MASSACHUSETTS INSTITUTE OF TECHNOLOGY

RE-56

STRAPDOWN NAVIGATION EQUATIONS FOR GEOGRAPHIC AND TANGENT COORDINATE FRAMES

by

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June, 1969

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ABSTRACT

Six coordinate frames relevant to the operation of a radar-aided strapdown inertial navigation system are defined and the relationships between these frames are established. Analytic expressions for the specific force are derived for the cases of computation in the local geographic frame and in the tangent coordinate frame. An algorithm for the solution of the direction cosine matrix is indicated. Approximate analytic relations are derived which relate the change in latitude and longitude to the radar coordinates.

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STRAPDOWN NAVIGATION EQUATIONS FOR GEOGRAPHIC AND TANGENT COORDINATE FRAMES

1. INTRODUCTION

The initial navigation system testing performed by N.A.S.A. E.R.C. (Phase IB) utilized a space stabilized inertial navigation system and a ground based tracking radar. It was found convenient to perform the navigational computations in a tangent plane coordinate frame which has its origin at the radar site or at the takeoff or landing point. The Phase II test effort will involve a strapdown inertial navigation system with, at least temporarily, the same tracking radar. It is hoped, however, that in the future other navigational aids can be incorporated, such as D.M.E., I.L.S., Decca, etc.

The question naturally arises as to whether the tangent computational frame should continue to be used in the Phase II test effort. Since the tangent frame has been successfully used in the past, there is some feeling in favor of its continued use. On the other hand, since the Phase II effort involves the development and testing of a hybrid navigation system for a general class of VTOL aircraft, it is desirable to have area navigation capability. This would involve the use of a reference frame which is not directly tied into a base point. The local geographic reference frame is a likely candidate in this regard since it is mechanized in virtually all of the aircraft inertial navigation systems built to date.

It is the objective of this paper, therefore, to specify the navigational equations to be used in the Phase II test effort. These equations will be derived for both a local geographic and tangent plane computational frame.

2. FRAME DEFINITIONS

It will be convenient in the work which follows to define six orthogonal, right-handed coordinate frames:

(1) Inertial frame ∿ "i" ∿ Earth centered, inertially nonrotating frame.

(2) Geographic frame ∿ "n" ∿ local north-east-down frame with origin at the system's location.

(3) Body frame ∿ "b" ∿ frame mechanized by the inertial sensor's sensitive axes with origin at the system's location.

(4) Earth frame ∿ "e" ∿ Earth centered frame which is fixed to the Earth.

(5) Tangent frame ∿ "t" ∿ earth fixed frame with origin at the radar location which is aligned with the local geographic frame at the radar site.

(6) Radar frame ∿ "r" ∿ spherical coordinate frame defined by radar range, heading, and elevation with origin at the radar site.

Figure 2.1 illustrates the relationships between the inertial, geographic, earth, and tangent frames. Note that at t = 0, the inertially fixed reference meridian, earth frame meridian, and local meridian are coincident. Thus we have that

$$\lambda = \ell - \ell_o + \omega_{ie} t \tag{2.1}$$

where

λ ∿ celestial longitude

ℓ ∿ terrestrial longitude from Greenwich

ℓ_o ∿ initial terrestrial longitude from Greenwich

ω_{ie} ∿ Earth's angular velocity

t ∿ time

(N, E, D)	∿ geographic
(x, y, z)	∿ tangent
(x_i, y_i, z_i)	∿ inertial
(x_e, y_e, z_e)	∿ Earth

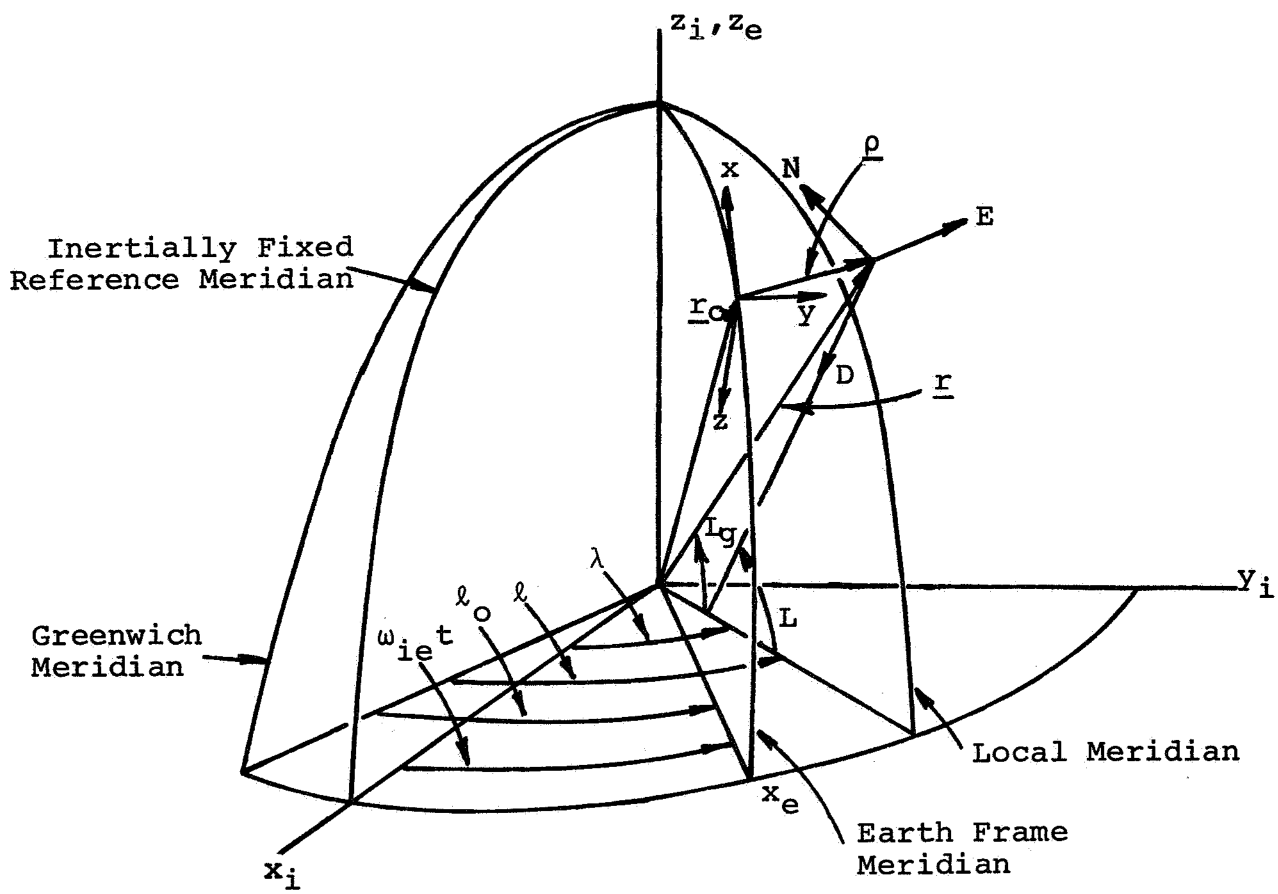

Figure 2.1 ∿ Coordinate Frame Geometry

Figure 2.2 shows the relationship between the radar and tangent frames.

