ADMISSIBILITY AND NONLINEAR VOLTERRA INTEGRAL EQUATIONS

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I. Introduction

The purpose of this paper is to study the behavior of solutions of a nonlinear system

$$(1) \qquad x(t) = f(t) + \int_0^t a(t,s)(x(s) + g(s,x(s)))ds$$

given certain information concerning the corresponding linear system

$$(2) \qquad y(t) = f(t) + \int_0^t a(t,s)y(s)ds.$$

These equations will be studied in the abstract form

$$(N) \qquad x = f + T(x + g(x))$$

and

$$(L) \qquad y = f + Ty$$

where x, y and f are elements of a Frechet space $\mathcal{F}$, $T: \mathcal{F} \to \mathcal{F}$ is a continuous linear map and $g: \mathcal{F} \to \mathcal{F}$ is a nonlinear map. Let

X be a linear subspace of $\mathscr{F}$. Assume that (L) is an admissible w.r.t. (X,X) that is for each $f \in X$ equation (L) admits a solution $y \in X$. The problem is to show that (N) also has a solution x in X.

For example, it is easy to give conditions on f, a and g which insure that equation (1) admits a unique continuous solution $x(t)$. It is much harder to show that $x(t)$ is bounded on the interval $R^+ = \{t: 0 \leq t < \infty\}$. This problem may be placed in the abstract setting above if one defines $\mathscr{F} = C(R^+)$ (with the topology of uniform convergence on compact subsets of R^+), $X = BC(R^+) = \{\varphi \in C(R^+): \varphi$ is bounded on $R^+\}$ and

$$(3) \qquad T\varphi(t) = \int_0^t a(t,s)\varphi(s)ds, \qquad \forall\, \varphi \in \mathscr{F}.$$

The key assumption that (L) is admissible w.r.t. (X,X) means that for each f in $BC(R^+)$ the solution $y(t)$ of (2) is in $BC(R^+)$. In other words, one has admissibility if and only if the linear system (2) is "bounded input - bounded output stable".

Admissibility has been studied by many authors. The idea seems to have originated with Massera and Schaffer [1] and has been applied to integral equations by Corduneanu [2,3] and Antosiewicz [4]. The results in this paper are closely related to the results of Corduneanu but are more easily and more widely applicable to certain problems of the form (1).

2. Main Results

Let $\mathscr{F}$ be a Frechet space, that is $\mathscr{F}$ is both a vector space and a complete metric space with metric ρ such that

a. vector addition and scaler multiplication are ρ-continuous, and

b. ρ is additively invariant, i.e. $\rho(x,y) = \rho(x-y,0)$.

Let X_1 and X_2 be linear subspaces of $\mathscr{F}$ which admit norms $\|\|_1$ and $\|\|_2$. Assume

(A1) X_i is a B-space under the norm $\|\|_i$. Moreover, $\|\|_i$ is stronger than the topology induced from $\mathscr{F}$ in the sense that if $\|x_n - x\|_i \to 0$ as $n \to \infty$ then $x_n \to x$ in $\mathscr{F}$.

(A2) $T: \mathscr{F} \to \mathscr{F}$ is a continuous linear map such that if I = identity, map then $(I-T): \mathscr{F} \to \mathscr{F}$ is both one to one and onto.

(A3) $f \in \mathscr{F}$ and $g: \mathscr{F} \to \mathscr{F}$.

Lemma 1. If (A1-3) are satisfied then equation (N) is equivalent to

$$\text{(V)} \qquad x = y - Rg(x)$$

where $R = I - (I-T)^{-1}$ is a continuous linear map of $\mathscr{F}$ into $\mathscr{F}$ and $y = f - Rf$ is the solution of (L).

<u>Proof</u>. From the definition of R it follows that $(I-R) = (I-T)^{-1}$. Thus $y = (I-T)^{-1}f = (I-R)f$ solves (L). Since $I - T$ is continuous, it is a closed map on $\mathscr{F} \times \mathscr{F}$. Thus $(I-T)^{-1}$ is closed, linear and everywhere defined on $\mathscr{F}$. By the closed graph theorem $I - R$ (and so also R) is continuous on $\mathscr{F}$.

