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COMPUTATION FRAMES FOR STRAPDOWN INERTIAL SYSTEMS

by

William Robert Killingsworth, Jr.

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Degree of Master of Science

MEASUREMENT SYSTEMS LABORATORY

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Submitted to the Department of Aeronautics and Astronautics in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

Three configurations of a strapdown navigation system are analyzed to determine the effect of mechanizing different computation frames. For systems computing in the navigation frame and in an inertial frame, a linear analysis is developed. These linear theories are verified as accurate analytical descriptions of the systems by a computer solution of the system differential equations by numerical methods. Gyro drift and torquing uncertainty are found to be the predominant error sources. In the system computing in the geographic frame, these two error sources result in bounded latitude error but unbounded longitude error. For inertial frame computation these two error sources result in unbounded errors for both longitude and latitude.

Thesis Supervisor: Winston R. Markey, Sc.D.
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1. INTRODUCTION

Any inertial navigation system must solve the acceleration equation to obtain position and velocity from measured values of specific force and time. The basic problem in system design is the choosing of some optimum mechanization including the choices of mechanical arrangement, the coordinate systems used, and the methods of making calculations. One method of differentiating between classes of inertial navigation systems is the physical arrangement of the instruments used to measure acceleration and attitude. The majority of inertial navigation systems today use two or three single-axis accelerometers mounted on a stabilized platform which is maintained at the desired orientation by the attitude reference gyros and platform servo system.(1,2,3) The platform is either kept non-rotating with respect to inertial space or precessed so that the input axis of the accelerometers are kept coincident with some slowly rotating set of coordinate axes, such as local geographic coordinates.

In the strapped-down systems, which constitute the other principle class, the accelerometers are mounted to the vehicle and hence the system is gimballess.(2,4,5) Single axis accelerometers measure the specific force and resolve it into body fixed coordinates. These strapped-down navigation systems are the subject of increasing interest and investigation because recent advances in computers and gyroscopic technology have enabled gimballess systems to compete successfully with gimballed systems for a variety of missions. (4,5) The gimballess inertial measuring unit potentially possesses many advantages over a gimballed inertial measuring unit. System weight, volume, power, cost, packaging flexibility, reliability, and maintainability are possible advantages. Many of these advantages are the direct result of removing the necessity of a gimbal assembly.(6)

However, by removing the gimbals, serious problems are generated. No longer is there a physically instrumented reference frame and no longer do the inertial measuring instruments operate in a comparatively benign environment. Now the measurement of accelerations and angular rates is done in the body coordinate frame. Since the computations are generally performed in another frame, the computer must provide the transformation between the body coordinate frame and the inertial coordinates. Hence, by eliminating the gimbal assembly, the burden of storing and updating the inertial orientation information is placed upon the guidance computer. The inertial measuring instruments also present a very serious problem in a strapped-down system because of the environment in which they must perform.(1,3,6) Most previous gyro applications used the device as a null instrument operating in a non-rotating or slowly rotating reference frame(1,2,3), however, in the strapped-down system, the gyros are required to operate in the generally rotating body frame. Since the accuracy obtainable from a gyroscopic instrument depends upon the angular rates to which it is subjected, the gyros will need to be of significantly better quality. However, the development of gyros and accelerometers which can be more accurately restrained through pulse torquing and the development of compact high speed digital computers gives reason to believe that the strapped-down system is feasible as an accurate navigation system.(4,5,6)

A linear analysis of a navigation system usually gives valuable insight into the performance of the system, especially concerning the response of the system to error sources. The major objective of this paper is the development and verification of a linear analysis for a system computing in the navigation frame and a system computing in an earth centered inertial frame. To verify the validity of these developments, the systems will be simulated on a digital computer. Error sources to be considered are accelerometer bias, gyro drift, torquing uncertainty, and initial matrix misalignments. A comparison of the two systems will be made concerning the response to the various error sources and to mechanization advantages or disadvantages. An analytical study is made of the system which computes in the body frame in order to investigate its requirements for mechanization.

2. SYSTEM MECHANIZATION

In this section, some of the possible mechanizations of a strapped-down navigation system are presented. The characteristic which is common to all strapped-down navigation systems is the fact that the accelerometers are mounted directly to the vehicle and, hence, measure the specific force vector in vehicle body coordinates. What is then done with this specific force information is the major point of difference in the various system mechanizations. The general configuration of a strapped-down navigation system may take either of two(6) basic forms. The first system uses the angular rate information from the body mounted gyros to compute a continuously up-to-date transformation matrix which relates the body coordinate frame to some stabilized coordinate frame. The specific force acting on the vehicle, as indicated in body coordinates by the body-mounted accelerometers, is then transformed to the stabilized coordinates through the transformation matrix. The navigation computation is then carried out in that frame just as if a physically instrumented stabilized frame had been employed. The second system solves the entire problem directly in the body coordinate frame. Solving the navigation or guidance problem in body coordinates requires the use of Coriolis terms in the differential equations which account for the changes in the components of a vector along body axes which are due to the rotation of the body.

There are basically three different computation configurations which may be used to mechanize a strapped-down navigation system. The three coordinate frames used as computation frames are the inertial frame, the non-inertial navigation (or local vertical) frame, and the body fixed coordinate frame.

The non-inertial navigation frame configuration is computationally analagous to the three gimbal local vertical navigation system.(1,2) The transformation matrix is computed and then used to transform the measured specific force, which is coordinatized in the body frame, into the inertial frame. This is then transformed into the navigation frame using latitude and longitude information. Acceleration compensations are then made and the resulting signal is integrated once for velocity information and again for position information.

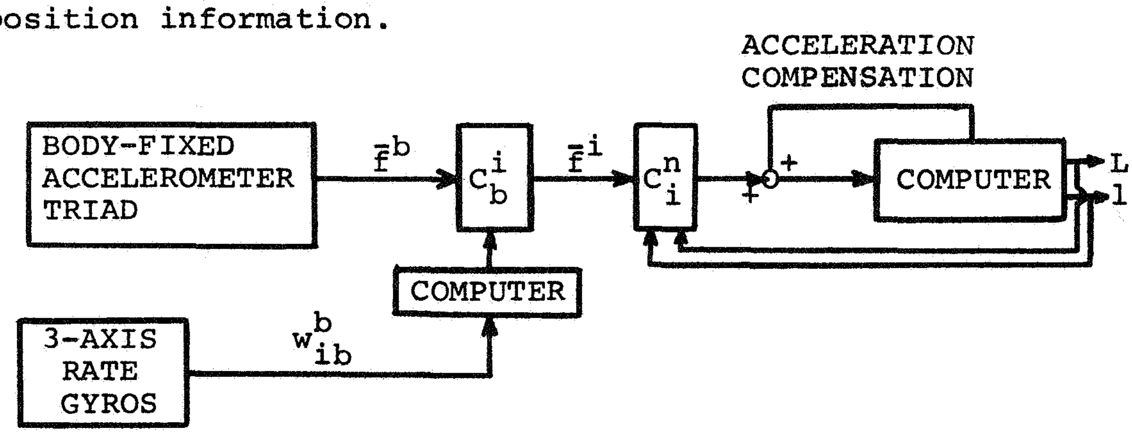


Figure 1. Computation in navigation coordinates

The strapped-down system which computes in inertial coordinates is computationally analagous to a space-stabilized system. Again the gyro output information is used to compute the direction cosine transformation from body coordinates to inertial coordinates. The inertial acceleration is obtained by compensating the gravitational force from the transformed specific force. This inertial acceleration is then integrated twice to obtain navigation information.

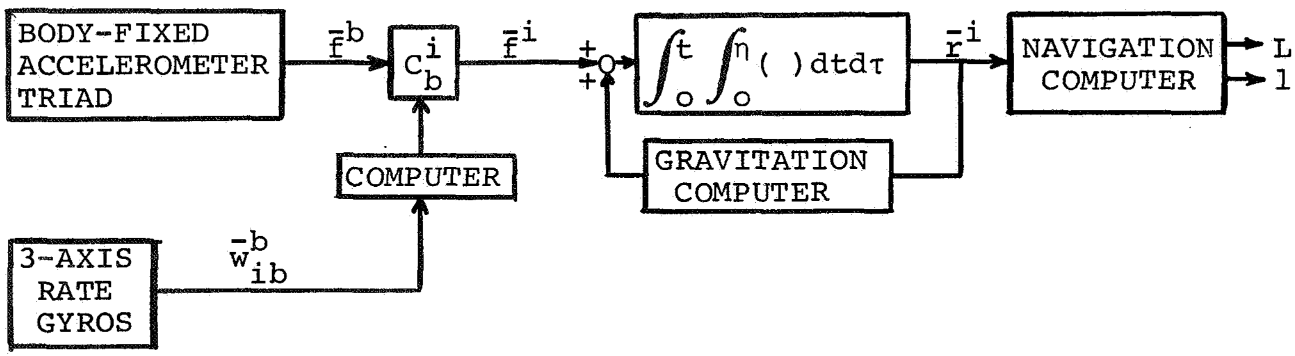


Figure 2. Computation in inertial coordinates

The system which computes in the body fixed coordinate frame is the most radically different of the three configurations. In this system the specific force vector is operated on in the body fixed frame; it is not transformed into another coordinate frame. This body resolved vector is updated using the standard form of the Law of Coriolis.(7) If two frames have a common origin and differentiation for any vector $\bar{A}$ it is true that

$$\dot{\bar{A}}_I = \dot{\bar{A}}_B + \bar{W} \times \bar{A}_B$$

where $\bar{W}$ is the angular velocity of the moving frame with respect to the stationary frame. If the above equation is to be practically applied, all components must be in the same frame. Assuming that all the components are available in the moving (body) frame, the vector $\bar{A}$ resolved into moving axes may be obtained by integration

$$\bar{A} = \int \dot{\bar{A}}_I dt - \int (\bar{W} \times \bar{A})\, dt$$

For example, if $\dot{\bar{A}}_I$ is the output of an orthogonal triad of body fixed accelerometers and $\bar{W}$ is the output of a corresponding set of gyros, the velocity vector resolved into body axes may be computed

$$\dot{\bar{V}}_B = \bar{a} - \bar{W} \times \bar{V}_B$$

where $\bar{a}$ is the output of the three body fixed accelerometer. However, to use this equation to obtain position information, gravity must be resolved into the body frame to compensate the measured specific force. Thus it seems that a transformation matrix is necessary for navigation even though it is not used in the same manner as the previous configurations.

For navigation information, a second integration must be performed to obtain the position vector in body coordinates.

$$\dot{\bar{R}}_B = \bar{V}_B - \bar{W} \times \bar{R}_B$$

However this information is in the body frame and, to be useful, it must be transformed so that latitude and longitude information is available. Hence, it appears that the orientation of the body frame to a stabilized frame must be computed in all three configurations.

3. SYSTEM EQUATIONS

The vector output of an accelerometer triad is proportional to the nonfield specific force coordinatized in the mechanized frame. The output signal of the accelerometer triad may be written as(1,2)

$$\bar{f}^m = C_i^m(\ddot{\bar{r}} - \bar{G})^i$$

where $\bar{f}^m$ is the nonfield specific force vector in the mechanized frame, m; C_i^m is the coordinate transformation matrix relating the inertial axes to the mechanized axes; $\ddot{\bar{r}}$ is the inertially referenced acceleration; and $\bar{G}$ is the gravitational acceleration due to the earth. The mechanized frame for a strapdown inertial system is a body fixed frame, b, such that

$$\bar{f}^b = C_i^b(\ddot{\bar{r}} - \bar{G})^i$$

For the system which computes in the body frame, the variables needed to obtain navigation information are coordinatized in the body frame. To obtain an expression for $\ddot{\bar{r}}^i$ which may be used in the above equation, $\bar{r}^i$ as a function of $\bar{r}^b$ is differentiated twice.

$$\bar{r}^i = C_b^i\,\bar{r}^b$$

Differentiating with respect to time yields

$$\begin{aligned}\dot{\bar{r}}^i &= C_b^i\,\dot{\bar{r}}^b + \dot{C}_b^i\,\bar{r}^b \\ &= C_b^i(\dot{\bar{r}}^b + C_i^b\,\dot{C}_b^i\,\bar{r}^b)\end{aligned}$$

It is known that

$$C_i^b \dot{C}_b^i = \Omega_{ib}^b = \begin{bmatrix} 0 & -W_z & W_y \\ W_z & 0 & -W_x \\ -W_y & W_x & 0 \end{bmatrix}$$

where the elements of the matrix, Ω_{ib}^b , are the elements of the angular velocity of the body frame relative to the inertial frame coordinatized in the body frame.

$$W_{ib}^b = \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix}^b$$

Making this substitution yields

$$\dot{\bar{r}}^i = C_b^i(\dot{\bar{r}}^b + \Omega_{ib}^b \bar{r}^b)$$

Differentiating again yields

$$\ddot{\bar{r}}^i = C_b^i(\ddot{\bar{r}}^b + 2\Omega_{ib}^b \dot{\bar{r}}^b + \dot{\Omega}_{ib}^b \bar{r}^b + \Omega_{ib}^b \Omega_{ib}^b \bar{r}^b)$$

So that for a strapped-down system

$$\bar{f}^b = C_i^b(\ddot{\bar{r}}^i - \bar{G}^i) .$$

becomes

$$\bar{f}^b = \ddot{\bar{r}}^b + 2\Omega_{ib}^b \dot{\bar{r}}^b + \dot{\Omega}_{ib}^b \bar{r}^b + \Omega_{ib}^b \Omega_{ib}^b \bar{r}^b - \bar{G}^b$$

Integrating by means of the method developed in the previous section yields $\bar{r}^b$. However, the vector $\bar{r}^b$ is generally a complicated function of time as are the acceleration compensation terms. In addition, the vector $\bar{r}^b$ as such yields limited information for navigation purposes and must be related to another coordinate frame to obtain meaningful information. Also, the explicit calculation of $\bar{G}$ in the body frame is a complex operation.

Hence, the non-field specific force in the body frame, $\bar{f}^b$, is usually transformed into another frame before calculation proceeds. For the case of a system computing in the navigation frame, the signal $\bar{f}^b$ is transformed into the new frame, and a similar development yields an expression for the non-field specific force in the navigation frame.

$$\bar{f}^n = \ddot{\bar{r}}^n + 2\Omega^b_{in}\ \dot{\bar{r}}^n + \dot{\Omega}^n_{in}\ \bar{r}^n + \Omega^n_{in}\ \Omega^n_{in}\ \bar{r}^n - \bar{G}^n$$

The variables in this expression are much more manageable than those in the expression for $\bar{f}^b$ because of the great complexity of the equations which describe $\bar{r}^b$, $\dot{\bar{r}}^b$, and $\ddot{\bar{r}}^b$ as functions of time. Hence, the navigation frame is a valid choice for a possible computation frame.

For the system which is mechanized to compute in the earth centered inertial frame, the signal $\bar{f}^b$ is transformed into this inertial frame, and now, the basic system computation equation is given simply by

$$\bar{f}^i = \ddot{\bar{r}}^i - \bar{G}^i$$

or

$$\ddot{\bar{r}}^i = \bar{f}^i + \bar{G}^i$$

Navigation information is readily obtained from $\bar{r}^i$, and the only obvious disadvantage is the necessary explicit calculation of $\bar{G}$ in the inertial frame so that the inertial frame is another valid choice as a possible computation frame.

4. THE TRANSFORMATION MATRIX

There are essentially three methods of representing rotational transformations(5). The most common of these is the matrix of direction cosines. This matrix may have its elements written explicitly in terms of the nine direction cosines or in terms of a set of Euler angles. The second method is quaternions which use four parameters to describe a rotation. The third transformation uses a vector to represent the rotation, and performs the transformation by means of purely vector and algebraic operations.