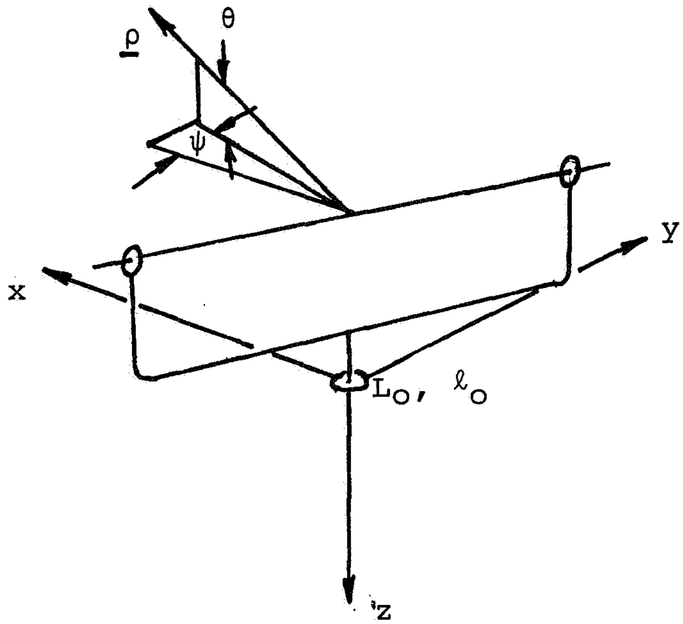

Figure 2.2 ∿ Radar and Tangent Frame Relationship

Thus the location of the aircraft in tangent coordinates is given by:

$$\underline{\rho}^t = \begin{bmatrix} \rho \cos\theta \cos\psi \\ \rho \cos\theta \sin\psi \\ -\rho \sin\theta \end{bmatrix} = \begin{bmatrix} \rho_x \\ \rho_y \\ \rho_z \end{bmatrix} \qquad (2.2)$$

We now list the relationships between the various coordinate frames:

2.1 Inertial-Geographic

$$\underline{\omega}^n_{in} = \{\dot{\lambda} \cos L, -\dot{L}, -\dot{\lambda} \sin L\} \qquad (2.3)$$

$$\underline{C}_i^n = \begin{bmatrix} -\sin L \cos \lambda & -\sin L \sin \lambda & \cos L \\ -\sin \lambda & \cos \lambda & 0 \\ -\cos L \cos \lambda & -\cos L \sin \lambda & -\sin L \end{bmatrix} \tag{2.4}$$

where:

$\underline{\omega}_{in}^n$ ∿ angular velocity of the geographic frame with respect to the inertial frame

L ∿ geographic latitude

$\underline{C}_i^n$ ∿ coordinate transformation matrix relating inertial coordinates to geographic coordinates

$(\underline{\ })^n$ ∿ superscript on vector denotes coordinatization in that reference frame

2.2 Inertial-Tangent

$$\underline{\omega}_{it}^t = \{\omega_{ie} \cos L_o, 0, -\omega_{ie} \sin L_o\} \tag{2.5}$$

$$\underline{C}_i^t = \begin{bmatrix} -\sin L_o \cos \omega_{ie}t & -\sin L_o \sin \omega_{ie}t & \cos L_o \\ -\sin \omega_{ie}t & \cos \omega_{ie}t & 0 \\ -\cos L_o \cos \omega_{ie}t & -\cos L_o \sin \omega_{ie}t & -\sin L_o \end{bmatrix} \tag{2.6}$$

where:

L_o ∿ geographic latitude at origin of tangent plane

2.3 Tangent-Geographic

$$\underline{\omega}_{tn}^n = \{\dot{\ell} \cos L, -\dot{L}, -\dot{\ell} \sin L\} \tag{2.7}$$

$$\underline{C}_t^n = \begin{bmatrix} \sin L \sin L_o \cos(\ell-\ell_o) + \cos L \cos L_o & -\sin L \sin(\ell-\ell_o) & \sin L \cos L_o \cos(\ell-\ell_o) - \sin L_o \cos L \\ \sin L_o \sin(\ell-\ell_o) & \cos(\ell-\ell_o) & \cos L_o \sin(\ell-\ell_o) \\ \sin L_o \cos L \cos(\ell-\ell_o) - \sin L \cos L_o & -\cos L \sin(\ell-\ell_o) & \cos L \cos L_o \cos(\ell-\ell_o) + \sin L \sin L_o \end{bmatrix} \quad (2.8)$$

The above transformation matrix can be approximated through series expansion to apply to situations where the origin of the tangent and geographic frames are separated by only a short distance. The second order approximation to equation (2.8) is given by:

$$\underline{C}_t^n \cong \begin{bmatrix} 1 - \frac{\Delta L^2}{2} - \sin^2 L_o \frac{\Delta\ell^2}{2} & -\Delta\ell(\sin L_o + \Delta L \cos L_o) & \Delta L - \frac{\Delta\ell^2}{4} \sin 2L_o \\ \Delta\ell \sin L_o & 1 - \frac{\Delta\ell^2}{2} & \Delta\ell \cos L_o \\ -\Delta L - \frac{\Delta\ell^2}{4} \sin 2L_o & -\Delta\ell(\cos L_o - \Delta L \sin L_o) & 1 - \frac{\Delta L^2}{2} - \frac{\Delta\ell^2}{2}\cos^2 L_o \end{bmatrix} \quad (2.9)$$

where:

$$\Delta L = L - L_o$$

$$\Delta\ell = \ell - \ell_o$$

It follows that the linear approximation to equation (2.8) is given by:

$$\underline{C}_t^n \cong \begin{bmatrix} 1 & -\Delta\ell \sin L_o & \Delta L \\ \Delta\ell \sin L_o & 1 & \Delta\ell \cos L_o \\ -\Delta L & -\Delta\ell \cos L_o & 1 \end{bmatrix} \quad (2.10)$$

3. GEOGRAPHIC FRAME MECHANIZATION

Strapdown systems are characterized by their lack of gimbal support structure. The system is mechanized by mounting three gyros and three accelerometers directly to the vehicle for which the navigation function is to be provided. An onboard digital computer keeps track of the vehicle's attitude with respect to some reference frame based on information from the gyros. The computer is thus able to provide the coordinate transformation necessary to coordinatize the accelerometer outputs in a reference frame. Navigation computations proceed in exactly the same fashion as for platform systems.

