X-643-69-297

PREPRINT

NASA TM X- 63621

DESCRIPTIVE EXAMPLES OF GRAVITY ASSISTED TRAJECTORIES

JAMES B EADES, JR.

JULY 1969

GSFC — GODDARD SPACE FLIGHT CENTER —
GREENBELT, MARYLAND

X-643-69-297

DESCRIPTIVE EXAMPLES OF GRAVITY ASSISTED TRAJECTORIES

by

James B. Eades, Jr.

JULY 1969

Goddard Space Flight Center
Greenbelt, Maryland

PRECEDING PAGE BLANK NOT FILMED.

DESCRIPTIVE EXAMPLES OF GRAVITY ASSISTED TRAJECTORIES

by

James B. Eades, Jr.*
Special Projects Branch

ABSTRACT

Two approximate methods for calculating the planetary Swing-By maneuver are presented. These schemes may be used to determine, analytically, the path of motion, time of motion, energy and velocity change as experienced by a satellite prior, during and often after the encounter.

Also, an examination for the extremals of the energy and velocity changes is discussed as a part of the overall Swing-By problem.

*NAS-NASA Resident Research Associate, on leave from the Virginia Polytechnic Institute, Blacksburg, Va.

PRECEDING PAGE BLANK NOT FILMED.

CONTENTS

LIST OF FIGURES

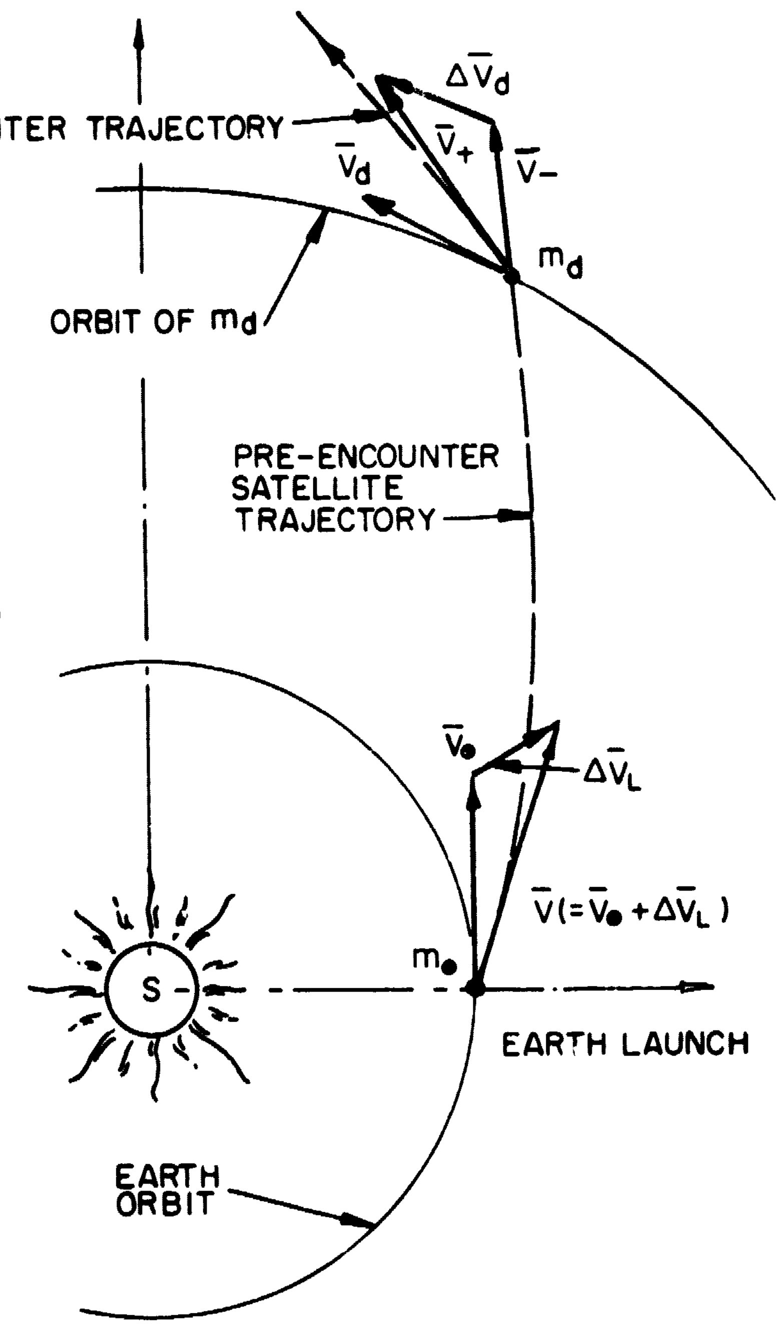
DESCRIPTIVE EXAMPLES OF GRAVITY ASSISTED TRAJECTORIES

I. INTRODUCTION

Gravity assisted trajectories, or "swing-by" maneuvers, are produced by the passage of a satellite (man-made or otherwise) close to a massive celestial body. As a consequence of this the satellite's flight path is perturbed, significantly, due to the mass attraction of the larger body. For situations such as these the flight path develops something akin to a "kink;" the vehicle will have its energy of motion altered, and its velocity vector will undergo a change in direction. These several happenings are all due to the gravitational attraction of one moving massive body on a second smaller moving mass, (say) the satellite.

As an example of the problem type which will be described herein, consider Figure I.1 where an earth launched satellite is depicted to pass close to one of the massive outer planets. Because of the close proximity of the spacecraft to the attracting planet, the vehicle's path is altered.

It is not difficult to visualize that one might be able to "control" the effects of such a swing-by operation; especially if it would be possible to "control" how close the satellite approached to the planet, and if some "control" could be maintained over the speed of the vehicle. As a matter of interest previous theoretical studies, related to space travel, have indicated that significant fuel savings and/or increased payloads – as well as shortened flight times – could be achieved by a proper utilization of this technique.



$\bar{V} \triangleq$ S/C HELIOCENTRIC VELOCITY, AT EARTH LAUNCH.

$\bar{V}_\bullet \triangleq$ HELIOCENTRIC VELOCITY OF $m_\bullet$

$\bar{V}_d \triangleq$ HELIOCENTRIC VELOCITY OF SWING-BY PLANET (m_d).

$\bar{V}_- \triangleq$ HELIOCENTRIC VELOCITY OF S/C BEFORE SWING-BY.

$\bar{V}_+ \triangleq$ S/C HELIOCENTRIC VELOCITY AFTER SWING-BY

$\Delta\bar{V}_L \triangleq$ S/C VELOCITY INCREMENT FOR LAUNCH.

$\Delta\bar{V}_d \triangleq$ S/C VELOCITY INCREMENT DUE TO SWING-BY

Figure I.1–Sketch Depicting the Coplanar Circular Orbits of Earth and the Swing-By Planet, with effects of Swing-By (ΔV_d and Path Change) Illustrated.

II. BASIC ASSUMPTIONS

In order to simplify the mathematics for this investigation, and to aid in providing analytic solutions, several assumptions will be made. In general, these are as follows:

1. All motions will be treated as two-body situations; thus, when the satellite is in the vicinity of a disturbing planet it will be assumed that the planet provides the principal influence on the trajectory. Consequently, all other bodies are neglected and the problem is referred to planetocentric space. However, for motion in interplanetary space it will be assumed that the primary influence is the Sun and the study is (then) referred to heliocentric space.

These assumptions naturally lead to the conceptual idea of a "Sphere of Influence," which will be employed when the gravitational influence of an attracting body, on the flight path, is that of the disturbing mass.

2. During that time period when the vehicle is in passage through the sphere of influence, the attracting planet is presumed to move only negligibly along its own orbit; hence, the "swing-by" is presumed to occur <u>at</u> a fixed heliocentric location for the attracting body.

3. The trajectory flown by the satellite through the Sphere of Influence will be a hyperbola. This will be treated as a constant energy path; thus the vehicle will be influenced solely by the attracting planet. For this condition, the trajectory will be described by its constant specific energy and a fixed specific angular momentum (relative to the attracting planet).

<u>NOTE:</u> The interplanetary trajectories, under the same restrictions as noted above, are two-body flight paths, where the satellite moves about under the singular influence of the Sun. In this regard the vehicle's trajectory, outside of the attracting planet's sphere of influence, is made up of heliocentric arcs characterized by constant energy and angular momentum (so long as the only force considered is that of the Sun's gravitational attraction.)

It should be mentioned that the solution technique employed here is by no means exact; it is an approximation utilized to alleviate some of the computational difficulties which necessarily arise when the true multi-body aspects of this problem are introduced. The case studies included here are variations of the patched conic mode of solution; the two modes of approach which constitute this investigation are included to indicate the degrees of complexity which can be introduced into this simpler analytic approach. Needless to say, when extreme accuracy is desired these approximate methods should be abandoned and a fully developed mathematical program, computer solved, should be employed.

III. EXAMPLE OF PLANETARY SWING-BY FLIGHT

The example presented here will be concerned with the flight of a spacecraft from an Earth orbit to (say) an outer planetary orbit. During this maneuver the spacecraft will pass close by an intermediate (massive) planet, executing a "swing-by" maneuver; thereby having its ballistic trajectory altered by the gravitational influence of this intermediate body.

In this study the ballistic path of the spacecraft is obtained from a restricted two-body solution, with the sun serving as a primary mass. For this problem, then, the intermediate planet effectively acts as a perturbing influence, altering the two-body path, and "bending" the ballistic space trajectory according to the mode of passage by the disturbing body.

In the analysis of this flight problem, two methods of approach will be presented. For the first, and simplest, case the effect on the trajectory produced by the perturbing planet will be assumed to occur at a given, fixed radial distance (namely, at that heliocentric distance locating the disturbing planet itself). Thus, the spacecraft will fly to the perturbing planet's orbit where the "swing-by" will be treated as a local occurrence – "bending" the flight path at this position in space – then, the spacecraft will be assumed to move on along a new ballistic trajectory. This is a rather uncomplicated approach to the "swing-by"; one which is easy to follow and to understand, yet one which provides the basic information needed to describe this type of a perturbed flight track.

In the second case study, a more rigorous and complicated approach is undertaken. For this investigation a finite size of the perturbing planet's "sphere of influence" is assumed, and the "bending" of the space track is accounted for in a more realistic fashion. Here, the geometry of the (close-in) planetocentric space trajectory is determined, and evolves, as a finite dimensioned situation. In this regard the events arising from the perturbation are relegated to the "vicinity" of the planet rather than to a point in heliocentric space. The consequence of this latter approach is that the problem becomes more complicated, and needs additional analysis and computational effort; but it does lead to a more realistic evaluation of the physical problem. Whether or not the added effort, involved here, is worth the additional accuracy achieved is a matter to be decided by the investigator. His need for computational accuracy and for geometric description will dictate which of these two approaches (if either) should be undertaken.

III.A. AN INTERPLANETARY TRANSFER

In this example it is proposed that an earth launched spacecraft will travel to one of the outer planets, aided (enroute) by a planetary "swing-by" maneuver. For the sake of specifics, suppose that a mission is designed whereby Pluto is reached via a Jupiter swing-by, following the earth orbit launch.

Having the need for an initial and comparison trajectory, then a first consideration will be to describe a basic ballistic transfer path; subsequently the swingby operation will be viewed as a modification to this reference trajectory.

For a fundamental reference path suppose that a Hohmann* transfer from (an assumed) earth circular orbit to an assumed circular outer planetary orbit is selected. Thus this basic flight path will be a nominal, least energy transfer arc, and the subsequent swing-by will result in a modification of this conic.

III.B.1. THE HOHMANN TRANSFER

As a means for describing the Hohmann trajectory, its several parameters will be listed below – as these relate to this problem. It should be recalled that this trajectory is part of an ellipse which is cotangential to both of the assumed circular orbits (e.g. those of earth and the terminal planet). Thus, here, the earth's orbit radius will match the Hohmann pericentric distance while the size of the terminal circular orbit will correspond to the apocentric radius.

The orbital (two-body) parameters:

(a) eccentricity of the Hohmann ellipse:

$$\epsilon_H \triangleq \frac{-r_{peri} + r_{apo}}{r_{peri} + r_{apo}} = \frac{r_f - r_\oplus}{r_\oplus + r_f}\,; \tag{III.1}$$

(b) major axis of the ellipse:

$$2a_H \triangleq r_{apo} + r_{peri} = r_f + r_\oplus\,; \tag{III.2}$$

*The Hohmann ellipse was selected simply as a matter of convenience. Any convenient initial path could be chosen; because of its relative simplicity, in definition, the Hohmann transfer serves as an easily understood reference arc.

(c) focal parameter for the **trajectory**:

$$p_H \stackrel{\Delta}{=} 2 \frac{r_{peri}\, r_{apo}}{r_{peri} + r_{apo}} = 2 \frac{r_\oplus\, r_f}{r_\oplus + r_f} . \qquad \text{(III.3)}$$

III.B.2. DYNAMICAL PROPERTIES OF THE HOHMANN ELLIPSE

(a) the specific angular momentum:

$$h_H \stackrel{\Delta}{=} |\bar{r} \times \bar{V}| = r_{peri}\, V_{peri}$$

(measured at the pericentric position on the ellipse; the earth's radial distance from the sun, 1 a.u.) wherein,

$$r_{peri}\ (\equiv r_\oplus) \stackrel{\Delta}{=} \frac{p_H}{1 + \epsilon_H} ;$$

and, from the specific energy equation, the speed at pericenter is

$$V_{peri} = \sqrt{2 \frac{\mu_\odot}{r_{peri}} - \frac{\mu_\odot}{a_H}} = \sqrt{\frac{\mu_\odot}{r_\oplus}} \left[\frac{2 r_f / r_\oplus}{1 + r_f / r_\oplus}\right]^{1/2} .$$

Here $\mu_\odot$ is the gravitational parameter $(G m_\odot)$ for the sun. One should keep in mind the fact that this particular ballistic path is a heliocentric trajectory.

Combining these definitions, it is evident that the momentum constant for the Hohmann path can be expressed as;

$$h_H = r_\oplus V_{c_\oplus} \left[\frac{2 r_f}{r_f + r_\oplus}\right]^{1/2} , \qquad \text{(III.4)}$$

wherein $V_{c\oplus}$ (circular orbit speed at a distance equivalent to the earth's heliocentric radius) $= \sqrt{\mu_\odot / r_\oplus}$.

(b) The level of energy, for this heliocentric trajectory, can be described from the specific energy expression; thus,

$$E_{1_H} \triangleq \left(\frac{V^2}{2} - \frac{\mu}{r}\right)_H = -\frac{\mu_\odot}{2a_H} = \frac{-\mu_\odot}{r_\oplus + r_f}$$

or, after manipulation,

$$E_{1_H} = \frac{-V_{c_\oplus}^2}{1 + r_f \ r_\oplus} . \tag{III.5}$$

These characteristics, above, are adequate for describing the Hohmann arc. There is one parameter which should also be defined – the time for motion – this is deferred for the moment, it appears in a following section.

III.B.3. CHARACTERISTIC SPEEDS AND VELOCITY IMPULSE SCHEDULE

On the Hohmann transfer ellipse the two speeds which are of primary importance, immediately, are those corresponding to departure (at pericenter) and to flight terminus (apocenter, at the terminal planet radius). Manipulating the definition for specific energy it is easy to show these two characteristic speeds are:

$$(V_{peri})_H = (V_\oplus)_H = V_{c_\oplus} \left[\frac{2r_f}{r_f + r_\oplus}\right]^{1/2} , \tag{III.6a}$$

and

$$(V_{apo})_H = (V_f)_H = V_{c_f} \left[\frac{2r_\oplus}{r_f + r_\oplus}\right]^{1/2} , \tag{III.6b}$$

wherein $V_{c_f} = \sqrt{\mu_\odot / r_f}$.

Next, in order to begin the transfer, at pericenter a ΔV (impulse) would be needed to shift the vehicle from its earth circular orbit onto the transfer ellipse; and, in a like fashion, to get off the ellipse (at apocenter) and onto a terminal circular path, a second ΔV is needed. In this regard, then, the ΔV schedule needed to achieve these two actions can be determined as: the impulse at,

(1) departure from pericenter;

$$\Delta V_{\oplus} = \left(\sqrt{\frac{2r_f}{r_f + r_{\oplus}}} - 1\right). \tag{III.7a}$$

and

(2) terminus, at apocenter;

$$\Delta V_f = V_{c_f} \left(1 - \sqrt{\frac{2r_{\oplus}}{r_f + r_{\oplus}}}\right). \tag{III.7b}$$

It should be evident that these ΔV_i values are in the (±) direction relative to the local velocity vectors, respectively.

III.B.4. THE TIME FOR TRANSFER

The Hohmann transfer, which occurs over one-half of the ellipse, would require a transit time equal to half of the orbital period; thus, the transfer time along this idealized flight path would be:

$$\Delta t_H = \frac{P_H}{2} = \pi \sqrt{\frac{a_H^3}{\mu_{\odot}}},$$

or, using the Hohmann parameters,

$$\Delta t_H = \frac{\pi}{2} \sqrt{\frac{(r_f + r_{\oplus})^3}{2\mu_{\odot}}}. \tag{III.8}$$

In the present problem this quantity is not needed since the <u>full extent</u> of the Hohmann geometry is not utilized, per se; however, it is included here since it is one of the parameters describing the reference path.

III.C. SPHERE OF INFLUENCE

For those calculations which describe the "swing-by," per se, it is necessary to define the perturbing planet's "sphere-of-influence." This is the region in which the maneuver occurs. Consequently a ficticious boundary which defines the "sphere" can be obtained from:

$$r_{IN} = R\left(\frac{m_d}{m_\odot}\right)^{2/5}, \qquad \text{(III.9)}$$

wherein r_{IN} describes the planetocentric radius of the "sphere of influence;" while R is the radius to the planet (m_d), measured from $m_\odot$ (the sun). Also, the quantities $m_\odot$ and m_d are the solar and disturbing planet masses, respectively.

III.D.1. THE SWING-BY MANEUVER; MODE I

The simpler mode of analysis, used to describe the swing-by maneuver, will be designated herein as Mode I. For this operation the flight begins as (say) a Hohmann transfer, and continues as such until the spacecraft is in the very immediate vicinity of the "swing-by" planet. Then, in the neighborhood of this perturbing body – actually, within its sphere of influence – the swing-by occurs. Due to the planet's gravitational attraction the flight path is altered so that the subsequent heliocentric trajectory is no longer a continuation of the Hohmann ellipse, in general, but becomes another (and possibly different) heliocentric conic.

In this first analysis the effects of the swing-by are assumed to occur at a fixed radius; that one which locates the perturbing planet in heliocentric space. Because of this assumption the swing-by is assumed to alter the trajectory locally, producing a change in the vehicle's heliocentric kinetic energy. However, the idea of this assumption does not allow any apparent change in the vehicle's potential energy (due to the encounter).

Now then, of necessity, when one does attempt to ascertain the change in direction for the spacecraft's velocity vector, an analysis which determines the flight path through the sphere of influence must be undertaken.

In addition, the sphere of influence may be utilized as an aid in accurately estimating the transit time through this region of (planetocentric) space during the encounter. This particular topic will be discussed later!

III.D.2. A DESCRIPTION OF INTERCEPT WITH THE SWING-BY PLANET'S ORBIT

The assumptions used for a Mode I swing-by consider the disturbance to occur at the planet's heliocentric radius rather than in its sphere of influence. Consequently this intersecting of the paths of motion is an important factor in the analysis; and one which necessitates an evaluation of this intercept. In this regard one needs to know conditions, parameters, and a description of the spacecraft's path, at the swing-by planet's heliocentric radius ($\bar{r}_d$).

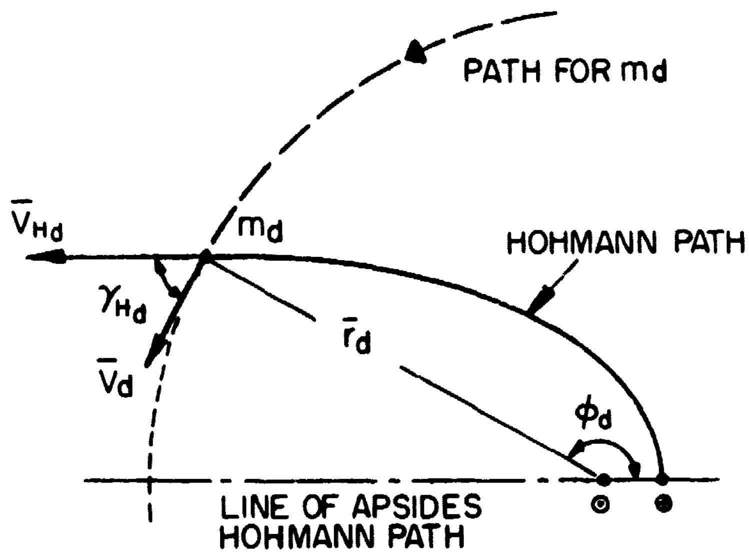


Figure III.1–Sketch Showing Intercept of the Hohmann Path with the Orbit of m_d.

Since the characteristics of the Hohmann path are known, explicitly, then at encounter with the disturbing planet's orbit, the spacecraft's heliocentric position (angle) can be acquired from the conic equation. A manipulation of this expression, expressedly written to describe the Hohmann ellipse (at r_d), gives

$$(\varphi_H)_d = \cos^{-1}\left[\frac{a_H}{r_d\,\epsilon_H}\,(1 - \epsilon_H^2) - \frac{1}{\epsilon_H}\right]. \tag{III.10}$$

To define the spacecraft's speed, at "m_d", use will be made of the specific energy Eqn. (solved by V_{H_d}); that is, the speed at r_d, for the vehicle on its Hohmann track is;

$$(V_H)_d = \left[2\,\frac{\mu_\odot}{r_d}\left(1 - \frac{r_d}{r_f + r_\oplus}\right)\right]^{1/2} . \qquad \text{(III.11)}$$

Also, for subsequent use, the heliocentric speed of the planet (m_d), at encounter, can be assumed to be

$$V_d = \sqrt{\frac{\mu_\odot}{r_d}} \, , \qquad \text{(III.12)}$$

since the path of "m_d" is presumedly circular.

In order to describe the elevation angle (γ_{H_d}), at r_d, for the spacecraft's velocity vector one could proceed in the following manner:

In view of the conservation of (two-body) moment of momentum, write,

$$V_{H_d} \cos\gamma_{H_d} = \frac{|\bar{h}|}{r_d} = r_d\, \dot{\varphi}_H \, .$$

Recalling that $h_H = \sqrt{p_H \mu_\odot}$; then, accounting for the conic equation, rewrite the above expression as

$$V_{H_d} \cos\gamma_{H_d} = \sqrt{\frac{p_H \mu_\odot}{r_d}} \equiv \sqrt{\frac{\mu_\odot}{r}\,(1 + \epsilon_H \cos\varphi_{H_d})} \, ;$$

and, therefore,

$$\gamma_{H_d} = \cos^{-1}\left[\frac{\mu_\odot}{r_d}\;\frac{(1 + \epsilon_H \cos\varphi_{H_d})}{V_{H_d}^{\;2}}\right]^{1/2} . \qquad \text{(III.13)}$$

(Note, there is no need to find the $\sin^{-1}$ function (here) since the physics of this problem locates γ_{H_d} in the first or second quadrants of the orbit's geometry. Similarly, any other transfer situation would physically describe γ so that no ambiguity would exist.)

In view of the assumptions made for the Mode I solution the spacecraft meets the swing-by planet – and subsequently departs from it – at the heliocentric position (r_d, φ_{H_d}). Notwithstanding this simplification, in order to evaluate the influence of the planet, m_d, on the spacecraft, during the "encounter," the overall analysis must include the essentials of a swing-by maneuver in order to ascertain the degree of trajectory perturbation introduced by this occurrence. For the purpose of determining a spacecraft track, relative to the encountered planet, the swing-by problem is treated (here) as a two-body situation, but with "m_d" serving as the primary mass, now.

III.D.3. TIME OF FLIGHT TO THE SPHERE OF INFLUENCE (MODE I)

Before continuing with this analysis, and prior to considering the flight path inside the sphere of influence, it would be advisable to ascertain the time required to perform the first segment of the overall mission. This will account for the time lapse from the initial point, on the Hohmann arc, to the point of intercept with the m_d-orbit (the perturbing planet's heliocentric orbit). It should be evident that the arc in question is from pericenter (at one earth heliocentric radius) to the intercept, at one m_d heliocentric radius.

Knowing the position of this intercept (r_d, φ_{H_d}), see Eqn. (III.10); and knowing the orbital parameters for the (Hohmann) arc, then it is quite apparent that the time required to reach the intercept position is given by

$$\Delta t_1 \triangleq t_{H_d} - t_0 = \sqrt{\frac{a_H^3}{\mu_\odot}} \left\{ - \epsilon_H \tan \gamma_{H_d} + 2 \tan^{-1} \left[\sqrt{\frac{1 - \epsilon_H}{1 + \epsilon_H}} \tan \frac{\varphi_{H_d}}{2} \right] \right\} \qquad \text{(III.14)}$$

with ϵ_H, a_H described in Eqns. (.1, .2), and γ_{H_d} obtained from Eqn. (III.13).

III.D.4. CONDITIONS AT ENTRY INTO THE SPHERE OF INFLUENCE

At that point in time and space when the spacecraft reaches the "boundary" for the sphere of influence, it will be necessary to describe the trajectory conditions relative to the attracting mass (m_d). This is needed so that one can shift from a heliocentric track to a planetocentric flight path.

At the imaginary boundary enclosing the sphere of influence, the vehicle's velocity vector(s) are related by

$$\bar{V}_d + \bar{V}_2 = \bar{V}_{H_d} \, .$$

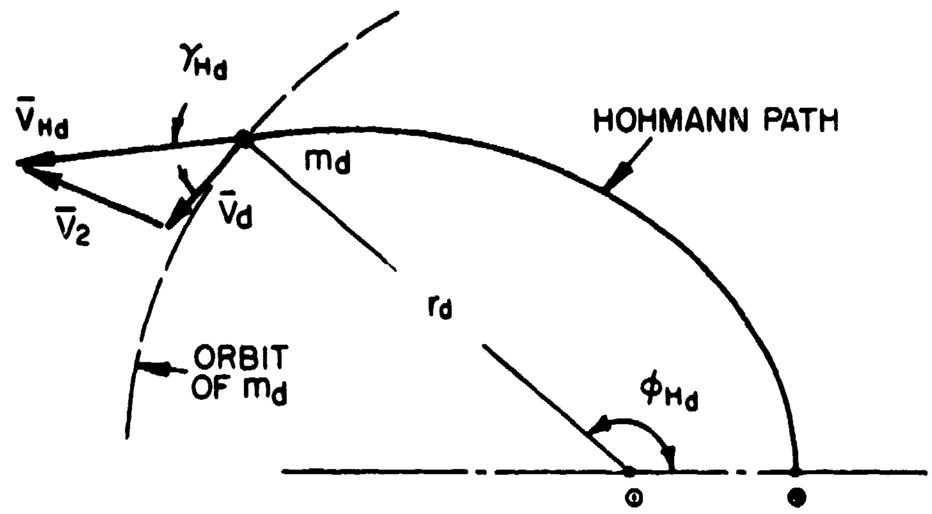


Figure III.2–Geometric and Kinematic Conditions at Intercept of the Hohmann Path with the Orbit of m_d

where $\bar{V}_2$ is the velocity of the vehicle relative to m_d; $\bar{V}_d$ is the velocity of m_d relative to the sun (⊙); and, $\bar{V}_{H_d}$ is the velocity of the vehicle, at the distance r_d, on the Hohmann path.

Solving for $|\bar{V}_2|$ one obtains a measure of the speed for the vehicle, relative to m_d, when it reaches the "sphere of influence." Thus,

$$|\bar{V}_2| = \sqrt{V_d^2 + V_{H_d}^2 - 2\,\bar{V}_d \cdot \bar{V}_{H_d}}\ ;$$

or, noting the previous sketch,

$$V_2 = \sqrt{V_d^2 + V_{H_d}^2 - 2V_d\, V_{H_d} \cos \gamma_{H_d}}\ . \tag{III.15}$$

III.D.5. THE TRAJECTORY INSIDE THE SPHERE OF INFLUENCE

Once the vehicle enters the sphere of influence its flight path is considered to be a planetocentric two-body trajectory. It is most likely, here, that the previously described heliocentric ellipse becomes a planetocentric hyperbola. Of course this will depend on the magnitude of V_2 in relation to (say) the planetocentric circular speed (or escape speed), at the radius of entry.

Inside the sphere of influence, where the only gravitational attraction is that due to mass m_d, the flight path is (again) characterized by constant specific energy and moment of momentum. In view of the assumptions used for the Mode I calculations the planetocentric study will treat this part of the problem as if the entry conditions were the same as the "infinity" values for a two-body hyperbola. That is, from the energy expression in planetocentric space,

$$E = \frac{V^2}{2} - \frac{\mu_d}{r} = + \frac{\mu_d}{2a}$$

(which presumes a hyperbolic geometric figure), the energy relation – corresponding to <u>large</u> radii – leads to the approximation

$$E \simeq \frac{V_\infty^2}{2} \simeq \frac{\mu_d}{2a} .$$

Allowing this approximation to be a description of <u>entry into the "sphere of influence,"</u> it follows that the planetocentric trajectory has, as its major dimension (approximately),

$$a \simeq \frac{\mu_d}{V_2^2} ; \qquad \text{(III.16)}$$

vith V_2 defined by Eqn. (III.15) above.

As an aid to obtaining pertinent information for this "hyperbolic encounter," vith the perturbing mass (m_d), one can employ (first) the conic equation. Thus, description of the planetocentric position angle, relative to the planetocentric ericenter, at entry into the sphere of influence, will come from

$$r_I = \frac{p_d}{1 + \epsilon_d \cos \varphi_\infty} .$$

here φ_∞ implies the "infinity," or large radius, (entry) position. Following with e idea that r_I is very large, then

$$\frac{p_d}{r_I} \simeq 0 = 1 + \epsilon_d \cos \varphi_\infty$$

(hence φ_∞ essentially defines the approach – and/or receding – asymptote); or, in terms of the complimentary angle (ψ), where $\psi = \pi - \varphi_\infty$ (see the sketch below),

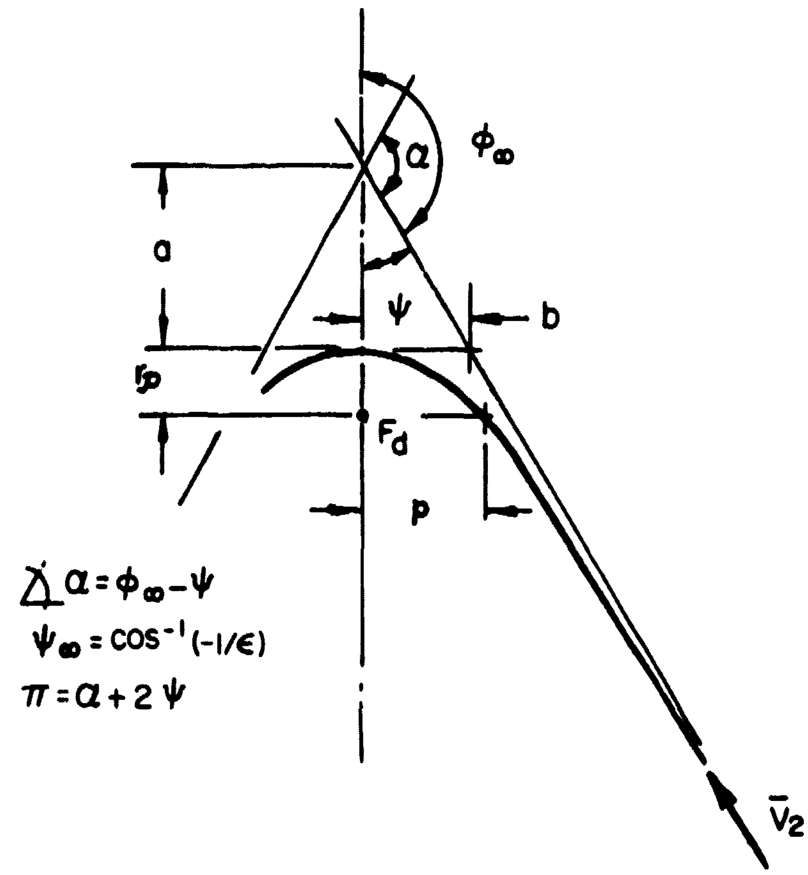


Figure III.3–Sketch Illustrating the Planetocentric Hyperbola

$$\frac{p}{r_I} \simeq 0 = 1 + \epsilon_d \cos(\pi - \psi),$$

and

$$\cos\psi = +\frac{1}{\epsilon_d}. \qquad \text{(III.17)}$$

From this expression the eccentricity of the planetocentric conic can be obtained as:

$$\epsilon_d = \sec\psi. \qquad \text{(III.18)}$$

A geometric parameter of major importance to the swing-by is the angle "α" shown on the accompanying sketch. This angle describes the magnitude of the rotation which is given to the spacecraft's velocity vector as a consequence of the "swing-by" maneuver. Denoting the vehicle's final velocity vector as $\bar{V}_3$ in planetocentric space – just before it departs from the sphere of influence – then the angle between $\bar{V}_2$ and $\bar{V}_3$ is "α". Thus, in mathematical parlence,

$$\alpha = \cos^{-1}\left(\frac{\bar{V}_2 \cdot \bar{V}_3}{|\bar{V}_2|\ |\bar{V}_3|}\right); \qquad \text{(III.19)}$$

also, it is the angle which describes the "size" of the "kink" produced in the heliocentric space trajectory due to the "swing-by" operation.

A description for α can be acquired in the following manner: Since $\alpha = \varphi_\infty - \psi$ (see the sketch on preceding page), then

$$\cos\alpha = \cos(\varphi_\infty - \psi) = \cos\varphi_\infty \cos\psi + \sin\varphi_\infty \sin\psi;$$

and

$$\sin\alpha = \sin\varphi_\infty \cos\psi - \cos\varphi_\infty \sin\psi;$$

hence, after substitution and clearing,

$$\cos\alpha = \frac{\epsilon_d^2 - 2}{\epsilon_d^2}, \quad \sin\alpha = \frac{2\sqrt{\epsilon_d^2 - 1}}{\epsilon_d^2}, \qquad \text{(III.20a)}$$

taking into account Eqn. (III.17); and/or

$$\tan\alpha = \frac{2\sqrt{\epsilon_d^2 - 1}}{\epsilon_d^2 - 2}. \qquad \text{(III.20b)}$$

A study of this last expression will indicate that the angle α is rather strongly dependent on the eccentricity of the planetocentric path – and, as an a priori statement, the figure is a hyperbola. Also, it is apparent that ϵ_d, Ψ and α are related according to the tabulations noted on following page:

When ϵ_d =	Values for ψ =	Values for α =
1.0	0	π
$1 < \epsilon_d < \sqrt{2}$	$0 < \psi < \pi/4$	$\pi > \alpha > \pi/2$
$\sqrt{2}$	$\pi/4$	$\pi/2$
$> \sqrt{2}$	$\pi/4 < \psi < \pi/2$	$\pi/2 > \alpha > 0$

The "size" of this hyperbolic path about the disturbing mass (m_d) can be defined by means of its focal parameter (p_d). That is, since

$$p_d \triangleq a(\epsilon_d^2 - 1);$$

then according to the assumptions set forth for this problem's study – namely, that $V_2 \simeq V_\infty = \sqrt{\mu_d/a}$; also, since $\tan \alpha/2 = 1/\sqrt{\epsilon_d^2 - 1}$ – it follows that

$$p_d \simeq \frac{\mu_d}{\left(V_2 \tan \frac{\alpha}{2}\right)^2} . \tag{III.21}$$

In general, the characteristics of the planetocentric (hyperbolic) trajectory are known, now, and this geometry is described, at least in principle.

The conditions (speed, orientation of $\bar{V}$, etc.) at exit from the sphere of influence have not been determined but will be obtained below. Also, even though the size of the sphere of influence is not considered – on the heliocentric scale of distances – it may be desirable to estimate the time required to move through this figurative region in space. To this end, in a subsequent paragraph a "time needed for the planetocentric transit" will be defined.

III.D.6. FLIGHT CONDITIONS AT EXIT FROM THE SPHERE OF INFLUENCE

The assumptions made for the Mode I swing-by computational procedures dictated that the discontinuity in the spacecraft's trajectory, produced by the encounter, would occur at the fixed position locating the disturbing planet in heliocentric space. This would imply a vanishingly small sphere of influence (on a heliocentric scale); however, in order to view and describe flight conditions, at exit from the sphere, it is necessary to give this region some finite size (as a planetocentric space). In this manner the orientation, etc. of the vehicle's "exiting" velocity vector can be defined. The pair of sketches below will indicate conditions which are of interest here.

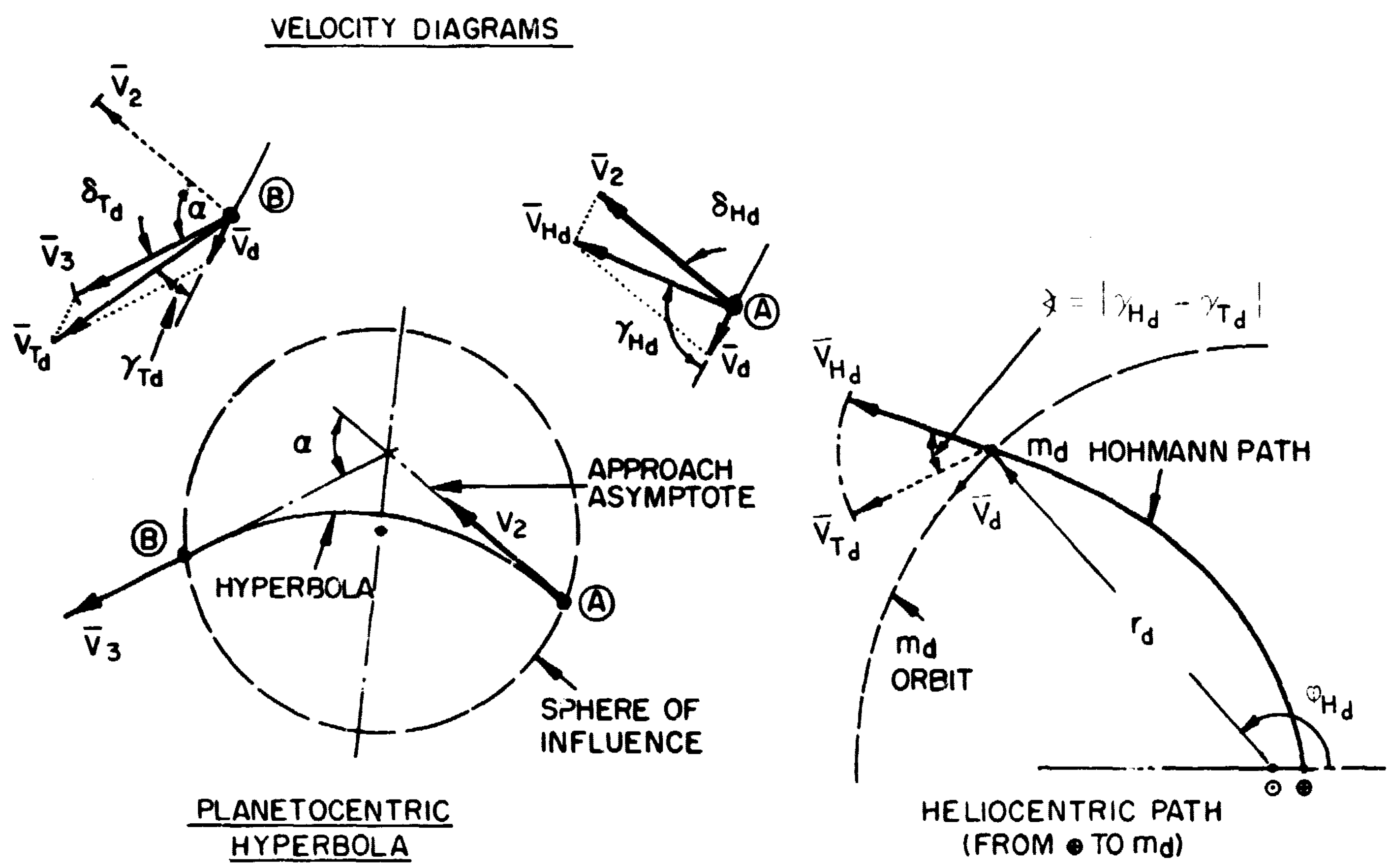


Figure III.4–Geometrics for the Encounter at m_d

The sketches (above) illustrate geometric conditions which fit the ideas of the Mode I calculations in the region of the encounter with m_d. For the sake of clarity the nomenclature used here is described below:

(1) At the position of m_d, the heliocentric velocity for the vehicle is $\bar{V}_{H_d}$; the velocity of m_d (itself) in this space is $\bar{V}_d$; also, according to the following vector expression these are related by

$$\bar{V}_2 + \bar{V}_d = \bar{V}_{H_d} ,$$

with $\bar{V}_2$ being the planetocentric velocity of the spacecraft at entry (A) into the Sphere of Influence.

(2) Point (B), at exit from the Sphere of Influence, is noted to have the following vector relation satisfied:

$$\bar{V}_3 + \bar{V}_d = \bar{V}_{T_d} . \qquad \text{(III.22)}$$

Here $\bar{V}_3$ is the planetocentric velocity of the spacecraft, and $\bar{V}_{T_d}$ is the heliocentric vector (the subscript T refers to the terminal heliocentric arc).

(3) In heliocentric space the elevation angles (γ_{H_d}, γ_{T_d}) locate $\bar{V}_{H_d}$ and $\bar{V}_{T_d}$ relative to $\bar{V}_d$ (the planet's local velocity vector, paralleling the heliocentric horizon). The corresponding planetocentric angles are (γ_{H_d} + δ_{H_d}) and (γ_{T_d} + δ_{T_d}), locating the planetocentric velocity vectors ($\bar{V}_2$, $\bar{V}_3$) relative to $\bar{V}_d$. (One should note that the δ_{jk} angles position the $\bar{V}_2$, $\bar{V}_3$ relative to the $\bar{V}_{jk}$ vectors).

(4) Since the hyperbola is a symmetric figure in planetocentric space, then it follows that

$$V_3 \equiv V_2 \left(\stackrel{\Delta}{=} \sqrt{\frac{\mu_d}{a}} \right). \qquad \text{(III.23)}$$

Also, according to the assumptions in use here the heliocentric speed of m_d is described by

$$V_d = \sqrt{\frac{\mu_\odot}{r_d}} .$$

(5) In planetocentric space the vehicle flies a hyperbolic track, about m_d, so that the effect of the encounter is to alter the direction of the spacecraft's velocity vector by an (angular) amount, α – in that space.

In heliocentric space the influence of the encounter is to alter the direction of $\bar{V}$ (spacecraft velocity) according to the following scheme: Relative to $\bar{V}_d$ the approach velocity ($\bar{V}_{H_d}$) is located by the elevation angle (γ_{H_d}). After the encounter the angular position of the vehicle's heliocentric velocity vector ($\bar{V}_{T_d}$), relative to $\bar{V}_d$, is (γ_{T_d}).

From the sketch (above) it is evident that $\bar{V}_2$ is described relative to $\bar{V}_d$ by two relations; namely,

$$\measuredangle\ (\bar{V}_2, \bar{V}_d) = \gamma_{H_d} + \delta_{H_d} = \gamma_{T_d} + \delta_{T_d} + \alpha . \qquad \text{(III.24a)}$$

Next, in heliocentric space the angle between the vectors $\bar{V}_{T_d}$ and $\bar{V}_{H_d}$ is

$$\measuredangle\ (\bar{V}_{T_d}\ ,\ \bar{V}_{H_d}) = |\gamma_{H_d} - \gamma_{T_d}|\ ;$$

or, from the expressions above,

$$\measuredangle\ (\bar{V}_{T_d}\ ,\ \bar{V}_{H_d}) = |\alpha + \delta_{T_d} - \delta_{H_d}|\ , \tag{III.24b}$$

denoting that the velocity change in direction is not the same in heliocentric space as it is in planetocentric space.

III.D.7. A DESCRIPTION OF THE VELOCITY, AFTER ENCOUNTER

At exit from the sphere of influence the spacecraft has a velocity $\bar{V}_3$, relative to m_d, and a velocity $\bar{V}_{T_d}$ in heliocentric space. (Note: For clarity the subscript "T" will be employed hereafter to identify the post-encounter, or terminal, heliocentric arc.)

This heliocentric vector can be fully described by: (1) defining its magnitude; and, (2) indicating its direction (say) in terms of its elevation angle (γ_{T_d}). The speed (V_{T_d}) is acquired by manipulating Eqn. (III.22); that is,

$$V_{T_d} \triangleq |\bar{V}_{T_d}| = \sqrt{V_3^2 + V_d^2 + 2\,\bar{V}_3 \cdot \bar{V}_d}\ ,$$

or

$$V_{T_d} = \sqrt{V_3^2 + V_d^2 + 2\,V_3\,V_d \cos\,(\gamma_{T_d} + \delta_{T_d})}\ , \tag{III.25}$$

wherein (as noted previously)

$$V_3 = V_2 \triangleq \sqrt{\frac{\mu_d}{a}}\ ,\ \text{and}\ \ V_d \triangleq \sqrt{\frac{\mu_\odot}{r_d}}\ ;$$

also, the angle ($\gamma_{T_d} + \delta_{T_d}$) is described from Eqn. (III.24) in terms of α, γ_H, δ_H.

In order to describe γ_{T_d}, explicitly, the geometry shown in the sketch above should be consulted to ascertain the following relations:

On the "velocity diagram" one notes that

$$|\bar{V}_3| \sin(\delta_{T_d} + \gamma_{T_d}) = |\bar{V}_{T_d}| \sin(\gamma_{T_d}),$$

but $(\delta_{T_d} + \gamma_{T_d}) = [(\delta_{H_d} + \gamma_{T_d}) - \alpha]$ from Eqn. (III.24); so,

$$\sin(\gamma_{T_d}) = \frac{V_3 \sin[(\delta_{H_d} + \gamma_{T_d}) - \alpha]}{V_{T_d}} \; ; \qquad \text{(III.26a)}$$

and, by manipulating Eqn. (III.22), one can show that

$$\cos(\gamma_{T_d}) = \frac{(V_{T_d}^2 + V_d^2) - V_3^2}{2V_{T_d} V_d} \, . \qquad \text{(III.26b)}$$

A combination of these two expressions leads directly to the arc tangent expression; or,

$$\gamma_{T_d} = \tan^{-1}\left[\frac{2V_d V_3 \sin[(\delta_{H_d} + \gamma_{T_d}) - \alpha]}{V_{T_d}^2 + V_d^2 - V_3^2}\right]. \qquad \text{(III.26c)}$$

Equations (III.25) and (III.26) should provide sufficient information to properly define the velocity ($\bar{V}_{T_d}$) in heliocentric space. With this as a known quantity, plus a knowledge of the position ($\bar{r}_d$), then according to previous assumptions one should be able to establish the nature of the terminal arc. Having a description of this (last) segment of the space track, the analyst would be able to complete his computations for the entire mission.

Prior to beginning a description for this last phase of the journey the side issue of "time to fly through the sphere of influence" will be mentioned.

III.D.8. TIME OF FLIGHT THROUGH THE SPHERE OF INFLUENCE

In many instances it may be desirable to estimate a time required for the spacecraft to complete the hyperbolic encounter, per se. Recognizing that this is the time of transit through the sphere of influence (from point (A) to point (B) shown on the sketch above), then it is readily evident that the time lapse for this segment of the flight track can be defined from

$$(\Delta t)_{IN} = 2\sqrt{\frac{a^3}{\mu_d}}\left[\frac{\epsilon_d \sqrt{\epsilon_d^2 - 1}\ \sin \varphi_2}{1 + \epsilon_d \cos \varphi_2} - 2 \tanh^{-1}\left(\sqrt{\frac{\epsilon_d - 1}{\epsilon_d + 1}}\ \tan \frac{\varphi_2}{2}\right)\right], \qquad \text{(III.27)}$$

where $(\Delta t)_{IN}$ signifies the time lapse in traversing the Sphere of Influence. In this expression (ϵ_d, a) are parameters describing the hyperbolic arc; μ_d is the gravitational parameter for the perturbing planet (m_d); and φ_2 is the planetocentric position angle corresponding to the actual entry (position) into the sphere of influence. The multiplier, "2", is included here to describe the full transit time; Eqn. (III.27), without the multiplier, is indicative of the time of motion from pericenter to the sphere of influence boundary (described by φ_2).

In order to obtain a realistic time measure, here, it is necessary to relax the assumption-used earlier-regarding the "size" of the sphere of influence. In the present instance, a true value for r_{IN} is needed so that a proper estimate of φ_2 (the position angle, corresponding to r_{IN}, on the hyperbolic path) can be obtained. Thus, using the conic equation – having determined r_{IN} from (III.9) and ϵ_d from Eqn. (III.18) – it follows that

$$\varphi_2 = \cos^{-1}\left[\frac{1}{\epsilon_d}\left(\frac{p_d}{r_I} - 1\right)\right] \qquad \text{(III.28)}$$

with p_d found from Eqn. (III.21).

In keeping with the various assumptions used for the Mode I calculations, it is quite likely that for the simpler case studies, the time to fly through the sphere of influence could be neglected (in many instances); yet, on a small (time) scaled mission it may be prudent to include this time fraction in the overall estimates. Such a decision must be made as a part of an evaluation for the overall problem.

III.D.9. CHARACTERISTICS OF THE TERMINAL TRAJECTORY

After the spacecraft exits from the influence of m_d, and continues on its way to m_f (the "final mass" and its space position), via a free flight heliocentric trajectory, one should ascertain the characteristics of this flight path. As an aid to this description the flight path can be identified simply in terms of how the vehicle's heliocentric speed, after swing-by, compares to (say) the heliocentric escape speed. If the spacecraft's speed is less than the escape value then the final path is an ellipse. However, if the flight speed is above the parabolic value (corresponding to the local value of r) then the vehicle escapes from m_d and progresses on toward m_f along a hyperbolic "free path."

In order to describe the character of this final path the vehicle's speed condition may be examined according to (the specific energy equation):

$$V_{T_d}^2 = 2\,\frac{\mu_\odot}{r_d} + \frac{\mu_\odot}{a_T}\;;$$

but $\mu_\odot / a_T = V_{\infty_\odot}^2$, so that

$$V_{\infty_\odot}^2 = V_T^2 - V_{esc_\odot}^2\,, \tag{III.29}$$

wherein $V_{esc_\odot}^2 = 2\,\mu_\odot / r_d$ (the escape speed squared, in heliocentric value), and $V_{\infty_\odot}$ is the heliocentric hyperbolic excess speed.

If Eqn. (III.29) is negative then the "escape" condition has not been realized and the flight path "T" is elliptic; however, if $V_{\infty_\odot}^2 \geq 0$, then escape has been attained and the heliocentric trajectory is a hyperbola – incidentally, one should recall that the parabolic path is a special case of the hyperbola, but one where $a_T \to \infty$ and $V_{\infty_\odot}^2$ vanishes.

Necessarily, the Terminal Flight Path would not be expected to have the same (or a parallel) line of apsides compared to the Hohmann Path. Likewise, these figures (in heliocentric space) are not likely to have a same eccentricity, or size. So, in order to complete the description of the terminal path these orbital parameters must be obtained.

The eccentricity of the terminal path (ϵ_T) is easily shown to be expressed by

$$\epsilon^2 = 1 + \frac{2Eh^2}{\mu^2}\quad ,$$

or

$$\epsilon_T = \left[\left(\frac{r_d\, V_{T_d}^2}{\mu_\odot} - 1\right)^2 \cos^2\gamma_{T_d} + \sin^2\gamma_{T_d}\right]^{1/2} \tag{III.30}$$

for the situation considered here (Mode I transfer geometry); and at exit from the sphere of influence, but with the exit assumed to occur at r_d.

The size of the terminal path can be described by the parameter (p_T); or, by the major dimension (a_T). Since $|E_1| = \mu/2a$ (for either figure), then it follows that

$$a_T = \left| \frac{\mu_\odot}{2E_1} \right| \equiv \left| \frac{\mu_\odot}{V_{T_d}^2 - 2\,\frac{\mu_\odot}{r_d}} \right| \tag{III.31}$$

where the absolute value is employed to leave a_T as a positive definite parameter regardless of the geometry describing the path (T).

In order to account for the "shift" (angular displacement) of the line of apsides (between the original Hohmann, and the terminal path) one should calculate the (new) position angle (φ_{T_d}) and then describe this displacement according to the relation

$$\Delta\varphi_d = \varphi_{H_d} - \varphi_{T_d}\,, \tag{III.32}$$

where φ_{H_d} is obtained from (III.10), and where φ_{T_d} is obtained from

$$\varphi_{T_d} = \cos^{-1}\left[\frac{p_T}{r_d\,\epsilon_T} - \frac{1}{\epsilon_T}\right], \tag{III.33}$$

In Eqn. (III.33) $p_T = a_T\ |\ 1 - \epsilon_T^2\,|$, accounting for all conic types (except the parabola).

Note: If (III.32) yields a negative resultant, then the line of apsides (terminal path) has moved in a retrograde direction relative to the original apsidal axis (for the Hohmann). A positive resultant indicates a posigrade rotation of the apsidal axis (wrt the Hohmann).

Some other quantities of interest, describing the terminal path, are noted below:

(1) Pericentric radius (of path T),

$$r_{\mathcal{P}_T} = a_T\ |1 - \epsilon_T|\ ; \qquad \text{(hyp. ellip.)}$$

(2) Apocentric radius,

$$r_{\mathcal{A}_T} = a_T\ (1 + \epsilon_T); \qquad \text{(ellip. only)}$$

(3) Period of Motion, for path T,

$$P = 2\pi\sqrt{\frac{a_T^3}{\mu_\odot}}\ ; \qquad \text{(ellip. only)}$$

(4) parameter, p_T ,

$$p_T = a_T \left| 1 - \epsilon_T^2 \right| ; \qquad \text{(ellip., hyp.)}$$

and

(5) parameter p_T (parabolic path),

$$p_T = 2 r_{\wp_T} . \qquad \text{(parab. only)}$$

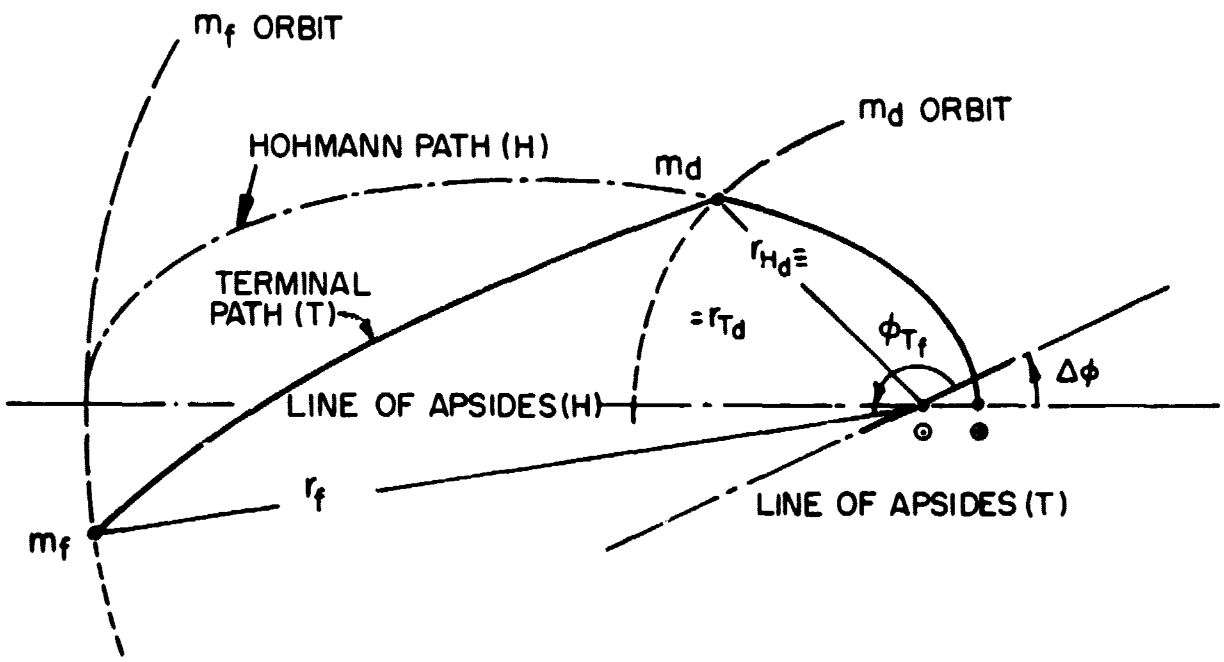


Figure III.5– Sketch Showing the Flight Paths for the Swing-By at m_d

At the terminus of the mission the spacecraft is assumed to have reached m_f (along the Terminal Path). In this regard the position of the intercept at the terminus is specified by (r_f, φ_f) where r_f is known, a priori. To calculate φ_{T_f} one can resort to the conic equation, obtaining

$$\varphi_{T_f} = \cos^{-1} \left[\frac{p_T}{r_f \, \epsilon_T} - \frac{1}{\epsilon_T} \right] \qquad \text{(III.34)}$$

wherein p_T and ϵ_T are known (see Eqns. (III.30, (III.33)). