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ANALYTICAL SOLUTIONS OF A CLASS OF OPTIMUM ORBIT MODIFICATIONS

by Nguyen X. Vinh and Christian Marchal

Prepared by
UNIVERSITY OF MICHIGAN
Ann Arbor, Mich.
for

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ABSTRACT

This paper presents the complete analytical solutions of several fundamental problems in orbital corrections. The initial orbit is represented by a given point in the phase space while the final orbit is constrained to stay in a given curve which can be bounded, unbounded, or is composed of a finite number of segments of different curves. The inclusion of atmospheric maneuver as part of the optimum process is discussed, and its analytical treatment can be carried out by modifying the final state to include the set of orbits having their perigees at the boundary of the atmosphere.

The selection of the apogee and perigee distances as state variables gives a symmetric form to the problem and results in a linear differential equation of the first order for the ratio of the adjoint variables. The introduction of a curve of comparison, called the separatrix, facilitates the discussion of the existence of a corner on an optimal trajectory.

1. INTRODUCTION

Consider a space vehicle initially in an orbit (E_o) around a spherical planet with center of attraction at 0. The initial orbit is defined by its semi major-axis a_o and its eccentricity e_o. It is proposed to bring the vehicle, by a series of orbital maneuvers, into a final orbit such that its elements, denoted by the subscript 1, satisfy a relation of the form

$$f(a_1, e_1) = 0 \tag{1}$$

We seek to minimize the total characteristic velocity for the maneuver. Since for a high-thrust propulsion system the characteristic velocity provides a direct measure of the fuel consumption, the optimal trajectory considered in this paper yields the minimum fuel expenditure.

We assume the planet is surrounded by a spherical atmosphere with center at 0 and radius R (Fig. 1). In the search of the absolute minimum fuel consumption we further assume that the duration of the maneuver is unlimited, and the thrust provided by the rockets on board the space vehicle is not bounded, that is it can produce impulsive changes in the velocity. For the case where the thrust magnitude is limited, it can be made impulsive by the process of fractioning. Thus the problem is of the class of time-free, impulsive, orbital transfers.

2. FORMULATION OF THE PROBLEM

The problem is formulated as an optimal control problem. At the time t, the state of the vehicle is characterized by the row vector (Fig. 1)

$$\vec{\xi} = (\alpha, \beta, \omega, u) \tag{2}$$

where α is the apogee distance, β the perigee distance, ω the longitude of the perigee, and u the characteristic velocity. The first three coordinates describe the osculating ellipse along which the vehicle is moving at the time t, that is, the Keplerian orbit which the vehicle would follow should the engine cease to operate at the time t. The parameter u is a measure of the latent velocity expended since the initial time and is defined by

$$u = \int_0^t \frac{T}{m}\, dt \geqslant 0 \tag{3}$$

where T is the instantaneous magnitude of the thrust and m the mass of the vehicle. The control is represented by the row vector

$$\vec{\eta} = (v, \phi) \tag{4}$$

where v is the true anomaly and ϕ the thrust direction with respect to the local horizon.

The equations of motion are derived from the classical equations of variations in celestial mechanics (Ref. 1). For a time-free problem, u is a convenient independent variable. We have

$$\frac{d\alpha}{du} = \frac{2\alpha^2}{nb(\alpha+\beta)}\left[\sin v \sin\phi + \frac{(\alpha-\beta)\cos^2 v + 2(\alpha+\beta)\cos v + (\alpha+3\beta)}{(\alpha+\beta)+(\alpha-\beta)\cos v}\cos\phi\right] = f_1(\vec{\xi},\vec{\eta})$$

$$\frac{d\beta}{du} = -\frac{2\beta^2}{nb(\alpha+\beta)}\left[\sin v \sin\phi + \frac{(\alpha-\beta)\cos^2 v + 2(\alpha+\beta)\cos v - (3\alpha+\beta)}{(\alpha+\beta)+(\alpha-\beta)\cos v}\cos\phi\right] = f_2(\vec{\xi},\vec{\eta})$$

$$\frac{d\omega}{du} = -\frac{4\alpha\beta}{nb(\alpha^2-\beta^2)}\left[\cos v \sin\phi - \frac{2(\alpha+\beta)+(\alpha-\beta)\cos v}{(\alpha+\beta)+(\alpha-\beta)\cos v}\sin v\cos\phi\right] = f_3(\vec{\xi},\vec{\eta})$$

(5)

$$\frac{du}{du} = 1 = f_0(\vec{\xi},\vec{\eta})$$

where

$$b = \sqrt{\alpha\beta} \quad , \quad n = \sqrt{\frac{8\mu}{(\alpha+\beta)^3}} \qquad (6)$$

respectively denote the semi minor-axis and the mean motion and where μ = GM is the gravitational constant. The end conditions are

$$u_0 = 0 \; , \alpha = \alpha_0 = a_0(1+e_0) \; , \beta = \beta_0 = a_0(1-e_0) \; , \omega_0 = 0$$
$$u = u_1 \; , \; \alpha = \alpha_1 \; , \; \beta = \beta_1 \; , \; \omega = \omega_1 \tag{7}$$

and α_1 , β_1 are such that they satisfy a specified relation

$$\theta \; (\alpha_1, \beta_1) = 0 \tag{8}$$

The problem is to find, at each instant u , the control $\vec{\eta}$ such that the characteristic velocity u_1 is a minimum. Using the maximum principle, we define an adjoint vector $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ such that its components satisfy the adjoint equations (Ref. 2)

$$\frac{d\lambda_1}{du} = -\frac{\delta H}{\delta \alpha} \; , \frac{d\lambda_2}{du} = -\frac{\delta H}{\delta \beta} \; , \frac{d\lambda_3}{du} = -\frac{\delta H}{\delta \omega} = 0 \tag{9}$$

where the Hamiltonian H is given by

$$H = \lambda_0 + \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 \tag{10}$$

with

$$\lambda_0 = -1 < 0 \tag{11}$$

The optimal trajectory is obtained by integrating the systems of equations (5) and (9), using the end conditions (7), (8) and (11), with the control parameters v and ϕ selected such that, at each instant, H is an absolute maximum.

3. ANALYSIS

3.1 Optimal Trajectories

ω is an ignorable coordinate. Hence if the final orientation of the orbit is not specified, $\lambda_3 = 0$ and the condition of optimality is the maximization of the reduced Hamiltonian

$$\bar{H} = \lambda_1 f_1 + \lambda_2 f_2$$

with respect to v and ϕ.

