X-643-69-305
PREPRINT

BOUND BOUND TRANSITIONS IN HYDROGEN LIKE ATOMS INDUCED BY FAST CHARGED PARTICLES

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JULY 1969

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

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Bound Bound Transitions in Hydrogen Like Atoms Induced by Fast Charged Particles

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ABSTRACT

Considering the transition from the $n1$ level in an hydrogen like atom to the $n'1'$ level induced by fast charged particles of mass M, charge z and energy E, when $(M/\mu)\,(\Delta E/E) << 1$, with μ the electronic charge and ΔE the excitation energy, the excitation cross section can be written as

$$Q_{BB}(n1, n'1') = (Mz/\mu Z)^2\, E^{-1}\, [A(n1, n'1')\, ln E + B(n1, n'1')],$$

where Z is the nuclear charge, and $A(n1, n1')$ and $B(n1, n'1')$ are some constants. In this paper an analytic expression for $A(n1, n'1')$ and $B(n1, n'1')$ is found. These coefficients after being averaged with respect to 1 and summed with respect to $1'$ are given numerically for $n = 1 - 9$ and a range of n' from $n + 1$ to 20. Also the Born cross sections $Q_B(n, n')$ for many transitions are given. Because of the shortage of space, cross sections not for all but for some transitions between different azimuthal quantum numbers are given here. Marked disagreement with some of the data of the previous authors is found.

Some of the results obtained can also be used with moderate accuracy for bound bound transitions in hydrogen like ions.

FORMULATION

A review on the collision induced bound bound transitions is given by Moiseiwitsch and Smith (1968). The total excitation cross section in the Born approximation when an atom collides with a projectile of charge ze and mass M is given by Bethe,[2]

$$Q_B(i,f) = \frac{8\pi}{a_0^2 k_1^2}\left(\frac{M}{\mu} z\right)^2 \int_{k_1-k_2}^{k_1+k_2} |\langle f | e^{i\mathbf{q}\cdot\mathbf{r}} | i\rangle|^2 \frac{dq}{q^3}, \tag{1}$$

with k_1 and k_2 the propagation vectors of the projectile before and after the collision, given in atomic units, $|i\rangle$ and $|f\rangle$ the initial and final states of the atom, μ the electronic mass, and a_0 the Bohr radius. $\mathbf{q}$ is the momentum transfered by the projectile. By taking $\hbar = 1, \mathbf{k}_1$, and $\mathbf{k}_2$ will be given through the conservation of energy by

$$k_1^2 - k_2^2 = \frac{M}{\mu}(E_f - E_i), \tag{2}$$

with E_i and E_f the initial and final energy of the atom in rydbergs. Similarly k_1^2 and k_2^2 are given numerically in rydbergs.

From the computational point of view it will be easier to evaluate the transition matrix using for $|i\rangle$ and $|f\rangle$ a set given in the parabolic coordinates, and then to find by a transformation the transition matrix when the initial and final states are given in the spherical coordinates. For a state specified by the quantum numbers nmn_1 we have the following set in the parabolic coordinates [3]

$$|n\,m\,n_1\rangle = \frac{\alpha^{m+3/2}}{\sqrt{\pi\, n}} \times \frac{[n_1!\; n_2!\,]^{1/2}\; exp\left[-\frac{\alpha}{2}(\xi+\eta)\right]}{[(n_1+m)!\;(n_2+m)!\,]^{3/2}}\;\; (\xi\,\eta)^{m/2}\, L^m_{n_1+m}(\alpha\,\xi)$$

$$\times\, L^m_{n_2+m}(\alpha\,\eta)\; e^{im\phi},\;\; \alpha = \frac{Z}{n\,a_0}. \qquad (3)$$

Here n and m are the principal, and the absolute value of the magnetic, quantum numbers, n_1 and $n_2 = n-1-m-n_1$ are the Stark levels quantum numbers. Z is the nuclear charge.

For evaluation of $\langle f|\, \exp(i\,\mathbf{q}\cdot\mathbf{r})\,|\,i\rangle$ we take the z-axis of the coordinate system along $\mathbf{q}$, then the matrix vanishes unless $|\,i\rangle$ and $|\,f\rangle$ have the same magnetic quantum numbers. With the help of (3) evaluation of the matrix is simple algebra. Without giving the details we find that

$$\langle n'\,m\,n_1'|\; e^{i\mathbf{q}\cdot\mathbf{r}}\;|n\,m\,n_1\rangle = C \sum_{\nu_1=0}^{n_1}\sum_{\nu_1'=0}^{n_1'}\sum_{\nu_2=0}^{n_2}\sum_{\nu_2'=0}^{n_2'} (-2\,\alpha)^{\nu_1+\nu_2}(-2\,\alpha')^{\nu_1'+\nu_2'}$$

$$\times \binom{n_1}{\nu_1}\binom{n_1'}{\nu_1'}\binom{n_2}{\nu_2}\binom{n_2'}{\nu_2'} \times \frac{(m+\nu_1+\nu_1')!}{(m+\nu_1)!\;(m+\nu_1')!} \times \frac{(m+\nu_2+\nu_2')!}{(m+\nu_2)!\;(m+\nu_2')!}$$

$$\times \frac{[(2\,m+2+\nu_1+\nu_1'+\nu_2+\nu_2')\,(\alpha+\alpha') - i\,(\nu_2-\nu_1+\nu_2'-\nu_1')\,q]}{(\alpha+\alpha'-i\,q)^{m+2+\nu_1+\nu_1'}\,(\alpha+\alpha'+i\,q)^{m+2+\nu_2+\nu_2'}},$$

$$C = \frac{(4\,\alpha\,\alpha')^{m+2}}{4\,Z}\left[\frac{(n_1+m)!}{n_1!}\times\frac{(n_2+m)!}{n_2!}\times\frac{(n_1'+m)!}{n_1'!}\times\frac{(n_2'+m)!}{n_2'!}\right]^{1/2}. \qquad (4)$$

The state $|\,nml\rangle$ with n and m defined before, and l the azimuthal quantum number of an optical level, is related to $|\,nmn_1\rangle$ by

$$|nml\rangle = \sum_{n_1=0}^{n-m-1} |nmn_1\rangle \langle nmn_1|nml\rangle , \tag{5}$$

where $\langle nmn_1|nml\rangle$ are elements of the transformation matrix. A recursion relation for these elements and their numerical values for $n = 1$ to 5 and all the possible values of m are given by Omidvar.[4] Consequently Barut and Kleinert,[5] and Hughes[6] have shown independently that these elements are related to the vector coupling coefficients through

$$\langle nmn_1|nml\rangle = (-)^m \sqrt{2l+1} \begin{pmatrix} (n-1)/2 & (n-1)/2 & l \\ (m-n_1+n_2)/2 & (m+n_1-n_2)/2 & -m \end{pmatrix}, \tag{6}$$

where the bracket on the right hand side is Wigner's $3j$ symbol. We then have that

$$\langle n'ml'|e^{iq\cdot r}|nml\rangle = \sum_{n_1, n_1'} \langle n'mn_1'|n'ml'\rangle$$

$$\times \langle nmn_1|nml\rangle \langle n'mn_1'|e^{iq\cdot r}|nmn_1\rangle . \tag{7}$$

Substitution of (4) into (7), and (7) into (1), and an integration with respect to q, will give $Q_B(nl, n'l')$.

