Kohrmann, W. "Der Stromanstieg einer Townsend Entladung unter dem einfluss der raumladung". Zeitschrift fur Naturforschung. Vol. 19A No. 2 245-253)February, 1964).

NI

THE INCREASE IN CURRENT IN A TOWNSEND-DISCHARGE UNDER THE INFLUENCE OF SPACE-CHARGING

Starting from the basic equation (Towsend-differential equations, supplementary - and Poisson-equation) an equation system is developed, from which the position coordinate x is eliminated, and which connects the desired currents and $J_+(t)$ via the variable $J_-(t)$. After conversion of this equation system the usual procedures for solving the differential equations can be used.

In contrast to an earlier computation by the author, the process can be used for a number of initial conditions (lightning, constant irradiation, weak irradiation, static breakthrough, over - and under-voltage). The treatment of examples gives a survey on the diverse possibilities and shows the usefulness of the process. The transition of increases in current with varying initial conditions into a unified course (asymptotic form) is of interest; this transition is expected in theory and is confirmed by experiments.

In experimental investigations of the increase in current of a Towsend-discharge in time [2-5] it was ascertained that a characteristic increase occurs with the attainment of a certain current intensity. Here first of all, only slight deviations from the normal course are shown. With increasing current intensity the rise then becomes increasingly steeper. A cause for this characteristic rise was found, by space-charging the positive ions, to be the distorted field in which the electronic avalanches attain an increased concentration of gas.

One of these experimental examples[4] was already treated theoretically earlier; here, from the basic equations (Townsend's differential equations, Poisson-equation) a differential equation for the increase in current was extablished[1].

In the meantime similarly measured increases[6] in current were treated theoretically with regard to space charging. In[7] the movement away of ions was considered when calculating the increase in current; this made possible a thorough description of the increase in current from the start of the released electrons to the steep rise leading to the breakthrough.

I. EQUATIONS FOR THE INCREASE IN CURRENT

The derivation of the basic equations is undertaken by using the following simplifications.

1. In the homogenous field of a plate spark area, support current or support density are distributed evenly over the discharge area in radial direction. Thus only x appears as the position coordinate.

2. The voltage set on the discharge area remains constant in time; according to this, the decrease in voltage on the outer resistance must be negligibly small.

3. When being confined to small E/p- values (large pd) the expression

$\alpha/p = A(E/p - B)^2$ can be used for α/p

The field intensity-changes (called forth by the space charging of ions) should be slight; under these conditions the drift velocities v_- of the electrons and v_+ of the ions are assumed to be constant. - As an extreme case (say in the $y_{max.}$ discussed in Section 5) - the following values may appear:

$$E(0)/E(d) \approx 1{,}15, \quad \alpha(0)/\alpha(d) \approx 3.$$

4. The change in time of α in an interval of length T_- should be slight.

As a supplementary mechanism we assume photo effect on the cathode exclusively, as corresponds to the experimental examples. Here the coefficient $\gamma_{ph} = \delta/\alpha$ is set as constant.

On the basis of these simplifications the following equation system (1), (2), (3) is derived in Appendix I from the basic equations (Townsend differential equations, supplementary equation.). It brings the current constituents (current density) of the electrons $J_-(t)$ and the ions $J_+(t)$ flowing in the outer circle, in relationship with the current of electrons $j_-(0,t)$ leaving the cathode. A clear explanation of equations (1), (2), and (3) is obtained from the following model: all the ions, electrons and photons are generated at point $x=d-\alpha_0^{-1}$. The passage time of the electronic avalanches up to this point (when the avalanche develops without delay) is $T_{ph}=T_-\{1-(\alpha_0 d)^{-1}\}$. Equation (1) describes the development in time of $j_-(0,t)$ here $j_F(t)$ is the foreign current released by foreign irradiation on the cathode. In equation (2) the electronic current J (t) is represented, whilst the passage time of the electrons from the place of generation up to the anode is ignored. Equation (3) describes the development in time of the ion current J (t); it is gained from the movement of ions (with velocity v_+) from the point $x=d-\alpha_0^{-1}$ up to the cathode; for this the time $T_i=T_+\{1-(\alpha_0 d)^{-1}\}$ is necessary.

$$j_-(0,t)=j_F(t)+\mu_{ph}\,j_-(0,t-T_{ph})\exp\left[\int_0^d \alpha(x,t-T_{ph})\,dx-\alpha_0 d\right],\quad \mu_{ph}=\gamma_{ph}[\exp\alpha_0 d-1],\qquad (1)$$

$$J_-(t) = \frac{\exp \alpha_0 d}{\alpha_0 d}\, j_-(0, t-T_{ph}) \exp\left[\int_0^d \alpha(x, t-T_{ph})\, dx - \alpha_0 d\right], \qquad (2)$$

$$\frac{dJ_+(t)}{dt} = \frac{\exp \alpha_0 d}{T_+} \left\{ j_-(0, t-T_{ph}) \exp\left[\int_0^d \alpha(x, t-T_{ph})\, dx - \alpha_0 d\right] \right.$$

$$\left. - j_-(0, t-T_i-T_{ph}) \exp\left[\int_0^d \alpha(x, t-T_i-T_{ph})\, dx - \alpha_0 d\right]\right\}. \qquad (3)$$