If the angles v and ϕ are not constrained, as they will be in the cases considered in this paper, then it is easy to verify that the stationary values of $\bar{H}$ correspond to $v = 0$ or $v = \pi$, and $\phi = 0$ or $\phi = \pi$. Therefore, along an extremal

$$\sin v = \sin\phi = 0, \cos v = \pm 1 = \epsilon_1, \cos\phi = \pm 1 = \epsilon_2 \tag{12}$$

Along an extremal, the Hamiltonian (10) reduces to

$$H = \lambda_0 + \frac{4\epsilon_2\left[(1+\epsilon_1)\alpha^2\lambda_1 + (1-\epsilon_1)\beta^2\lambda_2\right]}{nb\left[(\alpha+\beta) + \epsilon_1(\alpha-\beta)\right]} \tag{13}$$

and by elimination of u in (5) the equation of the optimal trajectory can be written as

$$\frac{(1+\epsilon_1)}{\beta^2}\, d\beta = \frac{(1-\epsilon_1)}{\alpha^2}\, d\alpha \tag{14}$$

when $\epsilon_1 = -1$, α = constant

when $\epsilon_1 = 1$, β = constant

In the (α , β) space, with $\alpha \geqslant \beta$, the optimal trajectories are the lines parallel to the axes (Fig. 2). The impulses are always applied tangentially at the apses.

3.2 Switching Curve

Along an optimal trajectory there may exist a corner S (or

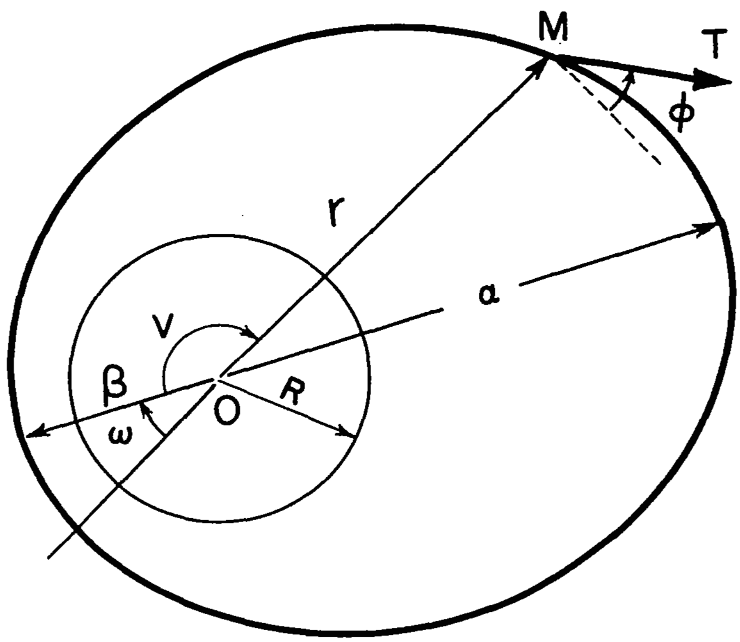

Fig. 1. The Osculating Orbit.

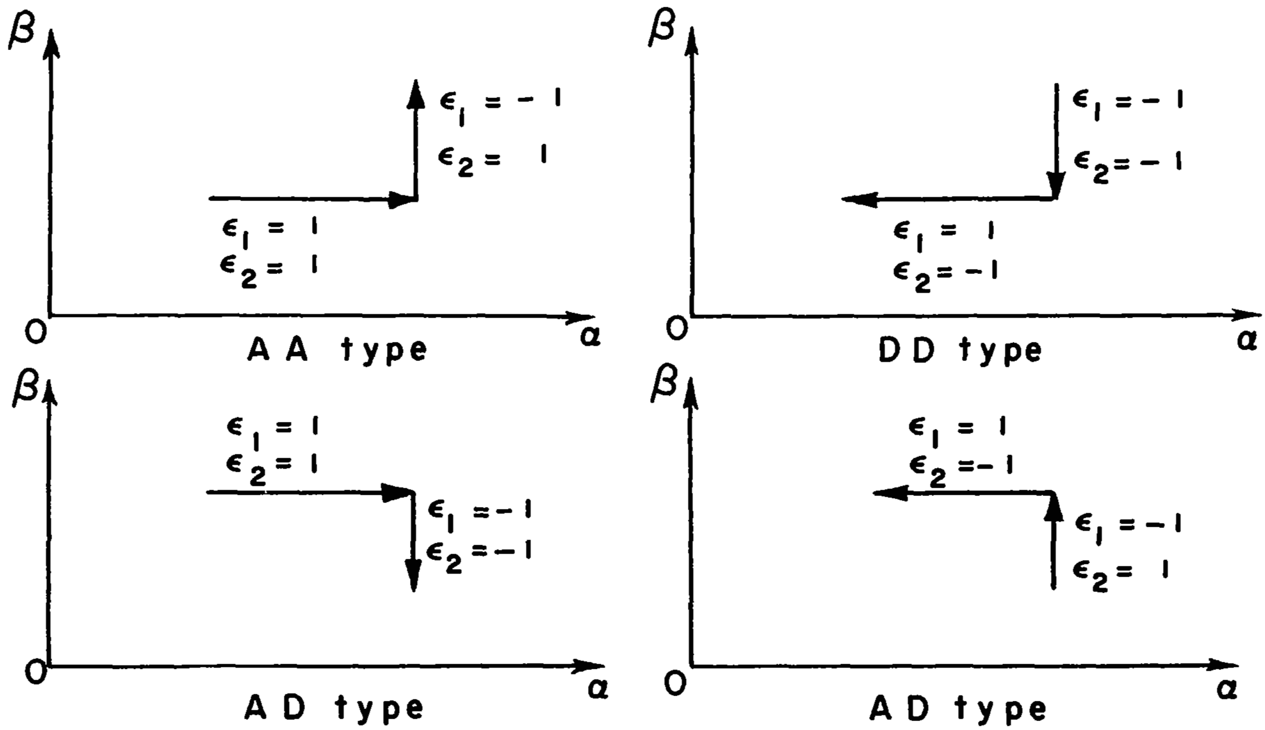

Fig. 2. Different Types of Switching.

switching) at which the trajectory changes direction. The direction of switching is of four possible types as shown in Fig. 2. Using the letter A to designate an accelerative impulse and D for a decelerative impulse we have

First type AA switching starting from the perigee. On the left of the corner S, $\epsilon_1 = 1$, $\epsilon_2 = 1$. On the right of the corner, $\epsilon_1 = -1$, $\epsilon_2 = 1$. Let

$$\Psi = \frac{\lambda_1}{\lambda_2} \tag{15}$$

By writing that the Hamiltonian (13) is continuous across a corner we have for the value of Ψ at the point S

$$\Psi_s = \frac{\beta}{\alpha} \tag{16}$$

where α and β are the coordinates of the point S.

Second type DD switching starting from the apogee. This is the inverse operation of the previous one. On the left of the corner S we have $\epsilon_1 = 1$, $\epsilon_2 = -1$. On the right we have $\epsilon_1 = -1$, $\epsilon_2 = -1$. In this case the value of Ψ at the point S is also given by (16).