To carry out the integration with respect to q we need the squared modulus of the transition matrix in spherical coordinates. Let us write (4) in the form

$$\langle n'mn_1'|e^{iq\cdot r}|nmn_1\rangle = C \sum_{\gamma} G(\gamma) \times \frac{\lambda_1(\alpha+\alpha'+iq) + \lambda_2(\alpha+\alpha'-iq)}{(\alpha+\alpha'-iq)^{\lambda_1+1} (\alpha+\alpha'+iq)^{\lambda_2+1}} , \tag{8}$$

where γ stands for $\nu_1 \nu_1' \nu_2 \nu_2'$, and

$$G(\gamma) = (-2\alpha)^{\nu_1+\nu_2} (-2\alpha')^{\nu_1'+\nu_2'} \times \binom{n_1}{\nu_1}\binom{n_1'}{\nu_1'}\binom{n_2}{\nu_2}\binom{n_2'}{\nu_2'}$$

$$\times \frac{(m+\nu_1+\nu_1')!}{(m+\nu_1)!\ (m+\nu_1')!} \times \frac{(m+\nu_2+\nu_2')!}{(m+\nu_2)!\ (m+\nu_2')!}\ ,$$

$$\lambda_1 = m+1+\nu_1+\nu_1', \qquad \lambda_2 = m+1+\nu_2+\nu_2'. \tag{9}$$

Then abbreviating $\langle nmn_1|nm1\rangle$ and $\langle n'mn_1'|n'm1'\rangle$ by a_{ln_1} and $a'_{l'n_1'}$ we have

$$|\langle n'm1'|e^{i\mathbf{q}\cdot\mathbf{r}}|nm1\rangle|^2 = \sum_{n_1 n_1' n_1'' n_1'''} a_{ln_1}\, a_{ln_1''}\, a'_{l'n_1'}\, a'_{l'n_1'''}$$

$$\times CC'' \sum_{\gamma,\gamma''} G(\gamma)\, G(\gamma'')\, R_e\ \{\ \},$$

$$\{\ \} = \frac{[\lambda_1(\alpha+\alpha'+iq)+\lambda_2(\alpha+\alpha'-iq)]\ [\lambda_1''(\alpha+\alpha'-iq)+\lambda_2''(\alpha+\alpha'+iq)]}{(\alpha+\alpha'-iq)^{\lambda_1+\lambda_2''+2}\ (\alpha+\alpha'+iq)^{\lambda_2+\lambda_1''+2}} \tag{10}$$

It is clear from this equation that $R_e\{\ \}$ contains only even powers of q. The integral in (1) corresponds to an integration over $R_e\{\ \}\, q^{-3}$. This integration can be done by elementary means. The final expression becomes particularly simple at the limit of high k_1^2 where by (2) we can write

$$k_1 - k_2 \sim \frac{M}{2\mu}\frac{\Delta E}{k_1}, \quad k_1 + k_2 \sim 2k_1, \quad k_1^2 >> 1, \tag{11}$$

with $\Delta E = E_f - E_i$ the excitation energy. In this way we find the cross section in the Born-Bethe approximation, valid when $(M/\mu)\ (\Delta E/k_1^2) \gg 1$,

$$Q_{BB}(n1, n'1') = [(Mz)/(\mu Z k_1)]^2\ [A(n1, n'1')\ ln\ (k_1/Z)^2 + B(n1, n'1')]\ (\pi a_0^2) \tag{12}$$

Here $A(nl, n'l')$ and $B(nl, n'l')$ are constant parameters independent of both the incident energy k_1^2 and the nuclear charge Z. They are given by

$$A(nl, n'l') = \sum_{n_1 n_1' n_1'' n_1'''} a_{ln_1}\, a_{ln_1''}\, a'_{l',n_1'}\, a'_{l',n_1'''}\, A(nn_1 n_1'',\ n'n_1' n_1''), \qquad (13)$$

$$B(nl, n'l') = \sum_{n_1 n_1' n_1'' n_1'''} a_{ln_1}\, a_{ln_1''}\, a'_{l',n_1'}\, a'_{l',n_1'''}\, B(nn_1 n_1'',\ n'n_1' n_1'''), \qquad (14)$$

where

$$A(nn_1 n_1'',\ n'n_1' n_1''') = \sum_{m=0}^{n-1} (2 - \delta_{m,0})\, \frac{1}{4} NN'' \sum_{\gamma,\gamma''} H(\gamma)\, H(\gamma'')$$

$$\times \sum_{\beta=0}^{2} D(\gamma\gamma'', \beta) \left[-\tau - \binom{\sigma_0}{2} \right], \qquad (15)$$

$$B(nn_1 n_1'',\ n'n_1' n_1''') = \sum_{m=0}^{n-1} (2 - \delta_{m,0})\, \frac{1}{4} NN'' \sum_{\gamma,\gamma''} H(\gamma)\, H(\gamma'')$$

$$\times \sum_{\beta=0}^{2} D(\gamma\gamma'', \beta) \left\{ -\left[\tau + \binom{\sigma_0}{2} \right] ln \left(\frac{2nn'}{n' - n} \times \frac{\mu}{M} \right)^2 + S(2, \tau, -1) \right.$$

$$\left. - \binom{\sigma_0}{2} S(1, \tau - 1, 0) - \sum_{\sigma=0}^{\sigma_0/2} \binom{\sigma_0}{2\sigma} S(0, \sigma - 2, -\tau + 1) \right\} \qquad (16)$$

In these expressions

$$N = \left[\frac{4nn'}{(n + n')^2} \right]^{m+2} \times \left[\frac{(n_1 + m)!}{n_1!} \times \frac{(n_2 + m)!}{n_2!} \times \frac{(n_1' + m)!}{n_1'!} \times \frac{(n_2' + m)!}{n_2'!} \right]^{1/2}, \qquad (17)$$

and N'' is obtained from N by replacing n_1 and n_1' by n_1'' and n_1''' respectively. γ and γ'' have been defined before, and

$$H(\gamma) = \left(\frac{-2n'}{n+n'}\right)^{\nu_1+\nu_2} \times \left(\frac{-2n}{n+n'}\right)^{\nu_1'+\nu_2'} \binom{n_1}{\nu_1}\binom{n_1'}{\nu_1'}\binom{n_2}{\nu_2}\binom{n_2'}{\nu_2'}$$

$$\times \frac{(m+\nu_1+\nu_1')!}{(m+\nu_1)!\,(m+\nu_1')!} \times \frac{(m+\nu_2+\nu_2')!}{(m+\nu_2)!\,(m+\nu_2')!}, \tag{18}$$

and a similar expression for $H(\gamma'')$. Similarly, the three values of $D(\gamma\gamma'', \beta)$, τ, and σ_0 are given by

$$D(\gamma\gamma'', 0) = 4\lambda_1''\lambda_2, \quad D(\gamma\gamma'', 1) = 2\,[\lambda_1''(\lambda_1 - \lambda_2) - \lambda_2(\lambda_1'' - \lambda_2'')],$$

$$D(\gamma\gamma'', 2) = -(\lambda_1 - \lambda_2)(\lambda_1'' - \lambda_2''), \quad \sigma_0 = \mu + (\lambda_1 - \lambda_2) - (\lambda_1'' - \lambda_2''),$$

$$\tau = \lambda_1 + \lambda_2'' + 2, \quad \text{if} \quad \lambda_1 - \lambda_2 \geq \lambda_1'' - \lambda_2''. \tag{19}$$

when $\lambda_1 - \lambda_2 \leq \lambda_1'' - \lambda_2''$, we let $\lambda_1 \rightleftarrows \lambda_2$, and $\lambda_1'' \rightleftarrows \lambda_2''$. Finally, the function $S(l, i, j)$ is defined by

$$S(l, i, j) = \sum_{i'=l}^{i} \binom{i}{i'} \frac{(-)^{i'}}{i'+j}. \tag{20}$$

It should be noted that $A(nl, n'l')$ is related to the optical oscillator strength, and it vanishes unless $l' = l \pm 1$. Although a simpler expression can be given for it, but this will not be done here.