In equations (1), (2), and (3) $\int_0 \alpha(x,t)\, dx$ is the modified gas concentration under the influence of the space charging; here we take over the relationship (4) already used, with $J_+(t)$ (see also [8]), which is obtained from the application of the Poisson-equation and the analytical expression for the α/p given above (K_+ see equation (3.1):

$$\int_0^d \alpha(x,t)\, dx - \alpha_0 d = K_+^2 J_+^2(t). \qquad (4)$$

Since the calculation with the functional equations (1), (2), and (3) cause several difficulties we wish to make a conversion to differential equations. For reasons of constancy we assume that $j_F(t)$ may only contain a δ-function with $t=0$ therefore we write $j_F(t) = q\,\delta(t) + j_F^*(t)$. By inserting (2) in (1) and using (4), first of all

$$J_-(t+T_{ph}) = J_-(t) \exp\{\varkappa + K_+^2 J_+^2(t)\} \qquad (5)$$
$$+ j_F^*(t) \frac{\exp \alpha_0 d}{\alpha_0 d} \exp\{K_+^2 J_+^2(t)\}.$$

is formed $(\varkappa = \ln \mu_{ph})$

By linear development of the term $J_-(t+T_{ph})$ and of the exponential function standing on the right at $J_-(t)$ one obtains equation (6). The factor next to $j_F^*(t)$ exp $\{K_+^2 J_+^2(t)\}$ may be made equal to 1, since the term $j_F^*(t)$ only plays a roll in the initial stage, in which no significant space charging has yet been formed. - The equation (7) results by introducing (2) and (3).

$$\frac{dJ_-(t)}{dt} = T_{ph}^{-1} J_-(t)\{\varkappa + K_+^2 J_+^2(t)\} + j_F^*(t)\frac{\exp a_0 d}{T_{ph}\, a_0 d}, \tag{6}$$

$$\frac{dJ_+(t)}{dt} = \frac{a_0 d}{T_+}\{J_-(t) - J_-(t-T_i)\}, \tag{7}$$

$$J_-(T_{ph}) = \frac{\exp a_0 d}{T_{ph}\, a_0 d}\, q, \quad J_+(T_{ph}) = 0. \tag{8}$$

The initial condition (8) for $J_+(t)$ is obtained from the model described above, which allows the generation of ions to be inserted at $t=T_{ph}$ The initial condition for $J_-(t)$ is obtained by the requirement that $j_-(0,t)$ for $t \to 0$ changes into the course which is free of space charging (see equation (2)).

In this way we have gained a system of differential equations for both functions which are of interest, $J_-(t)$ and $J_+(t)$ A deviation from the usual form of a differential equation is developed by the retarded term $J_-(t-T_i)$ in equation (7). - With (6) and (7) the theoretical curves applied in section 3 are calculated by using the method described in Appendix 3. With certain prerequisites, simple solutions are found, as is shown in the following section, Section II.

II. ASYMPTOTIC SOLUTIONS

It should be shown that all increases in current leading to a breakthrough with increasing current intensity pass over into a unified course. This course is independent of special conditions such as irradiationa and over-voltage.

We start out from the equation system (6), (7) and first of all we assume that $\varkappa \geqq 0$ It is seen that during the development of the discharge the electron current $J_-(t)$ becomes so large that the inequality

$$J_-(t) \gg J_-(t-T_i) \tag{9}$$

is fulfilled. Because of this

$$\frac{dJ_+(t)}{dt} = (\alpha_0 d/T_+)\, J_-(t)\,. \tag{10}$$

is developed from equation (7).

If an extremely unfavorable course of $j_F^*(t)$ is ignored, then the term with $j_F^*(t)$ in equation (6) can be neglected at this point in time. Then by introducing (10) in (6) the differential equation of 2nd degree for $J_+(t)$ result:

$$\frac{d^2J_+(t)}{dt^2} = T_{ph}^{-1}\frac{dJ_+(t)}{dt}\left\{\varkappa + K_+^2 J_+^2(t)\right\}. \tag{11}$$

In standardised form it goes (cp. Appendix 3)

$$\ddot{y}(\tau) = \dot{y}(\tau)\{\varkappa + y^2(\tau)\}. \qquad (12)$$

By integration via one obtains from (12) a differential equation of 1st degree.

$$\dot{y}(\tau) - \{\varkappa y(\tau) + \tfrac{1}{3} y^3(\tau)\}$$
$$= \dot{y}(\tau_1) - \{\varkappa y(\tau_1) + \tfrac{1}{3} y^3(\tau_1)\} = a. \qquad (13)$$