Third type AD switching starting from the perigee. On the left of the corner S, $\epsilon_1 = 1$, $\epsilon_2 = 1$. On the right we have $\epsilon_1 = -1$, $\epsilon_2 = -1$. The value of the ratio Ψ at the point S for this type of switching is

$$\Psi_s = -\frac{\beta}{\alpha} \tag{17}$$

Fourth type AD switching starting from the apogee. This is the inverse operation of the preceding one. On the left we have $\epsilon_1 = 1$, $\epsilon_2 = -1$. On the right we have $\epsilon_1 = -1$, $\epsilon_2 = 1$. The value of Ψ at the point S is also given by (17).

For a prescribed final state represented by the equation (8) the locus of the possible switching point S is a curve which we shall refer to as the switching. The switching is obtained by integrating the adjoint equations (9) along the last subarc SK (Fig. 3) and using the corner condition at the point S and the transversality condition at the point K. In the figure the final state is designated by Σ.

Integration along α = constant, $\epsilon_1 = -1$.

Explicitely we have

$$\frac{d\lambda_1}{du} = -\frac{2\epsilon_2\lambda_2}{nb} \cdot \frac{\beta(2\alpha-\beta)}{\alpha(\alpha+\beta)}$$

$$\frac{d\lambda_2}{du} = -\frac{2\epsilon_2\lambda_2}{nb} \cdot \frac{4\beta+\alpha}{(\alpha+\beta)}$$

$$\frac{d\beta}{du} = \frac{4\epsilon_2\beta}{nb}$$

Using β as the new independent variable we have the equation for $\Psi = \lambda_1 / \lambda_2$

$$\frac{d\Psi}{d\beta} - \frac{4\beta + \alpha}{2\beta(\alpha+\beta)}\Psi = \frac{\beta - 2\alpha}{2\alpha(\alpha+\beta)} \tag{18}$$

The general solution of this equation is

$$\Psi = C\sqrt{\beta(\alpha+\beta)^3} - \frac{\beta(2\alpha+\beta)}{\alpha^2} \tag{19}$$

where the constant of integration C is to be determined by the appropriate end conditions. Since the last subarc is along α = constant, the switching is of the first or the third type. For an AA type, using the value (16) for Ψ we have for the constant C , evaluated at the point S

$$C = \frac{\beta(3\alpha+\beta)}{\alpha^2\sqrt{\beta(\alpha+\beta)^3}} \tag{20}$$

where α and β are the coordinates of the point S .

For an AD type of switching the value (17) for Ψ at the point S is used to calculate the constant C . We have

$$C = \frac{\beta}{\alpha^2\sqrt{\beta(\alpha+\beta)}} \tag{21}$$

At the terminal point K , the vector (λ_1 , λ_2) is orthogonal to the curve $\theta\ (\alpha_1, \beta_1) = 0$ by the transversality condition. The value of Ψ at the point K(α_1 , β_1) is then

$$\Psi_K = \frac{\delta\theta/\delta\alpha_1}{\delta\theta/\delta\beta_1} \tag{22}$$

Using this value to calculate C in (19) we have

$$C = C(\alpha_1, \beta_1) = C(\alpha, \beta_1) \tag{23}$$

In the last relation, β_1 can be calculated in terms of α by solving

$$\theta\ (\alpha, \beta_1) = 0 \tag{24}$$

Finally if the value of C in (23) is equated to the value of C in (20) or (21), depending on the type of switching, we obtain a relation between α and β which is the equation of the switching curve.

Integration along β = constant , $\epsilon_1 = 1$.

If the last subarc SK is along a line β = constant the adjoint equations are integrated along this line, using α as independent variable. We obtain

$$\frac{1}{\Psi} = C\sqrt{\alpha(\alpha+\beta)^3} - \frac{\alpha(\alpha+2\beta)}{\beta^2} \tag{25}$$

Because of the symmetry of the state variables, this last relation can be easily obtained by replacing Ψ by $1/\Psi$ in Eq. (19) and interchanging α and β.

The switching now is of the second or the fourth type. For a DD switching the value of the constant C evaluated at the point S is

$$C = \frac{\alpha(\alpha+3\beta)}{\beta^2\sqrt{\alpha(\alpha+\beta)^3}} \tag{26}$$

For an AD switching

$$C = \frac{\alpha}{\beta^2\sqrt{\alpha(\alpha+\beta)}} \tag{27}$$

At the terminal point K , by using the transversality condition (22) we have for the constant C evaluated at $K(\alpha_1, \beta_1)$

$$C = C(\alpha_1, \beta_1) = C(\alpha_1, \beta) \tag{28}$$

In the last relation, α_1 can be evaluated in terms of β by solving

$$\theta(\alpha_1, \beta) = 0 \tag{29}$$

Finally if the value of C in (28) is equated to the value of C in (26) or (27), depending on the type of switching, we obtain the quation of the switching curve.

In the following, the quation of the switching curve is represented by

$$S(\alpha, \beta) = 0 \tag{30}$$

In deriving the equation of the switching we have assumed that no constraint has been put on the final state. If the final state is constrained then an optimal trajectory may have a corner which is not on the switching curve. In this case the final orbit is always at the boundary of the final state. Another type of corner on an optimal trajectory may arise when atmospheric drag is used in the optimal transfer. This type of corner will be discussed in section 3.4.

3.3 The Separatrix

The application of the maximum principle only gives the necessary conditions for optimality. Therefore, for a specified problem, even in the case where the switching is real in the space $\alpha \geqslant \beta > 0$

it only means that if the final state is not constrained, and if a corner exists on an optimal trajectory, this corner has to be on the switching. To avoid the difficult task of proving the sufficiency for optimality which requires the finding of the conjugate point we shall introduce a curve called the separatrix which can be used to rule out the existence of the corner in most cases. The separatrix is defined as a curve which delimits the domain where using a transfer via parabolic orbits is more economical than going directly to the final state by applying an impulse at one of the apses. Like the switching, the separatrix depends on the final state. The discussion is illustrated in Fig. 3. For the initial orbit E_0, the optimal trajectory to reach the final state Σ is the trajectory E_0H, obtained by applying a decelerative impulse at the apogee of E_0. For, the possible corner S is in the domain where it is more economical to follow the line SP to infinity rather than using the trajectory SK. The composite trajectory E_0SP, in turn, is less economical than the true optimal E_0H since E_0 is on the other side of the separatrix.