A functional block diagram for a strapdown system which computes in the geographic reference frame is shown in figure (3.1).

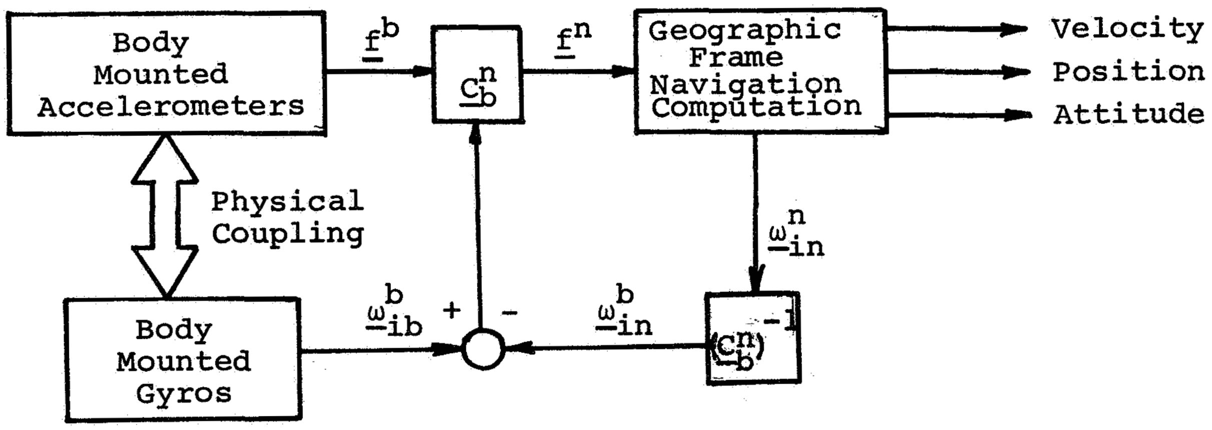

Figure 3.1 ∿ Geographic Frame Strapdown System

3.1 Navigation Equations

The outputs of the accelerometers are equal to the nonfield specific force coordinatized in body axes:

$$\underline{f}^b = \underline{C}_i^b\,(\ddot{\underline{r}}^i - \underline{G}^i) \tag{3.1}$$

where:

$\underline{f}$ ∿ nonfield specific force exerted on the instruments

$\underline{G}$ ∿ gravitational acceleration at the instrument location

$\ddot{\underline{r}}$ ∿ inertially referenced acceleration

After transformation into geographic axes, equation 3.1 can be written as a function of the geographic latitude, L, celestial longitude, λ, and the radii of curvature r_L and r_ℓ as follows:

$$\underline{f}^n = \begin{bmatrix} r_L\ddot{L} + \frac{1}{2}r_\ell(\dot{\lambda}^2-\omega_{ie}^2)\sin 2L + 2\dot{r}_L\dot{L} - \ddot{r}e\sin 2L - \xi g \\ r_\ell\,\ddot{\lambda}\cos L - 2r_\ell\dot{L}\dot{\lambda}\sin L + 2\dot{r}_\ell\dot{\lambda}\cos L + \eta g \\ -g - \ddot{r} - r_L\ddot{L}e\sin 2L + r_\ell(\dot{\lambda}^2 - \omega_{ie}^2)\cos^2 L + \frac{r_L^2}{r}\dot{L}^2 \end{bmatrix} \tag{3.2}$$

where:

$r_L \cong r(1 - 2e\cos 2L)$

∿ radius of curvature in meridian plane

$r_\ell \cong r\,(1 + 2e\sin^2 L)$

∿ radius of curvature in co-meridian plane

ξ ∿ meridian deflection of the vertical (positive about east)

η ∿ prime deflection of the vertical (positive about north)

e ∿ Earth's ellipticity $\cong$ 1/297

g ∿ magnitude of gravity

Equation 3.2 is an approximate expression which contains terms which are greater than $2 \times 10^{-5}g$ for the following maximum values of vehicle motion:

$$r\ddot{L}_{max} = r\ddot{\lambda}_{max} \leq 0.5g$$

$$\dot{L}_{max} = \dot{\ell}_{max} \leq 1.6 \times 10^{-5} \text{ rad/sec } (340 \tfrac{\text{ft}}{\text{sec}} \text{ velocity})$$

$$\dot{r}_{max} \leq 100 \text{ ft/sec}$$

$$\ddot{r}_{max} \leq 2g$$

Those limits correspond to those which one would expect to encounter in an aircraft application such as a helicopter. See ref. 1 for the detials of the derivation of eq. 3.2.

Navigational information is readily obtained from $\underline{f}^n$ since, if Coriolis and cross coupling compensation is provided in eq. 3.2, then

$$\underline{f}^n_{compensated} = \begin{bmatrix} r_L\ddot{L} \\ r_\ell\ddot{\lambda}\cos L \\ -\ddot{r} - g \end{bmatrix}$$

Latitude and longitude can then be found by a double time integration of the north and east specific force measurements, respectively.

Ref. 1. Britting, "Analysis of Space Stabilized Inertial Navigation Systems," M.I.T. E.A.L. Report RE-35, 1968.

Figure 3.2 illustrates the computation in detail.