As can be seen in equation (13) without a doubt, with increasing y at a point $\tau = \tau_2$ the case arises of the members $\dot{y}(\tau_2)$ or $\{\varkappa y(\tau_2) + \tfrac{1}{3} y^3(\tau_2)\}$ being large against a. From this, for $\tau > \tau_2$ we can make the left side of (13) equal zero. The development cescribed above is always the case for $\varkappa \geq 0$ for $\varkappa < 0$ it is possible that in some cases the conditions required for this are satisfied. Then the following considerations are obviously true. The solutions of equation (13) (left side made equal to zero) with the initial condition $y(\tau_2) = b$ are:

$$y = b(3\varkappa)^{1/2}\{(3\varkappa + b^2)\exp[2\varkappa(\tau_2 - \tau)] - b^2\}^{-1/2}$$
$$= (3\varkappa)^{1/2}\{\exp[2\varkappa(\tau_\infty - \tau)] - 1\}^{-1/2}, \quad (\varkappa \neq 0), \qquad (14)$$

$$y = b\{1 - \tfrac{2}{3} b^2(\tau - \tau_2)\}^{-1/2}$$
$$= (\tfrac{3}{2})^{1/2}\{\tau_\infty - \tau\}^{-1/2}. \quad (\varkappa = 0). \qquad (15)$$

The left side of equations (14) and (15) shows clearly that $y(\tau)$ becomes singular for a certain value τ_∞ On the right side the equations are rewritten as an argument to $\tau_\infty - \tau$.

The treatment of the functions for increasing y given by (14) should now be discussed. The analytical development of the exponential function is (14) shows that with $\tau_\infty - \tau \rightarrow 0$ (i.e., for $y \rightarrow \infty$) all functions with $\varkappa \neq 0$ approach the function $(\varkappa = 0)$ given by (15) asymptotically. This behavior can also be explained with the aid of the differential equation (12): the factor $\varkappa + y^2$ shows that with increasing y the influence of $\varkappa$ on the course of the function diminishes.

In the equations given above, only the ion constituent of the current in each case was considered in each case. From

$$\dot{y}_a(\tau) \equiv z_a(\tau) = K_- J_-(t);$$

one obtains the electron constituent; it is understandable that the differential quotient $\dot{y}_a(\tau)$ has the same asymptotic behavior as $y_a(\tau)$ Thus these considerations are not only true for the ion constituent, but also for the total current.

It would now be interesting to study this asymptotic behavior with an experimental example. In continuous

UV-radiation of the catnode a large number of increases in current was measured[2]; during tnis the over-voltage (i.e., $x = \ln \mu_{ph}$) as parameter, was varied. As opposed to the original task, where the voltage was plotted at the zero point of the time axis, the individual curves in Figure 1 have been shifted in the direction of time.

In this way a large number of curves approaching each other is created, which merge together with large intensities of current. In addition in Figure 1 the course calculated in the asymptotic form

$$I(t) = F\{K_+^{-1} y_a(\tau) + K_-^{-1} z_a(\tau)\},$$

$$y_a(\tau) = (\tfrac{3}{2})^{1/2} \{\tau_\infty - \tau\}^{-1/2}, \quad z_a(\tau) = (\tfrac{3}{\mu})^{1/2} \{\tau_\infty - \tau\}^{-3/2}, \qquad (16)$$

(F = cross-section of the discharge) is marked in with a dotted line; in the last phase this curve merges with the courses of the current measured.

A further application of the asymptotic solution is found in [6] for describing the increase in current; here it is shown that increases in current oscillating statistically in the beginning change into a unified course under the influence of the space discharge.

III. EXPERIMENTAL EXAMPLES

a) Measurements with δ-shaped irradiation

As an example we shall consider an increase in current measured in hydrogen[9]. In contrast to the example treated[7] with $\mu_{ph} \approx 1$ here μ_{ph} is considerably different from 1. The calculation of the increase in current for these conditions is an interesting example for the operation of space charging. On curves placed together in Figure 2, the inter-play between $J_+(t)$ and $J_-(t)$ according to Equation (6) and (7) can be studied. Thus first of all the electron current $J_-(t)$ falls (because of $\varkappa<0$) and correspondingly $J_+(t)$ has a curved course with negative 2nd differential quotient. With $t=T_i$ the ions movements away are noticeable by a sharp bend. In order to lead the increase in current to the breakthrough, the factor

$$\ln \mu_{ph} + K_+^2 J_+^2 (t) = \varkappa + y^2(\tau) \tag{17}$$

in equation (6) must become positive (see also right scale in Figure 2). This occurs in the curves a and b, not however in curve c, which therefore doesn't lead to the breakthrough. The importance of the factor (17) for realizing the breakthrough was already indicated in[9].

In the constituents $J_+(t)$ and $J_-(t)$ are added to the total current, a relatively smooth course results, which does not reveal the details just described.