It is easy to see from (13) and (14) that similar to (12) the cross section between the Stark levels is given by

$$Q_{BB}(nn_1, n'n_1') = [(Mz)/(\mu Z k_1)]^2\,[A(nn_1, n'n_1')\,ln\,(k_1/Z)^2 + B(nn_1, n'n_1')] \times (\pi a_0^2) \tag{21}$$

where (Cf. Eqs. (15) and (16))

$$A(nn_1, n'n_1') = A(nn_1n_1, n'n_1'n_1'), \quad B(nn_1, n'n_1') = B(nn_1n_1, n'n_1'n_1') \tag{22}$$

Finally for cross sections between principal quantum numbers we can write

$$Q_{BB}(n,n') = [(Mz)/(\mu Z k_1)]^2 \; [A(n,n') \; ln(k_1^2/Z^2) + B(n,n')] \; \pi a_0^2, \tag{23}$$

where

$$A(n,n') = n^{-2} \sum_{l=0}^{n=0} \sum_{l'=0}^{n'=1} A(nl, n'l'), \tag{24}$$

$$B(n,n') = n^{-2} \sum_{l=0}^{n-1} \sum_{l'=0}^{n'-1} B(nl, n'l'). \tag{25}$$

If we express the incident energy in the threshold units, corresponding to (23) we can write

$$Q_{BB}(n,n') = [(Mz)/(\mu Z k_T)]^2 \; [A_T(n,n') \; ln(k_T^2/Z^2)$$

$$+ B_T(n,n')] \; \pi a_0^2, \tag{26}$$

where A_T and B_T are related to A and B by

$$A_T(n,n') = (1/n^2 - 1/n'^2)^{-1} \; A(n,n'), \tag{27}$$

$$B_T(n,n') = (1/n^2 - 1/n'^2)^{-1} \; [B(n,n') + ln(1/n^2 - 1/n'^2) \; A(n,n')] \tag{28}$$

The same relationship between A and A_T, and B and B_T hold when Equations (12) and (21) are expressed in terms of the threshold units of energy.

II. RESULTS AND DISCUSSION

For inelastic collisions between charged particles and atoms the criterion for validity of the Born approximation, at least for low lying levels of the atom, is that $v/v_0 >> 1$, with v the velocity of the charged particle and v_0 the orbital velocity of the atomic electron.[7] The Bethe approximation consists in approximating (2) by (11), and is therefore valid when $(M/\mu)\,\Delta E/k_1^2 << 1$. Consequently when the incident particle is an electron, the validity criterion is the same for both approximations. From practical point of view this is useful, since agreement between Q_B and Q_{BB} can be taken as a measure of the validity of the Born approximation. From the graphs and the tables that are presented below it can be seen that in all cases the two approximations become the same at about 20 threshold units of energy.

In the case of inelastic collision between charged particles and hydrogen-like ions one can represent the incident particle by a plane wave instead of the Coulomb wave function. From the properties of the Coulomb wave function one can see that this representation is valid at large distances from the ionic center and for $k_1 >> Z/a_0$, with Z the ionic charge. The plane wave representation of an incident particle for scattering from neutral atoms is also valid at large distances from the atomic center. The condition $k_1 >> Z/a_0$ is the same as the condition for validity of the Born and the Bethe approximations for low lying levels of the atom. It then becomes plausible that when Born-Bethe approximation becomes valid for scattering from the neutral atoms, the plane wave approximation becomes also valid for ionic scattering. This conclusion will be supported by the experiment which will be discussed below. Below a comparison of our calculation is made with the experimental results available in the literature.

In Fig. 1 the experimental results of Hils, Kleinpoppen and Koschmieder,[8] and Dance, Harrison and Smith[9] for 1s-2s transition in H and He^+ are shown. Shown is also Q_{BB} (1s,2s). The experimental results on He^+ approach faster to their asymptote than those on H. It can be argued that in He^+ case the orbital electron is bounded more firmly to its nucleus compared to the hydrogen case with the result that the electronic orbit in He^+ is less perturbed by the incident electron, and the Born approximation is better satisfied. A striking feature of Fig. 1 is that the two sets of the experimental results with the particular scaling shown display similar behavior.

The measurements of Lichten et al,[10] and Stebbing et al[11] on 1s-2s transition in H precedes those of Hils et al., but they will not be reported here.

In Fig. 2 the experimental results of Fite and Brackmann[12] for 1s-2p transition in H is shown and compared to Q_{BB} (1s,2p).

In Fig. 3 Q_B and Q_{BB} are compared for transitions from $n=5$ to $n' = 9$, and from $n = 7$ to $n' = 8$. It is clear that above 20 threshold units the two approximations give the same results.

Transitions for $n = 1$ to $n' = 2 - 15$ are tabulated in Table I. Transitions for $n = 2$ to $n' = 3 - 12$ are tabulated in Table II. Transitions for $n = 3$ to $n' = 4 - 11$ and $n = 4$ to $n' = 5 - 10$ are tabulated in Table III. Transitions for $n = 5$ to $n' = 6 - 9$, $n = 6$ to $n' = 7, 8$, and $n = 7$ to $n' = 8$ are tabulated in Table IV. In all the tables the incident energies are in threshold units. Contribution to the Born calculation for the transitions discussed here has been made by many workers whose names are listed at the heading of each table. In addition, in a previous paper by the

author[13] cross sections for the transitions $n = 3$ to $n' = 5, 6, 7, 8$; $n = 4$ to $n' = 5, 6$; and $n = 5$ to $n' = 6$ are given, without giving the corresponding cross sections between the azimuthal quantum numbers. Similarly, Vainshteim[14] has tabulated cross sections for the s to s, s to p, and s to d transitions belonging to all the principal quantum numbers between and including $n = 1$ to $n' = 9$.

Because of space limitation it is not practical to give here the cross sectional values for transitions between all the azimuthal quantum numbers belonging to the principal quantum numbers listed in Tables I-IV. In addition, some of these transitions have been given by previous workers. Instead, with regard to the more important transitions, we tabulate here transitions between the azimuthal quantum numbers for $n = 3$ to $n' = 6$, $n = 5$ to $n' = 6$, $n = 6$ to $n' = 7$, and $n = 7$ to $n' = 8$. These are given in Tables V through VIII. In these tables for each particular transition the Bethe formula, Eq. (12), can be constructed by using two values for $Q_B(nl, n'l')$ for two of the energy parameters given and values for $Q_B(nl, n'l')$ for other incident energies can be obtained with moderate accuracy.

The results presented here in Tables I through VIII are in agreement with those of other workers except for the transitions 6s → 7p, 6s → 8p, 7s → 8p, 6s → 7d, 6s → 8d, and 7s → 8d, where the present results are in marked disagreement with those of Vainshtein. As an example, for 7s to 8d transition and $k_T^2 = 5$, the value for the cross section given here is $1.37 \times 10^4 \pi a_0^2$ (Cf. Table VIII),

while that of Vainshtein is $5.78 \times 10^2 \pi a_0^2$. In support of the results presented here it can be said that all the values have been obtained by a single general formula, and the chances of a systematic error is small. Furthermore, the consistency of the Born results with Bethe formula at sufficiently high energy for transitions between the principal quantum numbers has been verified.

It should be emphasized that Born cross sections given in Table I through VIII are for the particular case of an electron incident on an hydrogen atom. They are not valid for heavy charged particles, or nuclear charges different from 1. However, the Bethe formula whose coefficients for a number of transitions between the principal quantum numbers are given in Table IX is applicable to heavy charged particles and different nuclear charges. Nevertheless, the condition $(M/\mu)(\Delta E/E) \ll 1$ implies that the Bethe approximation becomes valid at much higher energies for the heavy charged particles in contrast to the electron case. It should be emphasized that neither in an exact solution nor in the Born approximation a simple dependence on the Z for the cross section can not be found. It is only in the Born-Bethe approximation that a simple Z dependence through (12), (21), and (22) is given.