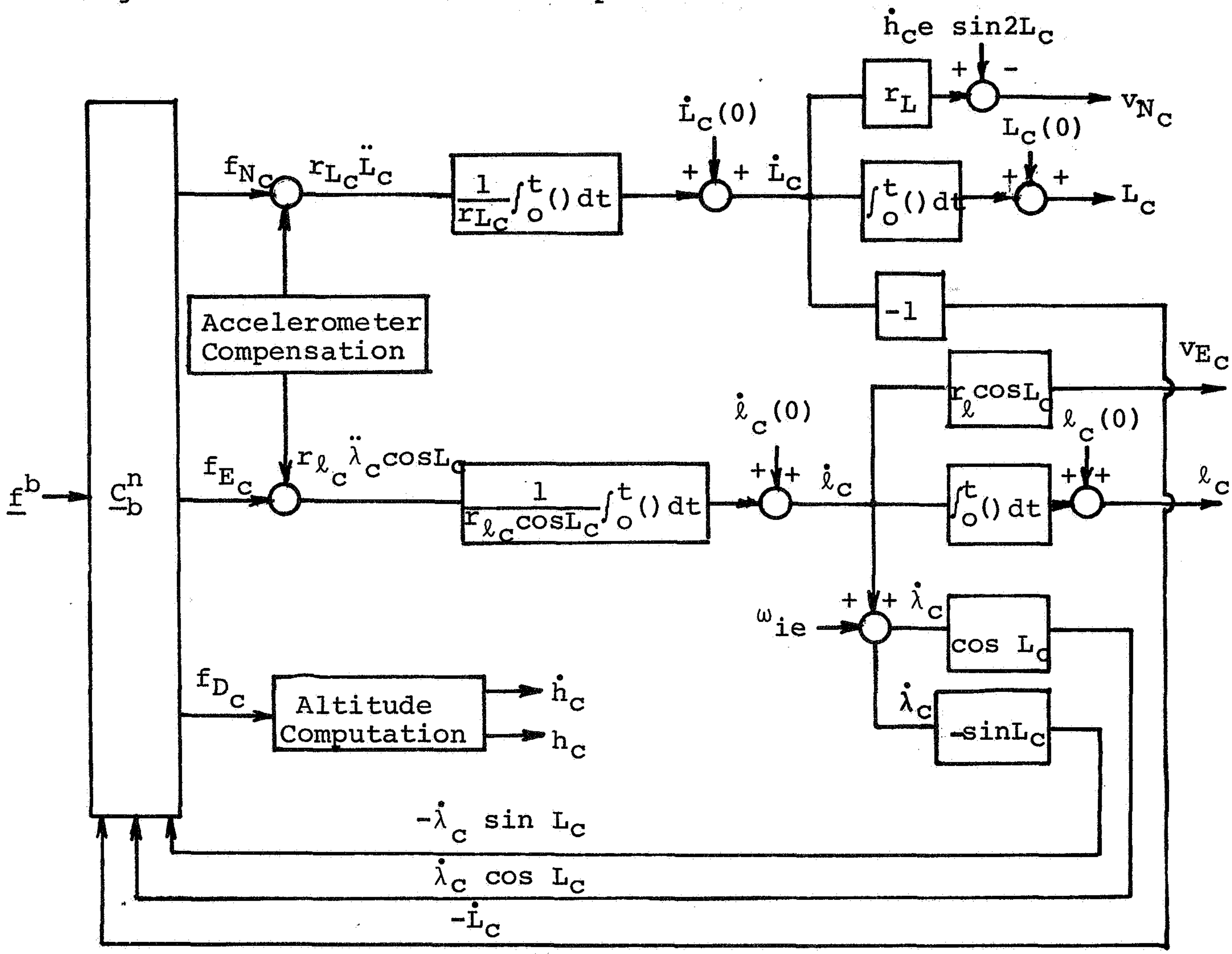

Figure 3.2 ∿ Geographic Computation Scheme

In Figure (3.2) the subscript "c" denotes a computed physical quantity. In addition, it was noted that the Earth referenced velocity, coordinatized in geographic axes is given by (to an accuracy of better than 0.1 ft/sec for aircraft altitudes):

$$\underline{v}^n \cong \begin{bmatrix} r_L \dot{L} - \dot{h}\ e \sin 2L \\ r_\ell\ \dot{\ell} \cos L \\ -\dot{h} \end{bmatrix} \tag{3.3}$$

As is shown in Figure (3.2), the computation scheme assumes that the effective north and east accelerometer outputs are given by the north and east components of equation (3.2). Thus the indication of latitude and longitude is found by subtracting off the Coriolis and cross coupling terms from the components of equation (3.2). Thus

$$r_{L_c}\ddot{L}_c = f_{N_c} - \frac{1}{2}\, r_{\ell_c}(\dot{\lambda}_c^2 - \omega_{ie}^2)\sin 2L_c - 2\dot{r}_{L_c}\dot{L}_c + \ddot{r}_c e \sin 2L_c \tag{3.4}$$

$$r_{\ell_c}\ \ddot{\lambda}_c \cos L_c = f_{E_c} + 2r_{\ell_c}\dot{L}_c\ \lambda_c \sin L_c - 2\dot{r}_{\ell_c}\ \dot{\lambda}_c \cos L_c \tag{3.5}$$

Note that the deflection of the vertical terms cannot be included in the above expression since no knowledge of their magnitudes is assumed.

Now the computed expressions for the radii of curvature are given by:

$$r_{L_c} = r_c(1 - 2e \cos 2L_c) \tag{3.6}$$

$$r_{\ell_c} = r_c(1 + 2e \sin^2 L_c) \tag{3.7}$$

and the calculated magnitude of the Earth radius vector is given by:

$$r_c = r_{o_c} + h_c \tag{3.8}$$

where

r_{o_c} ∿ calculated local geocentric Earth radius magnitude

$\simeq$ $r_e(1 - e \sin^2 L_c)$ [maximum error: 150 ft]

h_c ∿ estimated height above the reference Earth model's surface

r_e ∿ Earth's equatorial radius

The transformation from body to geographic axes is specified by using a combination of information from the strapdown gyros with information generated within the navigational computation loop (see figure 3.1). The transformation is generated by solving the matrix differential equation:

$$\dot{\underline{C}}^n_b = \underline{C}^n_b \, \underline{\Omega}^b_{nb} \tag{3.9}$$

where $\underline{\Omega}^b_{nb}$ is the skew symmetric representation of the angular velocity of the body frame with respect to the geographic frame:

$$\begin{aligned} \underline{\omega}^b_{nb} &= \underline{\omega}^b_{ib} - \underline{\omega}^b_{in} \\ &= \{\omega_x, \ \omega_y, \ \omega_z\} \end{aligned} \tag{3.10}$$

The angular velocity of the geographic frame with respect to the inertial frame is just the computed version of equation (2.3):

$$(\underline{\omega}^n_{in})_c = \{\dot{\lambda}_c \cos L_c, \ -\dot{L}_c, \ -\dot{\lambda}_c \sin L\} \tag{3.11}$$

Equation (3.9) can alternately be interpreted as a shorthand way of writing nine simultaneous differential equations in nine unknowns, as can be seen by writing it out in component form:

$$\begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} & \dot{C}_{13} \\ \dot{C}_{21} & \dot{C}_{22} & \dot{C}_{23} \\ \dot{C}_{31} & \dot{C}_{32} & \dot{C}_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_{12}\omega_z - C_{13}\omega_y & C_{13}\omega_x - C_{11}\omega_z & C_{11}\omega_y - C_{12}\omega_x \\ C_{22}\omega_z - C_{23}\omega_y & C_{23}\omega_x - C_{21}\omega_z & C_{21}\omega_y - C_{22}\omega_x \\ C_{32}\omega_z - C_{33}\omega_y & C_{33}\omega_x - C_{31}\omega_z & C_{31}\omega_y - C_{32}\omega_x \end{bmatrix}$$

The solution of equation (3.9) for the direction cosines can be done in a variety of ways. If single-degree-of-freedom, delta-modulated instruments are used, the gyro output angle is sampled and passed into a zero order hold circuit. Pulse torquing is then applied to the gyro float to null the instrument. Weiner (Ref. 2) shows that for this mechanization each output pulse is proportional to the integral of the input angular velocity. Thus the output of the instrument represents an angular rotation about the input axis equal to $\Delta\theta$. This property can be exploited in the solution for the direction cosine matrix if one considers a Taylor series expansion of $\underline{C}_b^n$ in Δt:

$$\underline{C}(t + \Delta t) = \underline{C}(t) + \dot{\underline{C}}(t)\ \Delta t + \frac{1}{2}\ \ddot{\underline{C}}(t)\ \Delta t^2 + \frac{1}{3!}\ \dddot{\underline{C}}\ \Delta t^3 + \ldots . \qquad (3.12)$$

where $\underline{C}_b^n = \underline{C}$ for notional simplicity.