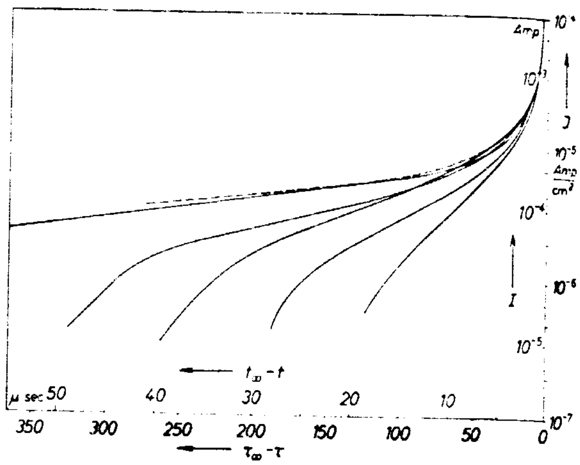
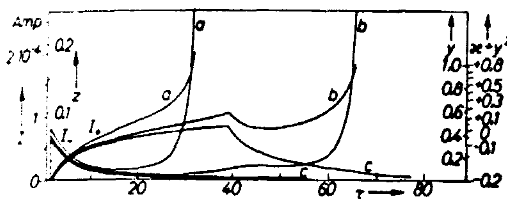


Figure 1

Experimental detection of the asymptotic behavior. Measurements in dry air[2] (Figure 2), data on the measurements. See Figure 4. By shifting the time scale the measurement curves ________ were brought for coverage in the last section. - - - - Current calculated in the asymptotic form according to Equation (16).

The curves a and b in Figure 3 reproduce the measured course qualitatively (cp. Figure 9 to 12 in [9]).

A direct comparison of the measured and calculated course of the current is rejected because of the difficulty in choosing the right parameter N and x. In addition in this case, with $y \approx 1$ the general validity range of our solutions is exceeded, so that the curves shown in Figure 3 only have a qualitative character in their last stretch.



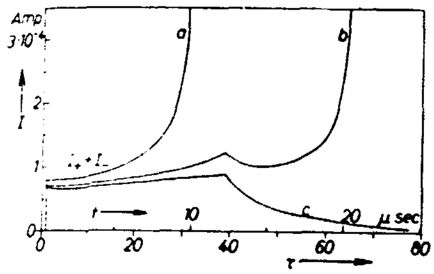


Figure 2 (above) and 3 (below)

Current calculated in hydrogen (p = 330 Torr, d = 2.5 cm. x = -0.2). Release of the primary electrons by lightning: $j_F(t) = N q_e \delta(t)$, a: $N = 6.5 \cdot 10^5$, b: $N = 5.8 \cdot 10^5$, c: $N = 5.5 \cdot 10^5$.

b) Measurements with constant irradiation

We will consider an increase in current measured in dry air[2]. On the cathode n_0 electrons were released per second by radiation with a UV-light source; from this $j_F(t) = j_0 = (n_0 q_e)/F$. The zero point in time in the oscillograms is given by switching on the voltage. Increases in current with varying Over-voltage in the vicinity of the static breakthrough voltage were used (cp. Figure 1). We will consider as an example the increase in current leading to the breakthrough after $35\,\mu sec$ In the half-logarithmic operations (Figure 4) the measured current rises steeply up to approximately T_- and then passes over to a flat section; from this the increase in current is developed, which becomes increasingly steeper and leads to the breakthrough.

In the following evaluation a generation interval enlarged against $T_0 = T_-\{1-(\alpha_0 d)^{-1}\}$ is used. This was necessary, since earlier investigations in dry air [4, 10] had shown a delayed-avalanche development. The value used here $T_{ph} = 1{,}5 \cdot 10^{-7}\,sec$ follows from measurements of build-up time[11].

First of all the course of the current $\hat{I}_+, \hat{I}_-$ and $\hat{I}_+ + \hat{I}_-$ is calculated without considering the effect of space charging, according to the formulas given in Appendix 2. Here we put $\varkappa = \ln \mu_{ph} = 0{,}009,$ whereby the most favorable adjustment to the initial course of the measurement curve (free of space loading) results. Then a thorough calculation of the course of the current is carried out by means of equations (6) and (7) according to the numerical process described in Appendix 3.

The results of this calculation are illustrated in Figure 4 and are compared with the experimental course.

The measurement and the calculation are in close agreement and show that as a secondary effect only photo electric release on the cathode is concerned, and not a fractional supplement with γ_{+}- effect, as was assumed, on the other side, as an explanation for the increase in current[12].

c) Measurements with weak irradiation.