In this paper, the transformation will be accomplished using the direction cosine matrix. The transformation itself can be found by simply identifying the orientation of each of the body fixed axes in terms of the direction cosines between these body axes and the inertial coordinates. Using direction cosines to describe the relative orientation of coordinate frames, the components of a vector in body fixed axes are then related to the components of the same vector in inertial space by

$$\begin{bmatrix} A_{ix} \\ A_{iy} \\ A_{iz} \end{bmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

where A_{bx} , A_{by} , and A_{bz} , are components of $\bar{A}$ in the body frame and A_{ix} , A_{iy} , and A_{iz} are the components of $\bar{A}$ in the inertial frame. The next step is to generate the elements of the transformation matrix using gyro information with the angular rates being in terms of body frame components. The transformation may be written

$$A_{ik} = \sum_j D_{kj} A_{Bj} \qquad k, j = 1, 2, 3$$

That is, A_{ik} is the K^{th} component of $\bar{A}_i$ and $\bar{A}_{Bj}$ is the j^{th} component of $\bar{A}_B$. The definition of D_{kj} is given by

$$D_{kj} = \hat{1}_{ik} \cdot \hat{1}_{Bj}$$

where $\hat{1}_{(\ )}$ is the unit vector in the direction indicated by the subscript.

Letting $\bar{W}_{IB}$ equal the angular velocity of the B frame with respect to the inertial frame I, then

$$\frac{d}{dt} D_{kj} = \dot{D}_{kj} = \hat{1}_{ik} \cdot (\bar{W}_{IB} \times \hat{1}_{Bj})$$

or in index form

$$\dot{D}_{kj} = D_{k,j+1} W_{IB(j+2)} - D_{k,j+2} W_{IB(j+1)}$$

The result may also be obtained by direct differentiation of $\bar{A}_j = D\bar{A}_b$

$$\dot{\bar{A}}_i = \bar{D}\dot{\bar{A}}_b + \dot{D} \bar{A}_b$$

$$= D(\dot{\bar{A}}_b + D^{-1} \dot{D} \bar{A}_b)$$

This may be interpreted as the Law of Coriolis in matrix form. It the term $D^{-1}\dot{D}\ \bar{A}$ is identified with $\bar{W}_{IB} \times \bar{A}_b$, and the angular velocity matrix Ω_{IB} be defined by

$$\Omega_{IB} = \begin{bmatrix} 0 & -W_z & W_y \\ W_z & 0 & -W_x \\ -W_y & W_x & 0 \end{bmatrix}$$

Note that these angular velocity components are measured in the body frame. Then

$$\bar{W}_{IB} \times \bar{A}_B = \Omega\ V = D^{-1}\ \dot{D}\ \bar{A}_b$$

Thus the differential equation for D is $\dot{D} = D\ \Omega$ which expands to give the previous result.

This represents nine separate equations: there are three sets of the form

$$\dot{D}_{k1} = D_{k2}\ W_z - D_{k3}\ W_y$$

$$\dot{D}_{k2} = D_{k3}\ W_x - D_{k1}\ W_z$$

$$\dot{D}_{k3} = D_{k1}\ W_y - D_{k2}\ W_x$$

Multiplication by dt and integration yields

$$D_{k1} = \int D_{k2}\, d\theta_z - \int D_{k3}\, d\theta_y$$

$$D_{k2} = \int D_{k3}\, d\theta_x - \int D_{k1}\, d\theta_z$$

$$D_{k3} = \int D_{k1}\, d\theta_y - \int D_{k2}\, d\theta_x$$

Hence, the terms of the direction cosine matrix may be obtained by numerical integration. The most common of these schemes are(8)

a) the rectangular rule

$$\int_{\theta(n-1)}^{\theta(n)} C(\theta)\, d\theta = \Delta\theta\; C[\theta(n-1)]$$

b) the trapezoidial rule

$$\int_{\theta(n-1)}^{\theta(n)} C(\theta)\, d\theta = \frac{\Delta\theta}{2}\left\{C[\theta(n-1) + C[\theta(n)]\right\}$$

c) Simpson's rule

$$\int_{\theta(n-1)}^{\theta(n)} C(\theta)\, d\theta = \frac{\Delta\theta}{3}\left\{C[\theta(n-2)] + 4C[\theta(n-1)] + C[\theta(n)]\right\}$$

5. ANALYSIS OF A STRAPDOWN SYSTEM COMPUTING IN THE NAVIGATION FRAME

The signal flow diagram for the system in shown below.

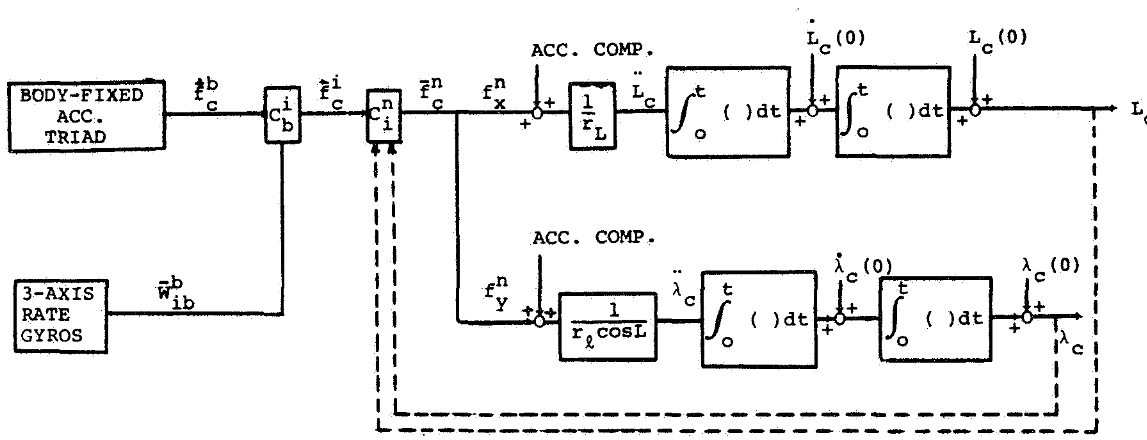


Figure 3. Signal Flow Diagram for Computation in Navigation Coordinates

In the analysis of this system, there are four points of interest: 1) the accuracy of the accelerometer signals, 2) the effect of gyro error sources, 3) accuracy of the coordinate transformation C_b^i , and 4) accuracy of the coordinate transformation C_i^n .

Accelerometer error can be primarily attributed to accelerometer uncertainty and scale factor error. Analytically the specific force signal in body coordinates may be expressed as

$$\bar{f}_c^b = \bar{f}^b + \delta\bar{f}^b + \begin{bmatrix} a_x & 0 & 0 \\ 0 & a_y & 0 \\ 0 & 0 & a_z \end{bmatrix} \bar{f}^b$$

where: $\bar{f}_c^b$ is the measured specific force

$\bar{f}^b$ is the specific force

$\delta\bar{f}^b$ is the accelerometer uncertainty

and a_i is the scale factor uncertainty for the i^{th} accelerometer. The gyro error sources to be considered are gyro drift and torquing uncertainty. Since the output of the gyro triad is used to up-date the transformation C_b^i , these errors directly effect the validity of this coordinate transformation. Another source of error is the accuracy of the initial alignment of the body frame with its determined orientation. The coordinate transformation C_i^n is kept up to date by feedback from the system output (indicated by dashed lines on the signal flow diagram), and hence, errors in the output generate errors in this matrix. Again, initial alignment determination is a source of error.

The actual orientation of the body frame is denoted by b; the ideal (non-rotating with respect to the body) orientation is denoted by b'. However, due to the gyro error sources and the inaccuracy in the initial determination procedure, there is an error between the actual orientation and the ideal orientation. Hopefully, the actual initial orientation of the body frame is coincident with its determined orientation, but if it is not, the misalignment matrix $\underline{M}$ is a skew symmetric matrix given by

$$\underline{M} = \begin{bmatrix} 0 & \zeta_z & -\zeta_y \\ -\zeta_z & 0 & \zeta_x \\ \zeta_y & -\zeta_x & 0 \end{bmatrix}$$

Thus the total error transformation between the actual and the ideal orientation is given $\underline{C}_{b'}^{b}$, such that

$$\underline{C}_{b'}^{b} = \underline{I} + \underline{\theta}_d + \underline{\theta}_T + \underline{\theta}_M$$

where $\underline{\theta}_d$ is a direction cosine transformation for the error in orientation due to gyro drift

$\underline{\theta}_T$ is a direction cosine transformation for the error in orientation due to torquing uncertainty

$\underline{\theta}_M$ is the misalignment matrix

Specializing now to the stationary system, the principal error sources are the accelerometer bias and gyro drift. The error matrix $\underline{C}_{b'}^{b}$ is a function of time because the gyro drifts which are denoted by W_x , W_y , W_z . As an angular velocity vector

$$\bar{W}_{bb'}^{b} = \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix}$$

This may be interpreted as the angular velocity of the b frame to the b' frame, with the rates coordinatized in the actual body frame b. The matrix $\underline{\theta}_d$ is defined by the matrix differential equation

$$\dot{\underline{\theta}}_d = \underline{\theta}_d \Omega$$

where

$$\underline{\Omega} = \begin{bmatrix} 0 & -W_z & W_y \\ W_z & 0 & -W_x \\ -W_y & W_x & 0 \end{bmatrix}$$

For the case which has been assumed of constant gyro drift, the solution of this matrix differential equation is given by

$$\underline{\theta}_d(t) = e^{\underline{\Omega}t} - \underline{I}$$

This, then, is the direction cosine matrix relating the time dependent orientation of b' to b due to the gyro drift. It is noted that the off diagonal terms are the principal elements of this matrix.

The error transformation is now seen to be an explicit function of time and is given by

$$\underline{C}_{b'}^{b} = e^{\underline{\Omega}t} + \underline{\theta}_M$$

The matrix exponential is analytically given by the expression(9)

$$e^{\underline{A}t} = \frac{1}{2\pi j} \int_{-j\infty + \sigma}^{+j\infty + \sigma} \left\{ \det(\underline{I}s - \underline{A}) \right\}^{-1} \left\{ \mathrm{Adj}(\underline{I}s - \underline{A}) \right\} e^{st} \, ds$$

Letting $Q = W_x^2 + W_y^2 + W_z^2$ $, A = \dfrac{W_x W_y}{W_z}$ $, B = \dfrac{W_x W_z}{W_y}$ $, C = \dfrac{W_y W_z}{W_x}$

the error transformation matrix as a function of time is given by

$$
\underline{C}_{b'}^{b}(t) = \begin{bmatrix}
\frac{W_x^2}{Q^2} - \frac{(W_x^2-Q^2)}{Q^2}\cos Qt & -W_z\left[\frac{-A}{Q^2} - \frac{(A^2+Q^2)^{1/2}}{Q^2}\cos\left(Qt+\tan^{-1}\frac{Q}{-A}\right)\right]+\zeta_z & W_y\left[\frac{B}{Q^2} - \frac{(B^2+Q^2)^{1/2}}{Q^2}\cos\left(Qt+\tan^{-1}\frac{Q}{B}\right)\right]-\zeta_y \\
W_z\left[\frac{A}{Q^2} - \frac{(A^2+Q^2)^{1/2}}{Q^2}\cos\left(Qt+\tan^{-1}\frac{Q}{A}\right)\right]-\zeta_z & \frac{W_y^2}{Q^2} - \frac{(W_y^2-Q^2)}{Q^2}\cos Qt & -W_x\left[\frac{-C}{Q^2} - \frac{(C^2+Q^2)^{1/2}}{Q^2}\cos\left(Qt+\tan^{-1}\frac{Q}{-C}\right)\right]+\zeta_x \\
-W_y\left[\frac{-B}{Q^2} - \frac{(B^2+Q^2)^{1/2}}{Q^2}\cos\left(Qt+\tan^{-1}\frac{Q}{-B}\right)\right]+\zeta_y & W_x\left[\frac{C}{Q^2} - \frac{(C^2+Q^2)^{1/2}}{Q^2}\cos\left(Qt+\tan^{-1}\frac{Q}{C}\right)\right]-\zeta_x & \frac{W_z^2}{Q^2} - \frac{(W_z^2-Q^2)}{Q^2}\cos Qt
\end{bmatrix}
$$

where the ζ_i's are the elements of the misalignments matrix. Hence, the specific force signal coordinatized in the navigation frame is given by

$$\bar{f}_c^n = C_i^n \, C_b^i \, C_{b'}^b \, \bar{f}_c^{b'}$$

where the transformation C_i^n is given by

$$
C_i^n = \begin{bmatrix}
-\sin L \cos\lambda & -\sin L \sin\lambda & \cos L \\
-\sin\lambda & \cos\lambda & 0 \\
-\cos L \cos\lambda & -\cos L \sin\lambda & -\sin L
\end{bmatrix}
$$

and the elements are continously computed as functions of the system output.

The expression for the specific force coordinatized in the navigation frame has been previously given as

$$\bar{f}^n = \ddot{\bar{r}}^n - \bar{G}^n + 2\underline{\Omega}_{in}^n \dot{\bar{r}}^n + \underline{\Omega}_{in}^n \underline{\Omega}_{in}^n \bar{r}^n + \dot{\Omega}_{in}^n \bar{r}^n$$

If acceleration compensation terms are calculated which when summed with the above expression remove the last three terms, that is, remove Coriolis terms and terms arising from $\bar{W} \times (\bar{W} \times \bar{R})$ operations, then

$$f_{nx} = r_L \ddot{L} - \xi g$$

and

$$f_{ny} = r_\ell \ddot{\lambda} \cos L + \eta g$$

where

$r_L = r(1-e\ 2\cos 2L)$ = radius of curvature in meridian plane

$r_\ell = r(1+2e \sin^2 L)$ = radius of curvature in co-meridian plane

ε = earth ellipticity $\simeq 1/297$

ξ = merdian deflection of the vertical (positive about east)

η = prime deflection of the vertical (positive about north)

Latitude and longitude may now be obtained by double integration of the north and east specific force measurements respectively. It is to be noted that the complex calculation of the acceleration compensation terms is required and is a disadvantage of the mechanization.

To analyze the affect of the various error sources and system performance, a simulation of the system was performed on a digital computer. The numerical integration was performed by means of a second-order Runge-Kutta algorithm. For a differential equation defined by

$$\dot{\chi} = f(x,t)$$

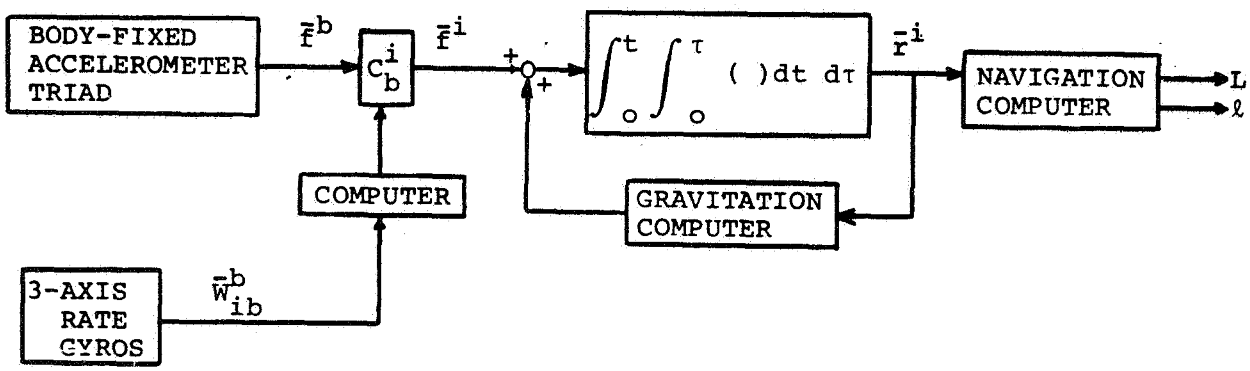
the second-order Runge-Kutta algorithm for the solution is given by(8,10)

$$\chi_{n+1} = \chi_n + \frac{h}{2} f(\chi_n, t_n) + \frac{h}{2} f[\chi_n + hf(\chi_n, t_n), t_n + h]$$

where h is the numerical integration step size time interval. In this simulation, the body frame was assumed to be initially aligned with the local navigation frame with, of course, the possibility of a small misalignment error.