But application of equation (3.9) yields:

$$\underline{C}(t + \Delta t) = \underline{C}(t)[\underline{I} + \underline{\Omega}\ \Delta t + \frac{1}{2}\ (\underline{\Omega}^2 + \dot{\underline{\Omega}})\Delta t^2 + \ldots .] \qquad (3.13)$$

If the first two terms of the expansion are used,

$$\underline{C}(t + \Delta t) = \underline{C}(t) + \underline{C}(t)\ \Delta\underline{\theta} \qquad (3.14)$$

where it was noted that:

$$\underline{\Omega}\ \Delta t = \Delta\underline{\theta}$$

and $\Delta\underline{\theta}$ is a skew symmetric matrix composed of the gyro outputs $\Delta\theta_k$, $k = x, y, z$. The result shown in equation (3.14) could, of course, have been shown by applying the definition of the derivative directly to equation (3.9).

If the computational algorithm of equation (3.14) is used, which corresponds to a rectangular integration scheme, then the

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Ref. 2. Weiner, "Theoretical Analysis of Gimballess Inertial Reference Equipment Using Delta Modulated Instruments," Sc.D. Thesis, Department of Aeronautics and Astronautics, 1962.

algorithm error (truncation error) is approximately given by the third term of equation (3.13):

$$\underline{\delta C} = \frac{1}{2}\,\underline{C}(\underline{\Omega}^2 + \dot{\underline{\Omega}})\,\Delta t^2 \tag{3.15}$$

Thus the time step, Δt, must be chosen such that the errors resulting from the vehicle angular velocity, $\underline{\Omega}$, and the vehicle angular acceleration, $\dot{\underline{\Omega}}$, satisfy the error budget. In addition, the finite computer word length causes the occurrence of "round-off" error.

3.2 Utilization of Radar Information

In order to use the radar information to update the inertial navigation system, the radar vector, equation (2.2), must be expressed as a function of latitude and longitude.

3.2.1 Latitude-Radar Relations

The development for the latitude relationship is motivated by first writing the system position vector in inertial coordinates (see figure 2.1):

$$\underline{r}^i = r \begin{bmatrix} \cos L_g \cos \lambda \\ \cos L_g \sin \lambda \\ \sin L_g \end{bmatrix} = \begin{bmatrix} r_{x_i} \\ r_{y_i} \\ r_{z_i} \end{bmatrix}$$

where:

$L_g \sim$ geocentric latitude

Then it follows that:

$$L_g = \sin^{-1} \frac{r_{z_i}}{r} \tag{3.16}$$

In order to solve eq. (3.16) for L_g, it will be necessary to express the r_{z_i} component of $\underline{r}^i$ in terms of $\underline{\rho}$. It is seen from figure 2.1 that:

$$\underline{r} = \underline{r}_o + \underline{\rho}$$

where

$\underline{r}$ $\sim$ system geocentric position vector

$\underline{r}_o$ $\sim$ radar geocentric position vector

$\underline{\rho}$ $\sim$ position vector from the radar (origin of tangent frame) to the system position

Thus the system position vector, coordinatized in the inertial frame, is given by:

$$\underline{r}^i = \underline{C}_t^i \, [\underline{r}_o^t + \underline{\rho}^t] \tag{3.17}$$

Now from equation (2.2)

$$\underline{\rho}^t = \{\rho_x, \rho_y, \rho_z\} \tag{3.18}$$

and from figure 2.1

$$\underline{r}_o^t = \{-r_o \sin D_o, 0, -r_o \cos D_o\} \tag{3.19}$$

where

$D_o \sim$ deviation of the normal at the base point

$D_o \approx e \sin 2L_o$

Thus substitution of equations (2.6), (3.18), and (3.19) into equation (3.17) yields the desired expression for r_{z_i}:

$$r_{z_i} = r_o \sin L_{g_o} + \rho_x \cos L_o - \rho_z \sin L_o \tag{3.20}$$

We solve for r in equation (3.17) by noting that from the expression for $\underline{r}$ in tangent coordinates, equations (3.18) and (3.19),

$$r^2 = (\rho_x - r_o \sin D_o)^2 + \rho_y{}^2 + (\rho_z - r_o \cos D_o)^2$$

or

$$r^2 \approx r_o{}^2 + \rho^2 - 2r_o \rho_z - 2r_o \rho_x e \sin 2L_o$$

Thus:

$$r^{-1} \cong r_o{}^{-1} \left(1 - \frac{1}{2}\frac{\rho^2}{r_o{}^2} + \frac{\rho_z}{r_o} + \frac{\rho_x}{r_o} e \sin 2L_o\right) \tag{3.21}$$

Substitution of equations (3.21) and (3.20) into equation (3.16) yields:

$$\sin L_g = \left(\sin L_{g_o} + \frac{\rho_x}{r_o}\cos L_o - \frac{\rho_z}{r_o}\sin L_o\right)\left(1 - \frac{1}{2}\frac{\rho^2}{r_o{}^2} + \frac{\rho_z}{r_o} + \frac{\rho_x}{r_o} e \sin 2L_o\right)$$

We can expand the above equation as a function of ΔL, where

$$\Delta L = L - L_o$$

by noting that:

$$L_g = L_{g_o} + \Delta L_g,$$

$$L_g = L - D,$$

$$D = e \sin 2L,$$

which yields:

$$\Delta L \left[1 + e(1 - 3\cos 2L_o)\right] = \frac{\rho_x}{r_o}\left(1 + \frac{\rho_z}{r_o} + 2e\sin^2 L_o - \frac{1}{2}\frac{\rho_x}{r_o}\tan L_o\right) - \frac{3}{2}\frac{\rho_z^2}{r_o^2}\tan L_o - \frac{1}{2}\frac{\rho_y^2}{r_o^2}\tan L_o + \frac{1}{2}\Delta L^2 \tan L_o$$

The presence of the tan L_o terms in the above equation points out the well known fact that geographic frames are inappropriate for use in polar applications. We see that except for operation in the polar regions, the dominant term is the north component of $\underline{\rho}$. Thus if we let $\Delta L^2 = \frac{\rho x^2}{r_o^2}$,

$$\Delta L \simeq \frac{\rho_x}{r_o}\left(1 + \frac{\rho_z}{r_o} + 2e\cos 2L_o\right) - \frac{1}{2}\frac{\rho_y^2}{r_o^2}\tan L_o \qquad (3.22)$$

We note that for $|\underline{\rho}| < 10$ n.m., we can calculate ΔL using

$$\Delta L = \frac{\rho x}{r_o} \tag{3.23}$$

with a maximum error of only about 4 $\widehat{\text{sec}}$.

