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UNIFIED ANALYSIS OF TURBULENT JET MIXING

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SYMBOLS

a	Initial jet radius or half-height
$b, b_{\frac{1}{2}}$	Widths of the mixing regions
C_D	Drag coefficient
d	Diameter
$g = P + \frac{\rho u^2}{2}$	Total pressure; $\tilde{g} \equiv g/g_e$ (nondimensional)
k, k_1, k_2	Constants
L	Characteristic length
P	Pressure
S	Transformed axial coordinate; $\overline{S} = S/\psi_a^2$ (nondimensional)
T	Temperature
u	Axial velocity; $\tilde{u} = u/u_e$ (nondimensional)
v	Normal velocity
x, y	Axial and normal coordinates
Y_i	Mass fraction of species "i"
ϵ	Eddy viscosity, defined by Equ. (1)
$\delta^*, \Delta^*, \delta_r^*$	Displacement thicknesses
$\mu_T = \rho\epsilon$	Turbulent viscosity
ρ	Density
τ	Shear
ψ	Stream function; $\overline{\psi} = \psi/\psi_a$ (nondimensional)

Subscripts

a	Conditions at $y = a$, $x = 0$
c	Conditions along jet centerline
e	Free stream conditions
j	Initial jet conditions

INTRODUCTION

The general flow field of interest is simply an inner stream of fluid injected parallel to a moving outer stream, and turbulent mixing between these streams is the subject of the present investigation. This is, of course, one of the oldest viscous flow problems that remains essentially unsolved, and the continuing interest in this general problem stems from the wide variety of practical devices whose intelligent design depends upon an understanding of such processes. Jet ejectors and combustion chambers are, perhaps, two of the most common examples.

In an area where so much work has been reported, it is fortunate, indeed, that several authors have endeavored to review previous studies. References (1) through (7) are the most useful of these as background information for what follows. Other work that has appeared more recently or that which is of particular interest here is also listed in the References. In Table I, the principal experimental studies are cited along with a very cryptic description of their main characteristics. Wake experiments also bear upon the present problem in a general way, but, as we shall see later, a direct quantitative application of such results must await further developments in our basic understanding of turbulent shear flows.

The history of the analytical treatments of this problem is also very lengthy; here, however, it is instructive to consider wake problems concurrently. The classical treatment (Refs. 2-6) of free mixing problems is based on a view of the flow field where there are two main regions of the flow: 1) the transitional or developing region and 2) the similarity regions, where suitably scaled profiles are self-preserving with axial distance. In the jet case, an initial region containing the potential core precedes the transitional region. Mixing in this initial region is normally taken as that for the well known half-jet problem (two unbounded parallel streams of different velocity initially separated by an infinitesimally thin plate). When the inner boundary of this mixing region intersects the axis of the jet, the initial region is deemed ended, and the next stage in the calculation must begin. At this point, however, a very crude assumption was commonly made: the transitional region was neglected and the similarity region was assumed to begin immediately. For wake problems, of course, one does not normally treat an initial region (the near wake) directly but the same assumption of neglecting the transitional region and taking similar profiles immediately, now at the "initial" station, has been commonly made.

This general procedure is open to criticism on two counts. First, the flow in the initial region is not well represented by the idealized half-jet problem since real initial profiles are far from ideal. The boundary layers on the inside and outside of the splitter plate and the thickness of the splitter plate itself serve to produce large distortions of the initial profiles which are known to produce large effects on the development of the near field. This manifests itself in the generally poor prediction of the length of the initial region by the classical theories (Ref. 4). Second, it is well known that the similarity region does not begin near the end of the initial region. Townsend (Ref. 3) indicates, for example, that the wake behind a circular cylinder does not become "similar" until an axial distance greater than about 100 diameters. These two criticisms are not merely academic; taken together they result in rather poor agreement between the classical theory and experiment. On the basis of these observations, it seems clear that the transitional or non-similar region of the flow field must be treated directly. Thus, we can consider either analytic approximations that do not employ a similarity assumption (Refs. 28, 29) or strictly numerical procedures (Ref. 30). Both of these choices will be discussed in detail in the present report.

To this point, our considerations of the analytical treatment of turbulent free mixing problems has dealt primarily with the question of solving the equations of motion. While being important, this is not the area of greatest concern. The major difficulty associated with these problems is the specification of some representation of turbulent transport processes. Our understanding of turbulence at this time is not such that one can treat the "turbulent" processes within a shear flow directly. Rather, the most that can be expected is some prediction of the "mean" flow properties. To this end, we relate the shear to the mean flow variables as following Boussinesq

$$\tau = (\rho\epsilon)u_y \qquad (1)$$

where ϵ is an "eddy viscosity". This is certainly a gross simplification of turbulence and can be attacked on physical grounds (Ref. 3), however, at the present time, it represents the only fruitful avenue of approach to turbulent shear flows. The eddy viscosity models that have been proposed for the main mixing region of free mixing flows are summarized in Table II. The initial region in the jet problem requires special consideration, and we shall discuss this point shortly. The earlier "mixing length" theories given as the first four cases in the table have been generally discredited (Ref. 34), and most workers in the field now favor some application of the concept of Prandtl's third model given as the fifth item in Table II.

We digress here to discuss the flow in the initial region. A general treatment of this part of the flow must be able to handle arbitrary boundary layers on the inside and the outside of the splitter plate where either or both may be laminar or turbulent. Further, the wake trailing from the finite thickness splitter plate and its interaction with both splitter plate boundary layers must be treated. This is a formidable fluid flow problem in itself and will require further extensive study before its solution can be confidently incorporated into a treatment of the transitional and far flow fields. There are, however, many technological applications of jet mixing (e.g. fuel injection) primarily concerned with the development of the flow field over distances that are long ($x \sim 10^2 a$) compared to the initial region ($x \sim 10a$). Under these circumstances it is to be expected that the details of the initial region will have a small effect on the region of interest, and some approximation there can be tolerated. If an approximation is to be made, a convenient one might as well be selected. In the greatest part of what follows, therefore, the eddy viscosity in the initial region has been based on the gross profile in the same manner as for the main region of the flow. This model is certainly not physically correct, but it will be shown later that it does, in fact, produce predictions at least as accurate as the classical half-jet treatment. In a few cases, the experimental data to be compared with theory do not extend far enough downstream of the initial region to make this approximation adequate. In those cases, it is necessary to start the analysis with an experimental "initial" profile just downstream of the initial region. This is exactly the course of action followed in the wake case where it is necessary to start with a profile at the end of the "near wake". The agreement between experiment and theoretical prediction in the main region of the jet is always improved by starting the calculation with an experimental "initial" profile at the end of the potential core region. This procedure has not been employed in general here as it is felt to be artificial. A useful theory must be able to provide reasonable predictions in the region of interest starting with the true initial conditions or some simplification thereof.

Returning now to Table II, it is important to note that the three entries following Prandtl's third model can be shown to be equivalent to it and each other. Consider first the Clauser model+

$$\epsilon = k_1 \, u_e \, \delta^* \tag{2}$$

where $k_1 \simeq .018$ and the Prandtl model using $b_{\frac{1}{2}}(x)$ as the width, b^5,

$$\epsilon = .037 \, b_{\frac{1}{2}}(x) \, |u_{max} - u_{min}| \tag{3}$$

For simple profile shapes such as a rectangular or triangular defect or excess, one finds that the two expressions agree exactly in form and to the extent of .036 versus .037 as the proportionality constant. Later, specific numerical results will be presented for jet solutions with both models and comparison with experiment made. This will only strengthen the assertion made here that Equ. (2) and (3) are equivalent. The "wake" models of Schlicting and Hinze can be reduced to the same form as the Clauser model. Nothing that

$$C_D \, d = 2 \, \theta \tag{4}$$

and taking a representative value of $C_D = 1.20$ for a circular cylinder in high Reynold's number flow, these expressions become respectively

$$\epsilon = .044 \, u_e \, \theta \tag{5}$$

and

$$\epsilon = .027 \, u_e \, \theta \tag{6}$$

+ Here δ^* must be interpreted to be based on the absolute value of $(1 - \frac{u}{u_e})$.

In the treatment of wake problems, it is common to neglect the factor (u/u_e) in the definition of θ since $(u/u_e) \simeq 1$. This does, however, render $\theta = \Delta^*$ $(\Delta^* = \int_{-\infty}^{\infty} \left| 1 - \frac{u}{u_e} \right| dy)$ so that, these formulas can as well be written

$$\epsilon = .044\ u_e\ \Delta^* \qquad (5a)$$

$$\epsilon = .027\ u_e\ \Delta^* \qquad (6a)$$

The Clauser model written in these terms is

$$\epsilon = .018\ u_e\ \Delta^* \qquad (2a)$$

where we have taken the displacement thickness appropriate to a "two-sided", planar free mixing problem rather than the "one-sided" boundary layer case considered by Clauser. As stated above, it will be shown that Equ. (2a) provides predictions in good agreement with jet experiments. The question arises as to why the constant for wakes $(\tilde{u}_j < 1)$ is so much larger than that for jets $(\tilde{u}_j > 1)$. Abramovich (Ref. 4, pp. 211) notes this effect and attributes it to increased turbulence caused by the separated base flow in the wake case. The writer is not aware of any constant density jet experiments with $\tilde{u}_j < 1$ that might help to clarify the matter. For our purposes here, however, the important result is that these free mixing eddy viscosity models are all equivalent in functional form.

The six models listed in Table II following the wake models are attempts to extend the basic Prandtl model to problems involving significant density variations. The Ting-Libby Model results from an attempt to apply transformation theory to turbulent free mixing; it has been shown to be unreliable (Ref. 18). Ferri's suggestion of utilizing a mass flow difference to replace the velocity difference in the Prandtl model has provided predictions of unreliable accuracy for the axi-symmetric case (Refs. 22, 23). However, when the mass flow difference was applied to the planar case, a good prediction (Ref. 29) was achieved for the one experiment that exists in the literature (Ref. 21). The Alpinieri model is contrary in form to any other model and must be viewed as essentially empirical, qualitatively as well as quantitatively. The simple Zakkay axi-symmetric model given in Table II is based on the presumption that the asymptotic decay of the centerline velocity and jet fluid concentration behave as the inverse of the square of the streamwise distance, i.e. x^{-n} where $n = 2$, in the axi-symmetric case. It is true that these quantities do behave as "x" to some negative exponent, n, but the value of "n" is at least a function of the jet to free stream mass flux ratio, $\rho_j u_j/\rho_e u_e$ as can be seen in Fig. No. 1. Zakkay asserts that an extended form of his model can be used for cases with $n \neq 2.0$. Examination of Ref. 23, however, reveals that it is necessary to know the decay exponent for both the centerline concentration and the velocity a priori before the extended model can be specified. This, in essence, requires knowledge of the behavior of the solution before the solution can be obtained. Thus, there is no unified picture of the treatment of flows with significant density variations on the basis of existing eddy viscosity models, particularly for the axi-symmetric configuration.

Since it was possible to demonstrate some unity of the models for planar, constant density cases, it is instructive to examine new means for extending these models to the compressible case. As these models are all equivalent, we may start with any one. Rather than the usual procedure of starting with the Prandtl model, start here with the Clauser model. It is simple, at least formally, to extend this to varying density, i.e.

$$\epsilon = .018\ u_e \Delta^* = .018\ u_e \int_{-\infty}^{\infty} \left| 1 - \frac{\rho u}{\rho_e u_e} \right| dy \qquad (7)$$

and to show that for simple profiles, such as a triangular velocity defect, that this expression is equivalent to the planar mass flow difference model given in Table II. Recalling that this model produced predictions in good agreement with experiment, it may be stated that the Clauser model can be viewed as an adequate representation of planar, free mixing flows with or without strong density variations. With the results of Ref. 36 the statement can be broadened to include the effects of strong streamwise pressure gradients.

In view of all this, one may ask what the difficulty is with the axi-symmetric case. It is the purpose of this paper to propose an answer to that question. First, some new results for the planar case will be presented to substantiate some of the assertions made above. Then, the axi-symmetric case will be treated in detail. All of the work is limited to two-dimensional (either planar or axi-symmetric), steady, fully turbulent flows without chemical reaction. Pressure gradient and three-dimensional effects are left to a later report.

ANALYSIS

The turbulent laws are taken here to be identical to the corresponding laminar laws when expressed in terms of mean flow quantities and turbulent transport coefficients. Further, the boundary layer form of the conservation laws is assumed to apply.

Planar Flows. In this section, some new results for planar flows will be developed and discussed. The primary aim of this work is to substantiate some of the assertions made in the Introduction; for this purpose constant pressure cases only will be considered.

It has been stated that it is necessary to treat the non-similar development of the flow field in the transitional region. The choice of a method for solving the flow equations thus reduces to either a strictly numerical procedure or an analytic approximation. Here, an analytic approximation of the linearized type (Ref. 37) will be employed. This type of approach has been shown to provide very good approximations for free mixing problems (Ref. 38). Specifically, the physical normal coordinate, y, is replaced by a stream function, $\psi(x,y)$,

$$\psi_y = u(x,y) \;; \quad -\psi_x = v(x,y) \tag{8}$$

and the total pressure, $g(x,\psi) \equiv P(x) + \rho u^2(x,\psi)/2$, is introduced as the dependent variable. With this, the Mass Continuity and Momentum Equ's. become

$$g_x = \epsilon u(x,\psi) g_{\psi\psi} \tag{9}$$

This is approximated as

$$g_x = \epsilon u_c(x) g_{\psi\psi} \tag{10}$$

where $u_c(x)$ is the velocity along the centerline of the jet. Introducing a new streamwise coordinate

$$S \equiv \int_o^x \epsilon(x')u_c(x')dx' \tag{11}$$

and using the boundary conditions

$$\begin{aligned} g(o,\psi) &= g_j = P + \rho u_j^2/2 \;; \quad 0 < \psi < \psi a \\ &= g_e = P + \rho u_e^2/2 \;; \quad \psi > \psi a \\ \lim_{\psi \to \infty} g(S,\psi) &= g_e \end{aligned} \tag{12}$$

the solution becomes

$$\tilde{g}(S,\psi) = 1 - \frac{1-\tilde{g}_j}{2} \left| \operatorname{erf} \frac{1-\bar{\psi}}{\sqrt{4\bar{S}}} + \operatorname{erf} \frac{1+\bar{\psi}}{\sqrt{4\bar{S}}} \right| \tag{13}$$

where $\tilde{g} = g/g_e$, $\bar{S} = S/\psi_a^2$, $\bar{\psi} = \psi/\psi_a$. The solution for $u(S,\psi)$ can be found by simple algebraic manipulation. It remains then to invert the S(x) and $\psi(y,x)$ transformations to obtain the solution in the physical plane.

It is at this point, inverting the S(x) transformation, that a model for the eddy viscosity must be specified. It has been asserted that the Clauser model represents a unified and adequate statement; it was used here. Some results of the analysis are compared with the experimental data presented in Ref. (14) in Fig. No. 2 on the basis of the centerline velocity decay for $u_j/u_e = 2.0$. Also shown is the Classical Theory (Ref. 4) prediction for each case. The first point of importance to note is the poor prediction of the initial region length by the Classical Theory even though some attempt at including the effects of initial non-uniformity was made in this case. It is true that the description of this region by the heuristic approximation on the eddy viscosity proposed herein is not truly adequate either, but it is certainly at least as accurate as that of Classical Theory. Secondly, the present theory does seem to predict the behavior of the flow in the non-similar transitional region more accurately. That is, the slope of the velocity decay in the region $25 \leq x/a \leq 75$ is more accurately predicted by the present solution. Lastly, the present and the Classical results become the same far downstream in the similarity region.

This has been accomplished using the generalized Clauser model for the eddy viscosity. To the author's knowledge this is the first time that this model has ever been applied to anything but a boundary layer flow. It has been previously asserted here that this model is equivalent to the Prandtl model commonly employed for free mixing problems, and it is proposed now to justify this assertion. This can be expeditiously accomplished by taking the solution developed here and calculating the eddy viscosity based on the Prandtl model from these profiles. The results of such a calculation are given in Fig. No. 3. Certainly, the two models are in very close quantitative and qualitative agreement. Also shown in the figure is the eddy viscosity in the initial region as predicted by the half-jet or free jet boundary. Taking this in conjunction with Fig. No. 2 one can conclude that the true eddy viscosity in the initial region lies somewhere between this prediction and the results of the heuristic approximation for this region employed here.

Axi-Symmetric Flows. Starting with the now substantiated conclusion that the Prandtl model, the Clauser model and the wake models are all equivalent in the planar case, one may choose to seek a corresponding axi-symmetric model using any of these three as the starting point. Schlicting (Ref. 5) simply used the planar form of the Prandtl model in the constant density case and Ferri (Ref. 18) then employed his mass flow difference concept to this. It will be shown here that neither step is adequate. Rather, an axi-symmetric equivalent of the planar models will be sought starting with the generalized form of Clauser's planar model. Rewrite this as

$$\mu_T(x) = \rho\epsilon = k_1 (\rho u_e \Delta^*) \tag{14}$$

where $\mu_T(x)$ is the turbulent viscosity. This can be read to say, "The turbulent viscosity is proportional to the mass flow defect (or excess) per unit width in the mixing region". One can carry this statement over into axi-symmetric flow by defining a new displacement thickness, δ^*_r, as

$$\pi \rho_e u_e \delta^{*2}_r = \int_0^\infty \left| \rho_e u_e - \rho u \right| 2\pi \, y \, dy \tag{15}$$

This gives

$$\mu_T(x) = k_2 (\rho_e u_e \pi \delta^{*2}_r)/L \tag{16}$$

The proportionality constant, k_2, will have to be determined by a comparison between theory and experiment for one case as is done with all eddy viscosity models. It will of course be a function of the choice of the characteristic length, L; here we take L = a, the initial jet radius. With this, the model becomes

$$\mu_T(x) = k_2(\pi\rho_e u_e \delta^{*2}_r)/a \tag{17}$$

The question of solving the equations of motion remains. The writer was fortunate to be able to apply some work of a previous colleague, S. L. Zeiberg, in this regard. The finite-difference program for treating hypersonic wake problems developed at the General Applied Science Laboratories (Ref. 30) was used directly, only minor alterations were made to include the new eddy viscosity model. The cooperation of the Special Projects Office of the U. S. Navy in agreeing to make the program available is gratefully acknowledged.

In order to make the one-time determination of the constant in Equ.(17) and to begin the comparison of theory and experiment, the constant density experiments of Forstall and Shapiro (Ref. 1) were employed. The particular case chosen was for $u_j/u_e = 2.0$ and the results are shown in Fig. No. 4. The value of $k_2\pi$ determined is 0.018. One can observe that the unified model gives excellent qualitative as well as quantitative agreement with the data; the asymptotic decay predicted by the Prandtl model is at variance with the data. Note that again, the Classical theory provides a poor prediction. Returning to the question of the asymptotic decay rate, it is interesting to put the data and the numerical results for the two eddy viscosity models on log paper since the asymptotic behavior should be a straight line (i.e. x^{-n}). This has been done in Fig. No. 5 where it can be clearly seen that the unified model predicts a decay rate in much better agreement with experiment than the Prandtl Model. The axial variation of the eddy viscosity as predicted by the two models is given in Fig. No. 6.

At this point, comparisons between theory and experiment can be extended to cover a wider range of parameters. The effects of density variations are now included, limited, for the moment, to those due to temperature variations. Ferri's extension of Prandtl's model is employed along with the unified model, and the experimental data of Landis and Shapiro (Ref. 12) is used as the standard against which the adequacy of the theories are measured. A constant Prandtl number of 0.75 in accordance with previous (Refs. 1, 12) suggestions is used throughout. Results for heated jets ($T_j/T_e = 1.19$) with $u_j/u_e = 2.0$ are given in Fig. No. 7 and those for ($T_j/T_e = 1.31$) with $u_j/u_e = 4.0$ are given in Figs. No. 8 and 9. In these cases also, the solution including the temperature field based on the unified model is generally in better agreement with the data than that based on Prandtl's model or Ferri's extension thereof. This conclusion is reached on the basis of the slope of the centerline velocity and temperature decays. The limited downstream extent of the data does allow some variety of opinion on this point especially for the case in Fig. No. 9.

The last variable that must be considered in the general treatment of this flow problem is jet and free stream composition. Many of the practical applications involve the injection of one fluid into a moving stream of another, and a useful eddy viscosity model must be able to handle such cases. The case of a jet of hydrogen injected into an air stream provides a stringent test of the theory since there is a very large density gradient across the mixing zone. This also represents a situation of current technological interest. Some tests of Alpinieri (Ref. 22) and Zakkay and Krause (Ref. 20) will be used to make the comparisons of theory and experiment, and the Ferri model and the unified model will both be used with a constant turbulent Schmidt number of 0.75 (Refs. 1, 12). Consider first the data of Zakkay and Krause (Ref. 20). Note that their external air stream was supersonic ($M \simeq 1.6$), but it will be seen that this does not prevent these results from fitting into the general picture that we are attempting to construct. Using the data for the case with $u_j/u_e = 1.14$ and $\rho_j u_j/\rho_e u_e = 0.14$, a comparison between theory and experiment is given in Fig. No. 10, on the basis of the centerline decay of hydrogen concentration. It is clear that both models underestimate the rate of decay in the far flow field. The unified model is, however, in better quantitative and qualitative agreement with the data.

It is interesting to consider a case with an even lower value of $\rho_j u_j/\rho_e u_e$, and for this purpose Alpinieri's results for $u_j/u_e = 0.67$ and $\rho_j u_j/\rho_e u_e = 0.04$ are employed. These experiments were made with a large jet (2" diameter) which reduces initial boundary layer effects but limited the non-dimensional axial length of the data to $x/a = 20$. In this short distance, it can be expected that the heuristic approximation of the eddy viscosity in the initial region employed previously will introduce serious error. The calculations were therefore begun using the measured profiles at $x/a = 5.0$ as "initial" conditions. The development of the flow field from $x/a = 5.0$ to $x/a = 20$ as predicted by the theory was then compared with the data as shown in Fig. No. 11. Both models give results in reasonable agreement with the data although the far downstream rate of decay is again under-estimated. Interestingly, the results for the two models are in very close agreement. Apparently, the two models tend to become the same in the limit of very small values of the parameter $\rho_j u_j/\rho_e u_e$.

One further aspect of the specification of an eddy viscosity model was considered. The Prandtl model and all its descendants, including that introduced here, are based upon the difference of some quantity across the mixing region. When the particular difference upon which a given model is based becomes zero, that model will predict a vanishing eddy viscosity. This, of course, is not in agreement with common experience, and one must simply accept this limitation upon the applicability of the given model. The actual extent of that limitation is, however, very important. In the present case, if the radius weighted integral of the mass flow difference becomes zero, the limitation has certainly been reached. The case of uniform initial mass flow (i.e. $\rho_j u_j/\rho_e u_e = 1.0$) will produce this result, so that the extent of the limitation of this model can be discussed in terms of the initial mass flow ratio. A case with $\rho_j u_j/\rho_e u_e = 0.63$ (Ref. 26) was considered, and a comparison between theory and experiment is presented in Fig. No. 12. Clearly, the present model is still applicable, and the limiting value of $\rho_j u_j/\rho_e u_e$ must lie closer to unity.

An additional case of this type, i.e. $\rho_j u_j/\rho_e u_e$ near unity, with hydrogen injection and u_j/u_e not near unity was studied. The experimental data of Chriss (Ref. 27) for his case 1A ($\rho_j u_j/\rho_e u_e = 0.56$, $u_j/u_e = 6.3$) were compared with analytical predictions using both the unified model and the Ferri model. The calculation was started with experimental profiles at $x/a = 5.9$. Results for centerline hydrogen concentration and velocity decay are given in Figs. No. 13 and 14, respectively. Here, as has been generally true, the unified model produces a superior prediction.

DISCUSSION

Analysis of the turbulent mixing of a jet in a co-flowing external stream has been considered in detail with particular emphasis placed on the form for a model for the eddy viscosity. A new interpretation was placed on Clauser's low speed, planar model for the "wake" region of a turbulent boundary layer that permitted the derivation of a model for cases with varying density and/or an axi-symmetric configuration. Extensive comparisons between experiment and calculations using the unified model as well as previous suggestions were made. Cases considered included planar and axi-symmetric flows, heated and unheated jets, similar and foreign gas injection, subsonic and supersonic external flows and a range of the mass flux ratio, $\rho_j u_j/\rho_e u_e$, from 0.04 to 4.0. Good quantitative agreement between theory and experiment was obtained over this range using the unified model for the eddy viscosity; the poorest agreement was at the lowest values of $\rho_j u_j/\rho_e u_e$ with hydrogen injection. Most importantly, the axial rate of decay of the centerline values of velocity, temperature and jet concentration in the axi-symmetric configuration as predicted by the unified model were in better agreement with experiment than previously suggested models. In this regard, it is interesting to note that the unified model and Ferri's model tend to give identical predictions in the limit as the mass flux ratio becomes very small. In summary, it may be stated that the present results in conjunction with those of (Ref. 36), demonstrate that the unified model provides a unified description of the eddy viscosity in free mixing flows including planar and axi-symmetric flows, varying temperature, varying composition and axial pressure gradients.

The formal statement of the unified model permits easy extension to truly three dimensional cases, and some studies in this area are planned. Also, further work with, both experimental and analytical, strong axial pressure gradients is needed.

REFERENCES

1. Forstall, W., Jr., Shapiro, A. H.: Momentum and Mass Transfer in Coaxial Gas Jets. Journal of Applied Mechanics, Vol. 72, pp. 339-408 (1950).

2. Pai, S. I.: Fluid Dynamics of Jets. Chapter V, VI and VII, D. Van Nostrand Co., Inc., New York (1954).

3. Townsend, A. A.: The Structure of Turbulent Shear Flow. Chapter 8, Cambridge Univ. Press (1956).

4. Abramovich, G. N.: The Theory of Turbulent Jets. MIT Press, Cambridge, Mass. (1963).

5. Schlichting, H.: Boundary Layer Theory. Fourth Edition, Chapter XXIII, McGraw-Hill Book Co., New York (1962).

6. Hinze, J. O.: Turbulence. Chapter 6, McGraw-Hill Book Co., New York (1959).

7. Halleen, R. M.: A Literature Review on Subsonic Free Turbulent Shear Flow. Rept. MD-11, Mechanical Engrg. Dept., Stanford University (1964).

8. Ledgett, L. A.: Mixing of Fluid Streams. ASME Summer Meeting, Berkeley, Calif. (1934).

9. Viktorin, K.: Investigation of Turbulent Mixing Processes. English Translation, NACA TM-1096 (1941).

10. Pabst, O.: Die Ausbreitung Heisser Grasstrahlen in Bewegter Luft, I Teil-Versuche in Kerngebeit. Deutsche Luftfahrtforschung, Aug. (1944).

11. Ferguson, C. K.: Mixing of Parallel Flowing Streams in a Pressure Gradient. Heat Transfer and Fluid Mechanics Institute (1949).

12. Landis, F. , and Shapiro, A. H.: The Turbulent Mixing of Co-axial Gas Jets. Heat Transfer and Fluid Mechanics Institute (Preprints and Papers) Stanford University Press (1951).

13. Helmbold, H. B., Luessen, G., and Heinrich, A. M.: An Experimental Comparison of Constant Pressure and Constant Diameter Jet Pumps. Univ. of Wichita, Engineering Rept. No. 147 (1954).

14. Weinstein, A. S., Osterle, J. F., and Forstall, W.: Momentum Diffusion from a Slot Jet into a Moving Secondary. Journal of Applied Mechanics, pp. 437-443, Sept. (1956).

15. Curtet, R.: Confined Jets and Recirculation Phenomena with Cold Air. Combustion and Flame, 2, pp. 383-411 (1958).

16. Mikhail, S.: Mixing of Coaxial Streams inside a Closed Conduit. J. of Mechanical Engrg. Sci., Vol. 2, No. 1, pp. 59-68 (1960).

17. Maczynski, J. F.: A Round Jet in an Ambient Coaxial Stream. J. of Fluid Mechanics, 13, pp. 597-608 (1962).

18. Ferri, A., Libby, P. A., and Zakkay, V.: Theoretical and Experimental Investigation of Supersonic Combustion. Third ICAS Conference, Stockholm (1962).

19. Becker, H. A., Hottel, H. C., and Williams, G. C.: Mixing and Flow in Ducted Turbulent Jets. Ninth Symposium on Combustion, Academic Press, pp. 7-21 (1962).

20. Zakkay, V., and Krause, E.: Mixing Problems with Chemical Reaction. 21st Meeting AGARD Combustion and Propulsion Panel, London (1963) also in Supersonic Flow, Chemical Processes and Radiative Transfer, Pergamon Press. (1964).

21. Marquardt Corp., TMC Report PR 297-2Q, Second Quart. Prog. Rept. AF Contract No. AF 33(657)-8431 (1963).

22. Alpinieri,L. J.,: Turbulent Mixing of Coaxial Jets. AIAA Journal, pp. 1560-1567 (1964).

23. Zakkay, V., Krause, E. and Woo, S. D. L.: Turbulent Transport Processes for Axisymmetric Heterogeneous Mixing. AIAA Journal, pp. 1939-1947 (1964).

24. Barchilon, M. and Curtet, R.: Some Details of the Structure of an Axisymmetric Confined Jet with Back Flow. Journal of Basic Engrg., pp. 777-787, Dec. (1964).

25. Bradbury, L. J. S. and Riley, J.: The Spread of a Turbulent Plane Jet into a Parallel Moving Airstream. Journal of Fluid Mechanics, Vol. 27, Pt. 2, pp. 381-394 (1967).

26. Henry, J. R.: Private Communication.

27. Chriss, D. E.: Experimental Study of Turbulent Mixing of Subsonic Axisymmetric Gas Streams. AEDC-TR-68-133, August (1968).

28. Libby, P. A.: A Theoretical Analysis of the Turbulent Mixing of Reactive Gases with Application to the Supersonic Combustion of Hydrogen. ARS Journal, pp. 388-395, March (1962).

29. Schetz, J.A.: Supersonic Diffusion Flames. In Supersonic Flow, Chemical Processes and Radiative Transfer, Pergamon Press (1964).

30. Zeiberg, S. L. and Bleich, G. D.: Finite Difference Calculation of Hypersonic Wakes. AIAA Journal, pp. 1396-1402, August (1964).

31. Clauser, F. H.: The Turbulent Boundary Layer. in Advances in Applied Mechanics, Vol. IV, Pergamon Press (1956).

32. Ting, L. and Libby, P. A.: Remarks on the Eddy Viscosity in Compressible Mixing Flows. Journal Aerospace Sci., pp. 797-798, October (1960).

33. Bloom, M. H. and Steiger, M. H.: Diffusion and Chemical Reaction in Free Mixing. IAS Annual Meeting, Paper No. 63-67 (1963).

34. Townsend, A.A.: The Eddy Viscosity in Turbulent Shear Flow. The Philosophical Magazine, pp. 890-906, September (1960).

35. Keagy, W. R. and Weller, A. E.: A Study of Freely Expanding Inhomogeneous Jets. Heat Transfer and Fluid Mechanics Inst., ASME, New York (1949).

36. Schetz, J. A. and Jannone, J.: Planar Free Turbulent Mixing with an Axial Pressure Gradient. Journal of Basic Engrg., December (1967).

37. Schetz, J. A. and Jannone, J.: A Study of Linearized Approximations to the Boundary Layer Equations. Journal of Applied Mechanics, pp. 757-764, December (1965).

38. Zeiberg, S. L. and Kaplan, B.: Approximate Analysis of Free Mixing Flow. AIAA Journal, pp. 552-554, March (1965).

TABLE I

SUMMARY OF EXPERIMENTS FOR A JET IN A CO-FLOWING STREAM

Author	Year	Ref. No.	Planar	Axi-symmetric	Variable Temperature	Foreign Gas Injection	Pressure Gradient
Ledgett	1934	8	x				x
Viktorin	1941	9		x			x
Pabst	1944	10		x	x		
Ferguson	1949	11	x				x
Forstall & Shapiro	1950	1		x		He	
Landis & Shapiro	1951	12		x	x	He, CO_2	
Helmbold, et al	1954	13		x			x
Weinstein, et al	1955	14	x				
Curtet	1958	15	x				x
Mikhail	1960	16		x			x
Maczynski	1962	17		x			x
Ferri, et al	1962	18		x	x	H_2	
Becker, et al	1962	19		x		Aerosol	x
Zhestov, et al	1963	4(pp.28,344)	x	x	x		
Yakovlevskiy & Pechenkin	1963	4(pp.274)		x	x		
Borodachev	1963	4(pp.29)		x	x		
Zakkay & Krause	1963	20		x	x	H_2	
Marquardt Corp.	1963	21	x		x	H_2	
Alpinieri	1964	22		x	x	H_2, CO_2	
Zakkay, et al	1964	23		x	x	H_2, He, CO_2, A	
Barchilon & Curtet	1964	24		x			x
Bradbury & Riley	1967	25	x				
Torrence & Eggers	1967	26		x	x		
Chriss	1968	27		x	x	H_2	

TABLE II EDDY VISCOSITY MODELS FOR MAIN MIXING REGION OF JETS AND WAKES

AUTHOR	YEAR	REFERENCE	PLANAR	AXI-SYMMETRIC	VARIABLE DENSITY	EXPRESSION	REMARKS
PRANDTL	1926	5 (pp. 477)	X	X		$\ell^2 \left(\frac{\partial u}{\partial y}\right)$	ℓ PROPORTIONAL TO THE WIDTH OF THE MIXING REGION
VON KARMAN	1930	5 (pp. 485)	X	X		$\kappa^2 \frac{(\partial u/\partial y)^4}{(\partial^2 u/\partial y^2)^2}$	
TAYLOR	1932	5 (pp. 482)	X	X		$\ell_w^2 \left(\frac{\partial u}{\partial y}\right)$	$\ell_w = \sqrt{2}\ell$
PRANDTL	1942	5 (pp. 481)	X	X		$\ell^2 \sqrt{\left(\frac{\partial u}{\partial y}\right)^2 + \ell_1^2 \left(\frac{\partial^2 u}{\partial y^2}\right)^2}$	REQUIRES TWO MIXING LENGTHS
PRANDTL	1942	5 (pp. 481)	X	X		$\kappa_1 b (u_{MAX} - u_{MIN})$	INTRODUCED "VELOCITY DIFFERENCE" CONCEPT; WITH b TAKEN AS $b_{1/2}$, κ_1 = 0.037 IN PLANAR JETS AND $\kappa_1 \cong 0.25$ IN AXI-SYMMETRIC JETS.
SCHLICHTING	1942	5	X			$0.0222\, u_e C_D d$	WAKE OF A CYLINDER OF ARBITRARY CROSS SECTION
CLAUSER	1956	31				$\kappa u_e \delta^* = \kappa \int_0^\infty \lvert u_e - u \rvert \, dy$	APPLIED TO "WAKE"-LIKE OUTER REGION OF A BOUNDARY LAYER, $0.016 < \kappa < 0.018$
HINZE	1959	6	X			$0.016\, u_e d$	WAKE OF A CIRCULAR CYLINDER
TING-LIBBY	1960	32		X	X	$\rho^2 \epsilon = \frac{2 \rho_c^2 \epsilon_o}{y^2} \int_0^y \frac{\rho}{\rho_c} y' \, dy'$	ϵ_o IS THE CONSTANT DENSITY EDDY VISCOSITY AND ρ_c IS THE CENTER-LINE DENSITY
TING-LIBBY	1960	32	X		X	$\rho^2 \epsilon = \rho_c^2 \epsilon_o$	ϵ_o IS THE CONSTANT DENSITY EDDY VISCOSITY AND ρ_c IS THE CENTER-LINE DENSITY
FERRI, ET AL	1962	18		X	X	$\rho \epsilon = 0.025 \left((\rho u)_{MAX} - (\rho u)_{MIN}\right)$	EXTENDED PRANDTL'S THIRD MODEL TO VARIABLE DENSITY, INTRODUCED "MASS FLOW DIFFERENCE" CONCEPT
BLOOM & STEIGER	1963	33		X	X	$\rho \epsilon = \kappa \delta' \rho_c (u_{MAX} - u_{MIN})$	ATTEMPT TO EXTEND PRANDTL'S THIRD MODEL TO VARIABLE DENSITY, δ' IS TRANSFORMED WAKE RADIUS
SCHETZ	1963	28	X		X	$\rho^2 \epsilon = 0.037 \rho_c \left((\rho u)_{MAX} - (\rho u)_{MIN}\right)$	SIMPLE APPLICATION OF "MASS FLOW DIFFERENCE" TO PLANAR FLOWS
ALPINIERI	1964	22		X	X	$\frac{\rho \epsilon}{\rho_j u_j} = 0.025\, b_{1/2} \left(\frac{\rho_e u_c}{\rho_j u_j} + \frac{\rho_e u_e^2}{\rho_j u_j^2}\right)$	
ZAKKAY, ET AL	1964	23		X	X	$0.011\, b_{1/2} u_c$	PRESUMES THAT CENTERLINE VELOCITY AND CONCENTRATION DECAY AS x^{-2}
SCHETZ	1968	PRESENT	X	X	X	"TURBULENT VISCOSITY PROPORTIONAL TO MASS FLOW DEFECT (OR EXCESS) IN THE MIXING REGION"	UNIFIED MODEL

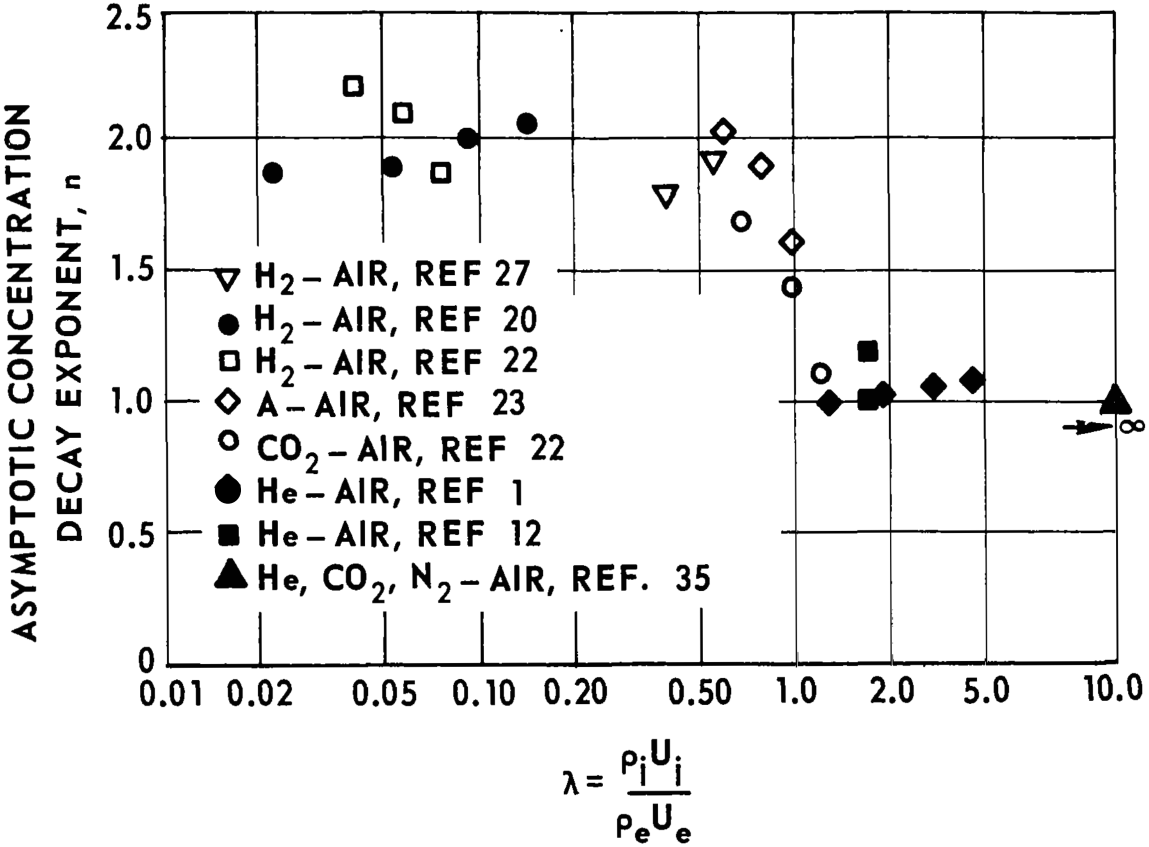

Fig. 1 EXPERIMENTAL DETERMINATION OF ASYMPTOTIC CONCENTRATION DECAY EXPONENT

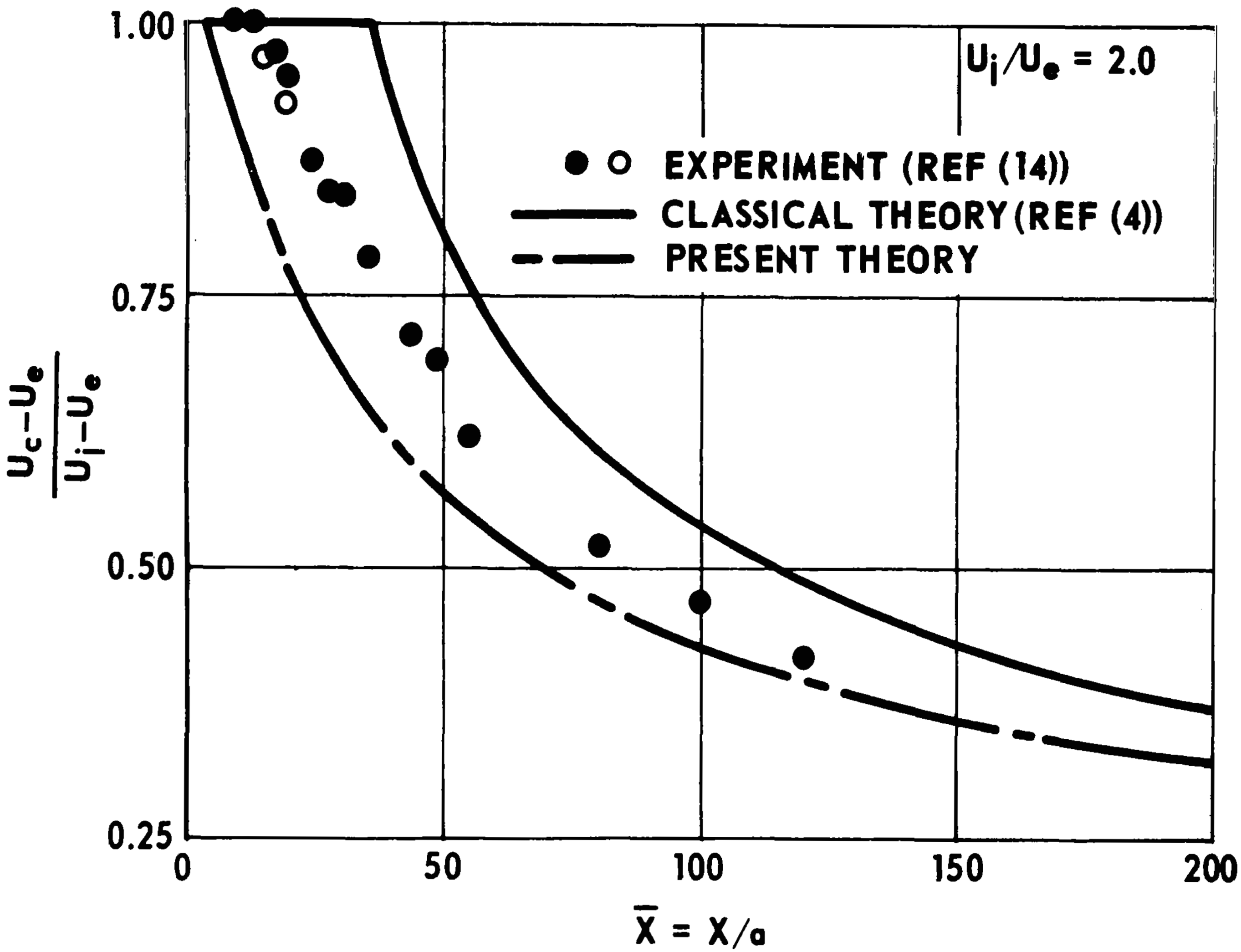

Fig. 2 THEORY AND EXPERIMENT FOR PLANAR, CONSTANT DENSITY JET MIXING, $U_j/U_e = 2.0$

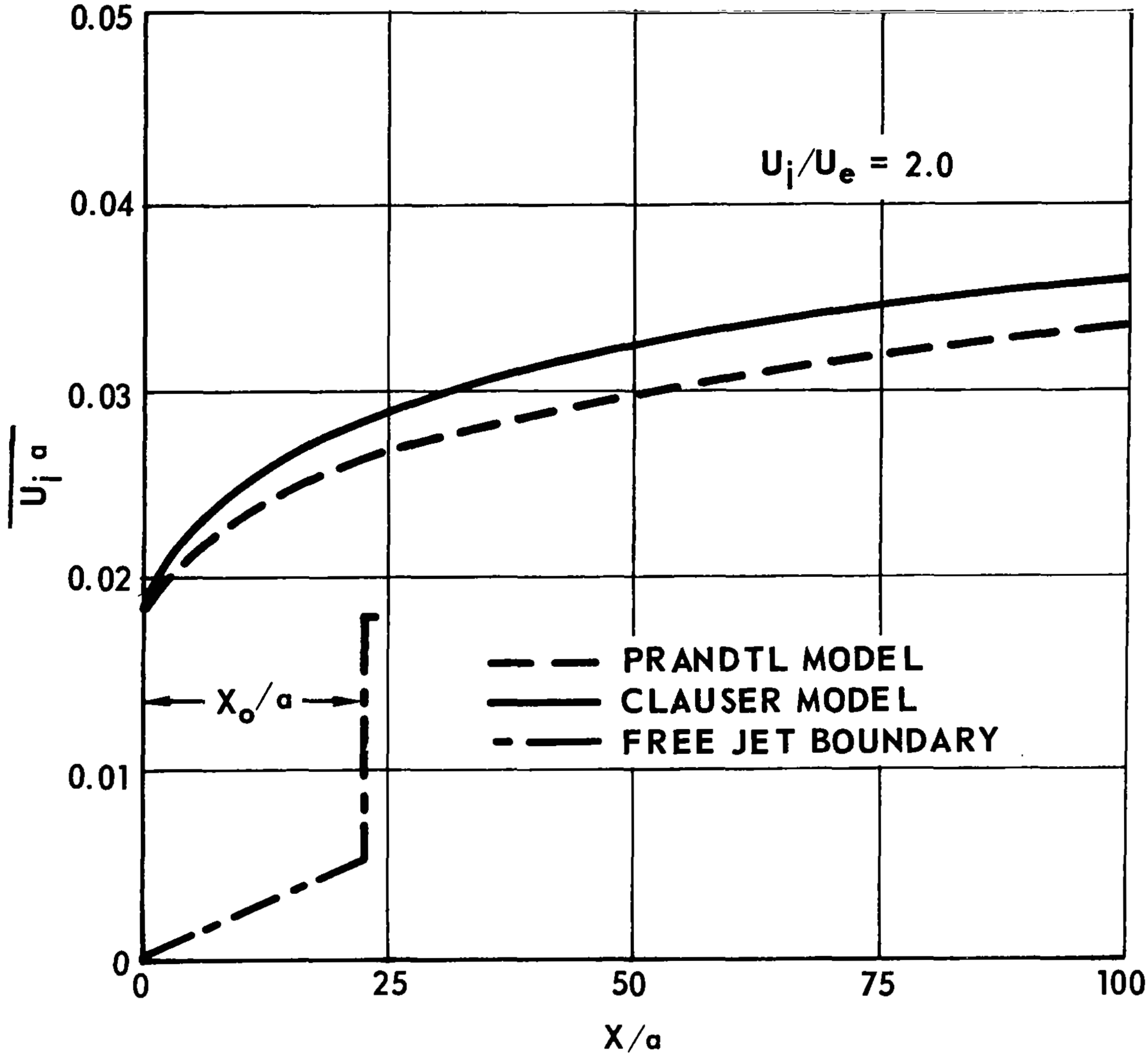

Fig. 3 EDDY VISCOSITY MODELS IN PLANAR JET MIXING

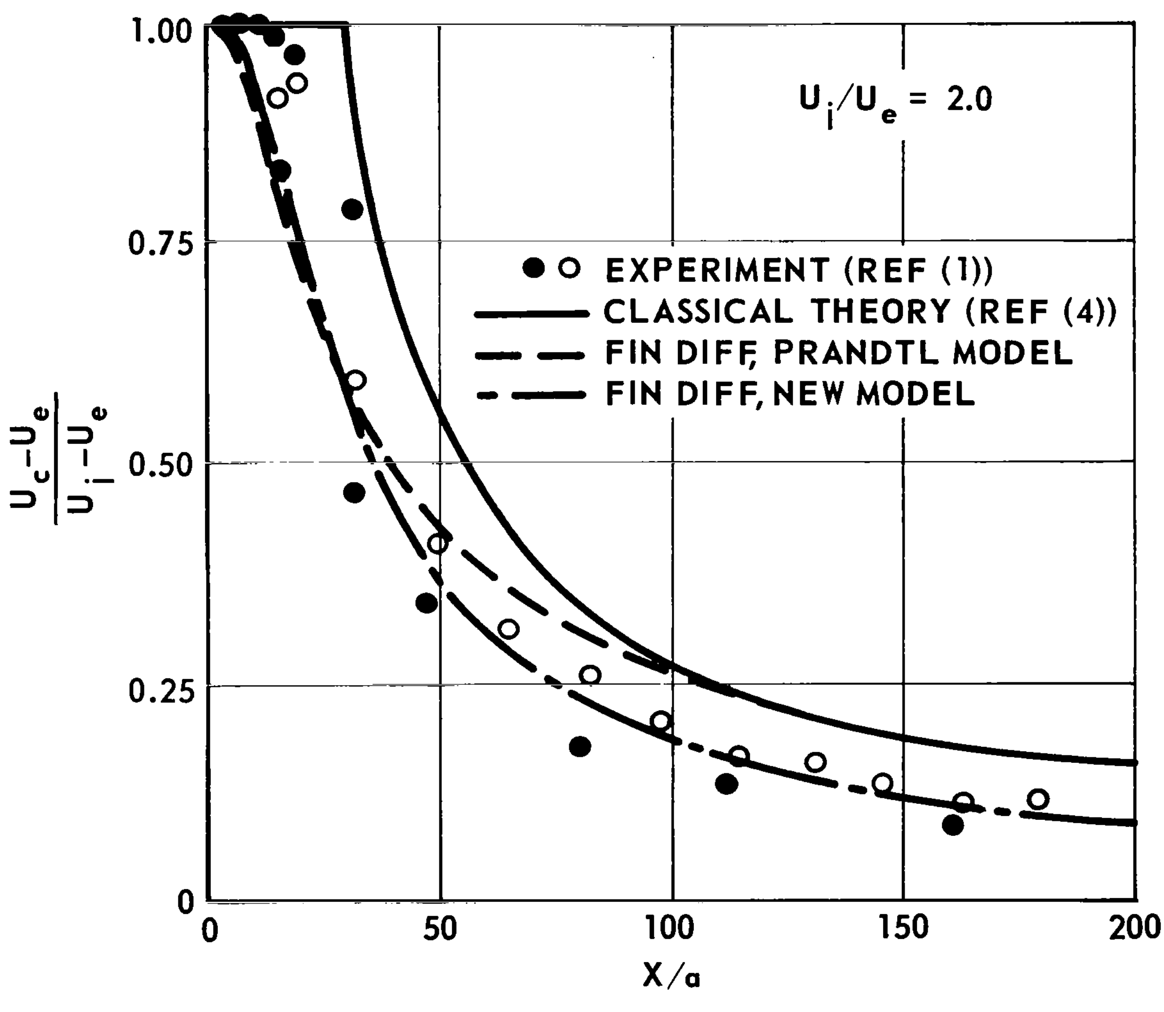

Fig. 4 THEORY AND EXPERIMENT FOR AXISYMMETRIC, CONSTANT DENSITY JET MIXING, $U_j/U_e = 2.0$

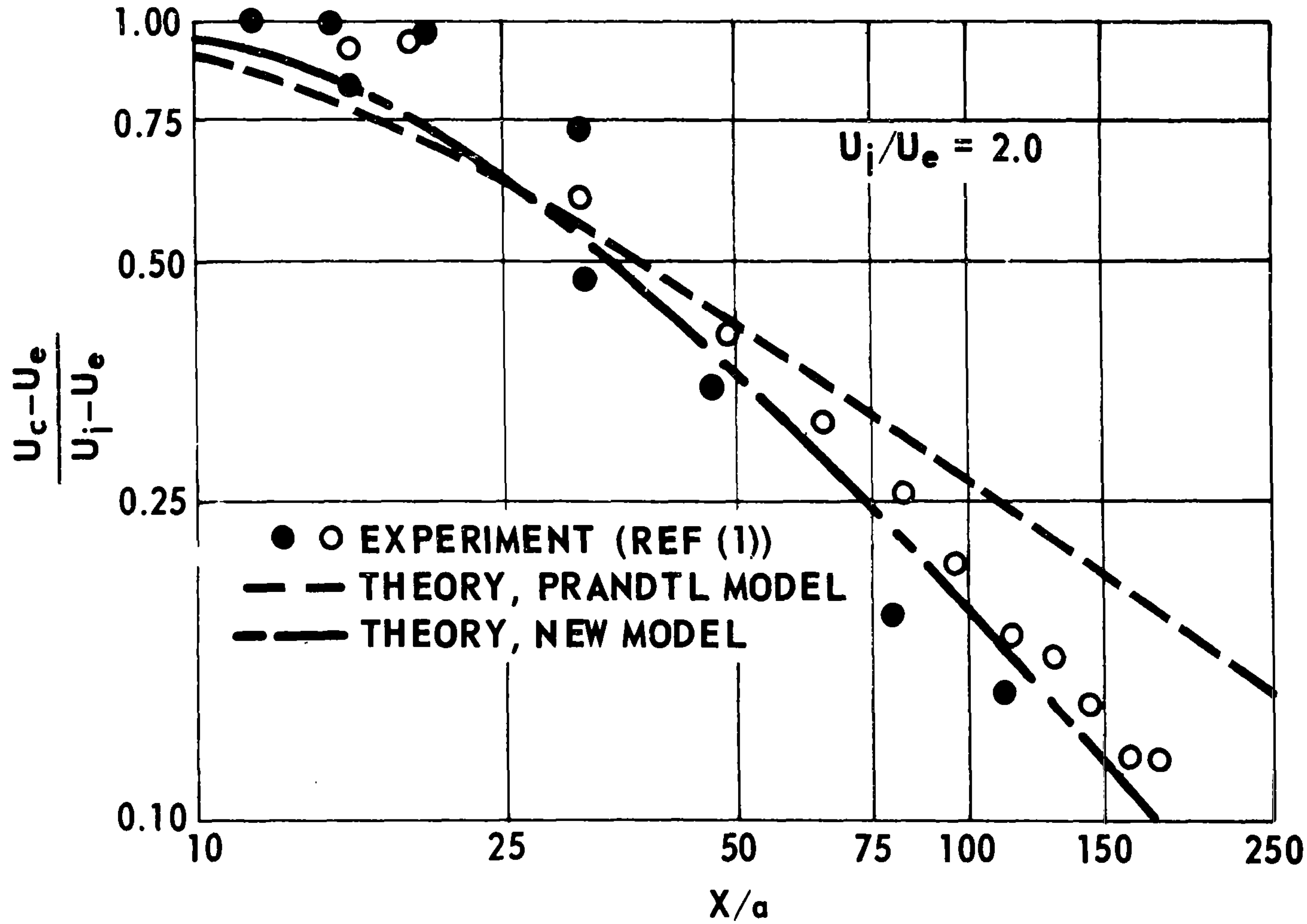

Fig. 5 THEORY AND EXPERIMENT FOR ASYMPTOTIC DECAY OF CENTERLINE VELOCITY FOR AXISYMMETRIC, CONSTANT DENSITY JET MIXING, $U_j/U_e = 2.0$

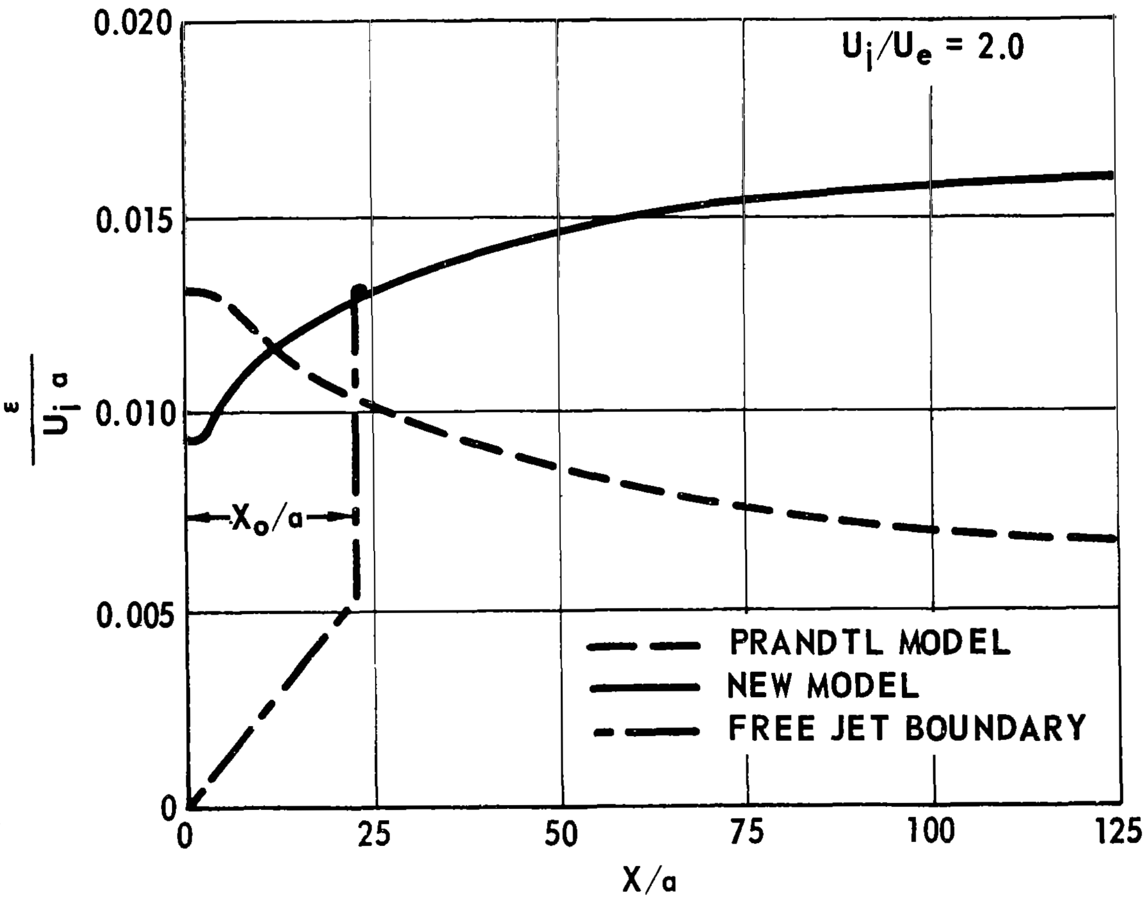

Fig. 6 EDDY VISCOSITY MODELS IN AXISYMMETRIC JET MIXING

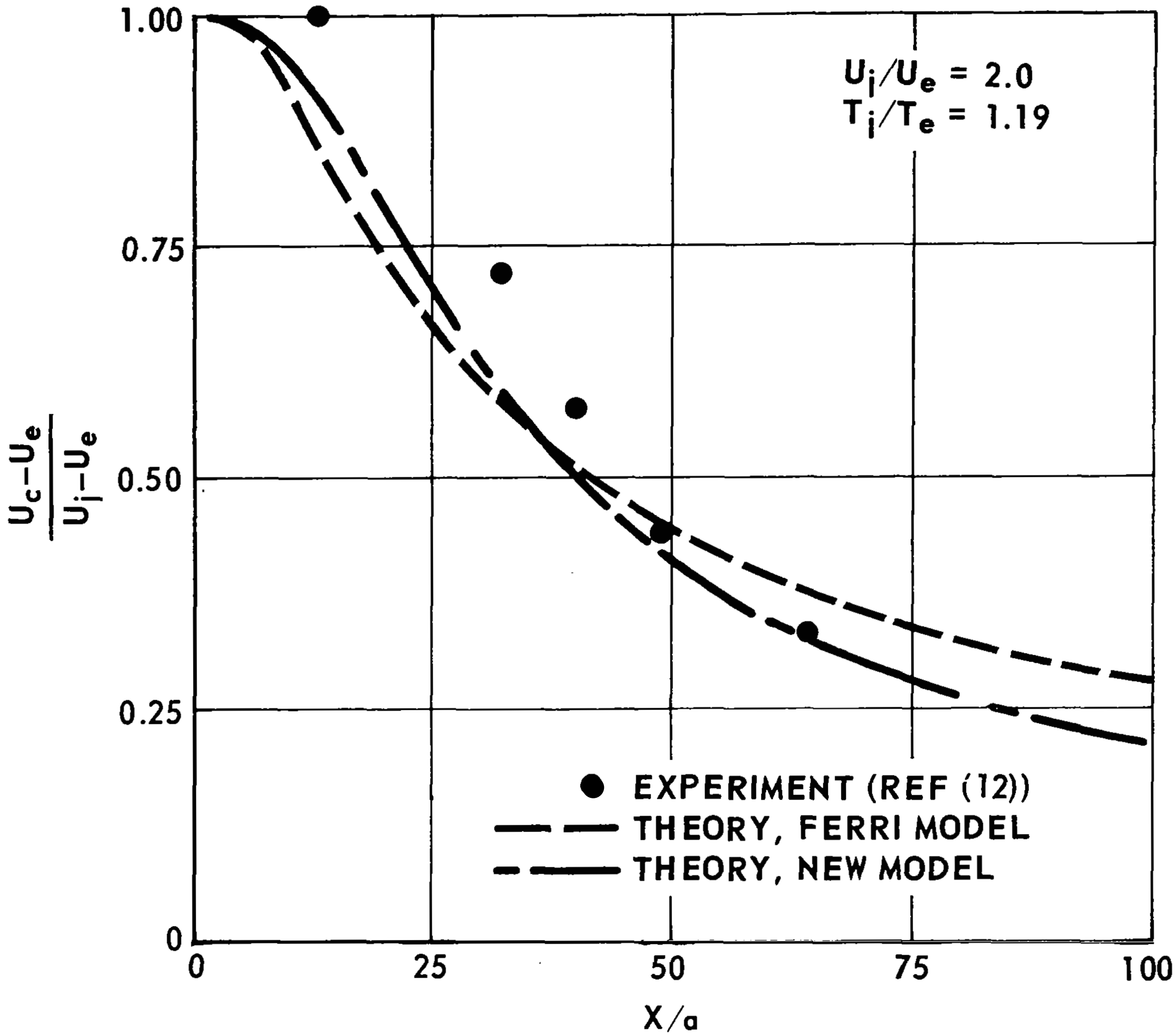

Fig. 7 THEORY AND EXPERIMENT FOR HEATED, AXISYMMETRIC AIR JET; $U_j/U_e = 2.0$, $T_j/T_e = 1.19$

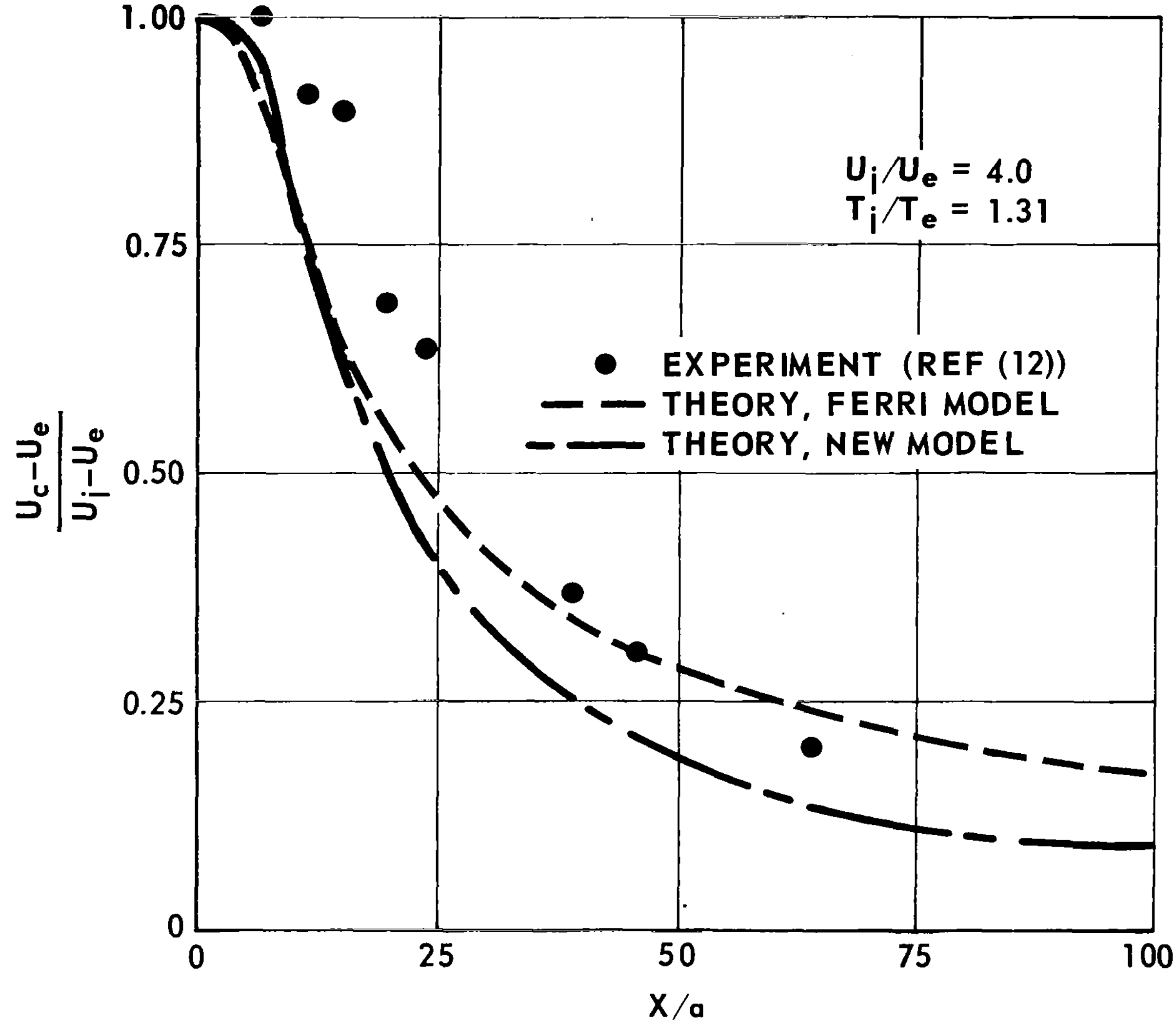

Fig. 8 THEORY AND EXPERIMENT FOR HEATED AXISYMMETRIC AIR JET; $U_j/U_e = 4.0$, $T_j/T_e = 1.31$

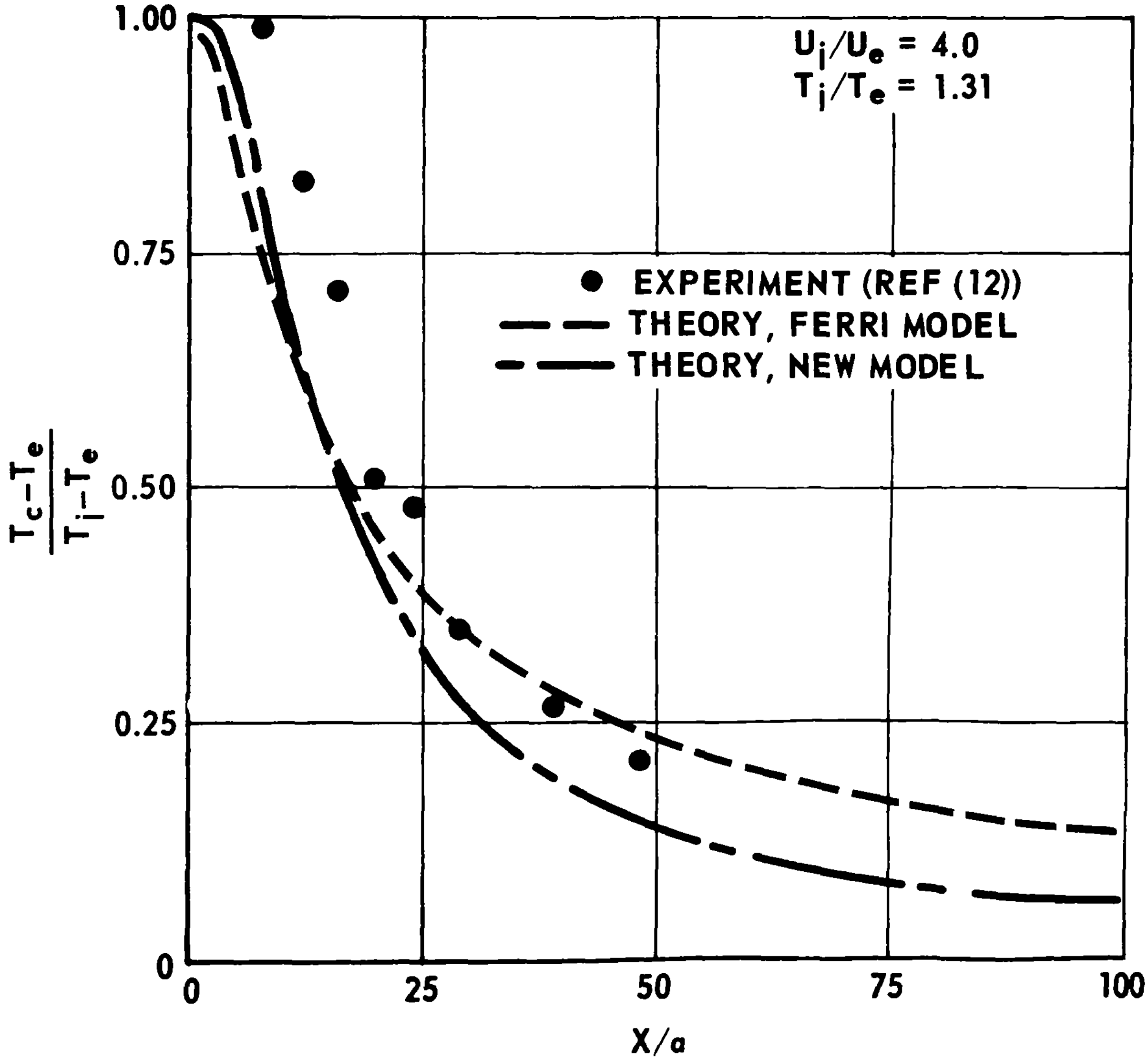

Fig. 9 THEORY AND EXPERIMENT FOR TEMPERATURE FIELD OF HEATED AXISYMMETRIC AIR JET; $U_j/U_e = 4.0$, $T_j/T_e = 1.31$

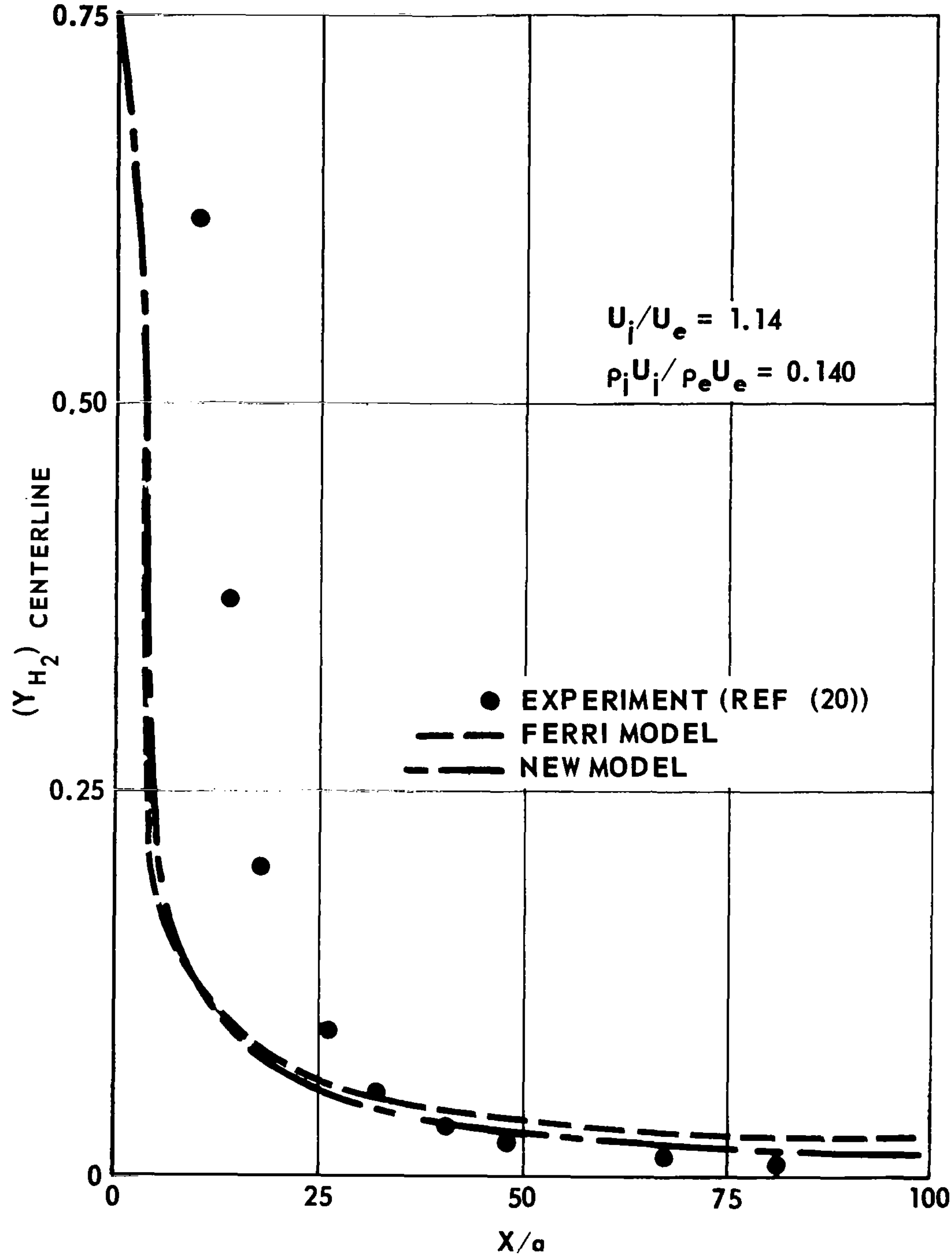

Fig. 10 THEORY AND EXPERIMENT FOR AXISYMMETRIC HYDROGEN JET MIXING IN AIR, $\rho_j U_j/\rho_e U_e = 0.140$

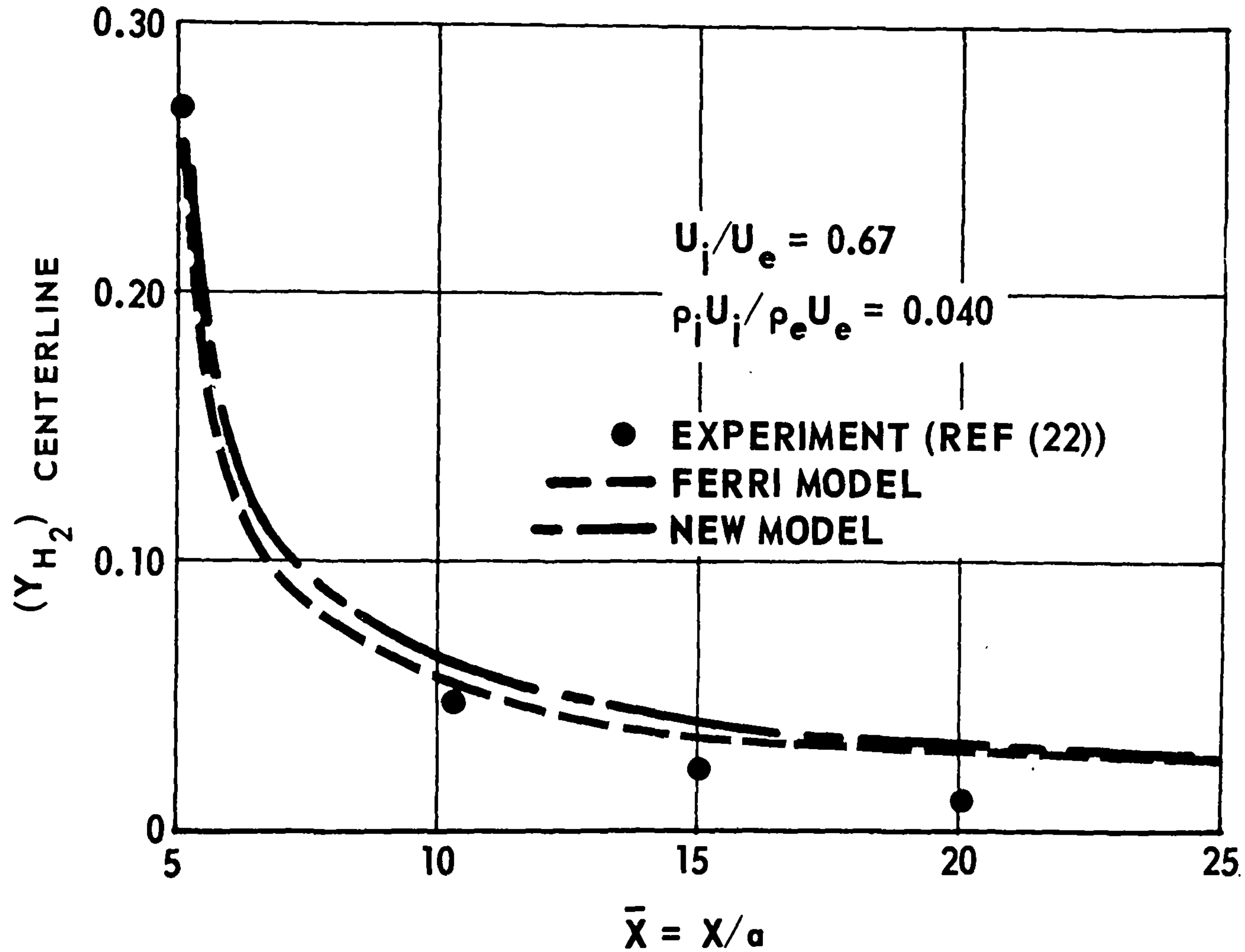

Fig. 11 THEORY AND EXPERIMENT FOR AXISYMMETRIC HYDROGEN JET MIXING IN AIR, $\rho_j U_j/\rho_e U_e = 0.040$

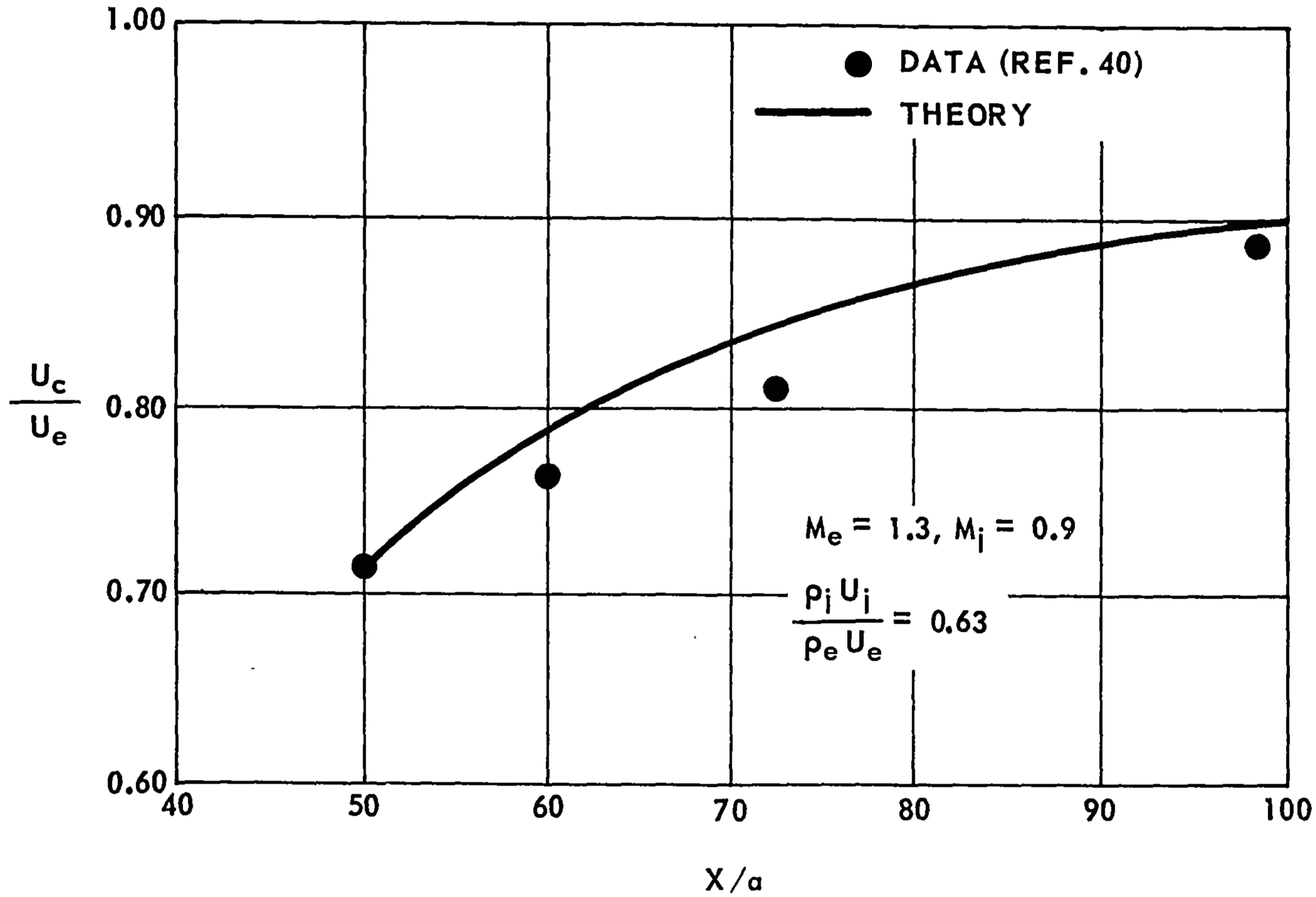

Fig. 12 THEORY AND EXPERIMENT FOR AIR JET MIXING IN A SUPERSONIC AIR SYSTEM, $\frac{\rho_j U_j}{\rho_e U_e} = 0.63$

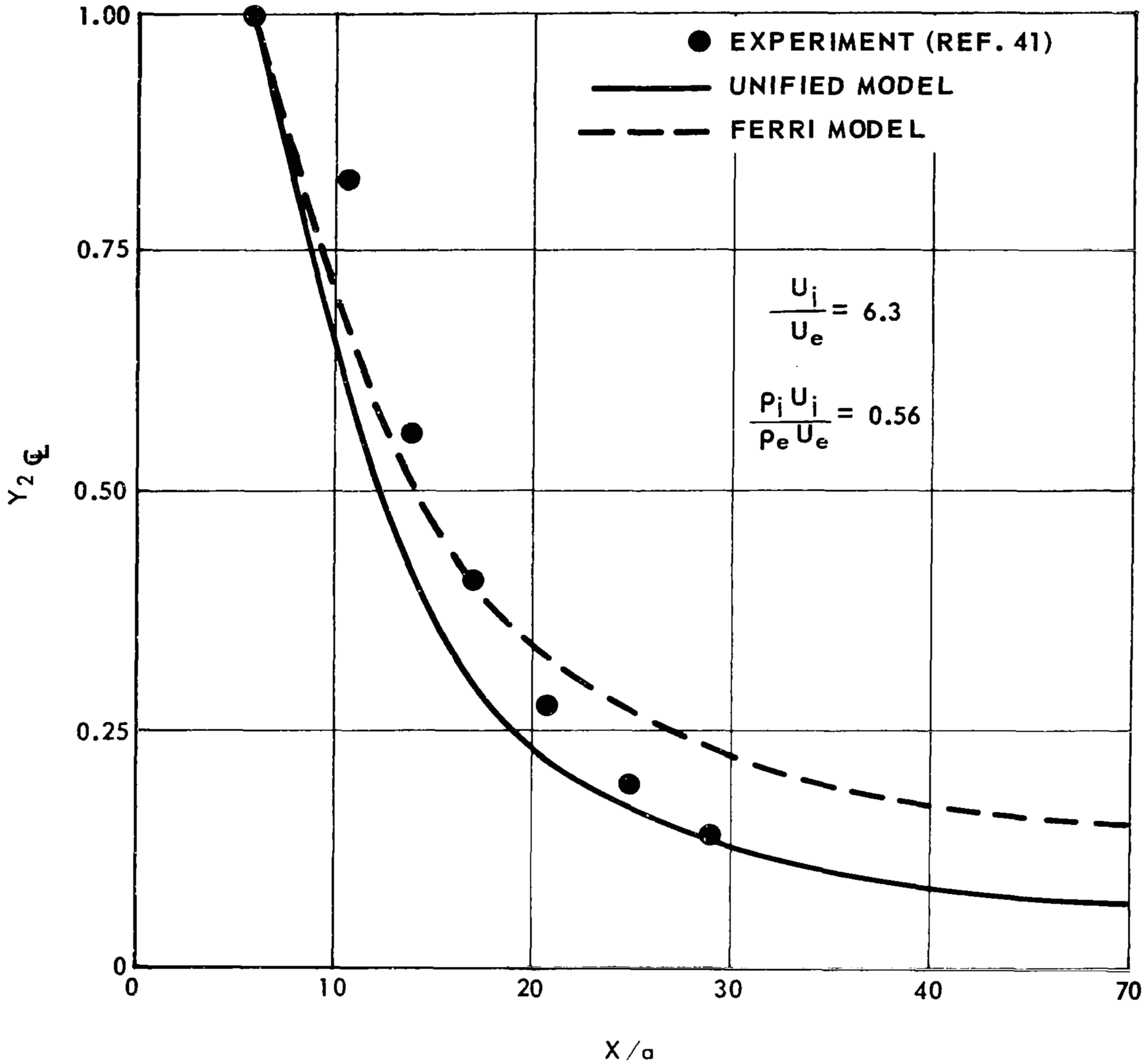

Fig. 13 THEORY AND EXPERIMENT FOR CONCENTRATION DECAY WITH HYDROGEN JET IN AIR, $\frac{\rho_j U_j}{\rho_e U_e} = 0.56$

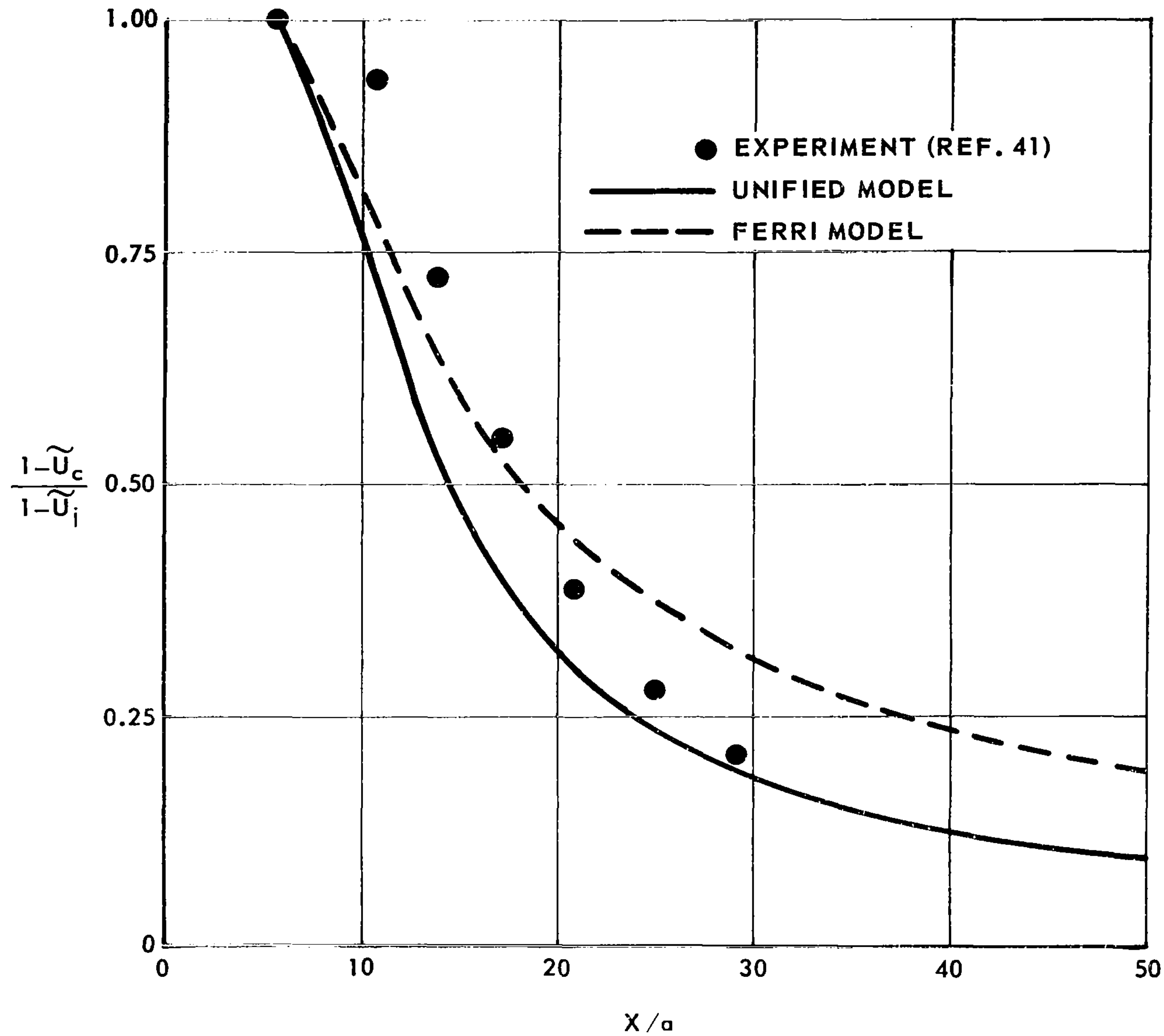

Fig. 14 THEORY AND EXPERIMENT FOR VELOCITY VARIATION WITH HYDROGEN JET IN AIR, $\frac{\rho_j U_j}{\rho_e U_e} = 0.56$