In these investigations [3-5] the cathode of the discharge area is radiated during the static breakthrough voltage $(\mu_{ph}=1)$ with a strongly reduced UV-light source, so that only individual photo-electrons with a large time-interval (10^{-3} to 1 sec.) are released. The breakthrough is created by an (infinitely large) avalanche chain which contains 2k-1 electron avalanches in the kth generation[13]. From this we can undertake the computation by means of the equations (6) and (7), by putting:

$$j_F(t) = j_0 = 2\,q_e/(T_{ph}\,F)\,.$$

In most cases, however, a simpler calculation is possible. Thus, in 1) the increase in current was calculated by ignoring the movement away from the ions (i.e., the term $J_-(t-T_i)$ in equation (7)). In 5) it was shown that the asymptotic solution (16) is useful for evaluating the increases in current, released by weak irradiation.

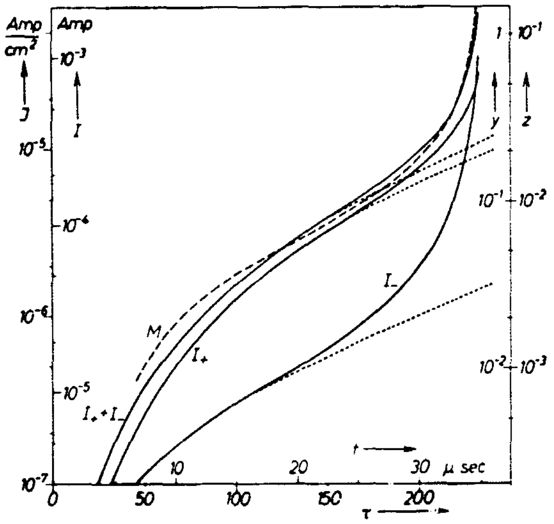


Figure 4

Rise in current in dry air[2] ($p = 724$ Torr, $d = 1$ cm, $U \approx 28{,}6$ kV, $n_0 =$ 60 electrons/µsec, $T_i = 15$ µsec), M = measurement curve, __________ calculated progress of current (- - - - without space charging).

IV. THE CALCULATION OF THE BUILD-UP TIME IN THE TOWNSEND SYNTHESIS

The measurement of the build-up time occurs in such a way that a voltage impulse is placed on the discharge space and the time span between applying the voltage and the collapse is measured with an oscillograph (cp [11, 14]). A calculation of the build up time is necessary in order to analyse the measurements.

This can be done in such a way that the increase in current is calculated by the voltage switch-on via the region of the increase which is free of space loading or influenced by space loading to the voltage decrease. In the following considerations we will limit ourselves to the supplement due to γ_{ph} effect.

First let us consider the course of the current in the section of the rise free of space charging. The formulas for this, given in Appendix 2, are simplified by working with over-voltage $(\varkappa > 0)$ By this means all the members fall with the exception of the exponential term in the equations (2.5) or (2.7) for $J_+(t)$ (apart from extremely small over-voltages). With equation (14) we have already described a course of current $y(\tau) = K_+ J_+(t)$ which rises exponentially in the region free of space charging.

$$\tilde{y}(\tau) = (3\varkappa)^{1/3} \exp\{\varkappa(\tau - \bar{\tau}_\infty)\}. \tag{18}$$

We can therefore use equation (14) to describe the course of the current in the region of space charging. In Figure 5 an example is illustrated for the development of the current from the switch-on of the voltage to the collapse. The discharge is started by electrons released on the cathode at time t = 0 by 10^2. The greatest part of the rise in current progresses without the influence of space charging. Only in the last section is the space charging effective; this part of the course of the current is illustrated with an extended time scale.

The course of the voltage on the discharge stretch results from the course of the total current $J_+(t)+J_-(t)$ if the given substitute diagram of connections is taken as a basis. On eht other hand the influence of the lowering of the voltage on the course of the current is not taken into consideration. Basically, because of this, the validity range of our analytical solution $y=K_+J_+$ is exceeded, so that the course of the current in the last section only has a qualitative character. If the time inter-value for the part of the rise free of space charging are compared with that part conditioned by space charging, it can be seen that a fault in the last part of the rise is of no significance for the length of the build-up time. - An exact calculation which is valid without reservation up to higher current intensities, and in which the fall in voltage is considered, is found in the following publication by the audhor[15] (see also Section 5).

As the final point of the build-up time T_A τ_∞ defined by equation (14) is applied: it can hardly be differentiated from the point in time at which the first noticeable drop in voltage can be observed. From equation

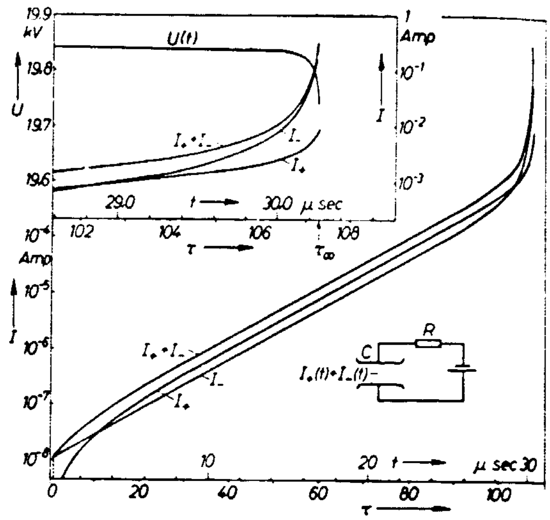


Figure 5

For calculating the build-up times. Calculated current and voltage progress in hydrogen (p = 500 Torr, d = 2cm U_D = 19.8kV), $\Delta U/U_D = 0{,}2\%$, $\varkappa = \ln \mu_{ph} = 0{,}105$. Substitute diagram of connections for the outer circuit discharge section:

$R = 485\ \Omega,\ C = 28$ pF.

(18) the following results (cp. also Figure 8 in[14])

$$\tilde{y}(\tau_\infty) = (3\varkappa)^{1/2}.$$

Thus now only the course of $\tilde{y}(\tau) = K_+ \tilde{J}_+(t)$ (constituent of the positive ions) needs to be calculated, without considering space charging. By applying

$$\tilde{j}_{+}(T_A) = (3\varkappa)^{1/3} K_{+}^{-1} = (3\varkappa)^{1/3} \left(\frac{A\,p\,d}{12}\right)^{-1/3} \frac{\varepsilon_0\,p}{T_{+}} \tag{19}$$

the build-up time T_A then follows. Until now only the current increase free of space charging was considered when calculating the build-up time; in a comparison with experimental build-up time a critical current density could be determined[16]. In contrast to the empirical criterion the formula (19) derived here is based on an explicit computation of the current in the region of space charging.

V. VALIDITY RANGE

An evaluation of the maximal current density, up to which the method applied here can be used, could be carried out by applying a critical consideration of the mathematical process. In particular, those approximations could be discurred which were made by deriving equations (1), (2) and (3) [17]. The evaluation of the maximal current density is, however, undertaken in another way.

A course of current calculated with equations (6) and (7) (method A) is compared with the calculation according to an exact method (N). In this method for a series of time point t_n the current densities $j_{-}(x, t_n)$ and $j_{+}(x, t_n)$ are calculated successively with the help of the Towsend differential equations. For this $\alpha(x, t_n)$, $v_{-}(x, t_n)$ and $v_{+}(x, t_n)$ are required, which are the functions of $E(x, t_n)$

The field intensity $E(x, t_n)$ is determined[15] for every point in time t_n by means of the Poisson-equation. The exact calculation is undertaken for the example presented in Figure 4. The comparison shows the following: the current of ions $J_+(t)$ has nearly the same course in both process; in the current of electrons $J_-(t)$ the course J_-^A calculated according to A is noticeably smaller than the current J_-^N calculated according to N. The cause of this can be found in the unfavorable approximation which was derived in Appendix I (Equation 2) for the current of the electrons. The following data should characterise this deviation:

$y = K_+ J_+$	$\int_0^d \alpha \, dx - \alpha_0 d = y^2$	J_-^N / J_-^A
0,2	0,04	1,06
0,4	0,16	1,3
0,7	0,49	1,5

The result obtained in the special example may be generalised without reservation, since according to Section 2 all increases in current with varying initial progress pass over into a unified course with large current intensities. The degree of the deviation J_-^N/J_-^A is obviously determined by the field distortion. Here we can use as a guide $y = K_+ J_+$ or $\int_0^d \alpha\,dx - \alpha_0 d$

If an error of 20% is assumed for the total current $J_+ + J_-$ the following maximal y values[18] result for the examples treated here:

Abb.	y_{max}
1 und 4	0,55
2, 3 und 5	0,65

APPENDIX I

DERIVATION OF THE FUNCTIONAL EQUATIONS

The following expressions for the solution satisfy the Towsend differential equations and the limit conditions appertaining to them $(v_-, v_+ = \text{const})$:

$$j_-(x, t) = j_-(0, t - x/v_-) \exp \int_0^x \alpha\left(x', t - \frac{x - x'}{v_-}\right) dx', \tag{1.1}$$

$$j_+(x,t) = \int_x^d j_-\left(x', t - \frac{x'-x}{v_+}\right) \alpha\left(x', t - \frac{x'-x}{v_+}\right) \mathrm{d}x' . \tag{1.2}$$

We will next consider the supplementary equation

$$j_-(0,t) = j_F(t) + \int_0^d j_-(x,t)\,\delta(x,t)\,\mathrm{d}x = j_F(t) + \gamma_{ph} \int_0^d j_-(x,t)\,\alpha(x,t)\,\mathrm{d}x . \tag{1.3}$$