Subtracting Tx from both sides of equation (N) and applying $(I-T)^{-1}$ one obtains

$$x = (I-T)^{-1}f + (I-T)^{-1}Tg(x) = (I-R)f + (I-T)^{-1}Tg(x).$$

Since $y = (I-R)f$ and $(I-T)^{-1}T = -R$ this reduces to equation (V). The entire calculation is reversible so that (V) and (N) are equivalent. Q.E.D.

Equation (V) is a "variation of constants" form of equation (N). For the Volterra integral equation (2) the map T is always a continuous map. The assumption that $I - T$ is one to one and onto is just the familiar theorem that linear Volterra integral equations have unique solutions. Moreover, the map R will have the form

$$R\varphi(t) = \int_0^t r(t,s)\varphi(s)ds, \tag{4}$$

where $r(t,s)$ is the resolvent kernel, i.e. $r(t,s)$ solves the equation

$$r(t,s) = -a(t,s) + \int_s^t a(t,u)r(u,s)du.$$

Once the variation of constants equation (V) is obtained, it is easy to apply various fixed point theorems to (V). First consider contraction maps. The next definition and Lemma follow Corduneanu [2].

Definition 1. Let (A1-2) be satisfied. Then the pair (X_1,X_2) is called admissible w.r.t. the map R if and only if for each $f \in X_1$, $Rf \in X_2$.

This admissibility is easily seen to be equivalent to the assumption that for each f in X_1 the solution y of (L) is in X_2 in the special case where $X_1 = X_2$.

Lemma 2. If (A1-2) hold and if (X_1,X_2) is admissible w.r.t. R then R is continuous as a linear map of X_1 into X_2, that is $\|R\| = \sup \{\|Rf\|_2 \colon \|f\|_1 = 1\} < \infty$.

Proof. Using the continuity of R as a map of $\mathcal{F}$ into $\mathcal{F}$ and assumption (A1) the conclusion follows immediately by the closed graph theorem. Q.E.D.

Theorem 1. Suppose (A1-3) are satisfied and in addition

(A4) $y \in X_2$, $g: X_2 \to X_1$ _and_ (X_1, X_2) _is admissible_ w.r.t. R.

(A5) _There exists_ $\alpha > 0$ _and_ $r (0 < r \leq +\infty)$ _such that if_ $z, w \in X_2$ _with_ $\|z\|_2, \|w\|_2 \leq r$ _then_ $\|g(z) - g(w)\|_1 \leq \alpha\|z-w\|_2$.

If $\alpha\|R\| < 1$ _and if_ $\|y\|_2 + \|R\|\|g(0)\|_1 \leq r(1-\alpha\|R\|)$ _then equation_ (N) _has a unique solution_ $x \in X_2$ _such that_ $\|x\|_2 \leq r$.

Proof. Under these conditions it is easy to see that the right hand side of equation (N) defines a contraction mapping on the set $\{z \in X_2: \|z\|_2 \leq r\}$. Q.E.D.

Theorem 2. _Let_ (A1-4) _be satisfied and assume in addition_

(A6) _For each_ $\epsilon > 0$ _there exists_ $\delta > 0$ _such that if_ $\|z\|_2, \|w\|_2 \leq \delta$ _then_ $\|g(z) - g(w)\|_1 \leq \epsilon\|z_1 - z_2\|_2$.

_If_ $g(0) = 0$ _then for each sufficiently small_ $r > 0$ _there exists_ $\eta > 0$ _such that if_ $\|y\|_2 \leq \eta$ _then equation_ (N) _has a unique solution_ $x \in X_2$ _with_ $\|x\|_2 \leq r$.

Proof. Pick $\epsilon_1 > 0$ such that $\epsilon_1\|R\| < 1$ and pick $\delta_1 > 0$ such that $\|g(z) - g(w)\|_1 \leq \epsilon_1\|Z-W\|_2$ if $\|z\|_2, \|w\|_2 \leq \delta_1$. If $0 < r \leq \delta_1$ and if $\eta = r(1-\epsilon_1\|R\|)$ then the two inequalities

$$\gamma\|R\| \leq \epsilon_1\|R\| < 1, \quad \|y\|_2 \leq r(1-\epsilon_1\|R\|)$$

of the last theorem are satisfied. Q.E.D.

If $X_1 = X_2$, Theorem 2 implies the following result.

