max |\phi^{(2)}| = 12.0$

$\max |\phi^{(3)}| = 57.1$

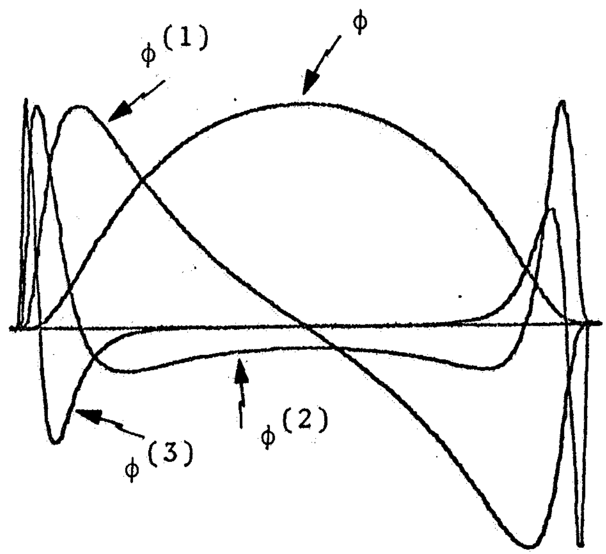


FIGURE 7

ϕ and the first three derivatives of ϕ on $[-1,1]$, $\phi = e^{-\frac{1}{1-t^2}}$

$\max\ |\phi| \quad = \quad .368$

$\max\ |\phi \quad | = \quad .798$

$\max\ |\phi \quad | = \quad 7.69$

$\max\ |\phi \quad | = 164.8$

$$\phi(t) = (t(t-T))^{M+1} \sum_{j=0}^{I} a_i (t-T/2)^{2i} \tag{1}$$

We use the symmetric testing function because it has the simplest form of all polynomial testing functions and because we evidently gain nothing by allowing asymmetry. Let us normalize ϕ so that

$$\max_t |\phi(t)| = 1 \tag{2}$$

Then the simplest choice of such a testing function is

$$\phi(t) = k(t(t-T))^{M+1} \tag{3}$$

This testing function clearly assumes its maximum at T/2; thus

$$k = (4/T)^{M+1} \tag{4}$$

In practice, the only problem encountered in using this testing f function was that $\phi^{(M)}$ was found to have a very steep slope near 0 and T, thus necessitating the use of an integration step size near 0 and T which was much smaller than that required over the rest of the interval [0,T]. Since data is usually sampled uniformly in time, either more data samples are required for each integration than might otherwise be needed if the slope of $\phi^{(M)}$ were better behaved, or else the data must be interpolated.

$\phi^{(M)}$ has a steep slope near 0 and T because its derivative is not constrained to be zero at 0 and T, whereas the slopes of all derivatives of ϕ which are of order less than M are constrained to be zero at 0 and T. Thus $\phi^{(M)}$ is the only one of the first M derivatives of ϕ such that its slope near 0 and T is significantly steeper than its maximum slope over the rest of the interval [0,T].

We may eliminate this problem by using

$$\phi(t) = (4/T)^{M+2}(t(t-T))^{M+2} \tag{5}$$

Then the first M derivatives of ϕ all have zero slopes at 0 and T.

We can show that the first M derivatives of this testing function are "well-behaved". In fact by Property (2) and (4) (See Section 3.2) and Corollary (4) (Section 4.1), the derivatives of this testing functions each have the least possible number of zeros and sign changes on [0,T], i.e., $\phi^{(k)}$ has k+2 zeros and k sign changes on [0,T]. Furthermore, by Corollary (4) all roots of $\phi^{(k)}$ are real and occur on [0,T]. 0 and T are roots of multiplicity M+2-k, and all other roots are simple. Because $\phi^{(k)}$ has no complex roots, it has no "wiggles" between its zeros. Thus ϕ and its derivatives are smooth functions well suited to numerical integration.

To make the integrands in the sequence generator all vary over approximately the same range, we first normalize the data (and the functions ϕ_k in the nonlinear case) (See Figure 1) by choosing k so that

$$\max_t |w(t)/k| = \max_t |v(t)| \text{ and setting } \hat{w} = \frac{w}{k} \tag{6}$$

where w and v are noise-corrupted observations of the system output and input respectively (See Figure 1). We use $\hat{w}$ and v in the identification algorithm and then divide the estimated differential equation coefficients of y and its derivatives by k.