3.4 The Use of Atmospheric Drag

For a transfer between a point and a final set which constitutes orbits outside the atmospheric sphere of radius R and when the change of orbital plane is not involved we must always have $\beta \geqslant R$, as shown in Fig. 4. The proof of the statement is very simple. Assume a

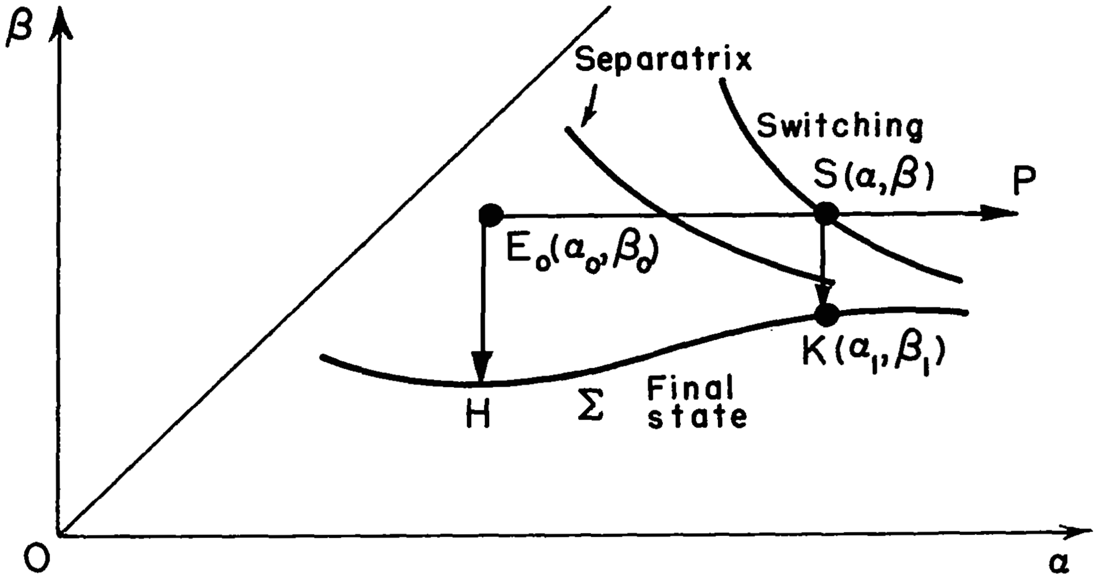

Fig. 3. The Switching and the Separatrix.

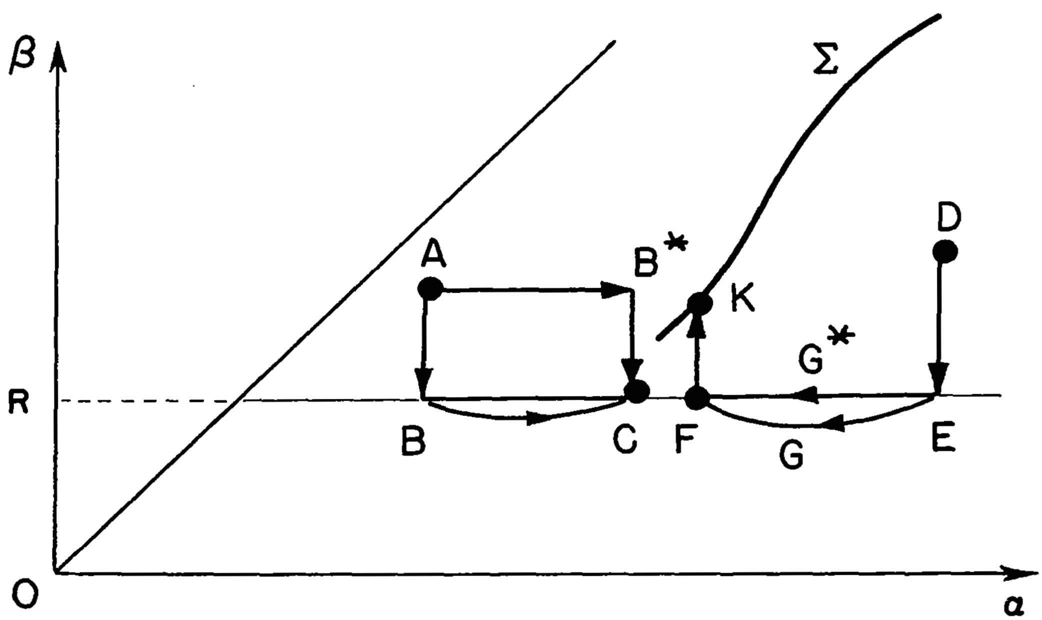

Fig. 4. The Use of Atmospheric Braking.

possible trajectory is the curve ABC . Then the Hohmann transfer AB*C is obviously superior. If the trajectory to be considered is the curve DEGF with G inside the atmosphere then it is better to use DEG*F since the portion from E to F via G* can be realized without fuel consumption by using atmospheric braking at the perigee distance $\beta = R$ of the orbit E . We can notice that atmospheric braking is used only in the direction of decreasing α .

If the final state Σ does not intersect the line $\beta = R$, using atmospheric braking as part of the optimal process, the last corner F can be found by minimizing the last impulse (to go from F to K)

$$\Delta U = \sqrt{\frac{2\mu\beta_1}{\alpha_1(\alpha_1+\beta_1)}} - \sqrt{\frac{2\mu R}{\alpha_1(\alpha_1+R)}} \tag{31}$$

subject to the constraint

$$\theta(\alpha_1, \beta_1) = 0 \tag{32}$$

If the final state intersects the line $\beta = R$, F and K coincide and the last impulse is infinitesimal, just enough to bring the perigee distance of the final orbit above the level R , thus stopping the atmospheric braking. In this case we can see that, reaching the final state in the quickest time (minimum fuel consumption here), is the same as optimally reaching the modified final state which is composed of Σ with

$\beta_1 \geqslant R$ and the line $\beta_1 = R$, $\alpha_1 \geqslant \alpha_F$

3.5 The Optimum Modes

The optimal trajectories in the (α, β) plane (with $\alpha \geqslant \beta \geqslant R$) are always of one of the four following modes (Fig. 5).

I. The "Hohmann Mode"

This mode generally has two impulses and one intermediary orbit E_S with

$$\alpha_S = \max(\alpha_0, \alpha_1)$$

$$\beta_S = \begin{cases} \beta_0 \text{ if } \alpha_S = \alpha_1 \\ \beta_1 \text{ if } \alpha_S = \alpha_0 \end{cases}$$

(α_0, β_0) and (α_1, β_1) being the initial and final points. The switching at (α_S, β_S) is either of the DD type ($\alpha_1 < \alpha_0$, $\beta_1 < \beta_0$) or of the AA type ($\alpha_1 > \alpha_0$, $\beta_1 > \beta_0$) or of one of the two AD types. The corner E_S is either on the switching or is such that E_1 is at one end of the final state when it is constrained.

We shall see that sometimes the Hohmann mode degenerates into a "one impulse at the perigee" mode ($\beta_0 = \beta_1$) or a "one impluse at the apogee mode" ($\alpha_0 = \alpha_1$) or a "parabolic mode" ($\alpha_1 = +\infty$).