Corollary 1. Let (A1-4) and (A6) be satisfied. If $g(0) = 0$ then for each sufficiently small $r > 0$ there exists $\eta > 0$ such that if $f \in X_1$ and $\|f\|_1 \le \eta$ then equation (N) has a unique solution $x \in X_2$ with $\|x\|_1 \le r$.

Proof. Since $y = (I-R)f$, $\|y\|_1 \le (1+\|R\|)\|f\|_1$. Thus $\|y\|_1$ is small if $\|f\|_1$ is small. Q.E.D.

As an application of Theorem 2 and Corollary 1 we note that all of the perturbation results in [5] and [6] are special cases. The various assumptions on the resolvent $r(t,s)$ are simply conditions which insure admissibility.

3. The Schauder Theorem

A <u>convex</u> Frechet space is a Frechet space $\mathcal{F}$ such that every neighborhood of the origin contains a convex subneighborhood. The only property of a convex Frechet space which will be needed here is that the Schauder Fixed Point Theorem is true in such spaces.

Theorem 3. *(A1-4) are satisfied, $\mathcal{F}$ is a convex Frechet space and the map $R: \mathcal{F} \to \mathcal{F}$ is a compact operator For any $r > 0$ define $S_i(v) = \{x \in X_i: \|x\|_i \leq r\}$ and let $\overline{S}_i(r)$ be the $\mathcal{F}$-closure of $S_i(r)$. Suppose for some positive numbers r and s, $g: \overline{S}_2(r) \to S_1(s)$ $\mathcal{F}$-continuously. If $y \in X_2$ and if $\|y\|_2 + \|R\|s \leq r$ then equation (N) has at least one solution $x \in X_2$ with $\|x\|_2 \leq r$.*

Proof. For any φ in $\overline{S}_s(r)$ define $M\varphi = y - Rg(\varphi)$. The assumptions easily imply that $M: \overline{S}_2(r) \to S_2(r) \subset \overline{S}_2(r)$ with M $\mathcal{F}$-continuous. It remains to show that $M(\overline{S}_2(r))$ is precompact in $\mathcal{F}$.

First note that the set $W = \{g(\varphi): \varphi \in \overline{S}_2(r)\}$ is $\mathcal{F}$-bounded. Indeed, let U be any $\mathcal{F}$-neighborhood of the origin. Since $\|\|_1$ is stronger than the $\mathcal{F}$-topology there exists a number $\delta > 0$ such that if $\|\varphi\|_1 \leq \delta$ then $\varphi \in U$. For any α with $|\alpha| \leq \delta/s$ if $\varphi \in W \subset S_1(s)$ then $\|\alpha\varphi\|_1 = |\alpha| \|\varphi\|_1 \leq \alpha s \leq \delta$.

Therefore, $\alpha W \subset U$ if $|\alpha| \le \delta/s$, i.e. W is $\mathcal{F}$-bounded.

Since W is bounded and $R: \mathcal{F} \to \mathcal{F}$ is compact, $R(W) = M(\overline{S}_2(r))$ is precompact. By Schauder's theorem M has a fixed point $x \in \overline{S}_2(r)$. Since $x = Mx$ and $M: \overline{S}_2(r) \to S_2(r)$ it follows that $x \in S_2(r)$. Q.E.D.

Notice that if R is the integral operator (4) then rather weak assumptions on $a(t,s)$ easily imply the compactness of R on $\mathcal{F} = C(R^+)$. At the same time for most subspaces X_i the compactness of R as a map of X_1 into X_2 is usually very difficult to prove (and is often false). Thus Theorem 3 seems to be a very natural and convenient application of Schauder's theorem for Volterra equations. Theorem 3 is motivated by and is closely related to Theorem 2 of Corduneanu [2].

As an application of Theorem 3 we shall give a generalization of an L^2-stability theorem of the type studied by Sandberg [7] and Zames [8]. Consider a system on n equations of the form

$$(5) \qquad x(t) = f(t) + \int_0^t a(t-s)g(s,x(s))ds. \qquad (t \ge 0)$$

Concerning (5) we assume

(K1) $g(t,x)$ is measurable in (t,x) for $t \ge 0$ and all x and $g(t,x)$ is continuous in x for each fixed t.