We then choose T so that ϕ and its derivatives all have as nearly as possible the same maximum magnitudes. Let $\hat{\phi}(t) = \phi(Tt)$, $t \in [0,1]$. Then

$$d^k/dt^k\ (\hat{\phi}(t)) = (T^k)\phi^{(k)}(Tt) \tag{7}$$

and ϕ is a testing function on [0,1]. Let

$$\max_t |\hat{\phi}^{(k)}(t)| = \hat{M}_k \tag{8}$$

$$\max_t |\phi^{(k)}(t)| = M_k \tag{9}$$

Then we wish to choose T so that $\max_{j,k \leq M} |M_k - M_j|$ is minimized. From Corollary (9) (See Section 3.) we know that $\hat{M}_k$ is monotone increasing with k. From (7), (8), and (9), $\hat{M}_k/T^k = M_k$ and we have chosen $M_0 = \hat{M}_0 = 1$. Thus we wish to choose T so that

$$\max |\hat{M}_k/T^k - \hat{M}_j/T^j| \ , \ \hat{M}_0 = 1 \quad j,k \leq M \tag{10}$$

is minimized.

For the testing function given in Equation (5), the maximum absolute values of ϕ and its first M derivatives were calculated for M = 1,...,4, and were found to satisfy the relation

$$M_i \leq (M_m)^{k/M}, \quad i = 0,\ldots,M \tag{11}$$

For these testing functions, the following assertion gives the value of T that mimimizes $\max_{j,k} |M_k/T^k - M_j/T^j|$, j, k $\leq$ M.

<u>Assertion</u> (12)

Suppose that

$$\hat{M}_i \leq (\hat{M}_M)^{i/M}, \qquad i = 0,\ldots,M$$

Then

$$\max_{k,j} \left| \frac{\hat{M}_j}{T^j} - \frac{\hat{M}_k}{T^k} \right|$$

is minimized by $T = (\hat{M}_M)^{1/M}$

<u>Proof</u>

Let $\hat{T}$ be the value of T that minimizes

$$\max_{k,j} \left| \frac{\hat{M}_j}{T^j} - \frac{\hat{M}_k}{T^k} \right|$$

Let $T_0 \triangleq (\hat{M}_M)^{1/M}$. Let $\hat{M}_i \leq (\hat{M}_M)^{1/M}$. Then

$$\max_{k,j} \left| \frac{\hat{M}_j}{T^j} - \frac{\hat{M}_k}{T^k} \right| = \max_j \frac{\hat{M}_j}{T^j} - \min_k \frac{\hat{M}_k}{T^k}$$

For $T = T_0$,

$$\max_k \frac{\hat{M}_k}{T^k} = 1$$

Thus

$$\min_{T \geq T_0} \max_{k,j} \left| \frac{\hat{M}_k}{T^k} - \frac{\hat{M}_j}{T^j} \right| = \min_{T \geq T_0} \left(1 - \min_j \frac{\hat{M}_j}{T^j} \right)$$

$$= 1 - \max_{T \geq T_0} \min_k \frac{\hat{M}_k}{T^k}$$

$$= 1 - \min_k \frac{\hat{M}_k}{T_0^k}$$

Thus

$$\hat{T} \leq T_0$$

Clearly, $\hat{T} \geq 1$, since for $T < 1$,

$$\min_T \max_{j,k} \left| \frac{\hat{M}_k}{T^k} - \frac{\hat{M}_j}{T^j} \right| = \frac{\hat{M}_M}{T^M} \geq \hat{M}_M - 1$$