II. The "Biparabolic Mode"

There exists an infinitesimal impulse at infinity to transfer the vehicle from one parabola to the other. The total characteristic velocity for the transfer is

$$U_{II} = \sqrt{\frac{2\mu}{\beta_0}} - \sqrt{\frac{2\mu\alpha_0}{\beta_0(\alpha_0+\beta_0)}} + \sqrt{\frac{2\mu}{\beta_1}} - \sqrt{\frac{2\mu\alpha_1}{\beta_1(\alpha_1+\beta_1)}} \qquad (33)$$

III. The "Parabolic Mode with Atmospheric Braking"

The apogee of the intermediary orbit is theoretically at infinity and the total characteristic velocity is

$$U_{III} = \sqrt{\frac{2\mu}{\beta_0}} - \sqrt{\frac{2\mu\alpha_0}{\beta_0(\alpha_0+\beta_0)}} + \sqrt{\frac{2\mu\beta_1}{\alpha_1(\alpha_1+\beta_1)}} - \sqrt{\frac{2\mu R}{\alpha_1(\alpha_1+R)}} \qquad (34)$$

The corner E_S is found by minimizing the last impulse subject to the constraint $\theta(\alpha_1, \beta_1) = 0$.

IV. The "Two-Impulse Mode with Atmospheric Braking"

This mode of course requires $\alpha_1 < \alpha_0$. The total characteristic velocity is

$$U_{IV} = \sqrt{\frac{2\mu\beta_0}{\alpha_0(\alpha_0+\beta_0)}} - \sqrt{\frac{2\mu R}{\alpha_0(\alpha_0+R)}} + \sqrt{\frac{2\mu\beta_1}{\alpha_1(\alpha_1+\beta_1)}} - \sqrt{\frac{2\mu R}{\alpha_1(\alpha_1+R)}} \qquad (35)$$

The corner E_S is found by minimizing the last impulse. This impulse is infinitesimal if the final state intersects the line $\beta = R$.

If (α_0, β_0) and (α_1, β_1) are known, it is easy to compare the four optimal possibilities. For example we may use the following conditions.

The mode I requires

$$\max(\beta_0, \beta_1) \leqslant 11.938 \min(\beta_0, \beta_1) \tag{36}$$

The mode II requires

$$\max(\beta_0, \beta_1) > 9 \min(\beta_0, \beta_1)$$

and

$$\beta_1 \geqslant \frac{4R(\alpha_1 + R)}{\alpha_1} \tag{37}$$

The mode III requires

$$\beta_1 \leqslant \frac{4R(\alpha_1 + R)}{\alpha_1} \Longrightarrow \beta_1 \leqslant (2 + 2\sqrt{2})R$$

and

$$\beta_0 > 4R\left[1 + \frac{R}{\max(\alpha_0, \alpha_1)}\right] \tag{38}$$

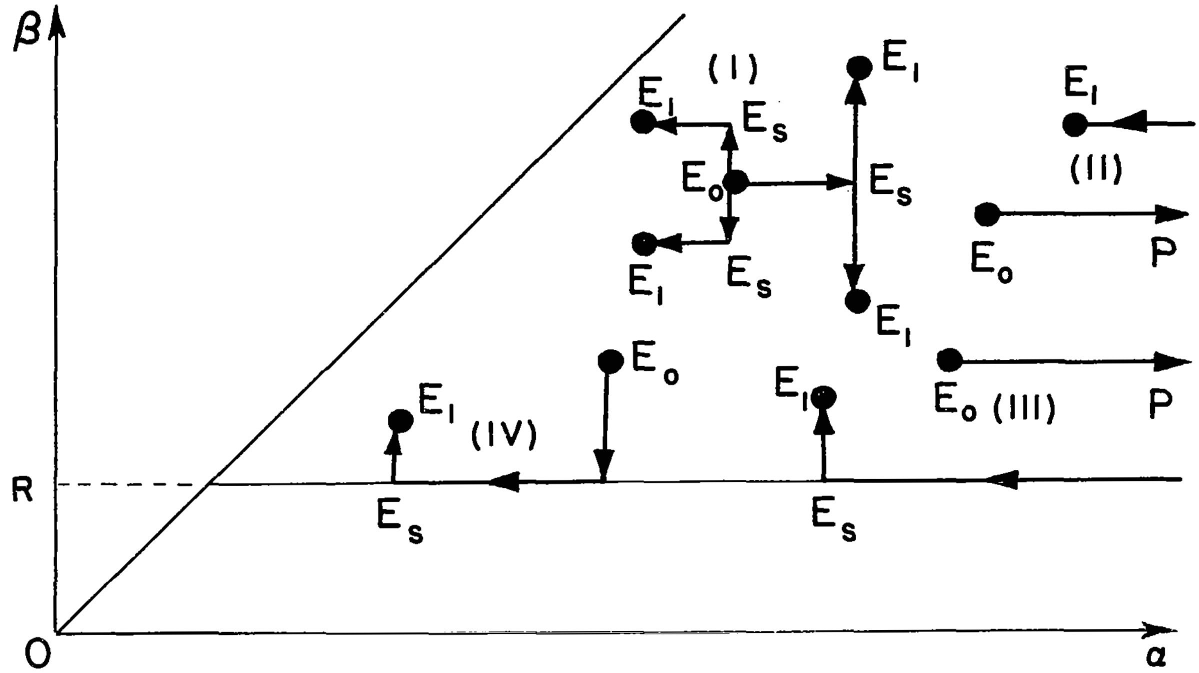

Fig.5. The Optimal Modes.

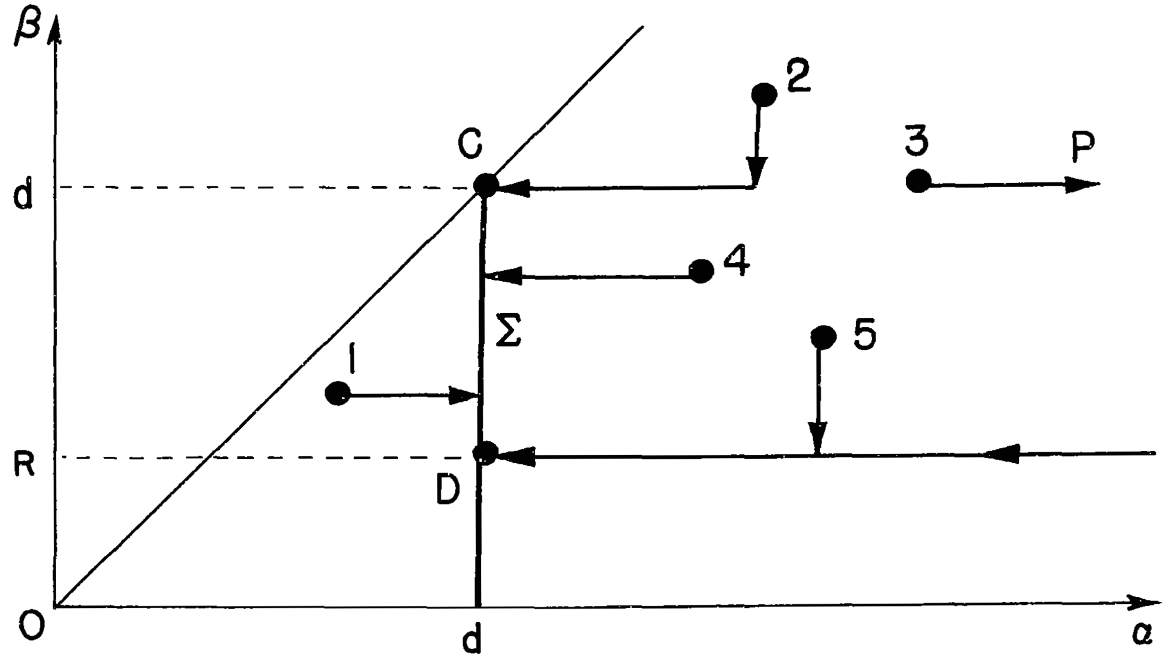

Fig.6. Change in the Apogee.

The mode IV requires

$$\alpha_1 < \alpha_0$$

$$\beta_0 \leqslant \frac{4R(\alpha_0 + R)}{\alpha_0} \implies \beta_0 \leqslant (2 + 2\sqrt{2})R \tag{39}$$

and $\alpha^2_1 (4R - \beta_1) + \alpha_1 R(4R + \beta_1) + R^2(R + \beta_1) > 0 \Rightarrow \beta_1 < 5.879\ R$

These conditions are always sufficient to compare at least the modes II, III and IV.

4. APPLICATIONS

The foregoing analysis is applied in this section to solve several problems of orbit corrections. The final state can be a portion of a curve, a curve with infinite branch or a composite curve. It is denoted by the symbol Σ . In the first three examples the existence of a corner on a switching curve is ruled out by using the separatrix as curve of comparison. In the last two examples a corner exists for certain types of transfer.

4.1 Change in the Apogee

Let d be the final apogee distance. The final state is a segment of straight line parallel to the β -axis (Fig. 6).

$$\alpha_1 = d \geqslant \beta_1 > R \tag{40}$$

If $\alpha_0 < d$ the optimum mode is the one-impulse accelerative at the perigee (orbit 1).

If $\alpha_0 > d$ there are four possible optimal trajectories

a) The Hohmann type bringing the vehicle to the circular orbit C of radius d . This transfer occurs only if $\beta_0 > d$ (orbit 2).

b) The parabolic mode with atmospheric braking (trajectory from 3 to D). This mode occurs only when

$$\beta_0 > \frac{4R(\alpha_0 + R)}{\alpha_0} \qquad (41)$$

c) The one-impulse decelerative at the perigee (orbit 4). This mode occurs only when

$$\alpha_0 > d \geqslant \beta_0 \qquad (42)$$

d) The two-impulse mode with atmospheric braking (trajectory from 5 to D).

4.2 Change in the Perigee

Let d be the final perigee distance. The final state is a ray parallel to the α -axis (Fig. 7)

$$\alpha_1 \geqslant \beta_1 = d > R \qquad (43)$$

If $\beta_0 > d$ there are two possible trajectories

a) If $\beta_0 \leqslant 4\,d\,(\alpha_0 + d)\,/\,\alpha_0$ the optimum mode is by one-impulse decelerative at the apogee (orbit 1).

b) If the inequality reverses the optimum mode is parabolic (orbit 2).

If $\beta_0 < d$ there are three possible trajectories

a) The one-impulse accelerative at the apogee (orbit 3). This mode is optimum when

$$\alpha_0 \geqslant d \tag{44}$$

and $\alpha_0 \beta_0 (3\alpha_0 - \beta_0)^2 - 4\,\alpha_0 d(\alpha_0 - \beta_0)(\alpha_0 - 3\,\beta_0) - 4\,d^2(\alpha_0 - \beta_0)^2 \geqslant 0$

If $\beta_0 \geqslant 4\,d/9$ the second inequality is automatically satisfied.

b) The Hohmann transfer (from 4 to C). This mode is optimum when

$$0.3026\ d \leqslant \beta_0 \leqslant \alpha_0 < d \tag{45}$$

c) The parabolic mode (orbit 5).

4.3 Change in the Eccentricity

Let e_1 be the final eccentricity. The final state is a straight line (Fig. 8)

$$\beta_1 = k_1 \alpha_1 \quad , \quad k_1 = \frac{1 - e_1}{1 - e_1} \tag{46}$$

If $e_0 < e_1$, the optimum mode is the one-impulse acceler-

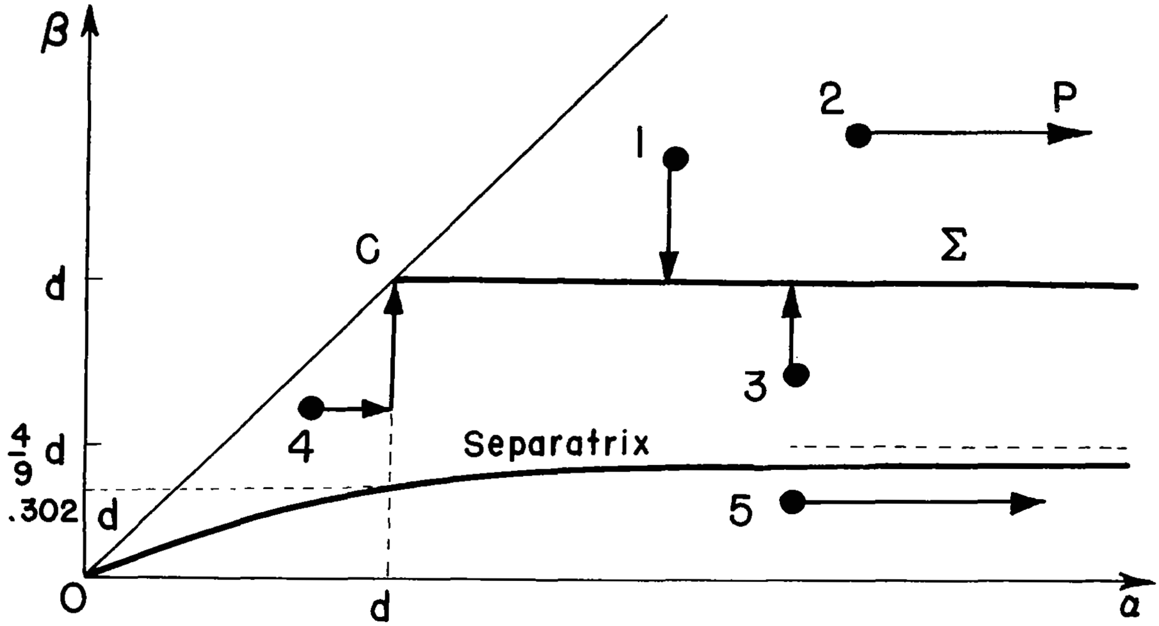

Fig. 7. Change in the Perigee.

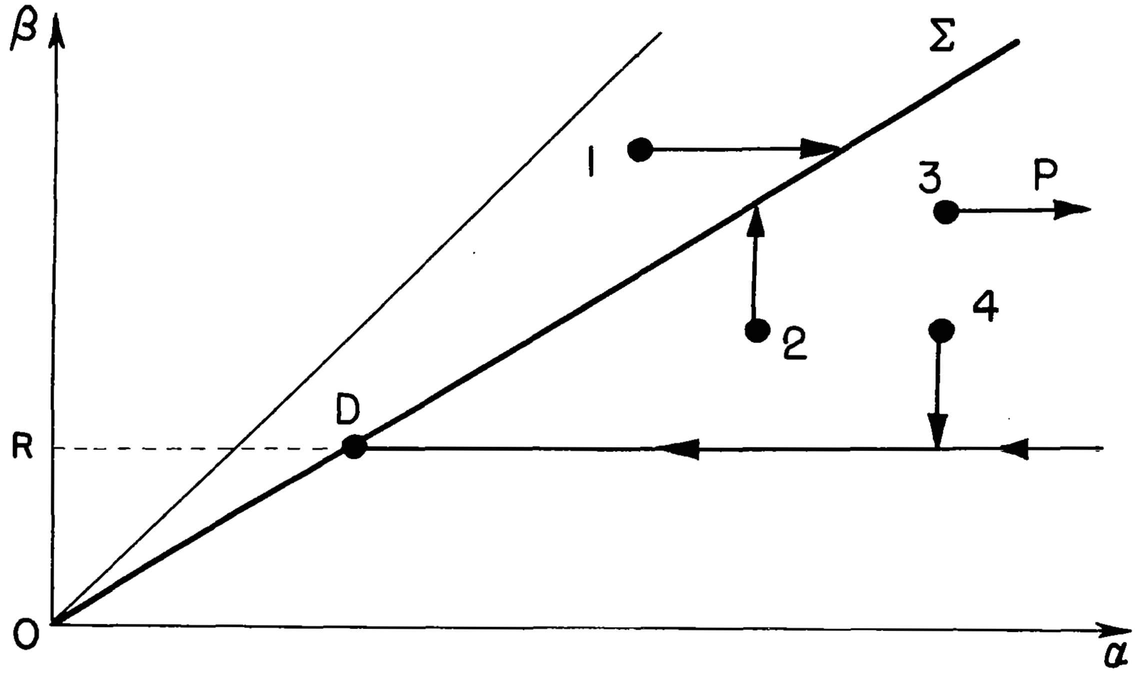

Fig. 8. Change in the Eccentricity.

ative at the perigee (orbit 1).

If $e_0 > e_1$ there are four possible trajectories.

a) The one-impulse accelerative at the apogee (orbit 2). This mode occurs only when

$$k_1 < \frac{(1+k_o)}{2(1-k_o)} \left[1 + \sqrt{1+k_o} \right] - 1 \tag{47}$$

where

$$k_o = \frac{\beta_o}{\alpha_o} = \frac{1-e_o}{1-e_o} \tag{48}$$

b) If the inequality reverses, which requires $k_0 < 0.3026$ ($e_0 > 0.53533$) the optimum mode can be the parabolic mode without atmospheric braking (since Σ has infinite branch) or parabolic mode with atmospheric braking (trajectory from 3 to D).

c) The two-impulse mode with atmospheric braking (orbit 4 to D). The second impulse is infinitesimal. This mode occurs only when

$$\beta_o \leqslant \frac{4R(\alpha_o + R)}{\alpha_o} \tag{49}$$

4.4 Change in the Major-Axis

Let d be the final major-axis. The final state is a segment of straight line (Fig. 9)

$$\alpha_1 + \beta_1 = d \quad , \quad \alpha_1 \geqslant \beta_1 > R \tag{50}$$

If $\alpha_0 + \beta_0 < d$, the optimum mode is the one-impulse accelerative at the perigee (orbit 1).

If $\alpha_0 + \beta_0 > d$ there are four possible trajectories.

a) The two-impulse mode with switching of the DD type (orbit 2). The perigee distance of the intermediary orbit is

$$\beta_s = \frac{-3\alpha_0(\alpha_0 - d) + 2\alpha_0\sqrt{d(2\alpha_0 + 3d)}}{9\alpha_0 + d} \tag{51}$$

This mode occurs only when

$$\beta_0 > \beta_s > R \tag{52}$$

The equality $\beta_0 = \beta_s$, written without subscript 0 is the equation of switching.

b) The one-impulse decelerative at the perigee (orbit 3). This mode occurs only when

$$R < \beta_0 \leqslant \beta_s \tag{53}$$

c) The two-impulse mode with atmospheric braking (orbit 4 and 5 to D). The second impulse is infinitesimal.

d) The parabolic mode with atmospheric braking (orbit 6 to D).

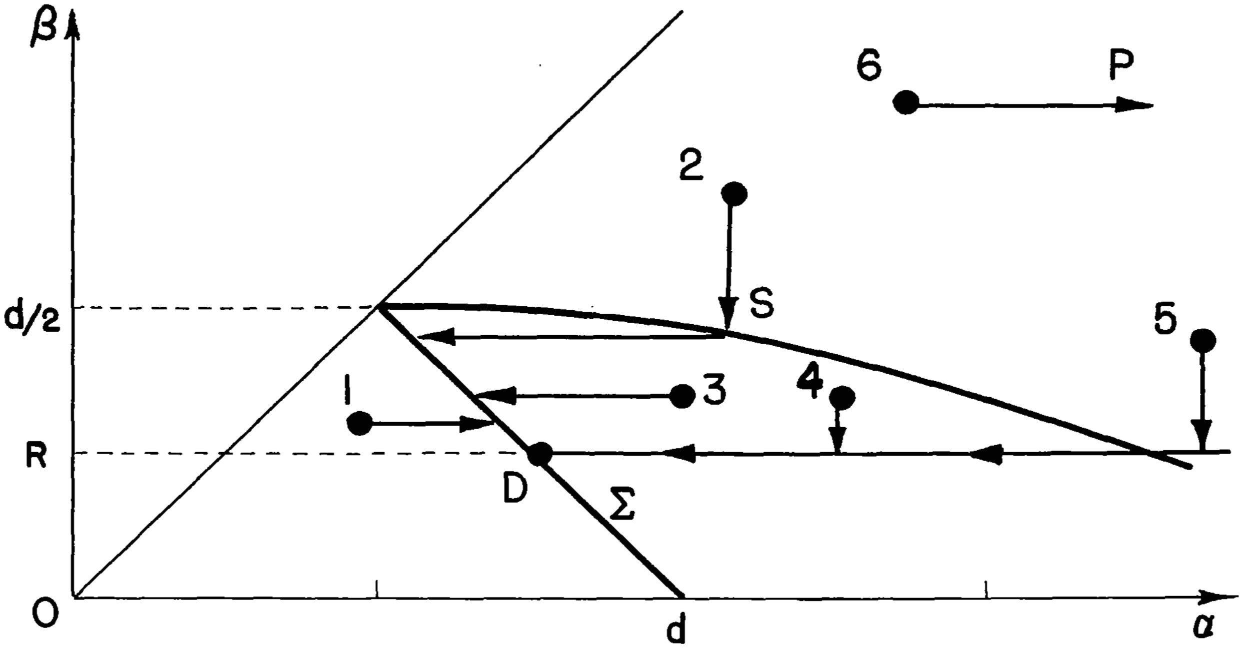

Fig.9. Change in the Major - axis.