In Table IX the coefficients $n'^3 A(n, n')$ and $n'^3 B(n, n')$ are tabulated when n ranges from 1 to 9, and n' as indicated. Theoretically these coefficients tend to a constant as n' becomes large. This can be seen by noticing that the asymptotic form of the radial part of an hydrogen eigenfunction is given by[3]

$$R_{nl}(Z\,r) \sim \frac{2\,(Z/n)^{3/2}}{\sqrt{Z\,r}}\,J_{2l+1}\left(\sqrt{8\,Z\,r}\right)\,,\quad r \ll n^2/Z,\quad l \ll n. \tag{29}$$

The largest contribution to the total cross section between n and n' is from those l and l' that their orbits have the largest overlap. As n' increases, the azimuthal quantum numbers that are much less than n' become more important and (29) indicates that the total cross section decreases as $1/n'^{3}$.

Study of Table IX indicates that as n and n' become large, the B coefficient becomes substantially larger than A. This implies that the non-optically allowed transitions become more important at high n and n'.

Concerning the numerical integration required in the Born approximation, it should be pointed out that this integration becomes more difficult for transitions in which $|l - l'| = 1$, where the existence of the $1/r^2$ scattering potential makes the integrand in (1) behave as $1/q$ for small values of q.

ACKNOWLEDGEMENTS

I am indebted to R. Baxter for programming of the numerical integration, and for the numerical evaluation of the A and B coefficients.

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FIGURE CAPTIONS

Fig. 1. A comparison of the Born-Bethe approximation for the 1s-2s excitation with the experimental results. The data of Hils et al. are for H, and those of Dance et al. are for He^+.

Fig. 2. A comparison of the Born-Bethe approximation for the 1s-2p excitation with the experimental results of Fite and Brackmann for H excitation.

Fig. 3. A comparison of the Born-Bethe, and Born, approximations for $n = 5$ to $n' = 9$, and $n = 7$ to $n' = 8$ transitions.

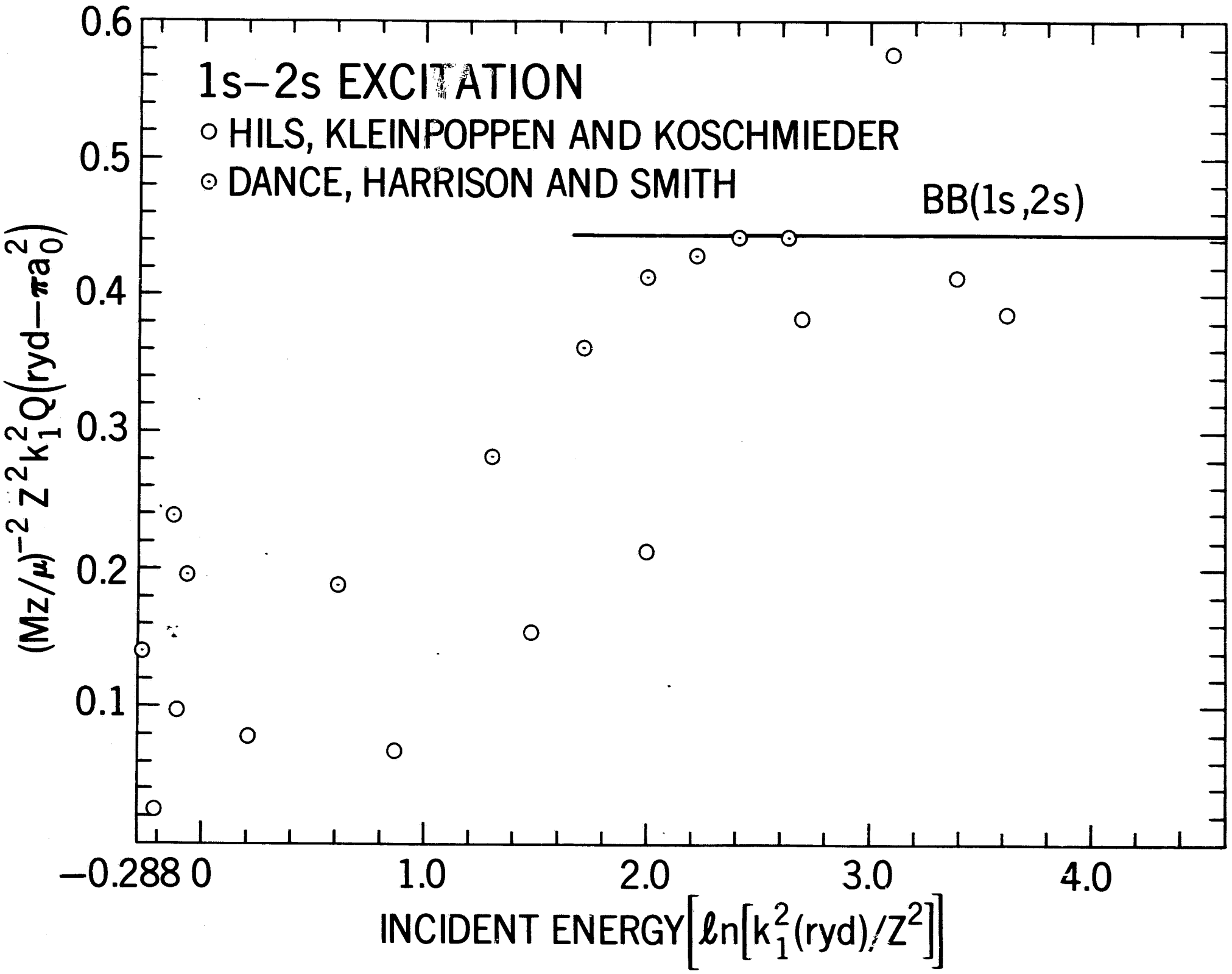
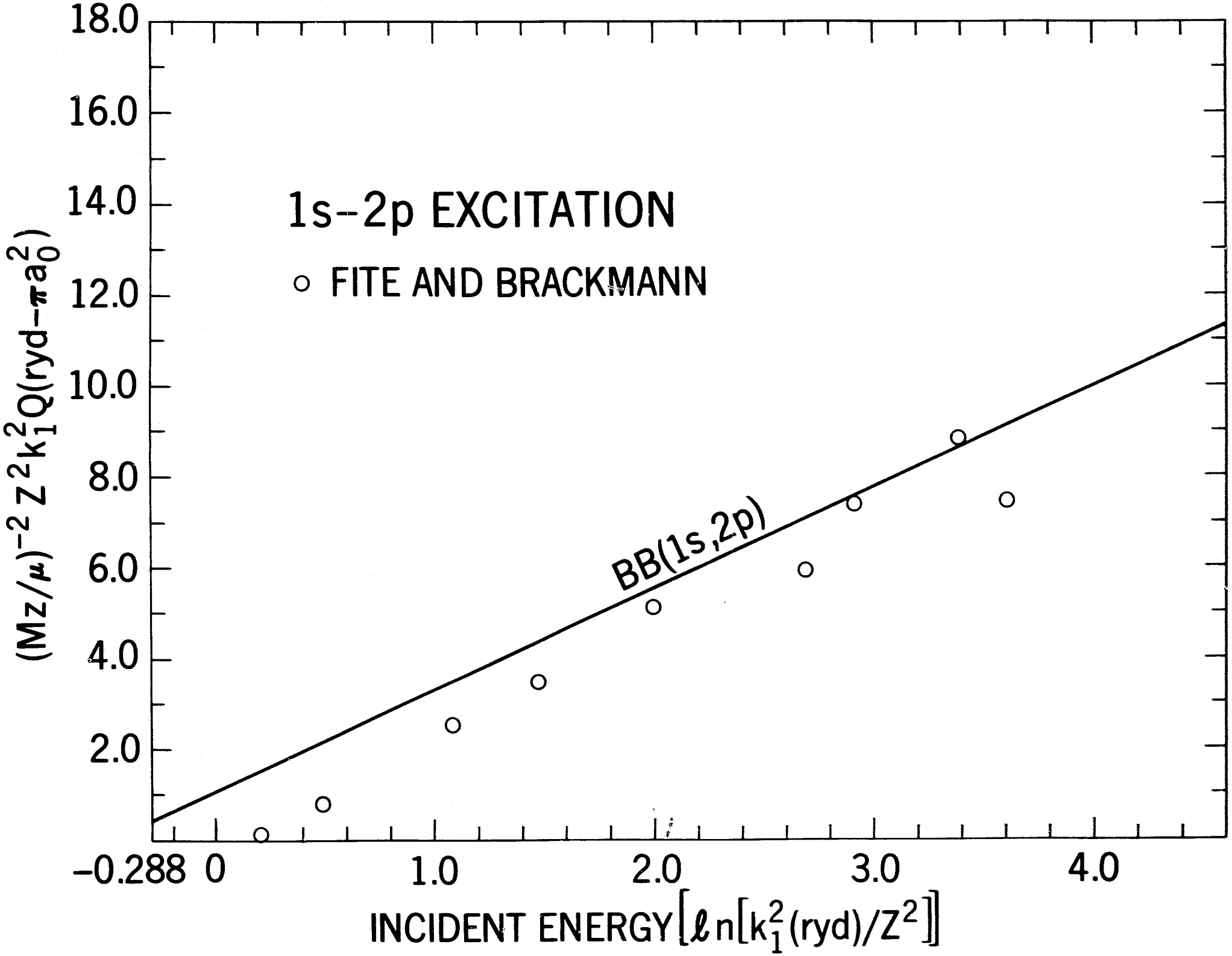