3.2.2 Longitude-Radar Relationships

The derivation of the longitude relationships is again motivated by geometrical considerations. From figure 2.1, consider the projections of $\underline{r}_o$, $\underline{\rho}$, and $\underline{r}$ in the equatorial plane:

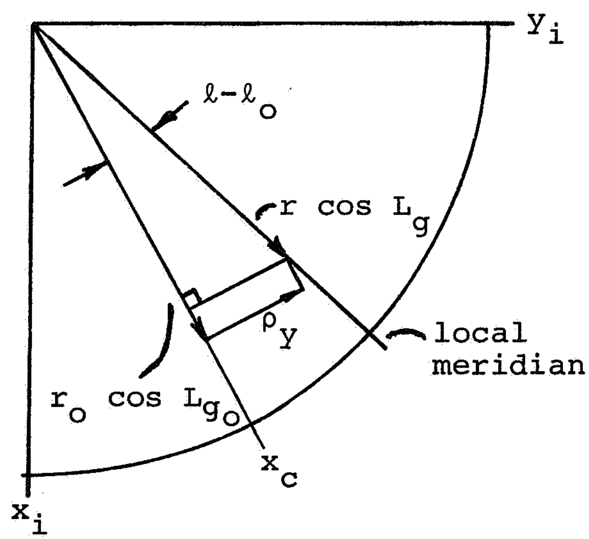

Figure 3.3 ∿ Equatorial Plane Projections

We see from the sketch that:

$$\sin(\ell - \ell_o) = \frac{\rho y}{r \cos L_g} \tag{3.24}$$

Letting

$$\ell - \ell_o = \Delta\ell$$

and

$$L_g = L - D = L_o + \Delta L - D$$

then

$$\Delta\ell = r^{-1}\,\rho_y\ \sec L_o\,[1 + (\frac{\rho_x}{r_o} - e\,\sin 2L_o)\,\tan L_o]$$

But from equation (3.21),

$$r^{-1} = r_o^{-1}\,(1 + \frac{\rho_z}{r_o} + \frac{\rho_x}{r_o}\,e\,\sin 2L_o - \frac{1}{2}\,\frac{\rho^2}{r_o^2}) \tag{3.25}$$

Thus

$$\Delta\ell = \frac{\rho_y}{r_o}\ \sec L_o\,[1 - 2e\,\sin^2 L_o + \frac{\rho_x}{r_o}\,\tan L_o + \frac{\rho_z}{r_o} - \frac{1}{2}\,\frac{\rho^2}{r_o^2}] \tag{3.26}$$

Note that for ρ_y and $\rho_x < 10$ n.m., the change in longitude can be calculated using:

$$\Delta\ell = \frac{\rho_y}{r_o}\ \sec L_o \tag{3.27}$$

with a maximum error of about 4 $\widehat{\text{sec}}$.

4. TANGENT FRAME MECHANIZATION

The functional block diagram for a strapdown system which computes in tangent coordinates is shown in Figure (4.1).

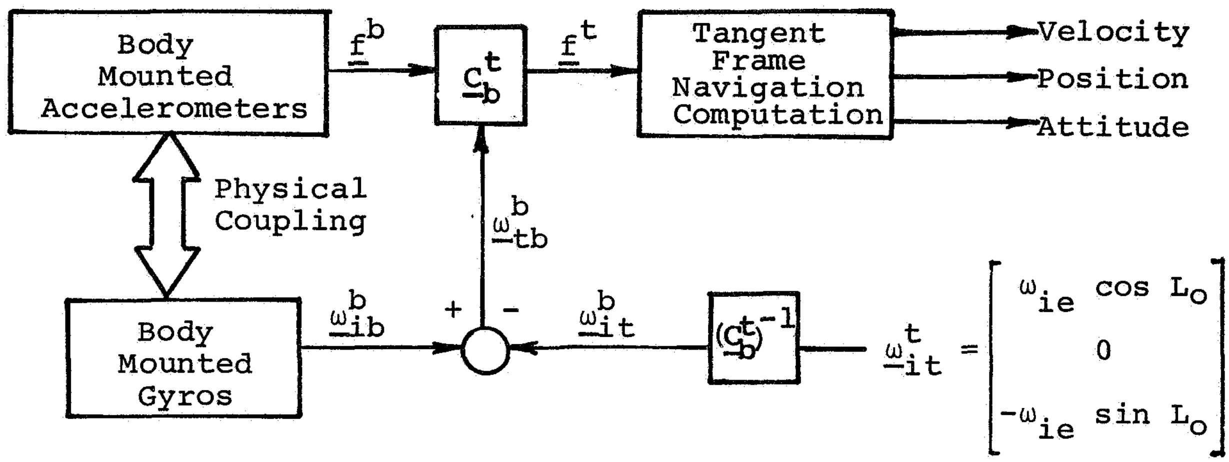

Figure 4.1 ∿ Tangent Frame Strapdown System

We note, of course, that the instrument outputs are identical to the geographic frame mechanization discussed previously since the body frame is instrumented by both systems. We also note that the angular velocity computation is independent of the navigation computations since the angular velocity of the tangent frame with respect to the inertial frame is a constant vector (see equation 2.5).