By inserting (1) we obtain

$$\begin{aligned} j_-(0,t) &= j_F(t) + \gamma_{ph}\, j_-(0, t - T_{ph}) \left\{ \exp \int_0^d \alpha(x, t - T_{ph})\,\mathrm{d}x - 1 \right\} \\ &= j_F(t) + \mu_{ph}\, j_-(0, t - T_{ph}) \exp \left\{ \int_0^d \alpha(x, t - T_{ph})\,\mathrm{d}x - \alpha_0 d \right\} \end{aligned} \tag{1}$$

with

$$T_{ph} = \frac{\int_0^d x/v_- \exp\{\int_0^x \alpha(x',t)\,\mathrm{d}x'\}\alpha(x,t)\,\mathrm{d}x}{\int_0^d \exp\{\int_0^x \alpha(x',t)\,\mathrm{d}x'\}\alpha(x,t)\,\mathrm{d}x} \approx d/v_- \{1 - (\alpha_0 d)^{-1}\} . \tag{1.4}$$

When converting (1.3) it is assumed that $j_-(0,t)$ acts linearly in an interval of the length $T_-/\alpha_0 d$ arranged according to size and that the change in time of $\alpha(x,t)$ is small (see hypothesis (4)). All $\alpha(x,t)$- values are related to the point in time $t-T_{ph}$ in this way consideration is taken that the gas-concentration of the avalanches started at time $t-T_{ph}$ is determined mainly by the heightened field on the cathode. For T_{ph} an expression is first formed in equation (1.4) which contains $\alpha(x,t)$ for the field ratios prevailing here the approximation value (constant in time) may be dependent on the transition to the second line of (1) is made by disregarding slight factors.

A relationship between $J_+(t)$ and $j_-(0,t)$ is obtained from the 2nd Townsend differential equation by integrating via X

$$\frac{dJ_+(t)}{dt} = \frac{1}{T_+}\left\{\int_0^d j_-(x,t)\,\alpha(x,t)\,dx - j_+(0,t)\right\}$$

$$= \frac{\exp \alpha_0 d}{T_+}\left\{j_-(0,t-T_{ph})\exp\left\{\int_0^d \alpha(x,t-T_{ph})\,dx - \alpha_0 d\right\}\right.$$

$$\left. - j_-(0,t-T_i-T_{ph})\exp\left\{\int_0^d \alpha(x,t-T_i-T_{ph})\,dx - \alpha_0 d\right\}\right\}. \qquad (3)$$

The conversion in the left term of the first line results analogously to the conversion in equation (1). The term $j_+(0,t)$ is rewritten according to equation (1.2);

it can only be compared with the left term in case $j_-(0,t)$ in the time space $t-T_+\ldots t$ has changed little. In this case $j_-(0,t)$ may be set as a function linear in time, the integration via x then produces the expression given.

In order to represent the current of electron we write

$$J_-(t) = d^{-1}\int_0^d j_-(0, t - x/v_-)\exp\left\{\int_0^x \alpha(x', t - T_{ph})\,dx'\right\}dx$$

$$= (\alpha_0 d)^{-1} j_-(0, t - T_{ph})\left\{\exp\left\{\int_0^d \alpha(x, t - T_{ph})\,dx\right\} - 1\right\}$$

$$= \frac{\exp \alpha_0 d}{\alpha_0 d}\, j_-(0, t - T_{ph})\exp\left\{\int_0^d \alpha(x, t - T_{ph})\,dx - \alpha_0 d\right\}. \qquad (2)$$

The integration in the first line can only be carried out after introducing the factor $\alpha_0^{-1}\cdot\alpha(x, t - T_{ph})$ in the expression to be integrated. This factor can deviate considerably from 1 with strong distortion of space charging; thus (2) is a relatively bad approximation. When inserting (2) in (3) (cp. (2)) the quality of the approximation does not, however, play any roll. Here only the abbreviated way of writing (2) is used for an expression which occurs in equation (3)

APPENDIX 2

EQUATIONS FOR THE INCREASING - CURRENT WHILST IGNORING SPACE CHARGING

The point of departure is the functional equation (1) with $\int \alpha\, dx = \alpha_0 d$:

$$\varphi(t) = j_F(t) + \mu_{ph}\, \varphi(t - T_{ph})\,. \qquad (2.1)$$

Für $j_F(t) = q\,\delta(t)$ ist $(\tau = t/T_{ph})$

$$\varphi(\tau) = q\, T_{ph}^{-1} \exp \varkappa \tau \qquad (2.2)$$

a solution of equation (2.1) with the exception of the time interval $0 < \tau < 1$. The solution (2.2) is largely identical with the "asymptotic solution" which resulted by calculating exactly the increase in current[19]. In particular, an exponential increase results from (2.2) with the time constants

$$\lambda = T_{ph}^{-1} \varkappa = T_{ph}^{-1} \{1 - (\alpha_0 d)^{-1}\}^{-1} \ln \mu_{ph}\,.$$

For $j_F(t) = j_0\ (t > 0)$ we obtain a solution from (2.1) (by way of approximation, a coefficient $\{\mu_{ph} - 1\}^{-1} \ln \mu_{ph}$ is substituted by 1):

$$\varphi(\tau) = j_0\, \varkappa^{-1} \{\exp \varkappa \tau - 1\}\,. \qquad (2.3)$$

By using (6), (7) and (8) the following expressions for $\tilde{J}_+(t)$ and $\tilde{J}_-(t)$ are calculated.

$$\varkappa \neq 0 \qquad\qquad \varkappa = 0$$

$j_F(t) = q\,\delta(t)$

$$\tilde{J}_- = H_1 \exp \varkappa(\tau-1) \qquad\qquad H_1 \tag{2.4}$$

$$\begin{matrix} (\tau \leqq \tau_i+1) \\ (\tau \geqq \tau_i+1) \end{matrix} \qquad \tilde{J}_+ = \begin{cases} H_2\,\varkappa^{-1}\{\exp \varkappa(\tau-1)-1\} & H_2\{\tau-1\} \\ H_2\,\varkappa^{-1}\{1-\exp(-\varkappa\,\tau_i)\}\exp \varkappa(\tau-1) & H_2\,\tau_i \end{cases} \tag{2.5}$$

$j_F(t) = j_0 \; (t>0)$

$$\tilde{J}_- = H_3\,\varkappa^{-1}\{\exp \varkappa(\tau-1)-1\} \qquad\qquad H_3\{\tau-1\} \tag{2.6}$$

$$\begin{matrix} (\tau \leqq \tau_i+1) \\ (\tau \geqq \tau_i+1) \end{matrix} \qquad \tilde{J}_+ = \begin{cases} H_4\,\varkappa^{-1}\{\varkappa^{-1}[\exp \varkappa(\tau-1)-1]-(\tau-1)\} & H_4\,\tfrac{1}{2}\{\tau-1\}^2 \\ H_4\,\varkappa^{-1}\{\varkappa^{-1}[1-\exp(-\varkappa\,\tau_i)]\exp \varkappa(\tau-1)-\tau_i\} & H_4\,\tau_i\{\tau-\tfrac{1}{2}\tau_i-1\} \end{cases} \tag{2.7}$$

$$H_1 = \frac{q \exp \alpha_0 d}{T_{ph}\,\alpha_0 d}, \quad H_2 = \frac{q \exp \alpha_0 d}{T_+}, \quad H_3 = \frac{j_0 \exp \alpha_0 d}{\alpha_0 d}, \quad H_4 = \frac{j_0\,T_{ph} \exp \alpha_0 d}{T_+}.$$

APPENDIX 3

METHODS OF SOLUTION

For the numerical calculations the transition to normal functions is useful. We introduce

$$\begin{aligned} \tau &= t/T_{ph}, & \tau_i &= T_i/T_{ph}, \\ y(\tau) &= K_+\,J_+(t), & K_+ &= \left(\frac{A\,p\,d}{12}\right)^{1/2} \frac{T_+}{\varepsilon_0\,p}, \\ z(\tau) &= K_-\,J_-(t), & K_- &= \left(\frac{A\,p\,d}{12}\right)^{1/2} \frac{T_{ph}\,\alpha_0\,d}{\varepsilon_0\,p}, \end{aligned} \tag{3.1}$$

ε_0 = di-electricity constant.

Then the following equation system is produced:

$$\dot{z}(\tau) = z(\tau)\,\{\varkappa + y^2(\tau)\} + K_- \frac{\exp \alpha_0 d}{\alpha_0 d}\, \dot{j}_F(\tau)\,, \tag{3.2}$$

$$\dot{y}(\tau) = z(\tau) - z(\tau - \tau_i)\,. \tag{3.3}$$

The initial conditions are:

$$z(1) = K_- \frac{\exp \alpha_0 d}{\alpha_0 d}\, T_{\mathrm{ph}}^{-1} q\,, \quad y(1) = 0\,. \tag{3.4}$$

In solving equation system (3.2), (3.3) the time axis is devided into sections of length τ_i The solutions are then each calculated in every section and the function $y(\tau), z(\tau)$ already known in the previous section, is inserted for $z(\tau-\tau_i)$ In this way the usual methods for solving a $z(\tau)$ differential equation are applied. In general one is referred to numerical processes in the calculation.

Thus the examples treated in Section 3 is calculated by the Runge-Kutta's method of stepwise approximation (formulas of 2nd order). It is found to be helpful that in this method the interval $\Delta\tau$ can be fitted to the $\Delta\tau$ character of the solutions; at the beginnings of the rise $\Delta\tau$ can be chosen to be relatively large, whereas for the last steep rise a small τ is appropriate.

The following table contains the values of F (cross - section of the discharge used in calculation:)

Abb.	1, 4	2, 3	5
F [cm²]	28,6	10,5	100

TRANSLATED BY
Translation and Interpretatation Division
INSTITUTE OF MODERN LANGUAGES, Washington, D. C.
Under contract with the NASA Goddard Space Flight Center,
Greenbelt, Md., 1968.