1s–2s EXCITATION
HILS, KLEINPOPPEN AND KOSCHMIEDER
DANCE, HARRISON AND SMITH
BB(1s,2s)
0.6
0.5
0.4
0.3
0.2
0.1
$(Mz/\mu)^{-2}\, Z^2 k_1^2 Q(ryd - \pi a_0^2)$
−0.288 0
1.0
2.0
3.0
4.0
INCIDENT ENERGY $\left[\ell n\left[k_1^2(ryd)/Z^2\right]\right]$

Fig. 1.

Fig. 2.


1s--2p EXCITATION
○ FITE AND BRACKMANN
BB(1s,2p)
$(Mz/\mu)^{-2} Z^2 k_1^2 Q(ryd-\pi a_0^2)$
INCIDENT ENERGY $[\ell n[k_1^2(ryd)/Z^2]]$
18.0
16.0
14.0
12.0
10.0
8.0
6.0
4.0
2.0
-0.288
0
1.0
2.0
3.0
4.0

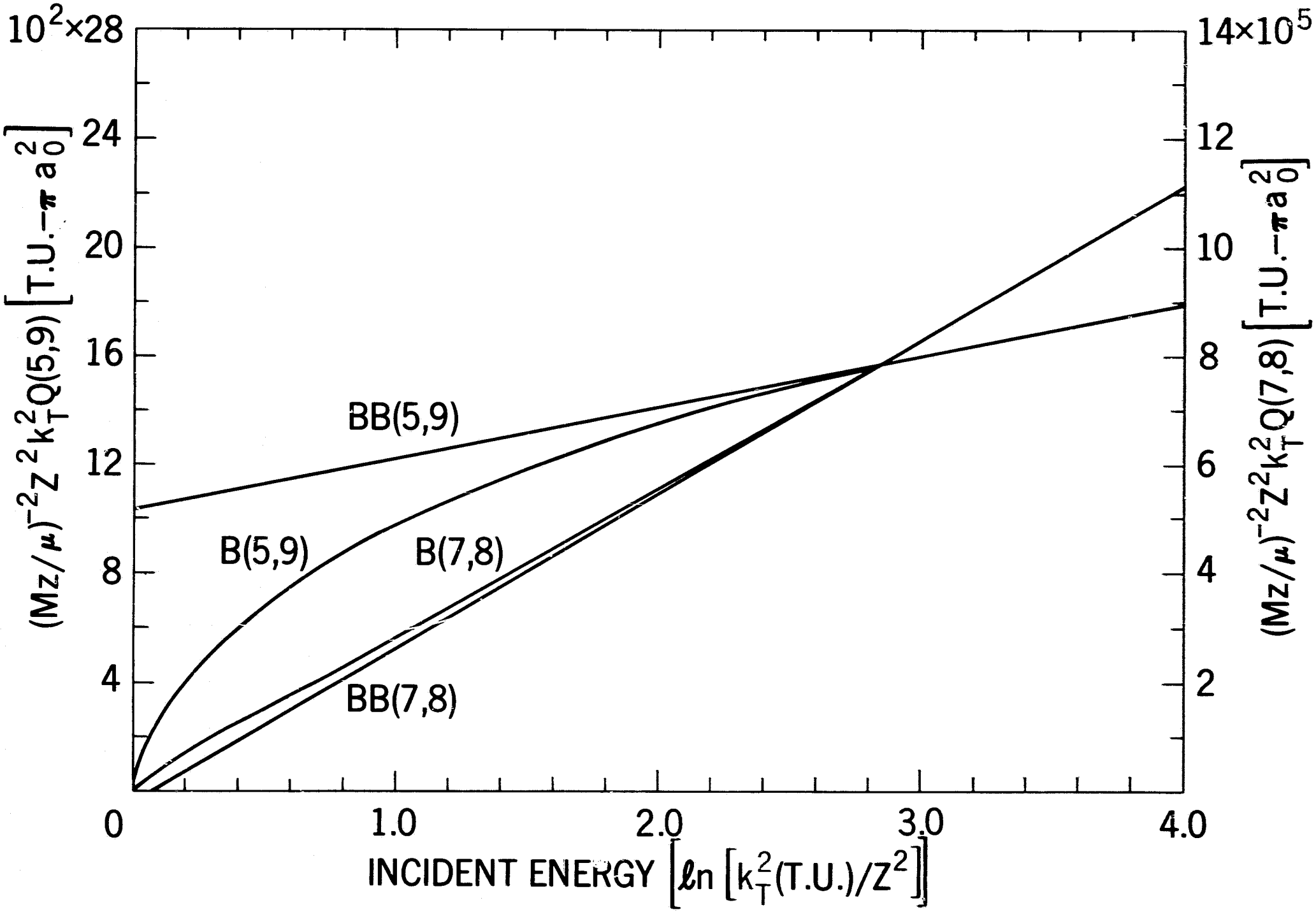

$10^2 \times 28$
24
20
16
12
8
4
14×10^5
12
10
8
6
4
2
BB(5,9)
B(5,9)
B(7,8)
BB(7,8)
$(Mz/\mu)^{-2} Z^2 k_T^2 Q(5,9) \left[T.U. - \pi a_0^2\right]$
$(Mz/\mu)^{-2} Z^2 k_T^2 Q(7,8) \left[T.U. - \pi a_0^2\right]$
0
1.0
2.0
3.0
4.0
INCIDENT ENERGY $\left[\ell n \left[k_T^2 (T.U.)/Z^2\right]\right]$

Fig. 3.