Thus

$$1 \leq \hat{T} \leq T_0$$

Now for $1 \leq T \leq T_0$ and $j < M$,

$$\frac{d}{dT}\left(\frac{\hat{M}_M}{T^M}\right) \Big/ \frac{d}{dT}\left(\frac{\hat{M}_j}{T^j}\right) = \frac{M}{j}\,\frac{\hat{M}_M}{T^M}\,T^{j-M}$$

$$= \frac{M}{j}\,\frac{\hat{M}_M}{T^M}\,(T_0/T)^M T^j > 1$$

Thus $\hat{M}_M/T^M$ decreases faster than $\hat{M}_j/T^j$ as T increases for $T \leq T_0$, $j < M$. Further,

$$\max_{\substack{k \\ T \leq T_0}} \frac{\hat{M}_k}{T^k} = \frac{\hat{M}_M}{T_0^M}$$

Thus

$$\min_{T \leq T_0}\ \max_{k}\ \frac{\hat{M}_k}{T^k} = \frac{\hat{M}_M}{T^M}$$

and for any $k < M$

$$\frac{\hat{M}_M}{T^M} - \frac{\hat{M}_k}{T^k}$$

is a monotone decreasing as T increases, for $T \leq T_0$. Thus

$$\min_{T \leq T_0}\ \max_{j,k} \left|\frac{\hat{M}_k}{T^k} - \frac{\hat{M}_j}{T^j}\right| = \min_{T \leq T_0}\left[\frac{\hat{M}_M}{T^M} - \min_j \frac{\hat{M}_k}{T^k}\right]$$

$$= \frac{\hat{M}_M}{T_0^M} - \min_j \frac{\hat{M}_j}{T_0^j}$$

<u>end proof</u>

Summary (13)

We have shown that a "good" M-differentiable polynomial testing function to use in the sequence generators is

$$\phi(t) = \begin{cases} (t(t-T))^{M+2}(4/T)^{M+2} & 0 \leq t \leq T \\ 0 \text{ elsewhere} & \end{cases}$$

and for $M \leq 4$, we set

$$T = \max_{t\varepsilon[0,1]} |\hat{\phi}^{(m)}(t)|^{1/M}$$

where

$$\hat{\phi}(t) = (t(t-1))^{M+2}(4)^{M+2} = \phi(t)\Big|_{T=1}$$

4.4 Identification in the Absence of Noise

We discuss only systems with the differential equation

$$L(p)y = u \tag{1}$$

where

$$L(p) = \sum_{i=0}^{M} a_i p^i \tag{2}$$

since the treatment of all other cases is analogous.

When the system input and output are observed in the absence of noise, we need only obtain a set of as many linear algebraic equations in the unknown constants as there are unknow constants, then solve these equations. To obtain these equations, we multiply (2) by a testing function ϕ and integrate the resulting equations, yielding

$$\sum_{i=0}^{M} a_i \int_{t_0}^{T+t_0} \phi(t-t_0) y^{(i)}(t)\,dt = \int_{t_0}^{T+t_0} \phi(t-t_0) u(t)\,dt \quad (3)$$

which by Corollary (30) (Section 1.6) becomes

$$\sum_{i=0}^{M} a_i (-1)^i \int_{t_0}^{T+t_0} \phi^{(i)}(t-t_0) y(t)\,dt = \int_{t_0}^{T+t_0} \phi(t-t_0) u(t)\,dt \quad (4)$$

which gives us one algebraic equation in the unknown constants. To get M+1 equations, we may use either M+1 linearly independent samples of y and u or M+1 linearly independent testing functions. In the first case we are simply using the first M+1 outputs of the sequence generator. In the second case, we need a set of M+1 linearly independent testing functions.

<u>Assertion</u> (5)

Let ϕ be a (2M-1)-differentiable testing function. Then $\{\phi, \phi^{(1)}, \ldots, \phi^{(M-1)}\}$ is a set of M M-differentiable linearly independent testing functions.

Proof

Clearly $\phi,\phi^{(1)},\ldots,\phi^{(M-1)}$ are all M-differentiable testing functions. Further, by Property (5) (Section 3.0), $\{\phi,\phi^{(1)},\ldots,\phi^{(2M-1)}\}$ is a linearly independent set on $[0,T]$.

end proof

Assertion (5) guarantees that any (2M)-differentiable testing function will suffice in the second case.