(K2) There exists $\gamma > 0$ and a nonsingular, constant, n by n

matrix A such that $|A^{-1}g(t,x) - x| \leq \gamma|x|$ for all (t,x).

(K3) $a(t)$ is $L^1(R^+)$ and the determinant $\det (I-a^*(s)A) \neq 0$ for $\mathrm{Re}\ s \geq 0$. Here I = identity matrix and $*$ denotes the Laplace transform.

(K4) f is in $L^2(R^+)$.

For any matrix W let $\Lambda(W) = \max \{|\lambda|^{\frac{1}{2}}: \lambda$ is an eigen-value of $W^*W\}$ be the spectral norm of W. Define

$$\alpha = \sup \{\Lambda(\{I-a^*(iw)A\}^{-1}a^*(iw)A): \ -\infty < w < \infty\}.$$

Theorem 4. If (5) satisfies (K1-4) and if $\gamma\alpha < 1$ then equation (5) has a solution $x(t) \in L^2(R^+)$ with $\|x\|_{L^2} \leq (1+\alpha)(1-\alpha\gamma)^{-1}\|f\|_{L^2}$.

Proof. Let $\mathscr{F}$ be the space of locally L^2 functions on R^+ with the topology of L^2 convergence on compact subsets of R^+. Let $X = X_1 = X_2 = L^2(R^+)$. Equation (3) may be written abstractly as

$$x = f + T[A^{-1}h(x)] = f + T[x + \{A^{-1}h(x) - x\}]$$

where $T\varphi(t) = \int^t a(t-s)A\varphi(s)ds$. Assumption (K3) implies that $a(t-s)A$ has a resolvent $r(t-s)$ of class $L^1(R^+)$, c.f. Paley and Weiener [9, p. 60]. Thus the pair $(L^2(R^+), L^2(R^+))$ is admissible w.r.t. the operator R where

$$R\varphi(t) = \int_0^t r(t-s)\varphi(s)ds. \quad (t \geq 0).$$

The Parseval equation and the convolution theorem for L^2-Fourier transforms imply $\|R\| \leq \alpha$.

In Theorem 3 let $s = \gamma$, $r = (1+\alpha)\|f\|(1-\alpha\gamma)^{-1}$ and let $g(\varphi) = A^{-1}h(\varphi) - \varphi$. Clearly $\bar{S}_2(r) = S_2(r)$ and if $\varphi \in S_2(r)$ then $g(\varphi) \in L^2(R^+)$ with $\|g(\varphi)\| \leq \gamma\|\varphi\|$. By Theorem 3 equation (5) has a solution $x \in L^2(R^+)$ with $\|x\| \leq r$. Q.E.D.

4. Extensions and Comparisons.

Corduneanu [3] studied the nonlinear equation

$$x = f + Tx + h(x) \tag{6}$$

on the Frechet space $\mathscr{F} = C(R^+)$. Here $h: \mathscr{F} \to \mathscr{F}$ is a nonlinear functional. If the pair (X_2, X_1) is admissible w.r.t. the map T the equation (N) above may be treated using his methods by setting $h(x) = Tg(x)$. The problem with this approach is that in many interesting applications the pair (X_2, X_1) is not admissible w.r.t. the map T.

Both points of view may be combined by studying nonlinear equations of the form

$$x = f + T(x + g(x)) + h(x) \tag{7}$$

where (A1-3) are true and $h: \mathscr{F} \to \mathscr{F}$. The variation of constants formula (V) implies that (7) is equivalent to

$$x = (I-R)f + (I-R)h(x) - Rg(x). \tag{8}$$

Various conditions may be given to insure that (8) has a solution in X_2. For example, the method of proof of Theorem 2 is easily applied in order to prove the following theorem.

Theorem 5. *Let* (H1-3) *be satisfied,* $f \in X_1$ *and let* $g, h: X_2 \to X_1$ *both satisfy* (H4). *If* (X_1, X_2) *is admissible w.r.t. both* R *and* I - R *then for each sufficiently small* $\epsilon > 0$ *there exists* $\eta > 0$ *such that if* $\|f\|_1 \leq \eta$ *then* (7) *has a solution* $x \in X_2$ *with* $\|x\|_2 \leq \epsilon$.

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