TABLE I. Total cross sections in the Born approximation and in units of πa_0^2 for electron impact excitation of atomic hydrogen for transitions n = 1 to n' = 2–15. k_T^2 is the impact energy in threshold units. The errors due to the numerical integration is much less, and always less, than 1%, except those cases indicated by the asterisks where the error may lie between 1 to 3 percent. 9.11-3 means 9.11×10^{-3}, etc. The n = 1 to n' = 2–6 of these transitions were first calculated by R. McCarroll, Proc. Phys. Soc. (London) A70, 460 (1957).

k_T^2 \ n'	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1.21	1.108	0.2162	0.0799	0.0386	0.0216	0.0134	8.84-3	6.16-3	4.46-3	3.34-3	2.56-3	2.01-3	1.61-3	1.30-3
1.44	1.383	0.2641	0.0969	0.0466	0.0261	0.0161	1.07-2	7.41-3	5.37-3	4.02-3	3.08-3	2.42-3	1.93-3	1.57-3
1.96	1.538	0.2835	0.1028	0.0492	0.0275	0.0169	1.12-2	7.78-3	5.63-3	4.21-3	3.23-3	2.53-3	2.02-3	1.64-3
2.56	1.506	0.2702	0.0972	0.0463	0.0258	0.0159	1.05-2	7.30-3	5.28-3	3.95-3	3.03-3	2.37-3	1.90-3	1.54-3
3.24	1.414	0.2483	0.0888	0.0422	0.0235	0.0144	9.53-3	6.62-3	4.79-3	3.58-3	2.75-3	2.15-3	1.72-3	1.39-3
4.00	1.306	0.2255	0.0802	0.0380	0.0211	0.0130	8.57-3	5.95-3	4.31-3	3.22-3	2.47-3	1.93-3	1.54-3	1.25-3
6.25	1.051	0.1760	0.0620	0.0293	0.0162	9.97-3	6.57-3	4.56-3	3.30-3	2.46-3	1.89-3	1.48-3	1.18-3	9.59-4
9.00	0.851	0.1396	0.0489	0.0230	0.0127	7.82-3	5.15-3	3.57-3	2.58-3	1.93-3	1.48-3	1.16-3	9.26-4	7.51-4
12.25	0.701	0.1131	0.0394	0.0185	0.0103	6.29-3	4.14-3	2.87-3						
16	0.586	0.0935	0.0325	0.0153	8.43-3	5.17-3	3.40-3	2.36-3	1.71-3*	1.28-3*	9.78-4*	7.66-4*	6.12-4*	4.96-4*
25	0.428	0.0672	0.0233	0.0109	6.01-3	3.68-3	2.42-3	1.68-3	1.22-3*	9.13-4*	7.00-4*	5.48-4*	4.38-4*	3.55-4*
36	0.327	0.0508	0.0175	8.18-3	4.51-3	2.76-3	1.82-3	1.26-3						
49	0.259	0.0399	0.0137	6.40-3	3.53-3	2.16-3	1.42-3	9.83-4						
64	0.211	0.0322	0.0111	5.16-3	2.84-3	1.74-3	1.14-3	7.91-4						
81	0.175	0.0266	9.11-3	4.25-3	2.34-3	1.43-3	9.40-4	6.52-4						
100	0.148	0.0224	7.66-3	3.57-3	1.97-3	1.20-3	7.90-4	5.47-4						
225	0.0765	0.0114	3.88-3	1.81-3	9.93-4	6.07-4	3.99-4	2.77-4						
400	0.0473	0.0070	2.38-3	1.11-3	6.07-4	3.70-4	2.43-4	1.69-4						

TABLE II. Definitions are the same as in Table I, but the transitions are n = 2 to n' = 3–12. The 2s to n' = 3–10 cases were first calculated by T. J. M. Boyd, Proc. Phys. Soc. (London) 72, 523 (1958), and the 2p to n' = 3 – 10 cases were first calculated by D. McCrea and T. V. M. McKirgan, Proc. Phys. Soc. (London) 75, 235 (1960).

k_T^2 \ n'	3	4	5	6	7	8	9	10	11	12
1.21	53.64	10.60	4.05	2.03	1.172	0.744	0.504	0.358	0.264	0.200
1.44	66.38	12.68	4.78	2.37	1.366	0.865	0.585	0.415	0.306	0.232
1.96	73.01	13.19	4.87	2.39	1.367	0.862	0.581	0.412	0.303	0.229
2.56	70.99	12.28	4.46	2.17	1.236	0.777	0.523	0.370	0.272	0.206
3.24	66.33	11.08	3.97	1.92	1.091	0.685	0.460	0.325	0.239	0.181
4.00	61.04	9.91	3.52	1.69	0.959	0.601	0.403	0.285	0.209	0.158
6.25	48.84	7.52	2.62	1.25	0.705	0.441	0.295	0.208	0.152	0.115
9.00	39.40	5.85	2.01	0.96	0.537	0.335	0.224	0.158	0.115	0.087
12.25	32.34	4.67	1.59	0.75	0.422	0.263	0.142	0.100	0.073	0.055
16	27.00	3.82	1.29	0.61	0.341	0.212	0.099*	0.069*	0.051*	0.038*
25	19.67	2.70	0.91	0.42	0.237	0.147				
36	15.01	2.02	0.67	0.31	0.175	0.109				
49	11.86	1.57	0.52	0.24	0.135	0.083				
64	9.64	1.26	0.41	0.193	0.107	0.066				
81	8.00	1.03	0.34	0.158	0.088					
100	6.76	0.87	0.28	0.132	0.073					
225	3.48	0.43	0.141	0.065	0.036					
400	2.15	0.26	0.085	0.039	0.022					

TABLE III. Definitions are the same as in Table I, but the transitions are n = 3 to n' = 4–11, and n = 4 to n' = 5–10. The n = 3 to n' = 4, and 3s to 5p, 3p to 5d, and 3d to 5f cases were first calculated by G. C. McCoyd et al., Phys. Rev. 119, 149 (1960). Similarly, the 4s → 5p, 4p → 5d, 4d → 5f, 4f → 5g, 4s → 6p, and 4f → 6g cases were first calculated by L. Fisher et al., Phys. Rev. 119, 153 (1960).

	n = 3								n = 4					
k_T^2 \ n'	4	5	6	7	8	9	10	11	5	6	7	8	9	10
1.2	513.7	99.3	38.00	19.15	11.20	7.191	4.925	3.536	2553	481.9	183.1	92.2	54.12	34.93
1.4	641.8	120.4	45.36	22.66	13.18	8.431	5.760	4.128	3263	599.5	224.0	111.7	65.09	41.80
1.6	700.9	127.9	47.57	23.59	13.66	8.713	5.940	4.250	3615	645.8	237.7	117.4	68.03	43.51
1.8	727.9	129.6	47.69	23.50	13.56	8.629	5.872	4.197	3792	660.1	239.8	117.6	67.78	43.21
2.0	737.5	128.5	46.85	22.98	13.22	8.393	5.704	4.073	3871	658.3	236.5	115.2	66.16	42.1
2.5	726.5	121.2	43.38	21.07	12.05	7.624	5.168	3.683	3860	625.6	219.9	105.9	60.33	38.2
3.0	696.6	112.4	39.68	19.14	10.90	6.881	4.656	3.314	3730	582.3	201.4	96.1	54.51	34.4
4.0	628.9	96.6	33.48	16.00	9.06	5.700	3.848	2.734	3399	502.3	169.9	80.2	45.1	28.4*
5.0	568.5	84.4	28.87	13.71	7.74	4.854	3.272	2.323	3091	439.2	146.3	68.5	38.4	24.1*
6.0	518.0	74.9	25.39	12.00	6.75	4.230	2.848	2.02	2828	390.0	128.5	59.8	33.4	20.9*
8.0	440.6	61.3	20.52	9.64	5.40	3.375	2.269	1.61	2419	319.4	103.6	47.8	26.6*	
10.0	384.7	52.2	17.28	8.08	4.52	2.818	1.892	1.34*	2119	271.4	87.0	40.0	22.2*	
15.0	295.4	38.4	12.53	5.81	3.24	2.013	1.351	0.96*	1636	199.4	62.8	28.6		
20.0	242.3	30.6	9.90	4.58	2.54	1.578	1.06*		1346	159.0	49.5	22.5*		
50.0	123.1	14.5	4.59	2.11*	1.16*	0.72*								
100.0	71.5	8.1	2.52	1.15*										
200.0	40.7	4.5*	1.38*											
500.0	18.9													
1000.0	10.5*													

TABLE IV. Definitions are the same as in Table I, but the transitions are n = 5 to n' = 6–9, n = 6 to n' = 7, 8, and n = 7 to n' = 8. The cases 5s → 6p, 5p → 6d, 5d → 6f, 5f → 6g, and 5g → 6h were first calculated by S. N. Milford et al., Phys. Rev. 120, 1715 (1960).