Finally, we note that if the set of testing functions given in Assertion (5) is used, we save considerable computation over the method of using one testing function and M samples of input and output since in the first case we need compute only 3M+2 integrals.

$$\int_0^T \phi^{(k)}(t)y(t)dt, \qquad k = 0,\ldots,2M$$

and

$$\int_0^T \phi^{(k)}(t)u(t)dt, \qquad k = 0,\ldots,M$$

whereas in the second case we need to compute $(M+1)^2$ integrals. (M+1 integrals for each sample of y and one integral for each sample of u, with M samples required).

4.5 Approximate Testing Functions

In a recent paper, Takaya[1] has proposed the following testing function.

Suppose that we have an M times differentiable function such that

$$\phi^{(i)}(0) \cong 0 \tag{1}$$

$$\phi^{(i)}(T) \cong 0 \qquad i = 0,\ldots,M \tag{2}$$

Then from the proof of Property (27), it is clear that

$$\int_0^T \phi(t)y^{(i)}(t)dt \cong (-1)^i\int_0^T \phi^{(i)}(t)y(t)dt \tag{3}$$

and if the approximations (1) and (2) are good enough, an identification using ϕ as an "approximate" testing function will be reasonably accurate. The function

$$g(t) = \exp(-t^2/2)H_0(t)/2 \tag{4}$$

where

$$g^{(k)}(t) = (-1)\exp(-t^2/2)H_k(t)/2 \tag{5}$$

$$H_k(t) = (-1)^k\exp(-t^2/2)d^k\exp(-t^2/2)/dt^k \tag{6}$$

(Hermite function)

on $[-r,r]$, for r sufficiently large, has been proposed by Takaya for such a testing function because for any given

M for r sufficiently large, H_0 is an approximate M-differentiable testing function and thus may be used for identification in the absence of noise as well as when noise is present.

Takaya carried out noise-free identification using H_0 which resulted in errors on the order of one or two percent. For a comparable system, errors in results obtained by using the polynomial testing function given in 4.3.13 resulted in errors on the order of less than .01 percent (the actual error is unknown since the results were printed out to only four decimal places and were all exactly correct as printed). Thus there appears to be no advantage in using this approximate testing function since it is both more difficult to evaluate and yields less accurate results than the polynomial testing function given in 4.3.13.

APPENDIX I

GENERATION OF AN INSTRUMENTAL VARIABLE SEQUENCE [1]

We have said that $\{E(S_k)\}$ is a satisfactory instrumental variable sequence. Clearly $\{E(S_k)\}$ cannot be computed directly when noise is present. However, the following procedure has been found in practice to give a sequence $\{\hat{S}_k\}$ which converges to $\{E(S_k)\}$ which simultaneously yielding a sequence of estimates of x.

1. Use any sequence $\{\hat{V}_k\}$ to get an initial estimate $\hat{x}_1$ of x. A sequence which is particularly convenient to use is given by

$$\hat{V}_k = (I_N \vdots \cdots \vdots I_N), \tag{1}$$

i.e., k adjoined NxN identity matrices, since using this sequence is equivalent to computing the matrix with i^{th} row

$$\sum_{i=1}^{k} s_{k(i-1)+j}, \quad s_i \text{ is the } i^{th} \text{ row of } S_k \tag{2}$$

i.e., using this instrumental variable sequence is equivalent to adding together rows of S_k to get an NxN matrix.

2. Using $\hat{x}_1$, we define the estimates $\hat{u}$ and $\hat{y}$ of u and y respectively, which are uncorrelated with the observation

noise as follows.

a. If the input observation noise N_1 is zero, let

$$\hat{u} = u \tag{3}$$

$$\hat{y} = \frac{\hat{M}_1(p)}{\hat{L}_1(p)} u \tag{4}$$

b. If the input observation noise N_1 is not zero and the output observation noise N_2 is zero, let

$$\hat{y} = y \tag{5}$$

$$\hat{u} = \frac{\hat{L}_1(p)}{\hat{M}_1(p)} y \tag{6}$$

where

$$\hat{L}(p)y = \hat{M}(p)u \tag{7}$$

is the differential equation of the system with coefficient $\hat{x}_1$.

3. Let $\{\hat{S}_k\}$ be the sequence obtained by computing $\{\hat{S}_k\}$ using $\hat{u}$ and $\hat{y}$ as though they were the observations v and w respectively. Then the forms of the components of $\hat{S}_k$ are as follows.

In the sampled linear filter case we have

a.

$$\int_0^{i\Delta T} h_j(i\Delta T-t)u(t)dt \tag{8}$$

and

$$\int_0^{i\Delta T} h_j(i\Delta T-t)\,\frac{\hat{M}_1(p)}{\hat{L}_1(p)}\;u(t)\,dt \tag{9}$$

if $N_1 = 0$ and $N_2 \neq 0$.

b.

$$\int_0^{i\Delta T} h_j(i\Delta T-\tau)\,y(t)\,dt \tag{10}$$

and

$$\int_0^{i\Delta T} \hat{h}_j(i\Delta T-\tau)\,u(t)\,dt \tag{11}$$

where

$$\hat{h}_j = \mathcal{L}^{-1}\,\frac{(p^{k_j}L_1(p))}{Q(p)M_1(p)}\,, \qquad k_j \leq M$$

If the testing function sequence generator is used, we have

a.

$$\int_{iT}^{(i+1)T} \phi^{(k)}(t-iT)u(t)\,dt \tag{12}$$

and

$$\int_{iT}^{(i+1)T} \phi^{(k)}(t-iT)\,\frac{\hat{M}_1(p)}{\hat{L}_1(p)}\,u(t)dt \tag{13}$$

if $N_1 = 0$ and $N_2 \neq 0$, and

b.

$$\int_{iT}^{(i+1)T} \phi^{(k)}(t)y(t)dt \tag{14}$$

and

$$\int_{iT}^{(i+1)T} \phi^{(k)}(t-iT)\,\frac{\hat{L}_1(p)}{\hat{M}_1(p)}\,y(t)dt \tag{15}$$

if $N_1 \neq 0$ and $N_2 = 0$.

Let degree $(\hat{L}_1)$ - degree $(\hat{M}_1) = m$ (16)

Then clearly for the sampled linear filter sequence generator we must have

degree $(Q) - M > m$ (17)

else we cannot calculate the integral (13) without knowledge of derivatives of u because $\hat{h}_j$ will contain delta functions for some j. For the same reason it is clear that for the testing function sequence generator ϕ must be an (M+m)-differentiable testing function.

4. Having completed the first three steps, we now use $\{\hat{S}_k{}^T\}$ as an instrumental variable sequence to get a new estimate $\hat{x}_2$ of x. We then repeat steps 2 and 3 using $\hat{x}_2$ in place

of $\hat{x}_1$, and so on.

In this manner we continue to obtain alternately new estimates of x and $\{S_k\}$. This method has been found to work rather well in practice. Wong and Polak give a method for improving the estimation of x by incorporating into the algorithm estimation of some noise statistics which are then used to get an asymptotically optimal instrumental variable sequence.

APPENDIX II

Time Scaling of Data

Time scaling is an important factor in the numerical stability of the computations in the identifications algorithms given in the first two chapters. We will discuss time scaling of data. Suppose that we have a system with the differential equation

$$\sum_{i=0}^{M} a_i p^i y = \sum_{j=0}^{M} b_j p^j u \tag{1}$$

Now suppose that we apply the indentification algorithm to the time scaled data

$$y\ (\alpha t) \text{ and } u\ (\alpha t) \tag{2}$$

By this we mean that wherever y(t) and u(t) appear in the identification algorithm, we use $y(\alpha t)$ and $u(\alpha t)$ respectively. Next, we note that for any scalar α,

$$y^{(i)}(\alpha t) \triangleq \frac{d^i}{d\xi^i} \ y(\xi)\Big|_{\xi=\alpha t} \tag{3}$$

and

$$\frac{d^i}{dt^i}\, y(\alpha t) = \frac{d^i}{d\xi^i}\, y(\alpha\xi)\Big|_{\xi=t} = \alpha^i y^{(i)}(\alpha t) \tag{4}$$

Then y and u satisfy the differential equation

$$\sum_{i=0}^{M} a_i y^{(i)}(\alpha t) = \sum_{j=0}^{M} b_j u^{(j)}(\alpha j) \tag{5}$$

or for $\alpha \neq 0$

$$\sum_{i=0}^{M} a_i \frac{1}{\alpha^i} \frac{d^i}{dt^i} y(\alpha t) = \sum_{j=0}^{M} b_j \frac{1}{\alpha^j} \frac{d^j}{dt^j} u(\alpha t) \tag{6}$$

Let us assume that $y(\alpha t)$ and $u(\alpha t)$ satisfy the differential equation

$$\sum_{i=0}^{M} \hat{a}_i \frac{d^i}{dt^i} y(\alpha t) = \sum_{j=0}^{M} \hat{b}_i \frac{d^j}{dt^j} u(\alpha t) \tag{7}$$

Then the coefficients estimated by using the time scaled data are $\hat{a}_i$ and $\hat{b}_i$ then

$$\hat{a}_i = a_i \frac{1}{\alpha^i} \tag{8}$$

$$\hat{b}_i = b_i \frac{1}{\alpha^i} \qquad i = 0,\ldots,M \tag{9}$$

and

$$\begin{aligned} a_i &= \hat{a}_i \alpha^i \\ b_i &= \hat{b}_i \alpha^i \end{aligned} \tag{10}$$

We further note that if, in the system differential equation we have $(f(y(t))^{(i)}$ in place of $y^{(i)}(t)$, then the above result holds without modification.

In order that the identification algorithm be as numerically stable as possible, we should choose a time scaling factor so that the estimated coefficients $\hat{a}_i$ and $\hat{b}_i$ are as nearly equal in magnitude as possible. Let α_0 be the scale factor. Let $a_0,\ldots,a_m$, $b_0,\ldots,b_m$ be the best a priori estimate or guess of the system coefficients. Then we choose α_0 so that

$$\max_i[|a_i/\alpha_0{}^i|,|b_i/\alpha_0{}^i|], \text{ the best a priori estimate of } \max_i[|\hat{a}_i|,|\hat{b}_i|], \tag{11}$$

is minimized.

When applying the instrumental variable method, successive estimates of the system coefficients may be used to get successive estimates of the best scale factor.

APPENDIX III

IDENTIFICATION OF MULTIPLE INPUT, MULTIPLE OUTPUT SYSTEMS

Consider a linear time invariant differential system S characterized by $(y,u,\{y,u\})$, where

$$y_1,\ldots,y_K \quad \text{and} \quad u_1,\ldots,u_J$$

are the outputs and inputs respectively,

$$y = (y_1,\ldots,y_K)^T$$

$$u = (u_1,\ldots,u_J)^T$$

and $\{y,u\}$ is the set of all input-output pairs of S. Let L(p) be a KxK matrix differential operator with generic element $\ell_{jk}(p)$ a polynomial in p of order N, where N is the assumed maximum possible degree of $\ell_{jk}(p)$, i.e.,

$$\ell_{jk}(p) = \sum_{i=0}^{N} a_{jki}p^i$$

Let M(p) be a KxJ matrix differential operator with generic element $m_{kj}(p)$ a polynomial in p of order N, i.e.,

$$m_{kj}(p) = \sum_{i=0}^{N} b_{kji}p^{i}$$

Definition

A vector differential equation

$$L(p)y = M(p)u$$

will be called canonical if L is square and has been reduced to triangular form by the standard elimination algorithm.[1]

Theorem

Given S characterized by $(y,u,\{y,u\})$, there is exactly one canonical differential equation

$$L(p)y = M(p)u \qquad (*)$$

such that

1. all pairs (y,u) which satisfy (*) lie in $\{y,u\}$

and

2. all pairs (y,u) in $\{y,u\}$ satisfy (*) for some set of initial conditions.