4.1 Navigation Equations

As before, the output of the accelerometers is given by:

$$\underline{f}^b = \underline{C}_i^b(\ddot{\underline{r}}^i - \underline{G}^i) \tag{4.1}$$

The computer transformation matrix, $\underline{C}_b^t$, is then used to transform the specific force into tangent axes:

$$\underline{f}^t = \underline{C}_i^t\,\ddot{\underline{r}}^i - \underline{C}_i^t\,\underline{G}^i \tag{4.2}$$

We can navigate with respect to the tangent frame origin by noting from Figure (2.1) that:

$$\underline{r} = \underline{r}_o + \underline{\rho} \qquad (4.3)$$

Substituting equation (4.3) into equation (4.2) yields:

$$\underline{f}^t = \underline{C}_i^t \, \ddot{\underline{r}}_o^i + \underline{C}_i^t \, \ddot{\underline{\rho}}^i - \underline{C}_i^t \, \underline{G}^i \qquad (4.4)$$

Since

$$\underline{r}_o^i = r_o \, \{\cos L_{g_o} \cos \omega_{ie} t, \; \cos L_{g_o} \sin \omega_{ie} t, \; \sin L_{g_o}\},$$

after differentiating twice with respect to time and transforming the resulting expression to the tangent frame using equation (2.6), we get:

$$\underline{C}_i^t \, \ddot{\underline{r}}_o^i = r_o \, \omega_{ie}^2 \begin{bmatrix} \sin L_o \cos L_{g_o} \\ 0 \\ \cos L_o \cos L_{g_o} \end{bmatrix}$$

Expansion of cos L_{g_o} in terms of L_o and D_o, the deviation of the normal at the base point, shows that:

$$\cos L_{g_o} = \cos L_o \, (1 + 2e \sin^2 L_o).$$

But the radius of curvature in the co-meridian plane at the base point is given by:

$$r_{\ell_o} = r_o \, (1 + 2e \sin^2 L_o)$$

Thus:

$$\underline{C}_i^t \, \ddot{\underline{r}}_o^i = r_{\ell_o} \, \omega_{ie}^2 \cos L_o \begin{bmatrix} \sin L_o \\ 0 \\ \cos L_o \end{bmatrix} \qquad (4.5)$$

An analytic expression for $\underline{C}_i^t \ddot{\underline{\rho}}^i$ is found by differentiating the quantity, $\underline{\rho}^i = \underline{C}_t^i \underline{\rho}^t$, twice with respect to time, noting that $\dot{\underline{C}}_t^i = \underline{C}_t^i \underline{\Omega}_{it}^t$ and that $\dot{\underline{\Omega}}_{it}^t = 0$:

$$\underline{C}_i^t \ddot{\underline{\rho}}^i = \ddot{\underline{\rho}}^t + 2\underline{\Omega}_{it}^t \dot{\underline{\rho}}^t + \underline{\Omega}_{it}^t \underline{\Omega}_{it}^t \underline{\rho}^t \tag{4.6}$$

We expand the above equation in component form by noting that:

$$\underline{\rho}^t = \{\rho_x, \rho_y, \rho_z\}$$

and

$$\underline{\omega}_{it}^t = \{\omega_{ie} \cos L_o, 0, -\omega_{ie} \sin L_o\}.$$

Thus:

$$\underline{C}_i^t \ddot{\underline{\rho}}^i = \begin{bmatrix} \ddot{\rho}_x + 2\omega_{ie} \dot{\rho}_y \sin L_o - \omega_{ie}^2(\rho_x \sin^2 L_o + \frac{1}{2}\rho_z \sin 2L_o) \\ \ddot{\rho}_y - 2\omega_{ie}(\dot{\rho}_x \sin L_o + \dot{\rho}_z \cos L_o) - \omega_{ie}^2 \rho_y \\ \ddot{\rho}_z + 2\omega_{ie} \dot{\rho}_y \cos L_o - \omega_{ie}^2(\frac{1}{2} \rho_x \sin 2L_o + \rho_z \cos^2 L_o) \end{bmatrix} \tag{4.7}$$

Since $\underline{G}$ takes on a convenient form when coordinatized in the geographic frame, we let

$$\underline{C}_i^t \underline{G}^i = \underline{C}_n^t \underline{G}^n$$

Now

$$\underline{G}^n = \underline{g}^n + \underline{\Omega}_{ie}^n \underline{\Omega}_{ie}^n \underline{r}^n,$$

But since

$$\underline{g}^n = \{\xi g, -\eta g, g\}$$

and

$$\underline{r}^n = \{-r \sin D, 0, -r \cos D\},$$

it follows that:

$$\underline{G}^n = \begin{bmatrix} r\omega_{ie}^2 \sin L \cos L_g + \xi g \\ -\eta g \\ g + r\omega_{ie}^2 \cos L \cos L_g \end{bmatrix} \tag{4.8}$$

We next want to transform equation (4.8) into the tangent frame such that we retain only terms which are greater than $2 \times 10^{-5}g$. The question then arises as to which version of the $\underline{C}_t^n$ matrix should be used. If we assume that $|\underline{\rho}| < 10$ n.m., then ΔL and $\Delta \ell$ are on the order of 1/344 rad. Thus, for this case, equation (4.8) can be transformed using the linearized version of $\underline{C}_t^n$ (eq. 2.10) since the error terms will be on the order of $\frac{\Delta L^2}{2} g$ or $0.5 \times 10^{-5}g$. Also, the centrifugal terms in eq. 4.8 will be effectively cancelled by the similar terms in eq. 4.5. In addition, the deflection of the vertical terms which have a maximum value of about $3 \times 10^{-4}g$ can be neglected when multiplied by either ΔL or $\Delta \ell$. Taking into account the above considerations, we find that for the case of $|\underline{\rho}| < 10$ n.m., that:

$$\underline{C}_i^t \, \ddot{\underline{r}}_o - \underline{C}_i^t \, \underline{G}^i = \begin{bmatrix} -\xi g + g\Delta L \\ \eta g + g\Delta\ell \cos L_o \\ -g \end{bmatrix} \tag{4.9}$$

where it was noted from equation (3.25) that:

$$r \cong r_o - \rho_z - \rho_x e \sin 2L_o + \frac{1}{2}\frac{\rho^2}{r_o} \tag{4.10}$$

For the case of $|\underline{\rho}| < 100$ n.m., eq. (4.9) is modified to read:

$$\underline{C}_i^t \, \ddot{\underline{r}}_o^i - \underline{C}_i^t \underline{G}^i = \begin{bmatrix} -\xi g + g(\Delta L + \frac{\Delta\ell^2}{4} \sin 2L_o) + r_o \, \omega_{ie}^2 \Delta L \sin^2 L_o \\ \eta g + g\Delta\ell(\cos L_o - \Delta\ell \sin L_o) + r_o \, \omega_{ie}^2 \, \Delta\ell \cos L_o \\ -g + g\,(\frac{\Delta L^2}{2} + \frac{\Delta\ell^2}{2} \cos^2 L_o) + \frac{1}{2} r_o \, \omega_{ie}^2 \, \Delta L \sin 2L_o \end{bmatrix} \tag{4.11}$$

If we now substitute eq. (4.9) and (4.7) into eq. (4.4), we have the desired expression for the specific force in tangent coordinates for the case of $|\underline{\rho}| < 10$ n.m.

$$\underline{f}^t = \begin{bmatrix} \ddot{\rho}_x + 2\omega_{ie} \; \dot{\rho}_y \sin L_o - \xi g + g \, \Delta L \\ \ddot{\rho}_y - 2\omega_{ie} \; (\dot{\rho}_x \sin L_o + \dot{\rho}_z \cos L_o) + \eta g + g \, \Delta \ell \cos L_o \\ \ddot{\rho}_z + 2\omega_{ie} \; \dot{\rho}_y \cos L_o - g \end{bmatrix} \qquad (4.12)$$

where it was noted that $\rho\omega_{ie}{}^2$ terms in equation (4.7) are less than $2 \times 10^{-5}g$ for $\rho = 10$ n.m.