	n = 5				n = 6		n = 7
k_T^2 \ n'	6	7	8	9	7	8	8
1.2	8.757 + 3	1.623 + 3	6.122 + 2	3.080 + 2	2.378 + 4	4.345 + 3	5.508 + 4
1.4	1.148 + 4	2.075 + 3	7.689 + 2	3.821 + 2	3.192 + 4	5.694 + 3	7.553 + 4
1.6	1.291 + 4	2.270 + 3	8.272 + 2	4.067 + 2	3.644 + 4	6.326 + 3	8.746 + 4
1.8	1.369 + 4	2.344 + 3	8.417 + 2	4.103 + 2	3.90 + 4	6.601 + 3	9.47 + 4
2.0	1.408 + 4	2.354 + 3	8.35 + 2	4.040 + 2	4.05 + 4	6.68 + 3	9.91 + 4
2.5	1.423 + 4	2.262 + 3	7.83 + 2	3.735 + 2	4.15 + 4	6.50 + 3	10.34 + 4
3.0	1.387 + 4	2.118 + 3	7.20 + 2	3.40 + 2	4.07 + 4	6.14 + 3	10.18 + 4
4.0	1.28 + 4	1.839 + 3	6.10 + 2	2.84 + 2*	3.79 + 4	5.38 + 3	9.57 + 4*
5.0	1.17 + 4	1.61 + 3	5.27 + 2*	2.43 + 2*	3.50 + 4	4.75 + 3	8.88 + 4
6.0	1.08 + 4	1.43 + 3	4.64 + 2*	2.13 + 2*	3.23 + 4*	4.25 + 3	8.25 + 4*
8.0	9.25 + 3	1.18 + 3	3.72 + 2*				
10.0	8.13 + 3	1.00 + 3*	3.13 + 2*		2.47 + 4		6.36 + 4*
15.0							
20.0	5.21 + 3				1.60 + 4		4.18 + 4*
50.0							
100.0							
200.0							
500.0							
1000.0							

TABLE V. Total cross sections in units of πa_0^2 and in the Born approximation for transitions from the optical levels of n = 3 to the optical levels of n = 6 in atomic hydrogen induced by electron impact. *l* and *l'* are the initial and final azimuthal quantum numbers.

l	0	0	0	0	0	0	1	1	1
l'	0	1	2	3	4	5	0	1	2
$k_T^2 = 5$	3.35	6.68	6.13	1.64	4.50	1.08	0.571	4.14	11.3
$k_T^2 = 10$	1.80	5.36	3.17	8.33	2.25	8.42	3.85	2.20	8.03
$k_T^2 = 20$	0.928	3.70	1.60	0.418	1.12	2.71	2.48	1.13	5.18
$k_T^2 = 50$	0.378	2.02	0.646	0.167	0.450	0.108	0.130	0.459	2.67
l	1	1	1	2	2	2	2	2	2
l'	3	4	5	0	1	2	3	4	5
$k_T^2 = 5$	2.97	6.01	1.08	0.125	0.562	3.29	18.0	8.81	0.881
$k_T^2 = 10$	1.49	3.04	0.544	0.065	0.306	1.71	11.5	4.89	0.455
$k_T^2 = 20$	0.744	1.52	0.272	0.033	0.168	0.871	6.90	2.57	0.230
$k_T^2 = 50$	0.298	0.609	0.109	0.013	0.076	0.352	3.35	1.05	0.092

TABLE VI. Total cross sections for transitions from the optical levels of n = 5 to the optical levels of n = 6. Definitions are the same as in Table V.

l	0	0	0	0	0	0	1	1	1	1	1	1	2	2	2
l'	0	1	2	3	4	5	0	1	2	3	4	5	0	1	2
$k_T^2 = 5$	1.79 + 3	1.38 + 3	2.21 + 3	1.63 + 3	1.06 + 3	4.75 + 2	2.39 + 2	2.16 + 3	1.94 + 3	2.56 + 3	1.60 + 3	6.59 + 2	8.44 + 1	3.23 + 2	2.11 + 3
$k_T^2 = 10$	1.06 + 3	1.46 + 3	1.35 + 3	8.33 + 2	5.32 + 2	2.38 + 2	1.98 + 2	1.28 + 3	1.83 + 3	1.49 + 3	8.12 + 2	3.30 + 2	5.03 + 1	2.08 + 2	1.22 + 3
$k_T^2 = 20$	5.69 + 2	1.20 + 3	7.39 + 2	4.19 + 2	2.66 + 2	1.19 + 2	1.52 + 2	6.92 + 2	1.42 + 3	7.98 + 2	4.07 + 2	1.65 + 2	2.74 + 1	1.37 + 2	6.54 + 2

l	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4
l'	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5
$k_T^2 = 5$	3.14 + 3	3.36 + 3	1.29 + 3	1.93 + 1	8.21 + 1	2.71 + 2	1.96 + 3	5.85 + 3	3.64 + 3	3.75 + 0	1.48 + 1	4.42 + 1	1.69 + 2	1.51 + 3	1.18 + 4
$k_T^2 = 10$	2.84 + 3	1.89 + 3	6.52 + 2	9.84 + 0	4.49 + 1	1.56 + 2	1.12 + 3	4.86 + 3	1.99 + 3	1.88 + 0	7.46 + 0	2.31 + 1	9.02 + 1	8.39 + 2	8.86 + 3
$k_T^2 = 20$	2.13 + 3	1.00 + 3	3.27 + 2	4.95 + 0	2.36 + 1	9.33 + 1	5.93 + 2	3.47 + 3	1.04 + 3	9.41 − 1	3.74 + 0	1.19 + 1	4.98 + 1	4.40 + 2	5.95 + 3

TABLE VII. Total cross sections for transitions from the optical levels of n = 6 to the optical levels of n = 7. Definitions are the same as in Table V.

l	0	0	0	0	0	0	0	1	1	1	1	1	1	1
l'	0	1	2	3	4	5	6	0	1	2	3	4	5	6
$k_T^2 = 5$	5.53 + 3	3.44 + 3	5.89 + 3	4.47 + 3	3.09 + 3	1.88 + 3	7.19 + 2	7.27 + 2	6.68 + 3	4.51 + 3	6.50 + 3	4.50 + 3	2.54 + 3	9.21 + 2
$k_T^2 = 10$	3.37 + 3	3.85 + 3	3.75 + 3	2.30 + 3	1.55 + 3	9.42 + 2	3.60 + 2	6.16 + 2	4.09 + 3	4.53 + 3	3.93 + 3	2.30 + 3	1.27 + 3	4.61 + 2
$k_T^2 = 20$	1.84 + 3	3.30 + 3	2.09 + 3	1.16 + 3	7.73 + 2	4.71 + 2	1.80 + 2	4.90 + 2	2.24 + 3	3.65 + 3	2.14 + 3	1.15 + 3	6.36 + 2	2.30 + 2
l	2	2	2	2	2	2	2	3	3	3	3	3	3	3
l'	0	1	2	3	4	5	6	0	1	2	3	4	5	6
$k_T^2 = 5$	2.96 + 2	1.07 + 3	6.59 + 3	6.46 + 3	8.43 + 3	4.62 + 3	1.53 + 3	8.06 + 1	3.29 + 2	9.92 + 2	6.40 + 3	1.07 + 4	1.06 + 4	3.43 + 3
$k_T^2 = 10$	1.83 + 2	7.08 + 2	3.95 + 3	6.38 + 3	4.95 + 3	2.35 + 3	7.66 + 2	4.15 + 1	1.86 + 2	5.86 + 2	3.78 + 3	9.89 + 3	6.07 + 3	1.74 + 3
$k_T^2 = 20$	1.01 + 2	4.81 + 2	2.14 + 3	5.03 + 3	2.67 + 3	1.18 + 3	3.83 + 2	2.09 + 1	9.88 + 1	3.65 + 2	2.04 + 3	7.49 + 3	3.22 + 3	8.70 + 2
l	4	4	4	4	4	4	4	5	5	5	5	5	5	5
l'	0	1	2	3	4	5	6	0	1	2	3	4	5	6
$k_T^2 = 5$	2.08 + 1	8.14 + 1	2.34 + 2	7.99 + 2	5.88 + 3	1.91 + 4	1.06 + 4	4.79 + 0	1.73 + 1	4.32 + 1	1.20 + 2	4.76 + 2	4.37 + 3	3.60 + 4
$k_T^2 = 10$	1.04 + 1	4.13 + 1	1.26 + 2	4.38 + 2	3.39 + 3	1.62 + 4	5.87 + 3	2.40 + 0	8.68 + 0	2.17 + 1	6.23 + 1	2.49 + 2	2.46 + 3	2.76 + 4
$k_T^2 = 20$	5.22 + 0	2.07 + 1	6.55 + 1	2.52 + 2	1.81 + 3	1.16 + 4	3.08 + 3	1.20 + 0	4.34 + 0	1.09 + 1	3.18 + 1	1.34 + 2	1.30 + 3	1.88 + 4