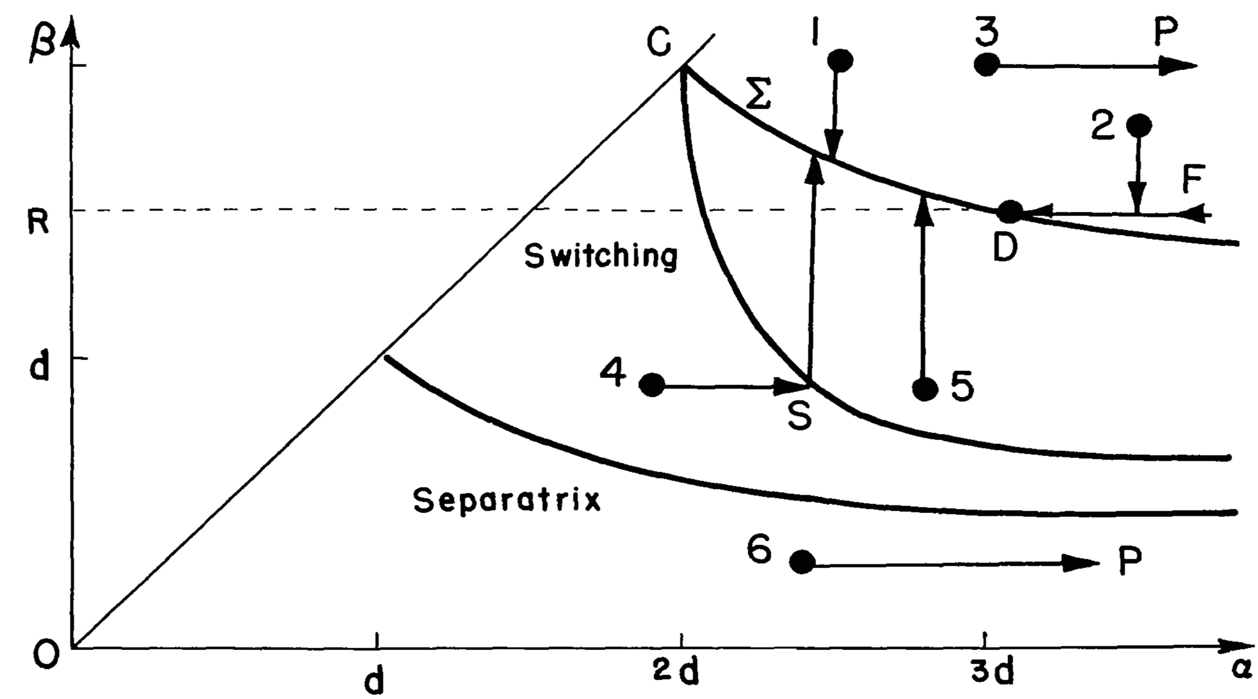

Fig.10. Change in the Angular Momentum .

This mode occurs only when

$$\beta_o > \frac{4R(\alpha_o + R)}{\alpha_o} \tag{54}$$

4.5 Change in the Angular Momentum

This is the same as changing the semi latus-rectum. Let 2 d be the final value of the semi latus-rectum. The final state is a branch of hyperbola (Fig. 10)

$$\beta_1 = \frac{\alpha_1 d}{\alpha_1 - d} \quad , \quad \alpha_1 \geqslant 2d \geqslant \beta_1 > d \tag{55}$$

If the initial angular momentum is larger than the final angular momentum, that is if

$$\beta_o > \frac{\alpha_o d}{\alpha_o - d} \tag{56}$$

there are four possible modes.

a) The one-impulse decelerative at the apogee (orbit 1).

b) The two-impulse mode with atmospheric braking (orbit 2 to D). This mode occurs and is optimum when

$$d < R < 2d$$

$$\alpha_o \geqslant \alpha_D = \frac{R\,d}{R - d} \tag{57}$$

and

$$\beta_o \leqslant \frac{4R(\alpha_o + R)}{\alpha_o}$$

c) The parabolic mode with atmospheric braking (orbit 3 to D). This mode occurs only when

$$d < R < 2d$$

and (58)

$$\beta_o \geqslant \frac{4R(\alpha_o + R)}{\alpha_o}$$

d) The parabolic mode without atmospheric braking. This mode occurs and is optimum when

$$R < d$$

and

$$\beta_o \geqslant \frac{4\alpha_o^2\,d}{(\alpha_o - d)^2} \tag{59}$$

We can notice that in the case where inequality (56) and the first of the inequalities (57) are verified, the strategy is to go in an optimum way from the initial orbit to the final state represented by the composite curve CDF .

If inequality (56) reverses, that is if the initial angular momentum is less than the final angular momentum there are three possible modes, none of them involving atmospheric braking.

a) The two-impulse mode with switching (orbit 4). This mode occurs, and is optimum when

$$\beta_o > \frac{4}{9} d$$

and (60)

$$S(\alpha_o, \beta_o) = (9\beta_o - 4d)\alpha_o^3 + 6\beta_o(\beta_o - 2d)\alpha_o^2 + \beta_o^2(\beta_o - 12d)\alpha_o - 4\beta_o^3 d \leqslant 0$$

The equality written without subscript 0 is the equation of the switching. The apogee distance of the intermediary orbit is obtained by solving $S(\alpha_s, \beta_o) = 0$.

b) When $S(\alpha_o, \beta_o) \geqslant 0$ the optimum mode is by one accelerative impulse at the apogee (orbit 5).

c) The parabolic mode (orbit 6). This mode occurs and is optimum when

$$R < \beta_o \leqslant \frac{4}{9} d \qquad (61)$$

5. CONCLUSION

This paper presents the complete analytical solutions of several fundamental problems in orbital corrections. The initial state is a given point in the phase space while the terminal state is a segment of a curve, a branch of a curve or a composite curve. The possible use of atmospheric braking is discussed, and by modifying the final state to include the line $\beta = R$, the problem again can be solved by the same method. The selection of the apogee and perigee distances as state variables gives a symmetric form to the problem and results in a linear differential equation of the first order for the ratio of the adjoint variables.

The applications of the solution derived in this paper are not restricted to the examples which have been selected. The solution can be applied to the problem of optimum disorbit (Ref. 3), optimum ascent into an orbit (Ref. 4) or optimum orbit correction involving more than two orbital elements (Ref. 5).

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