Proof

Assume there are at least two such equations,

a. $L_1(p)y = M_1(p)u$

b. $L_2(p)y = M_2(p)u$

$L_1 = L_2$;

Consider the homogeneous equations

$$\left.\begin{aligned} L_1 y &= 0 \\ L_2 y &= 0 \end{aligned}\right\} \qquad (**)$$

Clearly by proper choice of initial conditions, i.e., $y_k, \ldots, y_K$ and their derivatives are zero, successively for $k = 2, \ldots, K$, we can show that the diagonal elements of L_1 and L_2 are equal. Further, any solution to (**) is also a solution to

$$\hat{L}(p)y = (L_1 - L_2)y = 0$$

but the diagonal elements of $\hat{L}$ are zero and by proper choice of initial conditions (i.e., the same as above) it is clear that the superdiagonal elements of $\hat{L}$ are zero. We repeat the argument for each parallel diagonal of $\hat{L}$. Thus $\hat{L} = 0$, i.e., $L_1 = L_2$.

$M_1 = M_2$;

Since $L_1 y = L_2 y$ for all y in $\{y,u\}$, it must be true that

$$M_1 u = M_2 u \qquad \forall u \in \{y,u\} \qquad (***)$$

Then it is clear that we can pick $u, u^{(1)}, \ldots, u^{(M)}$ sufficiently varied such that (***) can be satisfied if and only if

$M_1 = M_2$.

end proof

Corollary

There are at most (K!)(J!) canonical differential equations for S.

Proof

Each such equation corresponds to a different ordering of inputs and/or outputs in the input and output vectors.

Comments

1. Given a canonical differential equation which describes S, there is a well known algorithm[2] for constructing a state equation for S with a minimal state (i.e., the state vector is of the least possible dimension of all state vectors of S).

2. By the theorem, the identification problem is now reduced to that of finding a canonical differential equation describes S.

We now give an algorithm which gives a canonical differential equation which describes S.

Assumptions

1. None of the outputs satisfies an equation of the form

$$\ell(p)y_k = 0$$

where $\ell(p)$ is a polynomial in p.

2. The constant term in $\ell_{kk}(p)$ is nonzero (i.e., is 1), $k = 1,\ldots,K$.

Algorithm

1. Using the error decomposition identification algorithm,[3] find the bottom row of L and M, i.e., find $\ell_{KK}(p)$ and $m_{Kj}(p)$, $j = 1,\ldots,J$ such that

$$\ell_{KK}(p)y_K = \sum_{j=1}^{J} m_{Kj}(p)u_j$$

2. Again using the error decomposition algorithm, find the K-1 row of L and M. If the inputs are observed noise free, we can find y_K using the K row equation so that we may use y_K as another noise free input while finding the K-1 row, thus getting faster convergence than if noisy observations of y_K were used.

3. We continue in this fashion with the K-2, K-2,...,1 rows, thus identifying the entire system, using noise free estimates of the outputs as in 2., whenever possible, to speed convergence.

Comments

3. Having found ℓ_{kk}, we may assume that the degree of ℓ_{jk}, $j < k$, is at most (degree ℓ_{kk})-1 since the differential equation is assumed to be estimated. In particular, if ℓ_{jk} is a constant (including zero) then $\ell_{jk} \equiv 0$, $j < k$.

4. Assumption 1. is not necessary if the error decomposition algorithm is used, but is necessary if the instrumental variable algorithm is used.

5. If we use the instrumental variable method[4] in place of the error decomposition algorithm, we must assume that in each row some specific constant in M or in an off-diagonal element of L is not zero. If in fact the assumption we make is wrong, then our estimates will not converge. In particular, a row equation may be homogeneous, i.e., a row of M may be zero, in which case we must assume that some specific constant in one of the off-diagonal elements of that row of L is not zero. Thus if we do not use the error decomposition algorithm we must institute a system of checks to determine whether a row equations is homogeneous and to find an appropriate nonzero constant in L or M, whichever is needed. This system of checks is the price we pay for lack of knowledge of observation noise statistics, knowledge of which enables us to use the error decomposition algorithm.

6. If we use the error decomposition algorithm, we implicitly assume that we know some observation noise statistics. If we assume that none of the noise processes associated with the inputs and outputs are correlated with one another, then knowledge of the autocorrelations of each such process is sufficient to implement the algorithm. If any two processes are correlated, we must know their cross-correlation function in order to implement the algorithm.

7. It is clear that results exactly analogous to the above results hold for systems described by finite difference equations, if we let p be the differencing operator, i.e.,

$$p(y(k)) = y(k+1) - y(k)$$

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