6. ANALYSIS OF STRAPDOWN SYSTEM COMPUTING IN THE INERTIAL FRAME

The signal flow diagram for the system is shown below.



The basic equation describing the operation of this system is given by

$$\bar{f}^i = \ddot{\bar{r}}^i - \bar{G}^i$$

Once the specific force vector $\bar{f}_c^b$, which is measured in the body frame, is transformed into the inertial frame, navigation information may be obtained by double integration of the signal

$$\ddot{\bar{r}}^i = \bar{f}^i + \bar{G}^i$$

No acceleration compensation terms need to be calculated. However, this mechanization has the disadvantage that gravity must be computed explicity.

The output of the system is the position vector coordinatized in an earth-centered inertial frame. As such, it gives limited information, however, latitude and longitude are obtained without difficulty by means of the expression below.

$$\lambda = \tan^{-1} \frac{rIx}{rIy}$$

and

$$L = \tan^{-1} \frac{rIz}{\sqrt{r^2 Ix + r^2 Iy}}$$

As in the system which computes in the navigation frame, there are four major points of interest. Three are identical, but the fourth is quite different. Again interest is directed toward: 1) the accuracy of the accelerometer signals, 2) the effect of the gyro error sources, and 3) accuracy of the coordinate transformation C_b^i . Now, instead of investigating the transformation C_i^n , this system requires that the means of gravity calculation be investigated. Resolved into the geocentric inertial frame, the gravitation vector, $\bar{G}$, can be expressed to first order as(2)

$$G_x = -\frac{Gm}{r^2} \left\{ 1 + \frac{3}{2} J_2 \left(\frac{r_e}{r}\right)^2 \left[1-5\left(\frac{r_t}{r}\right)^2 \right] \right\} \frac{r_x}{r}$$

$$G_y = -\frac{Gm}{r^2} \left\{ 1 + \frac{3}{2} J_2 \left(\frac{r_e}{r}\right)^2 \left[1-5\left(\frac{r_t}{r}\right) \right] \right\} \frac{r_y}{r}$$

$$G_z = -\frac{Gm}{r^2} \left\{ 1 + \frac{3}{2} J_2 \left(\frac{r_e}{r}\right)^2 \left[3-5\left(\frac{r_t}{r}\right)^2 \right] \right\} \frac{r_z}{r}$$

where $Gm \cong 1.4 \times 10^6 \ ft^3/sec^2$

and $\frac{3}{2} J_2 \cong 1.6 \times 10^{-3}$

For stability reasons, the inertially computed position vector is not used in the calculation of the gravity magnitude, but is used only to obtain the direction of the gravity vector. Instead, $\bar{r}$ is calculated as the sum of externally obtained altitude information and the elliptic geocentric radius,

$$r = r_o + h$$

where $r_o = r_e[1-\frac{e}{2}(1-\cos 2L)+\frac{5}{16}e^2(1-\cos 4L)]$

and e = earth ellipticity

Neglecting higher order terms, the expression for gravity may be written in vector form as

$$\bar{G}_c \cong - \frac{E}{(r_o+h)^3} \bar{r}_c$$

Hence, the system is now susceptible to error due to incorrect altitude information.

To analyze the performance of this system and its response to the various error sources, simulation was performed on a digital computer. For the stationary case, the error transformation between the actual body frame orientation and the ideal orientation is again given as a time dependent matrix

$$C_{b'}^{b}(t) = \theta_M + e^{\Omega t}$$

The basic equation describing the system may now be written as

$$\ddot{\bar{r}}^i = \bar{f}^i_c + \bar{G}^i_c$$

or

$$\ddot{\bar{r}}^i = C^i_b \, C^b_{b'} \, \bar{f}^b_c - \frac{E}{(r_o+h)^3} \, \bar{r}_c$$

Once the inertial position vector, $\bar{r}^i$, has been obtained, latitude and longitude may be calculated by means of the given equations.

7. LINEAR ANALYSIS OF STRAPDOWN SYSTEM COMPUTING IN THE NAVIGATION FRAME

In this section, a linear analysis will be performed for a strapdown inertial system which computes in the navigation coordinate frame. The result of this analysis will be two linear differential equations describing the position errors with the inertial sensor uncertainties as the forcing functions.

The output of the body-fixed accelerometer triad may be interpreted as the true specific force corrupted by accelerometer uncertainties. The system then transforms this signal into the geocentric inertial frame via the computed transformation C_b^i so that the signal is further corrupted by errors in the transformation C_b^i . Analytically this may be expressed as follows

$$\bar{f}_c^i = C_b^i \, \bar{f}_c^b$$

Obtaining differentials yields

$$\delta\bar{f}_c^i = \delta C_b^i \, \bar{f}_c^b + C_b^i \, \delta\bar{f}_c^b$$

so that

$$\bar{f}_c^i = \bar{f}_c^i + \delta\bar{f}_c^i = C_b^i \, \bar{f}_c^b + \delta C_b^i \, \bar{f}_c^b + C_b^i \, \delta\bar{f}_c^b$$

The $\delta \bar{f}_c^b$ term will be taken to include all error sources arising from the accelerometers, and δC_b^i will be the errors of the transformation matrix.

The transformation to the navigation frame may be expressed as

$$\bar{f}_{c'}^n = [C_i^n + \Delta C_i^n] \; [C_b^i \, \bar{f}_c^b + \delta C_b^i \, \bar{f}_c^b + C_b^i \, \delta \bar{f}_c^b]$$

where ΔC_i^n is the error in the determination of C_i^n . To find an expression for ΔC_i^n , the known matrix differential equation below will be used.

$$\dot{C}_i^n = C_i^n \, \Omega_{ni}^i$$

where

$$\Omega = \begin{bmatrix} 0 & -W_z & W_y \\ W_z & 0 & -W_x \\ -W_y & W_x & 0 \end{bmatrix}$$

and

$$\bar{W} = \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix}$$

is the angular velocity of the inertial frame with respect to the navigation frame coordinatized in the inertial frame.

This angular velocity is given by

$$\bar{W} = \begin{bmatrix} \dot{L} \sin\lambda \\ -\dot{L} \cos\lambda \\ \dot{\lambda} \end{bmatrix}$$

so that

$$\frac{\Delta C_i^n}{\Delta t} = C_i^n \begin{bmatrix} 0 & -\dot{\lambda} & -\dot{L} \cos\lambda \\ \dot{\lambda} & 0 & -\dot{L} \sin\lambda \\ \dot{L} \cos\lambda & \dot{L} \sin\lambda & 0 \end{bmatrix}$$

Multiplication by Δt yields the desired expression for ΔC_i^n

$$\Delta C_i^n = C_i^n \begin{bmatrix} 0 & -\delta\lambda & -\delta L \cos\lambda \\ \delta\lambda & 0 & -\delta L \sin\lambda \\ \delta L \cos\lambda & \delta L \sin\lambda & 0 \end{bmatrix}$$

Hence,

$$\begin{aligned} C_i^{n'} &= C_i^n + C_i^n[\Phi] \\ &= C_i^n[I + \Phi] \\ &= C_i^n[I + C_n^i \; \Delta C_i^n \end{aligned}$$

So that the computed specific force in the navigation frame is given by

$$\bar{f}^n_{c'} = C^n_i[I + C^i_n \Delta C^n_i] \; [\bar{f}^i_c + \delta C^i_b \; \bar{f}^b + C^i_{b'} \delta \bar{f}^b]$$

$$= \bar{f}^n_c + C^n_i \delta C^i_b \; \bar{f}^b + C^n_b \delta \bar{f}^b + \Delta C^n_i \; \bar{f}^i_c$$

Substituting the relations

$$\delta C^i_b = C^i_b[\theta_b] \quad \text{and} \quad \Delta C^n_i = C^n_i[\Phi_i]$$

yields

$$\bar{f}^n_c = \bar{f}^n_c + C^n_i \; C^i_b[\theta_b]C^b_n \; \bar{f}^n + C^n_{b'} \delta \bar{f}^b + C^n_i[\Phi_i]C^i_n \; \bar{f}^n$$

By means of the similarity theorem,

$$[C^n_b] \; [\theta_b] \; [C^b_n]\bar{f}^n = [\theta_N]\bar{f}^n$$

where $[\theta_N]$ indicates that the elements are now coordinatized in the navigation frame.

Similarly,

$$[C^n_i] \; [\Phi_i] \; [C^i_n] = [\Phi_N]$$

so that now,

$$\bar{f}^n_{c'} = [I + \Phi_N + \theta_N]\bar{f}^n_c + C^n_{b'} \delta \bar{f}^b$$

The matrix $[\Phi_N]$ is given by

$$\Phi_N = \begin{bmatrix} 0 & -\delta\lambda \sin L & \delta L \\ \delta\lambda \sin L & 0 & \delta\lambda \cos L \\ -\delta L & -\delta\lambda \cos L & 0 \end{bmatrix}$$

It is shown in the general analysis of this system, that if perfect acceleration compensation is provided, then

$$\bar{f}_c^n = \begin{bmatrix} r_L\ddot{L} - \xi g \\ r_\ell\ddot{\lambda}\cos L + \eta g \\ -\ddot{r} - g \end{bmatrix}$$

In the computation variables, the X or north component of computed specific force in the navigation frame is given by

$$f^n_{c',x} = r_{L_c}\ddot{L}_c$$

and the Y or east component is given by

$$f^n_{c',y} = r_{\ell_c}\ddot{\lambda}\cos L_c$$

where the variables are defined

$$r_{L_c} = r_L + \delta h$$

$$r_{\ell_c} = r_\ell + \delta h$$

$$L_c = L + \delta h$$

and

$$\lambda_c = L + \delta\lambda$$

So that now

$$f^n_{c',x} = r_L\ddot{L} + \ddot{L}\delta h + r_L\delta\ddot{L}$$

$$f^n_{c',y} = r_\ell\ddot{\lambda}\cos L + r_\ell\cos L\delta\ddot{\lambda} + \delta h\ddot{\lambda}\cos L - r_\ell\ddot{\lambda}\sin L\delta L$$

Obtaining the product $\Phi_N \, f_c^n$ and substituting the expressions for $f^n_{c',x}$ and $f^n_{c',x}$ yields for the X equation

$$\ddot{L}\delta h + \ddot{r}\delta L + \ddot{\lambda} r \sin 2L\delta\lambda + \ddot{r}\delta L + g\delta L = -\xi g + \left\{ C_b^n \theta_b C_n^b \bar{f}_c^n + C_{b'}^n \delta \bar{f}^b \right\}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and for the Y equation

$$-r\ddot{L}\sin L\delta\lambda + r\cos L\ddot{\delta\lambda} - \delta L r_\ell \ddot{\lambda}\sin L + \ddot{r}\cos L\delta\lambda + g\cos L\delta\lambda + \ddot{\lambda}\cos L\delta h$$

$$= \eta g + \left\{ C_b^n \theta_b C_n^b \, \bar{f}_c^n + C_{b'}^n \overline{\delta f} \right\}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

If the system is stationary, the equations become uncoupled and are given by

$$\ddot{\delta L} + W_s^2 \delta L = -W_s^2 \xi + \frac{1}{r_L} \left\{ C_b^n \theta_b C_n^b \, \bar{f}_c^n + C_{b'}^n \overline{\delta f}^b \right\}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\ddot{\delta\lambda} + W_s^2 \delta\lambda = W_s^2 \eta + \frac{1}{r_\lambda \cos L} \left\{ C_b^n \theta_b C_n^b \, \bar{f}_c^n + C_{b'}^n \overline{\delta f} \right\}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

In a simplified form, these expressions for a stationary system may be written

$$\ddot{\delta L} + W_s^2 \delta L = -W_s^2 \theta_{N_y} + W_s^2 \frac{\delta f N_x}{g} - W_s^2 \xi$$

$$\ddot{\delta\lambda} + W_s^2 \delta\lambda = \frac{1}{\cos L} \left[W_s^2 \frac{\delta f N_y}{g} + W_s^2 \theta_{N_x} + W_s^2 \eta \right]$$

where

$$\begin{bmatrix} \theta_{N_x} \\ \theta_{N_y} \\ \theta_{N_z} \end{bmatrix} = C_i^n C_b^i \begin{bmatrix} \theta_{b_x} \\ \theta_{b_y} \\ \theta_{b_z} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \delta f_{N_x} \\ \delta f_{N_y} \\ \delta f_{N_z} \end{bmatrix} = C_i^n C_b^i \begin{bmatrix} \delta f_{b_x} \\ \delta f_{b_y} \\ \delta f_{b_z} \end{bmatrix}$$

and for a stationary system

$$\bar{f}_c^n = \begin{bmatrix} -\xi g \\ \eta g \\ -g \end{bmatrix}$$

These two linear differential equations will now be solved with the various error sources as the forcing functions. These results will then be compared with the results from the digital computer simulation in order to analyze the accuracy and validity of the linear development.

For a stationary strapdown inertial system computing in the navigation frame, constant accelerometer bias results in responses given by

$$\delta L(t) = \frac{\delta f_{N_x}}{g} \quad (1-\cos W_s t)$$

and

$$\delta \lambda (t) = \frac{\delta f_{N_x}}{g} \quad (1-\cos W_s t) \sec L$$

These results of the linear theory agree very well with the results of the computer simulation which solved the system non-linear differential equations by numerical methods. This agreement is illustrated in Graph 1.

To evaluate the error for constant gyro drift, the Θ_N error matrix must be investigated. It is known that

$$\dot{C}_b^i = C_b^i \, \Omega_{ib}^b$$

However, for linear approximation the transformation may be regarded as

$$C_b^i = C_i^{i'} C_b^i$$

and

$$\Omega_{ib}^b = \Omega_{ii'}^b + \Omega_{ib}^b$$

This may be interpreted as

$$C_b^i = (I-\Theta_i) \; C_b^i$$

where Θ_i is a small perturbation misalignment matrix.

$$\Theta_i = \begin{bmatrix} 0 & -\Theta_z & \Theta_y \\ \Theta_z & 0 & -\Theta_x \\ -\Theta_y & \Theta_x & 0 \end{bmatrix}$$

Hence for the total derivative, the above yields

$$\frac{d}{dt} \left[(I-\Theta_i) C_b^i \right] = (I-\Theta_i) \; C_b^i \; (\Omega_{ib}^b + \Omega_{ii'}^b)$$

This simplifies to

$$-\dot{\Theta}_i C_b^i = C_b^i \Omega_{ii'}^b$$

Now $\Omega_{ii'}^b = C_i^b \Omega_{ii'}^i C_b^i$ so that

$$\dot{\Theta}_i = \Omega_{ii'}^i$$

In vector form this is given by

$$\dot{\overline{\Theta}}_i = C_b^i \, \overline{\delta\omega}$$

Transforming to the n frame

$$\overline{\Theta}_i = C_n^i \, \overline{\Theta}_n$$

Therefore

$$\dot{\overline{\Theta}}_i = \dot{C}_n^i \, \overline{\Theta}_n + C_n^i \, \dot{\overline{\Theta}}_n$$

$$= C_n^i \, \Omega_{in}^n \, \overline{\Theta}_n + C_n^i \, \dot{\overline{\Theta}}_n$$

Substitution yields

$$C_n^i \, \dot{\overline{\Theta}}_n + C_n^i \, \Omega_{in}^n \, \overline{\Theta}_n = C_b^i \, \overline{\delta\omega}_b$$

or

$$\dot{\overline{\Theta}}_n + \Omega_{in}^n \, \overline{\Theta}_n = C_b^n \, \overline{\delta\omega}_b$$

If the system is stationary, and the body frame is initially aligned with the navigation frame, then

$$C_b^n \, \overline{\delta\omega}_b = \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix}$$

and the differential equation describing $\overline{\Theta}_N$ is given below.

$$\begin{bmatrix} \dot{\Theta}_N \\ \dot{\Theta}_E \\ \dot{\Theta}_D \end{bmatrix} = \begin{bmatrix} 0 & -\omega_{ie} \sin L & 0 \\ \omega_{ie} \sin L & 0 & \omega_{ie} \cos L \\ 0 & -\omega_{ie} \cos L & 0 \end{bmatrix} \begin{bmatrix} \Theta_N \\ \Theta_E \\ \Theta_b \end{bmatrix} = \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix}$$

This may easily be solved for constant gyro drift and the resulting system output is the response of the system to these forcing functions.