The appropriate expression for $|\underline{\rho}| < 100$ n.m. is obtained by substituting eq. (4.11) and (4.7) into eq. (4.4), which yields:

$$\underline{f}^t = \begin{bmatrix} \ddot{\rho}_x + 2\omega_{ie}\dot{\rho}_y \sin L_o - \xi g + g(\Delta L + \frac{\Delta \ell^2}{4} \sin 2L_o) \\ \ddot{\rho}_y - 2\omega_{ie}(\dot{\rho}_x \sin L_o + \dot{\rho}_z \cos L_o) + \eta g + g\Delta\ell(\cos L_o - \Delta L \sin L_o) \\ \ddot{\rho}_z + 2\omega_{ie} \; \dot{\rho}_y \cos L_o - g + g \; (\frac{\Delta L^2}{2} + \frac{\Delta \ell^2}{2} \cos^2 L_o) \end{bmatrix}$$

(4.13)

where it was noted that $\Delta L \simeq \frac{\rho_x}{r_o}$ and $\Delta \ell \simeq \frac{\rho_y}{r_o} \sec L_o$.

The procedure to be used in solving for $\underline{\rho}$ is now quite straightforward. The transformation from body to tangent coordinates is computed by solving the matrix differential equation:

$$\dot{\underline{C}}_b^t = \underline{C}_b^t \, \underline{\Omega}_{tb}^b \qquad (4.14)$$

where the angular velocity between the tangent and body frames is computed from the relationship:

$$\underline{\omega}_{tb}^{b} = \underline{\omega}_{ib}^{b} - \underline{\omega}_{it}^{b} \qquad (4.15)$$

The computation scheme is based on the assumption that the output of the accelerometers is given by eq. (4.12) or (4.13), depending on the magnitude of $\underline{\rho}$. For the $|\underline{\rho}| < 10$ n.m. case, if we let the computed specific force coordinatized in tangent axes be denoted by:

$$\underline{f}_c^t = \{f_{x_c}, f_{y_c}, f_{z_c}\}$$

then

$$\ddot{\rho}_{x_c} = f_{x_c} - 2\omega_{ie} \; \dot{\rho}_{y_c} \sin L_{o_c} - g \, \Delta L_c \qquad (4.16)$$

$$\ddot{\rho}_{y_c} = f_{y_c} + 2\omega_{ie} \, (\dot{\rho}_{x_c} \sin L_{o_c} + \dot{\rho}_{z_c} \cos L_o) - g \, \Delta \ell_c \cos L_{o_c} \qquad (4.17)$$

$$\ddot{\rho}_{z_c} = f_{z_c} - 2\omega_{ie} \; \dot{\rho}_{y_c} \cos L_{o_c} + g_c \qquad (4.18)$$

One then performs a double integration of each of these equations. But for this case we have from equations (3.23) and (3.27) that $\Delta L_c = r_o^{-1} \rho_{x_c}$ and $\Delta \ell_c = r_o^{-1} \rho_{y_c} \sec L_{o_c}$. Thus:

$$\ddot{\rho}_{x_c} + \frac{g_c}{r_o} \rho_{x_c} + 2\omega_{ie} \, \dot{\rho}_{y_c} \sin L_{o_c} = f_{x_c} \qquad (4.19)$$

$$\ddot{\rho}_{y_c} + \frac{g_c}{r_o} \rho_{y_c} - 2\omega_{ie} \, (\dot{\rho}_{x_c} \sin L_{o_c} + \dot{\rho}_{z_c} \cos L_{o_c}) = f_{y_c} \qquad (4.20)$$

$$\ddot{\rho}_{z_c} - g_c + 2\omega_{ie} \, \dot{\rho}_{y_c} \cos L_{o_c} = f_{z_c} \qquad (4.21)$$

It is seen from the above equations that the gravity field vector magnitude, g, must be calculated. We therefore proceed to define an analytical expression for this quantity:

From equation (4.8),

$$g = G_D - r\,\omega_{ie}^{2} \cos L \cos L_g \qquad (4.22)$$

Now the vertical component of the gravitational field vector can be calculated with an accuracy of about 2×10^{-5} g using the following expression (see Ref. 1):

$$G_D = \frac{E}{r^2}\,[1 - J(1 - 3 \cos 2L)] \qquad (4.23)$$

where

E ∿ product of the Earth's mass with the universal gravitational constant

$J \cong 0.82 \times 10^{-3}$

We can substitute eq. (4.23) into (4.22) and expand the resulting expression in the region around the base point, yielding:

$$g = g_o[1 + 2\,\frac{\rho_z}{r_o} + 2(e - 3J)\,\frac{\rho_x}{r_o}\,\sin 2L_o - \frac{\rho^2}{r_o^2}] + \rho_x\,\omega_{ie}^{2}\sin 2L_o \qquad (4.24)$$

where

g_o ∿ gravity magnitude at base point

If $|\underline{\rho}| < 10$ n.m., then the above expression can be simplified to:

$$g = g_o\,(1 + 2\,\frac{\rho_z}{r_o}) \qquad (4.25)$$

Thus in equations (4.19) and (4.20), the gravity field magnitude can be calculated using eq. (4.25):

$$g_c = g_o\,(1 + 2\,\frac{\rho_{z_c}}{r_o}) \qquad (4.26)$$

while in eq. (4.21), equation (4.24) must be used:

$$g_c = g_o \left[1 + 2\,\frac{\rho_z^*}{r_o} + 2\,(e - 3J)\,\frac{\rho_{x_c}}{r_o}\,\sin 2L_o - \frac{\rho_c^{\,2}}{r_o^{\,2}}\right] + \rho_{x_c}\omega_{ie}^{\,2}\sin 2L_{o_c} \tag{4.27}$$

Note that an external indication of ρ_z, i.e., ρ_z*, is necessary in order to obtain a convergent calculation of ρ_{z_c}.

The radar information is, of course, very easily related to the navigation system's output:

$$\{\rho_{x_c},\ \rho_{y_c},\ \rho_{z_c}\},$$

since these quantities are identical to the radar coordinates defined by equation (2.2).