TABLE VIII. Total cross sections for transitions between the azimuthal quantum numbers of n = 7, and the azimuthal quantum numbers of n' = 8. Definitions and terminologies are the same as in Table V.

l	0	0	0	0	0	0	0	0	1	1	1	1
l'	0	1	2	3	4	5	6	7	0	1	2	3
$k_T^2 = 5$	1.44 + 4	7.58 + 3	1.37 + 4	1.07 + 4	7.44 + 3	5.07 + 3	2.81 + 3	9.37 + 2	1.88 + 3	1.74 + 4	9.55 + 3	1.46 + 4
$k_T^2 = 10$	9.05 + 3	8.74 + 3	9.16 + 3	5.60 + 3	3.73 + 3	2.53 + 3	1.40 + 3	4.68 + 2	1.59 + 3	1.10 + 4	9.86 + 3	9.18 + 3
l	1	1	1	1	2	2	2	2	2	2	2	2
l'	4	5	6	7	0	1	2	3	4	5	6	7
$k_T^2 = 5$	1.07 + 4	6.71 + 3	3.53 + 3	1.14 + 3	8.44 + 2	2.96 + 3	1.73 + 4	1.23 + 4	1.83 + 4	1.17 + 4	5.62 + 3	1.69 + 3
$k_T^2 = 10$	5.49 + 3	3.36 + 3	1.77 + 3	5.69 + 2	5.45 + 2	1.96 + 3	1.07 + 4	1.29 + 4	1.12 + 4	5.96 + 3	2.81 + 3	8.47 + 2
l	3	3	3	3	3	3	3	3	4	4	4	4
l'	0	1	2	3	4	5	6	7	0	1	2	3
$k_T^2 = 5$	2.60 + 2	1.02 + 3	2.88 + 3	1.71 + 4	1.85 + 4	2.36 + 4	1.15 + 4	3.18 + 3	7.53 + 1	2.96 + 2	8.15 + 2	2.56 + 3
$k_T^2 = 10$	1.35 + 2	5.96 + 2	1.72 + 3	1.04 + 4	1.87 + 4	1.40 + 4	5.84 + 3	1.59 + 3	3.78 + 1	1.51 + 2	4.53 + 2	1.42 + 3
l	4	4	4	4	5	5	5	5	5	5	5	5
l'	4	5	6	7	0	1	2	3	4	5	6	7
$k_T^2 = 5$	1.65 + 4	3.02 + 4	2.82 + 4	7.86 + 3	2.24 + 1	8.12 + 1	2.04 + 2	5.51 + 2	1.99 + 3	1.48 + 4	5.19 + 4	2.61 + 4
$k_T^2 = 10$	9.87 + 3	2.84 + 4	2.63 + 4	3.99 + 3	1.12 + 1	4.07 + 1	1.03 + 2	2.94 + 2	1.06 + 3	8.69 + 3	4.46 + 4	1.47 + 4
l	6	6	6	6	6	6	6	6				
l'	0	1	2	3	4	5	6	7				
$k_T^2 = 5$	5.96 + 0	2.03 + 1	4.48 + 1	1.01 + 2	2.77 + 2	1.14 + 3	1.07 + 4	9.26 + 4				
$k_T^2 = 10$	2.99 + 0	1.02 + 1	2.25 + 1	5.11 + 1	1.43 + 1	5.86 + 2	6.11 + 3	7.25 + 4				

TABLE IX. Numerical values for $n'^3A(n, n')$ and $n'^3B(n, n')$ (Cf. Eq. (23)). The coefficients A(n, n') and B(n, n') are used when the incident energy is in rydberg units. When the incident energy is given in threshold units, Eq. (26) should be used in which A_T and B_T are given in terms of A and B through (27) and (28). Below, for each n and n' two numbers are given. The top number is n'^3 A(n, n') and the bottom number is n'^3 B(n, n').

n \ n'	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1.776+1	9.611	7.916	7.260	6.931	6.742	6.623	6.543	6.486	6.445	6.413	6.389	6.370	6.355	6.342	6.332	6.323	6.316	6.309
	1.224+1	1.071+1	1.006+1	9.765	9.605	9.510	9.448	9.406	9.376	9.353	9.336	9.323	9.313	9.304	9.298	9.292	9.287	9.283	9.280
2		4.982+2	1.629+2	1.064+2	8.590+1	7.592+1	7.022+1	6.661+1	6.418+1	6.245+1	6.117+1	6.020+1	5.944+1	5.884+1	5.835+1	5.795+1	5.762+1	5.734+1	5.710+1
		1.226+3	5.647+2	4.233+2	3.670+2	3.381+2	3.210+2	3.100+2	3.025+2	2.971+2	2.931+2	2.900+2	2.876+2	2.857+2	2.841+2	2.828+2	2.818+2	2.809+2	2.801+2
3			4.435+3	1.059+3	5.790+2	4.188+2	3.439+2	3.022+2	2.761+2	2.587+2	2.463+2	2.372+2	2.302+2	2.248+2	2.205+2	2.170+2	2.141+2	2.117+2	2.097+2
			1.521+4	5.145+3	3.322+3	2.639+3	2.298+3	2.099+3	1.972+3	1.884+3	1.822+3	1.775+3	1.739+3	1.711+3	1.688+3	1.670+3	1.654+3	1.642+3	1.631+3
4				2.306+4	4.460+3	2.135+3	1.411+3	1.087+3	9.113+2	8.037+2	7.323+2	6.822+2	6.455+2	6.176+2	5.959+2	5.787+2	5.647+2	5.532+2	5.436+2
				9.413+4	2.586+4	1.490+4	1.102+4	9.137+3	8.058+3	7.373+3	6.906+3	6.571+3	6.321+3	6.130+3	5.979+3	5.858+3	5.760+3	5.678+3	5.610+3
5					8.704+4	1.449+4	6.258+3	3.844+3	2.805+3	2.255+3	1.926+3	1.710+3	1.560+3	1.451+3	1.368+3				
					3.976+5	9.388+4	4.949+4	3.449+4	2.741+4	2.343+4	2.093+4	1.924+4	1.803+4	1.714+4	1.645+4				
6						2.651+5	3.944+4	1.571+4	9.085+3	6.329+3	4.907+3	4.068+3	3.532+3						
						1.313+6	2.763+5	1.355+5	8.988+4	6.890+4	5.730+4	5.009+4	4.500+4						
7							6.918+5	9.437+4	3.521+4	1.936+4	1.296+4								
							3.650+6	7.009+5	3.241+5	2.060+5	1.506+5								
8								1.607+6	2.045+5	7.227+4									
								8.920+6	1.590+6	6.752+5									
9									3.408+6										
									1.969+7										