The solutions to the system linear differential equations are given by

$$\delta L(t) = -\omega_s^2(\omega_{ie} U_x \sin L + \omega_{ie} U_z \cos L) f_1(t) - \omega_s{}^2 U_y f_2(t)$$

$$\delta\lambda(t) = -\omega_s{}^2\omega_{ie} U_y \sin L \, f_1(t) + \omega_s{}^2 U_x f_2(t)$$

$$+ \omega_s{}^2 \omega_{ie}^2 (U_x \cos^2 L - U_z \sin L \cos L) f_3(t)$$

where

$$f_1(t) = \frac{1}{\omega_s^2}\left(\frac{1}{\omega_{ie}^2-\omega_s^2}\right)(1-\cos\,\omega_s t) + \frac{1}{\omega_{ie}^2}\left(\frac{1}{\omega_s^2-\omega_{ie}^2}\right)(1-\cos\,\omega_{ie} t)$$

$$f_2(t) = \frac{1}{\omega_s}\left(\frac{1}{\omega_{ie}^2-\omega_s^2}\right)\sin\,\omega_s t + \frac{1}{\omega_{ie}}\left(\frac{1}{\omega_s^2-\omega_{ie}^2}\right)\sin\,\omega_{ie} t$$

$$f_3(t) = \frac{1}{\omega_s^2\omega_{ie}^2}t + \frac{1}{\omega_s^3}\left(\frac{1}{\omega_s^2-\omega_{ie}^2}\right)\sin\,\omega_s t + \frac{1}{\omega_{ie}^3}\left(\frac{1}{\omega_{ie}^2-\omega_s^2}\right)\sin\,\omega_{ie} t$$

It is noticed that because of the large difference in magnitude between ω_s and ω_{ie}, the terms involving the earth rate mode greatly dominate the observed response. As shown in Graphs 2 and 3, these results agree well with the digital computer simulation. Obviously, the results of the linear theory will begin to diverge from the general simulation when the elements of $[\Theta_b]$ develop sufficient magnitude so that small angle approximation is not valid.

Uncertainty in the accuracy of the torquing mechanism of the rate gyros results in an uncertainty of the angular velocity of the body frame with respect to an inertial frame. This uncertainty may be expressed as

$$\delta\omega_{ib}^b = -\frac{\delta T}{T}\,\omega_{ib}^b$$

where $\frac{\delta T}{T}$ is the uncertainty of the torque generator. For a stationary system initially aligned with the navigation frame,

$$\delta\omega_{ib}^{b} = -\frac{\delta T}{T}\begin{bmatrix} \omega_{ie}\cos L \\ 0 \\ -\omega_{ie}\sin L \end{bmatrix}$$

Now using the relation derived in the gyro drift error analysis

$$\dot{\overline{\Theta}}_n + \Omega_{in}^{n}\,\overline{\Theta}_n = C_b^n\,\overline{\delta\omega}_b$$

The vector differential equation is then given by

$$\begin{bmatrix} \dot{\Theta}_N(t) \\ \dot{\Theta}_E(t) \\ \dot{\Theta}_D(t) \end{bmatrix} = \begin{bmatrix} 0 & \omega_{ie}\sin L & 0 \\ -\omega_{ie}\sin L & 0 & -\omega_{ie}\cos L \\ 0 & \omega_{ie}\cos L & 0 \end{bmatrix}\begin{bmatrix} \Theta_N(t) \\ \Theta_E(t) \\ \Theta_D(t) \end{bmatrix} + \begin{bmatrix} -\frac{\delta T}{T}\,\omega_{ie}\cos L \\ 0 \\ \frac{\delta T}{T}\,\omega_{ie}\sin L \end{bmatrix}$$

Substituting the solution into the system differential equations and solving the resulting linear equations yields,

$$\delta L(t) = -\omega_s^2\,(\omega_{ie}\,U_1\sin L + \omega_{ie}\,U_3\cos L)\;f_1(t)$$

$$\delta\lambda(t) = \omega_s^2\,U_1 f_2(t) + (\omega_s^2\omega_{ie}^2)(U_1\cos^2 L - U_3\,\sin L\cos L)\;f_3(t)$$

where

$$U_1 = -\frac{\delta T}{T}\,\omega_{ie}\cos L$$

$$U_3 = \frac{\delta T}{T}\,\omega_{ie}\sin L$$

and $f_1(t)$, $f_2(t)$, and $f_3(t)$ are as given in the gyro drift analysis. Again the earth rate modes dominate the response as is shown in Graphs 4 and 5, and the agreement with the digital simulation is very satisfactory.

Note that for both constant gyro drift and torquing uncertainty, the latitude error is bounded, but the longitude error is a ramp function and hence unbounded in time. The effect of the ramp is diminished in the simulation due to its coefficient being very small, but it is seen that

$$\frac{\delta\lambda(t)}{\cos^2 L - \sin L \cos L} \approx Ut$$

so that the effect is fairly strong in the long run analysis.

For initial body frame misalignment, the position error is given by

$$\delta L(t) = -\Theta_{N_y} (1-\cos\omega_s t)$$

and

$$\delta\lambda(t) = \Theta_{N_x} \sec L(1-\cos\omega_s t)$$

where $[\Theta_N)_M$ is given by $C_b^n[\Theta_b]_M C_n^b$, and $[\Theta_b]_M$ is the skew symmetric misalignment matrix. Graph 6 illustrates agreement with the computer simulation.

If is assumed that the acceleration compensation is in error, then the analysis is somewhat more involved, and the result is slightly different in that some relatively small coupling effects are brought to light.

The analytically derived expression for the specific force in the navigation frame can be shown to be given by (1,2)

$$\bar{f}^n = \begin{bmatrix} r_L\ddot{L} + \frac{1}{2}r_\ell(\dot{\lambda}^2 - W_{ie}^2)\sin 2L + 2\dot{r}_L\dot{L} - \ddot{r}e\sin 2L - 3er\sin 2L\dot{L}^2 - \xi g \\ r_\ell\ddot{\lambda}\cos L - 2r_\ell\dot{L}\dot{\lambda}\sin L + 2\dot{r}_\ell\dot{\lambda}\cos L + \eta g \\ -g - \ddot{r} - r_L\ddot{L}\,e\sin 2L + r_\ell(\dot{\lambda}^2 - W_{ie}^2)\cos^2 L + \frac{r_L^2}{r}\dot{L}^2 \end{bmatrix}$$

For computed variables, the x component is given by

$$f_{x'}^n = r_{L_c}\ddot{L}_c + \frac{1}{2}r_{\ell_c}(\dot{\lambda}^2 - W_{ie}^2)\sin 2L_c + 2\dot{r}_{L_c}\dot{L}_c$$

$$- \ddot{r}_c e\sin 2L_c - 3er_c\sin 2L_c\dot{L}_c^2 - \xi g$$

and the computed y component is given by

$$f_{y'}^n = r_{\ell_c}\ddot{\lambda}_c\cos L - 2r_{\ell_c}\dot{L}_c\dot{\lambda}_c\sin L_c + 2\dot{r}_{\ell_c}\dot{\lambda}_c\cos L_c + \eta g$$

Substituting the relations below

$$r_c = r + \delta h$$

$$r_{L_c} \simeq r_L + \delta h$$

$$r_{\ell_c} \simeq r_\ell + \delta h$$

$$L_c = L + \delta L$$

$$\lambda_c = \lambda + \delta\lambda$$

and providing compensation yields for the compensated computed x component

$$f^n_{x'} = \ddot{r}L + r\dot{\delta L} + 2\dot{L}\delta\dot{h} + r\dot{\lambda}\sin 2L\delta\dot{\lambda} + \ddot{L}\delta h$$

and for the compensated y component

$$f^n_{y'} = r\ddot{\lambda}\cos L + r\cos L\delta\ddot{\lambda} + 2[\dot{r}\cos L - r\dot{L}\sin L]\delta\dot{\lambda} - 2r\dot{\lambda}\sin L\delta\dot{L}$$

$$- r[\ddot{\lambda}\sin L + 2\dot{L}\dot{\lambda}\cos L]\delta L + 2\dot{\lambda}\cos L\delta\dot{h} + \ddot{\lambda}\cos L\delta h$$

It has been shown in this section that the computed specific force given by

$$\bar{\bar{f}}^n_{c'} = [I + \Phi_N + \theta_N]\ \bar{\bar{f}}^n_c + C^n_{b'}\ \delta\bar{\bar{f}}^b$$

Obtaining the product $\Phi_N\bar{\bar{f}}^n_c$ where $\bar{\bar{f}}^n_c$ is the ideally compensated specific force and substituting the expressions for $f^n_{c'_x}$ and $f^n_{c'_y}$ yields for the x equation

$$r\delta\ddot{L} + (\ddot{r}+g)\delta L + r\dot{\lambda}\sin 2L\delta\dot{\lambda} + \frac{1}{2}\, r\ddot{\lambda}\sin 2L\delta\lambda$$

$$= \ddot{L}\delta h + 2\dot{L}\delta\dot{h} - \xi g + \left\{C^n_b\ \theta_b\ C^b_n\ \bar{\bar{f}}^n_c + C^n_b\delta\bar{\bar{f}}^b\right\}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and the y equation

$$r\cos L\delta\ddot{\lambda} + 2[\dot{r}\cos L - r\dot{L}\sin L]\delta\dot{\lambda} + [(\ddot{r}+g)\cos L - r\ddot{L}\sin L]\delta\lambda$$

$$-2r\dot{\lambda}\sin L\delta\dot{L} - r[\ddot{\lambda}\sin L + 2\dot{L}\dot{\lambda}\cos L]\delta L$$

$$= -\ddot{\lambda}\cos L\delta h - 2\dot{\lambda}\cos L\delta\dot{h} + \eta g + \left\{C_b^n\,\theta_b\,C_n^b\,\bar{f}_c^n + C_b^n\,\delta\bar{f}^b\right\}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

The preceding equations give the latitude and longitude errors for arbitrary vehicle motion for the strapdown system which computes in the navigation frame. In the preceding development, in order to make the resulting equations both meaningful and manageable, certain assumptions have been made to delete terms which have small magnitudes relative to the other terms. These assumptions are based on a quantitative description of vehicle motion which is in accord with those to be encountered by the supersonic transport. Assuming that the following data represents the maximum values of vehicle motion

$$r\ddot{L}_{max} = r\ddot{\lambda}_{max} = 0.5g$$

$$\dot{r}_{max} = 100\text{ft/sec}$$

$$\dot{L}_{max} = \dot{\lambda}_{max} = 1.6 \times 10^{-4}\ \text{rad/sec}$$

$$\ddot{r}_{max} = 2g$$

and that the maximum error data is given by

$$\delta L_{max} = \delta\lambda_{max} = 10_{min} = 2.9 \times 10^{-3} \text{ rad}$$

$$\delta\dot{L}_{max} = \delta\dot{\lambda}_{max} = \delta L_{max} W_s = 3.6 \times 10^{-6} \text{ rad/sec}$$

$$\delta\ddot{L}_{max} = \delta\ddot{\lambda}_{max} = \delta L_{max} W_s^2 = 4.5 \times 10^{-9} \text{ rad/sec}$$

$$\delta h_{max} = 2000\text{ft.}$$

$$\delta\dot{h}_{max} = \delta h_{max} W_s = 2.5\text{ft/sec}$$

results in the given equations when all terms which are of magnitude less than 2×10^{-5} g are neglected.

For the case of a stationary system, the equations simplify to the coupled linear equations

$$\delta\ddot{L} + W_s^2 \delta L + W_{ie} \sin 2L \delta\dot{\lambda}$$

$$= -W_s^2 \xi + \frac{1}{r} \left\{ C_b^n \, \theta_b \, C_n^b \, \bar{f}_c^n + C_b^n \, \delta\bar{f}^b \right\}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\delta\ddot{\lambda} + W_s^2 \delta\lambda - 2W_{ie} \tan L \delta\dot{L}$$

$$= W_s^2 \sec L \eta + \frac{\sec L}{r} \left\{ C_b^n \, \theta_b \, C_n^b \, \bar{f}_c^n + C_b^n \, \delta\bar{f}^b \right\}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

However, it is noted that the cross-coupling is rather weak, and that the uncoupled equations obtained assuming perfect acceleration compensation give an accurate description of the system.

In conclusion of this section, the excellent agreement between the results of the linear analysis and the results obtained by numerical integration of the complete system differential equations indicate that the derived linear theory is indeed a valid analytical description of the system under stated limitations.

8. LINEAR ANALYSIS OF STRAPDOWN SYSTEM COMPUTING IN THE INERTIAL FRAME

In this section, a linear analysis will be performed for a strapdown inertial system which computes in geocentric inertial coordinates. The result of this analysis will be a linear differential equation describing the position errors in the inertial frame with the inertial sensor uncertainties as the forcing function.

This system transforms the output of the body-fixed accelerometers into the geocentric inertial frame and then solves the navigation problem in that frame. In the inertial frame, the definition of specific force yields

$$\ddot{\bar{r}}^i - \bar{G}^i = \bar{f}^i$$

where $\bar{r}^i$ is the position vector from the origin of the inertial frame to the origin of the system frame. This may be rewritten using computational variables as

$$\bar{f}^i_c + \bar{G}^i_c = \ddot{\bar{r}}_c$$

Now defining

$$\bar{r}^i_c = \bar{r}^i + \overline{\delta r}^i$$

and also defining

$$\ddot{\bar{r}}^i_c = \ddot{\bar{r}}^i + \ddot{\overline{\delta r}}^i$$

Defining the distance of the system from the center of the earth as the radius of the earth plus the alitute obtained from an altimeter yields $r_c = r_o + h_c = r_o + h + \delta h$

Gravity is computed using

$$\bar{G}_c = - \frac{E}{(r_o+h_c)^3} \bar{r}_c$$

Substitution yields

$$\bar{G}_c = - \frac{E}{(r_o+h+\delta h)^3} (\bar{r}^i + \delta\bar{r}^i)$$

Now making the approximation

$$(r_o+h+\delta h)^{-3} = (r_o+h)^{-3} \left(1+\frac{\delta h}{r_o+h}\right)^{-3} \simeq (r_o+h)^{-3} \left(1-3 \frac{\delta h}{r_o+h}\right)$$

substitution yields

$$\bar{G}_c = - \frac{E}{(r_o+h)^3} \left(1-3 \frac{\delta h}{r_o+h}\right) (\bar{r}^i+\overline{\delta r}^i)$$

$$\bar{G}_c = - \frac{E}{(r_o+h)^3} \bar{r}^i + 3 \frac{E}{(r_o+h)^4} \delta h\bar{r}^i - \frac{E}{(r_o+h)^3} \overline{\delta r}^i$$

The final result is given by

$$\bar{G}_c = \bar{G}^i + 3W_s^2 \delta h \hat{\ell}_R^i - W_s^2\overline{\delta r}^i$$

Substituting this into the specific force relation yields

$$\ddot{\bar{r}}^i + \ddot{\overline{\delta r}}^i = \bar{f}^i + \overline{\delta f}^i + \bar{G}^i + 3W_s^2 \delta h\hat{\ell}_R^i - W_s^2\overline{\delta r}^i$$

Noting that $\bar{f}^i = \ddot{\bar{r}}^i - \bar{G}^i$, then

$$\ddot{\overline{\delta r}}^i = \overline{\delta f}^i + 3W_s^2 \; \delta h_R^{\hat{\imath}} - W_s^2 \; \overline{\delta r}^i$$

$$\ddot{\overline{\delta r}}^i + W_s^2 \; \overline{\delta r}^i = \overline{\delta f}^i + 3W_s^2 \delta h_R^{\hat{\imath}}$$

Now investigating the $\overline{\delta f}^i$ term, it is known that

$$\bar{f}^i = C_b^i \; \bar{f}^b$$

Obtaining the differential yields

$$\overline{\delta f}^i = \delta C_b^i \; \bar{f}^b + C_b^i \; \overline{\delta f}^b$$

where

$$\delta C_b^i = C_b^i [\theta_b]$$

so that

$$\ddot{\overline{\delta r}}^i + W_s^2 \; \overline{\delta r}^i = C_b^i [\theta_b] \bar{f}^b + C_b^i \; \overline{\delta f}^b + 3W_s^2 \; \delta h_R^{\hat{\imath}}$$

Substituting for $\bar{f}^b$ yields

$$\ddot{\overline{\delta r}}^i + W_s^2 \; \overline{\delta r}^i = C_b^i [\theta_b] C_i^b \; \bar{f}^i + C_b^i \; \overline{\delta f}^b + 3W_s^2 \delta h_R^{\hat{\imath}}$$

By the similarity transformation theorem, the final result may be written as

$$\ddot{\overline{\delta r}}^i + W_s^2 \; \overline{\delta r}^i = [\theta_i] \bar{f}^i + C_b^i \; \overline{\delta f}^b + 3W_s^2 \delta h_R^{\hat{\imath}}$$

Latitude and longitude errors are given by

$$\delta L = \frac{\delta r_{Nx}}{r} \qquad \delta\lambda = \frac{\delta r_{Ny}}{r\cos L}$$

where $$[\overline{\delta r}_N] = C_i^N [\overline{\delta r}^i]$$

The differential equations describing this system will now be solved with constant gyro drift as the input; hence,

$$\ddot{\overline{\delta r}}^i + W_s^2 \overline{\delta r}^i = C_b^i [\theta_b]\ C_i^b\ \overline{f}^i$$

where the elements of the $[\theta_b]$ matrix are given by

$$\begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \begin{bmatrix} U_x t \\ U_y t \\ U_z t \end{bmatrix}$$

where the U's are the respective gyro drift rates.

Assuming that the body frame is stationary and initially aligned with the navigation frame, the solution of the above vector differential equation is given in component form as

$$\delta r_x^i(t) = U_y g \sin L[\cos\ell f_2(t) - 2W_{ie}\sin\ell f_1(t)] - U_x g[2W_{ie}\cos\ell f_1(t) + \sin\ell f_2(t)]$$

$$\delta r_y^i(t) = U_y g \sin L[2W_{ie}\cos\ell f_1(t) + \sin\ell f_2(t) + U_x g[\cos\ell f_2(t) - 2W_{ie}\sin\ell f_1(t)]$$

$$\delta r_z^i(t) = -U_y g \cos L[\frac{1}{W_s^2} t - \frac{1}{W_s^3}\sin W_s t]$$

where

$$f_1(t) = \frac{1}{(W_{ie}^2 - W_s^2)^2}\cos W_s t + \frac{1}{(W_s^2 - W_{ie}^2)}\,\frac{1}{2W_{ie}} t \sin W_{ie} t - \frac{1}{(W_s^2 - W_{ie}^2)^2}\cos W_{ie} t$$

$$f_2(t) = -\frac{W_s^2 + W_{ie}^2}{W_s(W_{ie}^2 - W_s^2)^2}\sin W_s t + \frac{1}{(W_s^2 - W_{ie}^2)} t \cos W_{ie} t + \frac{2W_{ie}}{(W_s^2 - W_{ie}^2)^2}\sin W_{ie} t$$

To obtain position errors

$$\delta L = \frac{\delta r_x^n}{r} \quad \text{and} \quad \delta\lambda = \frac{\delta r_y^n}{r \cos L}$$

where

$$\delta r_x^n = -\sin L \cos(W_{ie}t + \ell)\delta r_x^i - \sin L \sin(W_{ie}t + \ell)\delta r_y^i + \cos L \delta r_z^i$$

and

$$\delta r_y^n = -\sin(W_{ie}t + \ell)\delta r_x^i + \cos(W_{ie}t + \ell)\delta r_y^i$$

Hence,

$$\delta L(t) = -\sin L\cos(W_{ie}t+\ell)\{U_y W_s^2 \sin L[\cos\ell f_2(t)-2W_{ie}\sin\ell f_1(t)]$$

$$-U_x W_s^2[2W_{ie}\cos\ell f_1(t)+\sin\ell f_2(t)]\} -\sin L\ \sin(W_{ie}t+\ell)$$

$$\{U_y W_s^2 \sin L[2W_{ie}\cos\ell f_1(t)+\sin\ell f_2(t)] + U_x W_s^2[\cos\ell f_2(t)$$

$$-2W_{ie}\sin\ell f_1(t)]\} - \cos L\ U_y \cos L[t-\frac{1}{W_s}\sin W_s t]$$

and

$$\delta\lambda(t) = -\sec L\ \sin(W_{ie}t+\ell)\{U_y W_s^2 \sin L[\cos\ell f_2(t)-2W_{ie}\sin\ell f_1(t)]$$

$$-U_x W_s^2[2W_{ie}\cos\ell f_1(t)+\sin\ell f_2(t)]\} + \cos(W_{ie}t+\ell)\sec L$$

$$\{U_y W_s^2 \sin L[2W_{ie}\cos\ell f_1(t)+\sin\ell f_2(t)] + U_x W_s^2[\cos\ell f_2(t)$$

$$-2W_{ie}\sin\ell f_1(t)]\}$$

The error values given by these expressions agree closely with those obtained by numerical integration of the complete system differential equations. This result is illustrated in Graphs 7 and 8.

To analyze the effect or torquing uncertainty, the system differential equation is given by

$$\ddot{\overline{\delta r}}{}^i + W_s^2\overline{\delta r}{}^i = C_b^i[\theta_b]$$

where $[\theta_b]$ is the error transformation matrix resulting from the uncertainty of the angular velocity of the body frame with respect to an inertial frame. Assuming that the body frame is stationary and initially aligned with the navigation frame, the error differential equations are given in component form as

$$\delta\ddot{r}_x^i + W_s^2\delta r_x^i = -\frac{\delta T}{T} g \sin\lambda\cos L\, W_{ie} t$$

$$\delta\ddot{r}_y^i + W_s^2\delta r_y^i = \frac{\delta T}{T} g \cos\lambda\cos L\, W_{ie} t$$

The solutions to these differential equations are given by

$$\delta r_x^i(t) = -\frac{\delta T}{T} g \cos L\, W_{ie}[2W_{ie}\cos\ell f_1(t)+\sin\ell f_2(t)]$$

$$\delta r_y^i(t) = \frac{\delta T}{T} g \cos L\, W_{ie}[\cos\ell f_2(t)-2W_{ie}\sin\ell f_1(t)]$$

where $f_1(t)$ and $f_2(t)$ are the same as given in the analysis of gyro drift error, so that the solutions will be very similar in form as would be expected. The position errors are given by

$$\delta L = \frac{\delta r_x^n}{r} \quad \text{and} \quad \delta\lambda = \frac{\delta r_y^n}{r \cos L}$$

where

$$\delta r_x^n = -\cos\lambda\ \sin L\delta r_x^i - \sin\lambda\ \sin L\delta r_y^i$$

and

$$\delta r_y^n = -\sin\lambda\delta r_x^i + \cos\lambda\delta r_y^i$$

Hence, the latitude and longitude error solutions are given by

$$\delta L(t) = \frac{\delta T}{T} W_s^2\cos L\ \sin L\ W_{ie}\cos(W_{ie}t+\ell)[2W_{ie}\cos\ell f_1(t)+\sin\ell f_2(t)]$$

$$-\frac{\delta T}{T} W_s^2 \cos L\ \sin L\ W_{ie}\sin(W_{ie}t+\ell)[\cos\ell f_2(t)-2W_{ie}\sin\ell f_1(t)]$$

and

$$\delta\lambda(t) = \frac{\delta T}{T} W_s^2 W_{ie} \sin(W_{ie}t+\ell[2W_{ie}\cos\ell f_1(t)+\sin\ell f_2(t)]$$

$$+ \frac{\delta T}{T} W_s^2 W_{ie} \cos(W_{ie}t+\ell)[\cos\ell f_2(t)-2W_{ie}\sin\ell f_1(t)]$$

Graph 9 illustrates the agreement of thee values with those of the computer simulation.

If constant accelerometer bias is assumed to be the error source, the vector error differential equation is given by

$$\ddot{\overline{\delta r}}^i + W_s^2 \overline{\delta r}^i = C_b^i \delta \overline{f}^b$$

Assuming that the body frame is stationary and initially aligned with the navigation frame, the error differential equations are given in component form as

$$\delta\ddot{r}_x^i + W_s^2 \delta r_x^i = -\cos\lambda\ \sin L\delta f_x - \sin\lambda\delta f_y - \cos\lambda\ \cos L\delta f_z$$

$$\delta\ddot{r}_y^i + W_s^2 \delta r_y^i = -\sin\lambda\ \sin L\delta f_x + \cos\lambda\delta f_y - \sin\lambda\ \cos L\delta f_z$$

$$\delta\ddot{r}_z^i + W_s^2 \delta r_z^i = \cos L\delta f_x - \sin L\delta f_z$$

The solutions to these equations are given by

$$\delta r_x^i(t) = -[(\sin L\delta f_x+\cos L\delta f_z)\cos\ell+\delta f_y\sin\ell][\frac{\cos W_{ie}t-\cos W_s t}{W_s^2-W_{ie}^2}]$$

$$+[(\sin L\delta f_x+\cos L\delta f_x)\sin\ell-\delta f_y\cos\ell][\frac{1}{(W_{ie}^2-W_s^2)}\frac{W_{ie}}{W_s}\sin W_s t$$

$$+\frac{\sin W_{ie}t}{(W_s^2-W_{ie}^2)}]$$

$$\delta r_y^i(t) = -[(\sin L\delta f_x+\cos L\delta f_z)\cos\ell-\delta f_y\cos\ell][\frac{\cos W_{ie}t-\cos W_s t}{W_s^2-W_{ie}^2}]$$

$$-[(\sin L\delta f_x+\cos L\delta f_z)\cos\ell-\delta f_y\sin\ell][\frac{1}{(W_{ie}^2-W_s^2)}\frac{W_{ie}}{W_s}\sin W_s t$$

$$+\frac{\sin W_{ie}t}{(W_s^2-W_{ie}^2)}]$$

$$\delta r_z^i(t) = \frac{1}{W_s^2}(\cos L\delta f_x-\sin L\delta f_z)\sin W_s t$$

Latitude and longitude errors are given by

$$\delta L = \frac{\delta r_x^n}{r} \quad \text{and} \quad \delta\lambda = \frac{\delta r_y^n}{r\cos L}$$

where

$$\delta r_x^n = -\delta r_x^i \sin L\cos(W_{ie}t+\ell) - \delta r_y^i \sin L\sin(W_{ie}t+\ell) + \cos L\delta r_z^i$$

$$\delta r_y^n = -\delta r_x^i \sin(W_{ie}t+\ell) + \delta r_y^i \cos(W_{ie}t+\ell)$$

Graph 10 illustrates the agreement of these values with those of the computer simulation.

For initial misalignment of the body frame, the error differential equation is given by

$$\ddot{\overline{\delta r}}{}^i + W_s^2\overline{\delta r}^i = C_b^i[\theta_b]\bar{f}^b$$

where $[\theta_b]$ is the skew misalignment matrix

$$[\theta_b] = \begin{bmatrix} 0 & \zeta_z & -\delta_y \\ -\zeta_z & 0 & \zeta_x \\ \zeta_y & -\zeta_x & 0 \end{bmatrix}$$

If the system is stationary and the body frame initially aligned with the navigation frame, the component solutions of the above differential equations are given by

$$\delta r_x^i(t) = -[(\sin L\zeta_y g)\cos\ell - (\zeta_x g)\sin\ell][\frac{\cos W_{ie}t - \cos W_s t}{W_s^2 - W_{ie}^2}]$$

$$+[(\sin L\zeta_y g)\sin\ell + (\zeta_x g)\cos\ell][\frac{1}{(W_{ie}^2 - W_s^2)}\frac{W_{ie}}{W_s}\sin W_s t$$

$$+\frac{\sin W_{ie}t}{(W_s^2 - W_{ie}^2)}]$$

$$\delta r_y^i(t) = -[(\sin L\zeta_y g)\sin\ell + (\zeta_x g)\cos\ell][\frac{\cos W_{ie}t - \cos W_s t}{W_s^2 - W_{ie}^2}]$$

$$-[(\sin L\zeta_y g)\cos\ell - (\zeta_x g)\sin\ell][\frac{1}{(W_{ie}^2 - W_s^2)}\frac{W_{ie}}{W_s}\sin W_s t$$

$$+\frac{\sin W_{ie}t}{(W_s^2 - W_{ie}^2)}]$$

$$\delta r_z^i(t) = [\zeta_y \cos L_g]\frac{1}{W_s^2}\sin W_s t$$

Latitude and longitude errors are given by

$$\delta L = \frac{\delta r_x^n}{r} \quad \text{and} \quad \delta\lambda = \frac{\delta r_y^n}{r \cos L}$$

where

$$\delta r_x^n = -\delta r_x^i \sin L \cos(W_{ie}t+\ell) - \delta r_y^i \sin L \sin(W_{ie}t+\ell) + \cos L \delta r_x^i$$

and

$$\delta r_y^n = -\delta r_x^i \sin(W_{ie}t+\ell) + \delta r_y^i \cos(W_{ie}t+\ell)$$

Graph 11 illustrates the agreement of these values with those of the computer simulation.

In conclusion of this section, the excellent agreement between the results of the linear analysis and the results obtained by numerical integration of the complete system differential equations indicate that the derived linear system under stated limitations.

CONCLUSION

The solution of the general system differential equations by numerical methods verified that the derived linear theory is a valid analytical description of the systems under stated limitations regarding the magnitude of certain rotation angles. It was found that accelerometer bias and initial misalignment result in position errors which are very nearly oscillatory at the Schuler frequency. Gyro drift and torquing uncertainty were found to be the predominant error sources. For the system which computes in the navigation frames, a constant gyro drift resulted in a bounded latitude error but an unbounded longitude error. This is very similar to the performance of a local vertical inertial navigation system (11). In fact, the systems have the same characteristic function but forcing functions of different form. Hence, the similarity on response to error sources. The strapdown system which computes in the geocentric inertial frame has unbounded errors in both latitude and longitude when driven by a constant gyro drift. This performance is very similar to that of the space stabilized inertial navigation system (12). Again, in this case, the systems have the same characteristic equation but different forms of driving functions. From these results, it would seem that the system computing in the navigation frame possesses an advantage in error stability over the system computing in the inertial frame.

Each of the systems presents advantages and disadvantages for system mechanization. The principal disadvantages are computation of acceleration compensation terms for the system computing in the navigation coordinates and explicit computation of gravity for the system computing in an inertial frame. For the system computing in the body frame, the disadvantages are that gravity must be computed explicitly and that position information is coordinatized in the body frame and requires a transformation to become useful information.

REFERENCES

(1) Markey, W.R., and Hovorka, J., The Mechanics of Inertial Position and Heading Indication, Methuen, London, 1961.

(2) Broxmeyer, C., Inertial Navigation Systems, McGraw-Hill, New York, N.Y., 1964.

(3) Pitman, G.R., Inertial Guidance, John Wiley & Sons, New York, N.Y., 1962.

(4) O'Donnell, C.F., Inertial Navigation Analysis and Design, McGraw-Hill, New York, N.Y., 1964.

(5) Wiener, T.F., Theoretical Analysis of Gimballess Inertial Reference Equipment Using Delta-Modulated Instruments, Instrumentation Laboratory, Massachusetts Institute of Technology, Cambridge, Mass., Report T-300, 1962.

(6) Vandervelde, W.E., and Bumstead, R.M., Navigation and Guidance Systems Employing a Gimballess I.M.U., Instrumentation Laboratory, Massachusetts Institute of Technology, Cambridge, Mass., Report 412A.

(7) Halfman, R.L., Dynamics, Addison-Wesley, Reading, Mass., 1962.

(8) Hildebrand, F.B., Introduction to Numerical Analysis, McGraw-Hill, New York, N.Y., 1956.

(9) Ogata, K., State Space Analysis of Control Systems, Prentice-Hall, Englewood Cliffs, N.J., 1967.

(10) Macon, N., Numerical Analysis, John Wiley & Sons, New York, N.Y., 1963.

(11) Britting, K.R., "Analysis of Local Vertical Inertial Navigation Systems," M.I.T. EAL Report 54. To be published.

(12) Britting, K.R., "Analysis of Space Stabilized Inertial Navigation Systems," M.I.T. EAL Report 35, 1968.

APPENDIX I

ERROR CURVES FOR SYSTEM COMPUTING IN NAVIGATIONAL COORDINATES

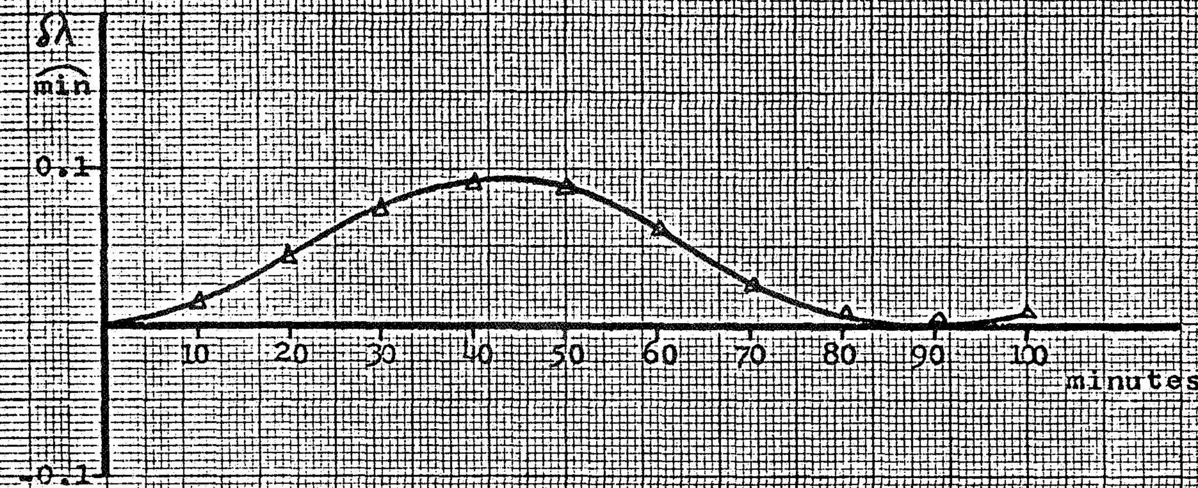
Position errors due to constant uncertainties in inertial sensors.

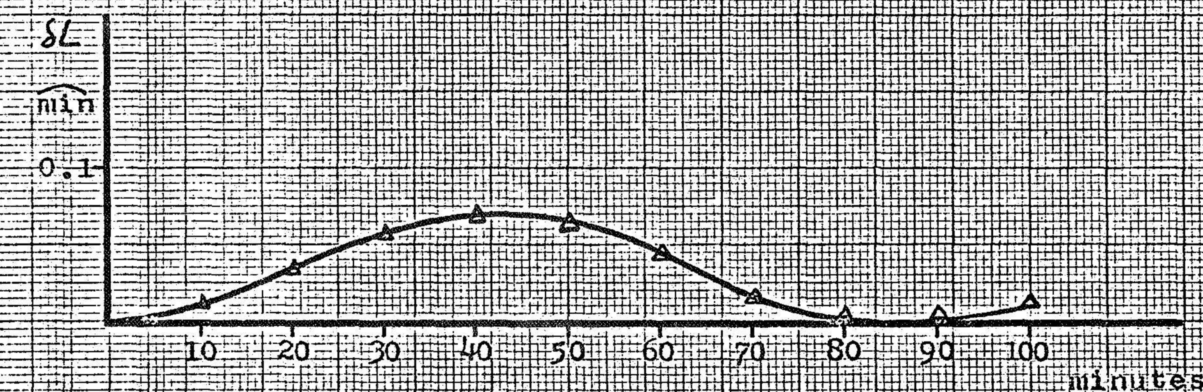
Error Sources

Accelerometer Bias	10^{-5}g
Misalignment Angle	10^{-5}rad
Gyro Drift	1.5×10^{-7}rad/sec
Torquing Uncertainty	$= 1.5 \times 10^{-7}$rad/sec

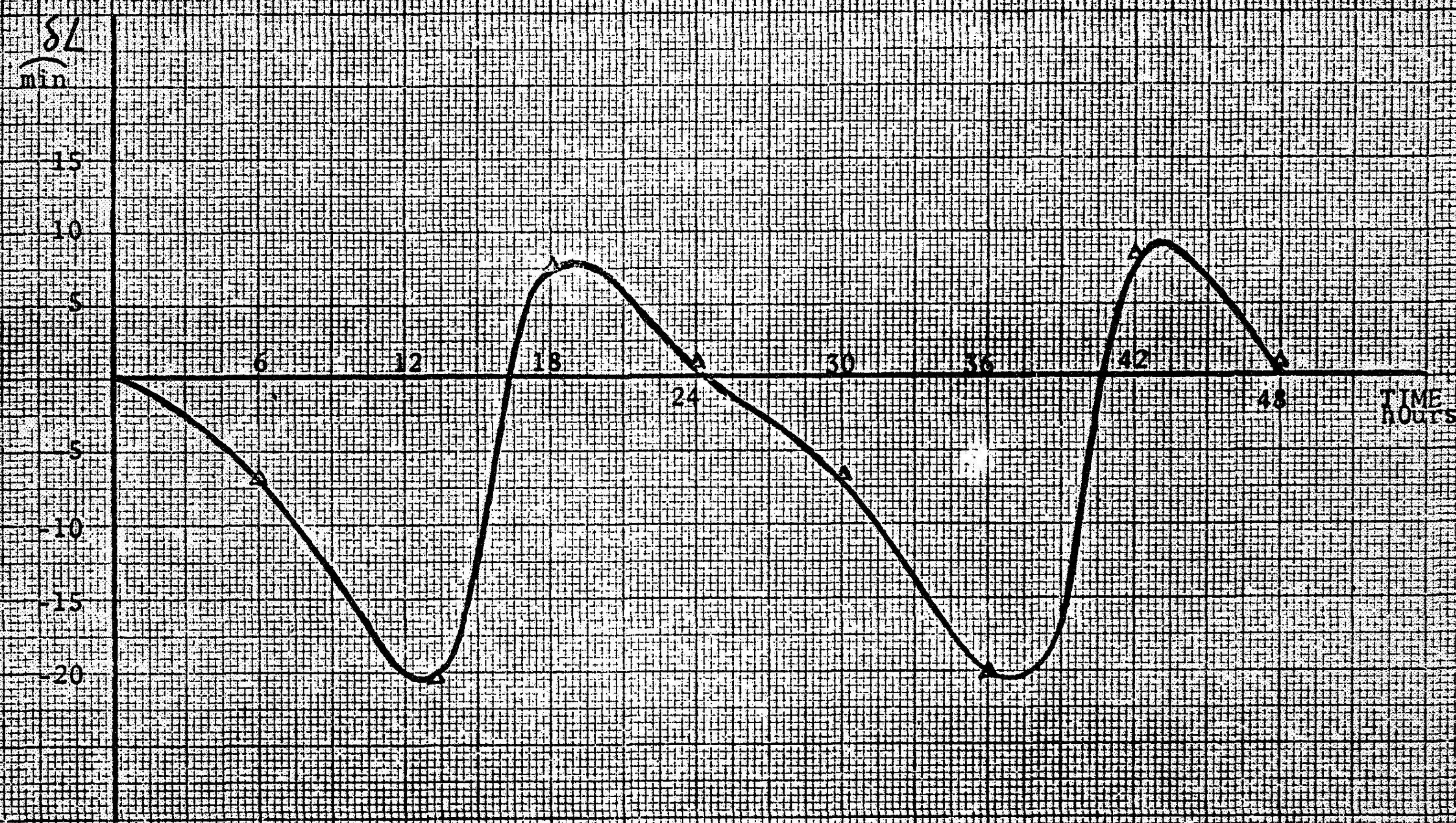
Note: All plots at the approximate location of Cambridge, Massachusetts, 42 degrees north latitude and 71 degrees west longitude.

On all plots, the smooth curve is the result of the linear theory and the indicated points (Δ) are the computer simulation results.

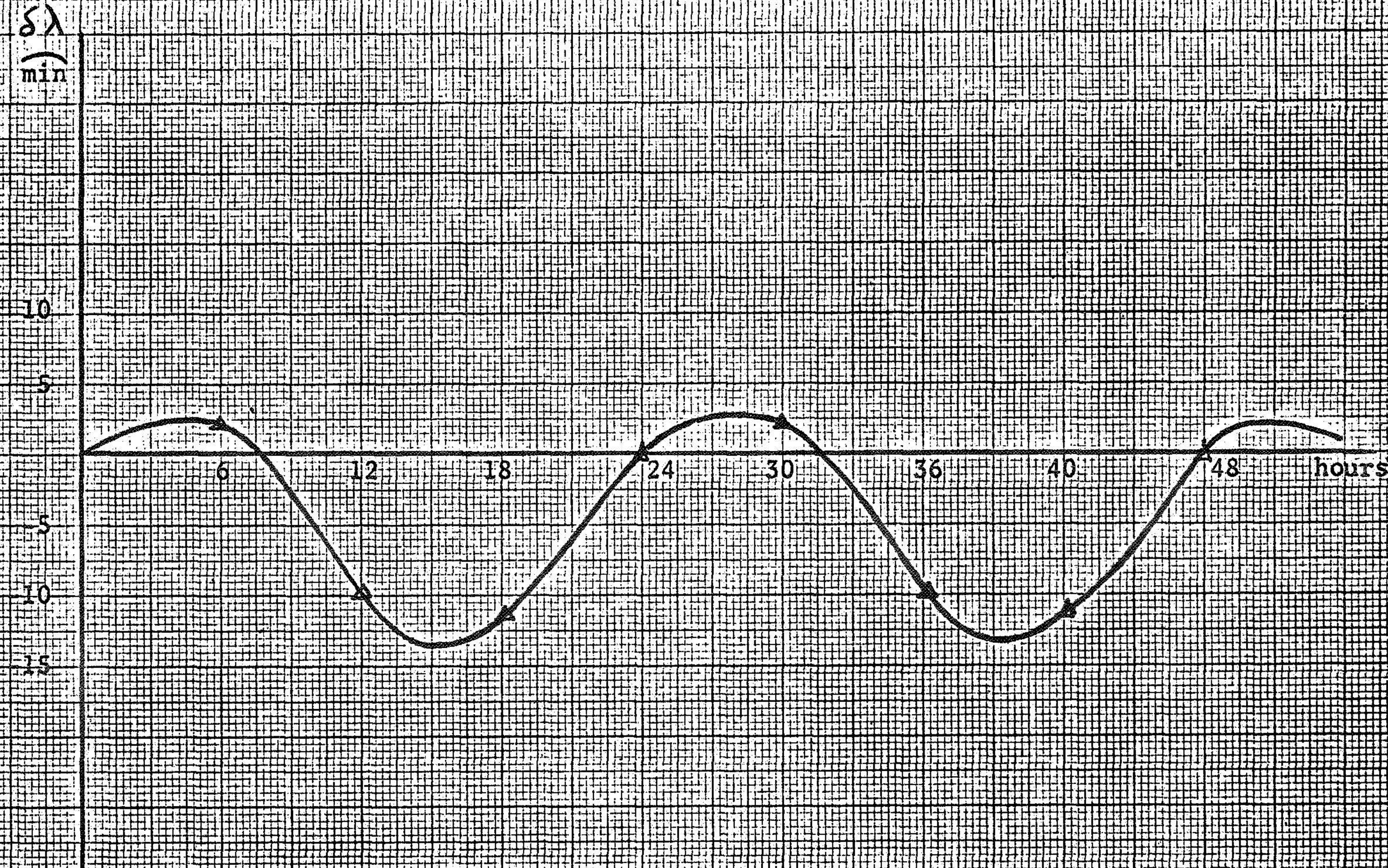




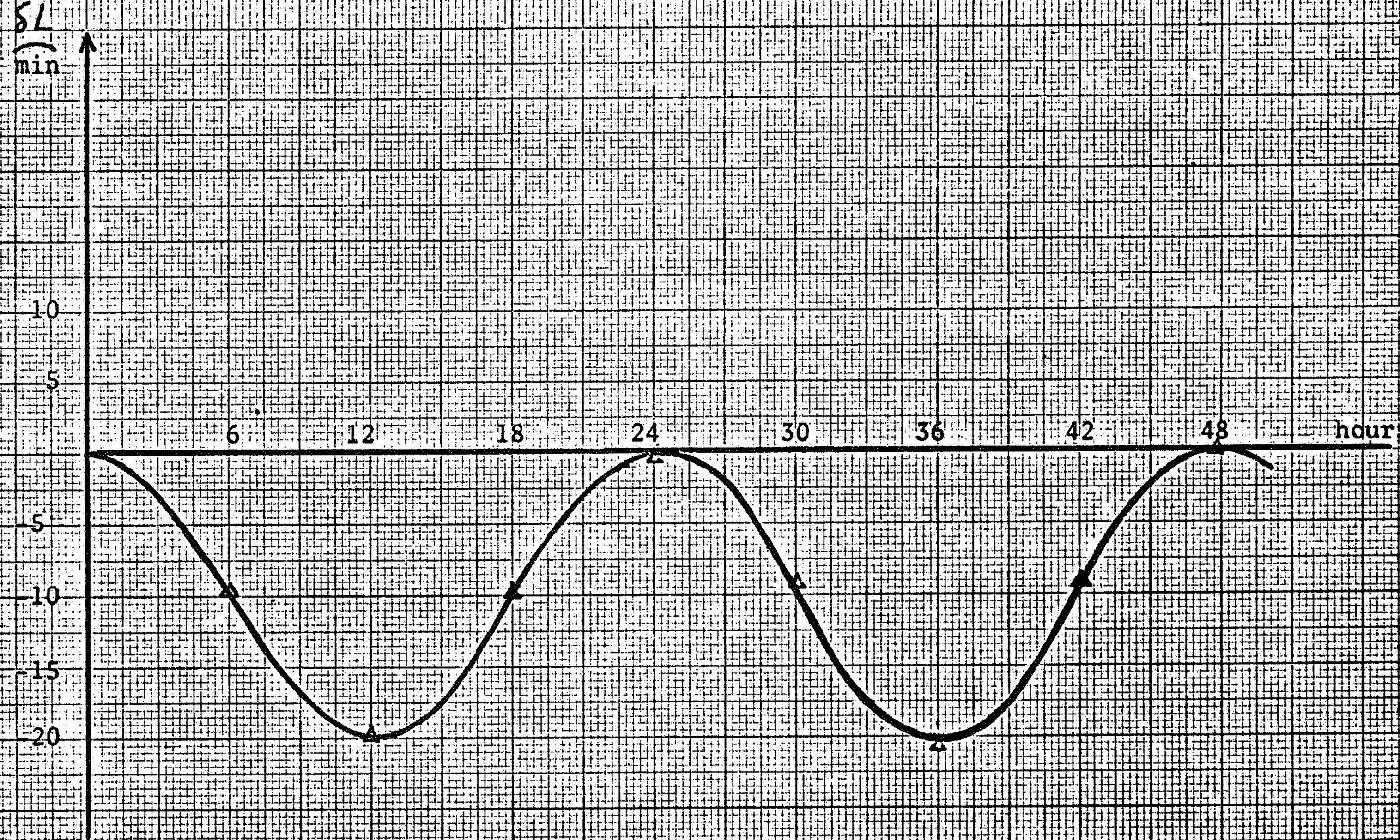
Graph 1: Position errors due to accelerometer bias



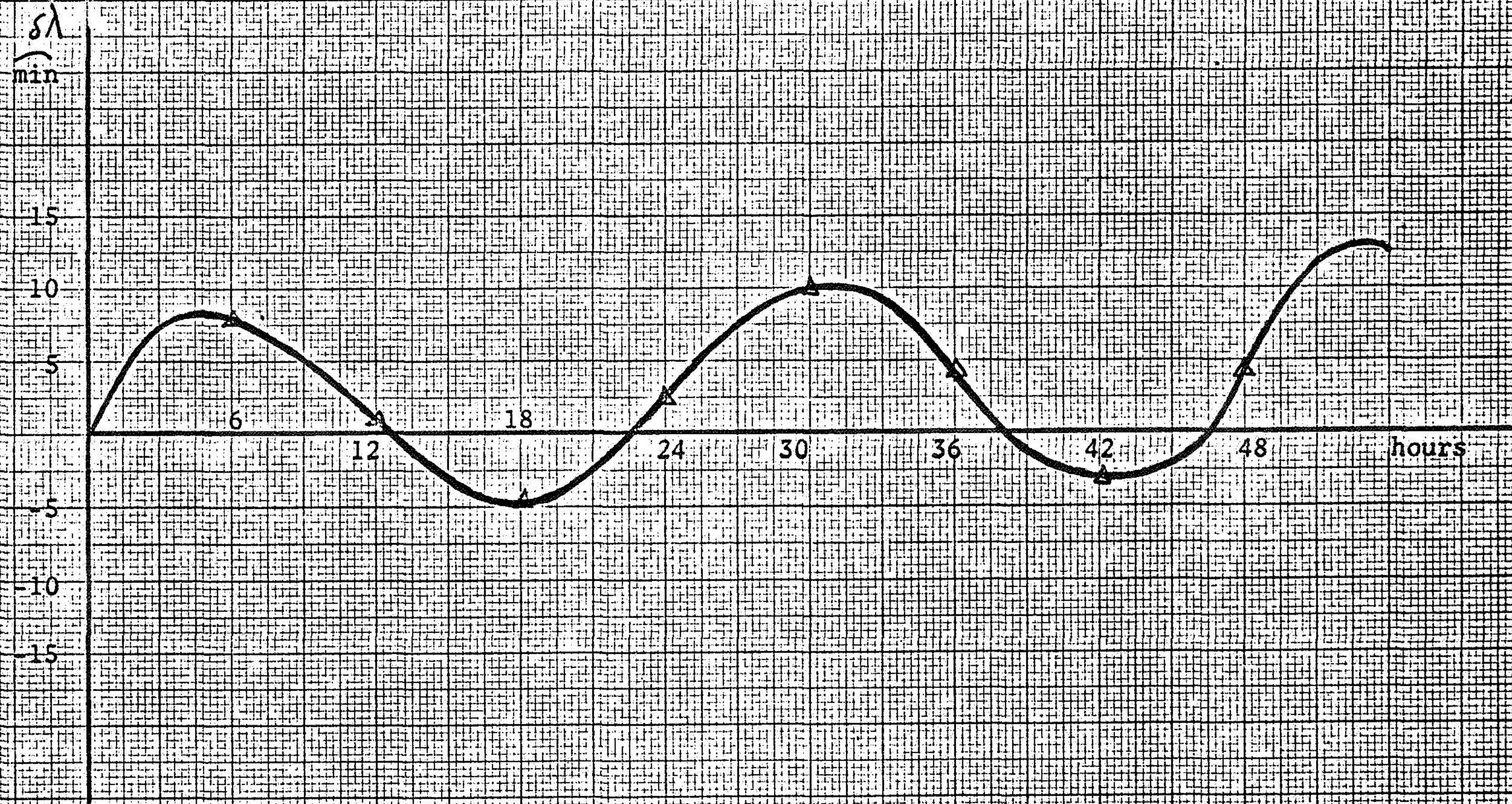
Graph 2: Latitude error due to gyro drift



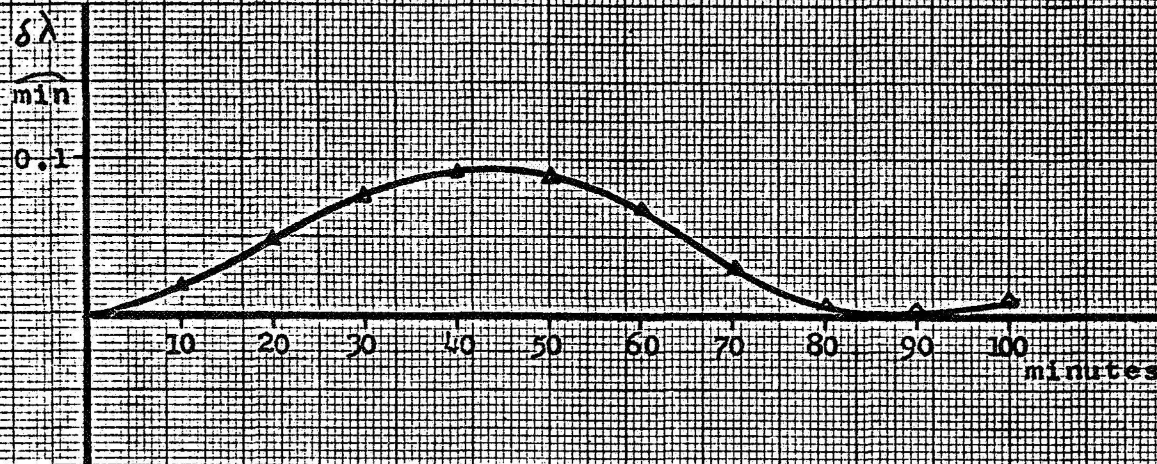
Graph 3: Longitude error due to gyro drift

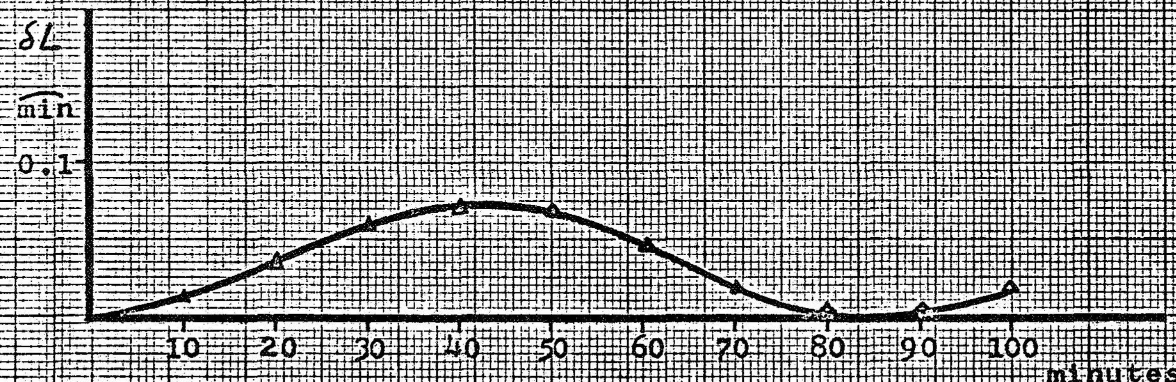


Graph 4: Latitude error due to torque uncertainty



Graph 5: Longitude error due to torque uncertainty





Graph 6. Position errors due to initial misalignment

APPENDIX II

ERROR CURVES FOR SYSTEM COMPUTING IN GEOCENTRIC INERTIAL COORDINATES

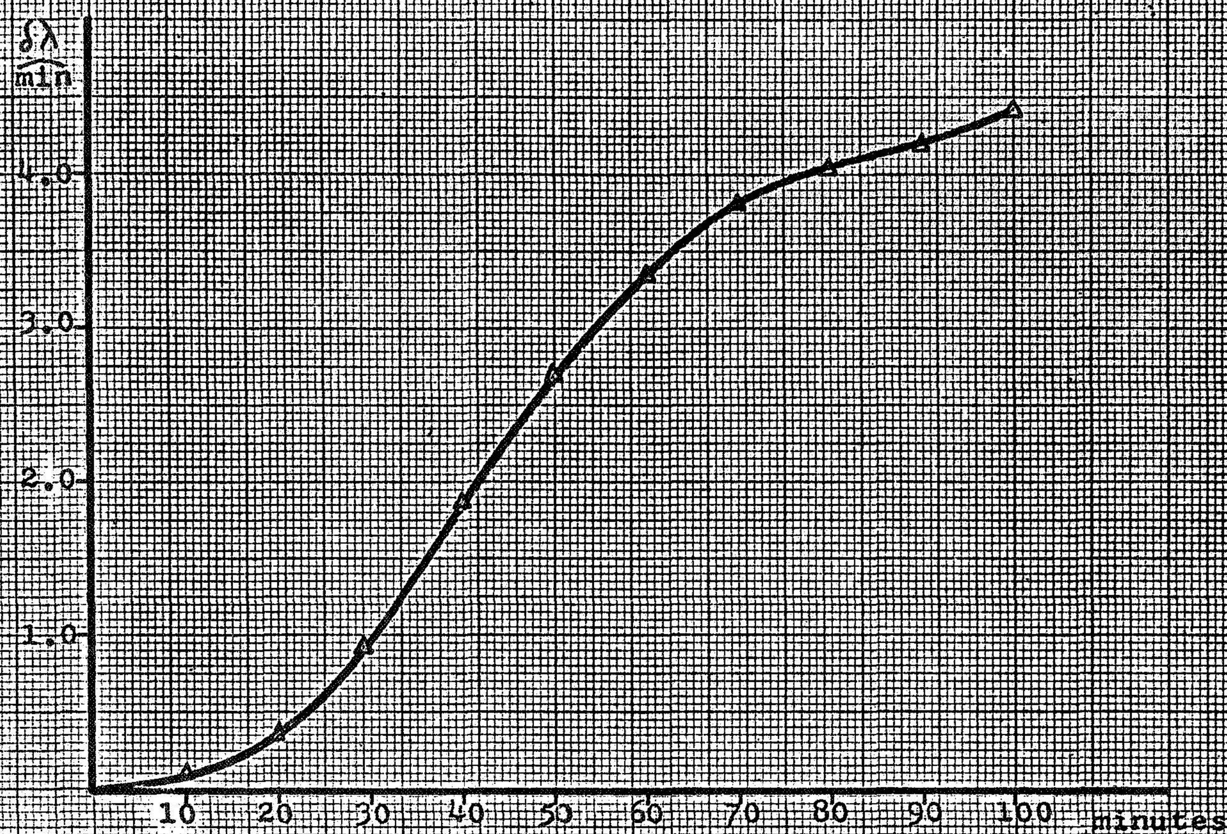
Position errors due to constant uncertainties in inertial sensors.

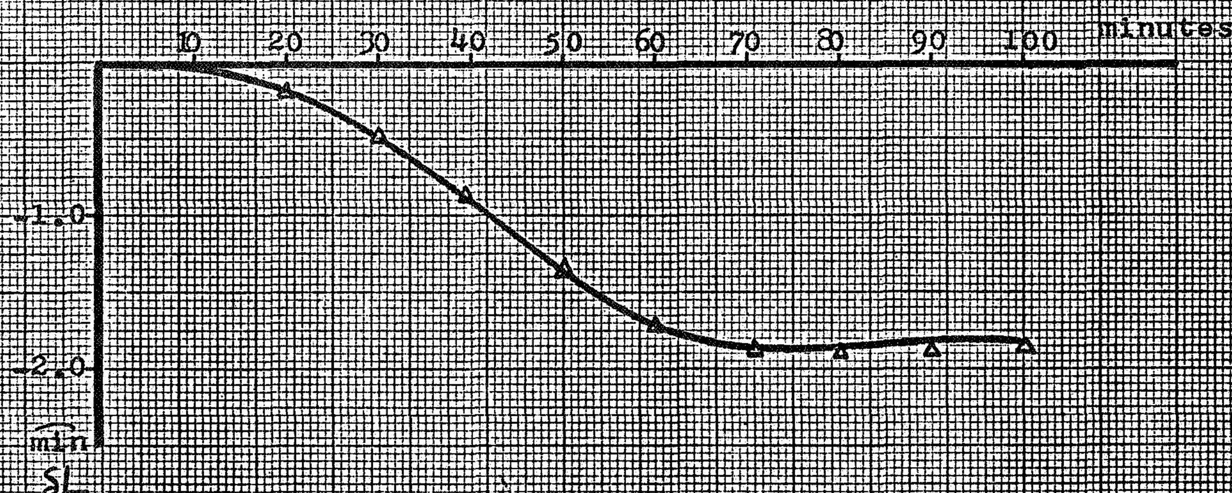
Error Sources

Accelerometer Bias	10^{-5}g
Misalignment Angle	10^{-5}rad
Gyro Drift	1.5×10^{-7}rad/sec
Torquing Uncertainty	$= 1.5 \times 10^{-7}$rad/sec

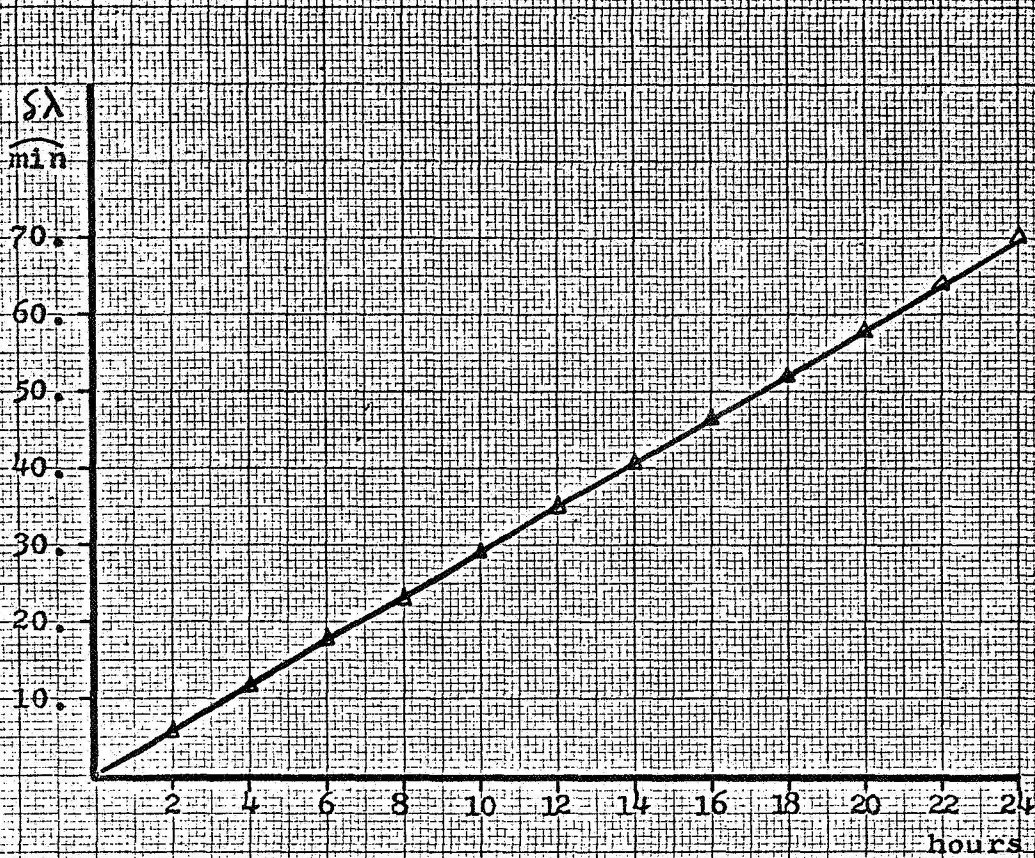
Note: All plots at the approximate location of Cambridge, Massachusetts, 42 degrees north latitude and 71 degrees west longitude.

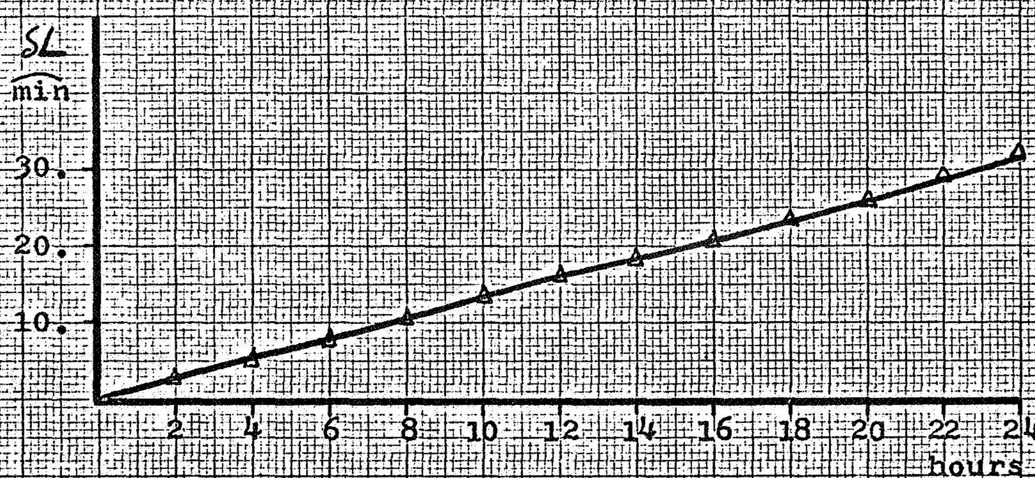
On all plots, the smooth curve is the result of the linear theory and the indicated points (Δ) are the computer simulation results.



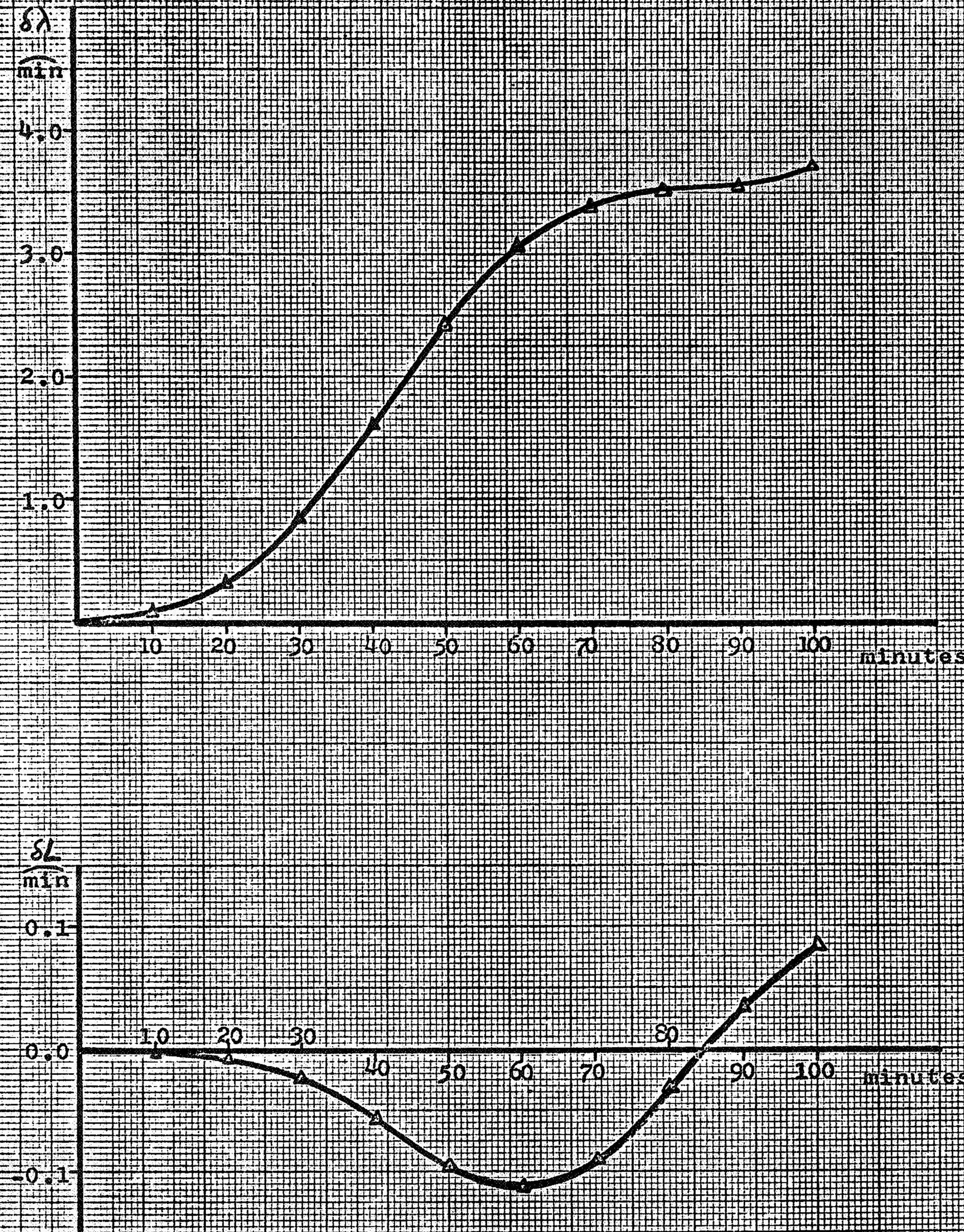


Graph 7: Position errors due to gyro drift

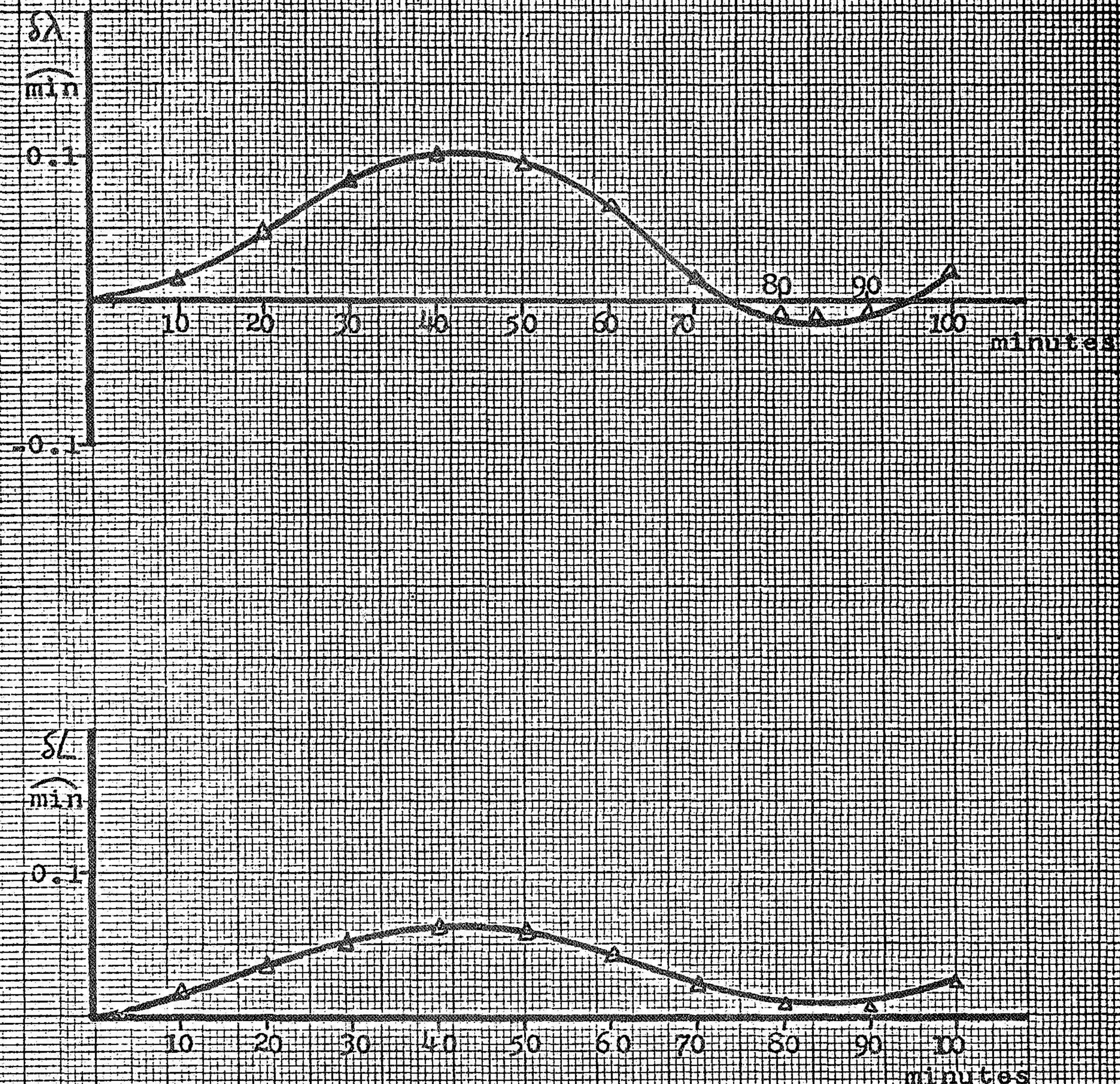




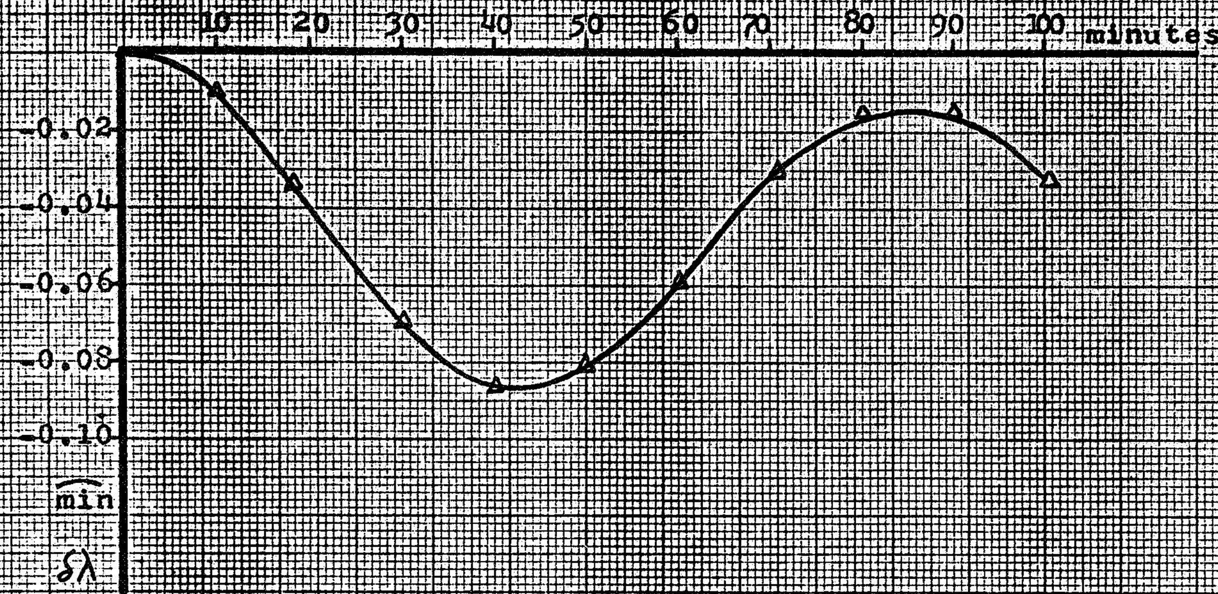
Graph 8: Position errors due to gyro drift
System computing in inertial frame

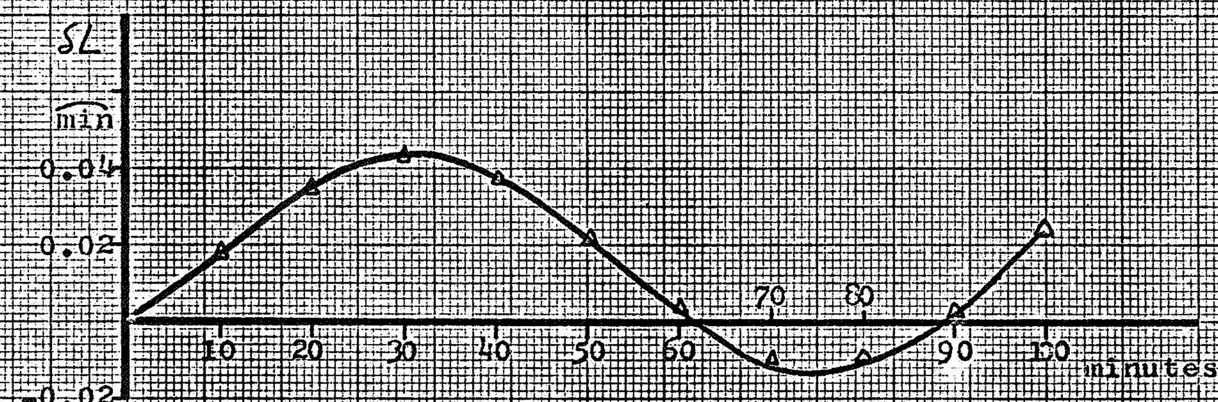


Graph 9: Position errors due to torquing uncertainty



Graph 10: Position errors due to accelerometer bias





Graph 11: Position errors due to initial misalignment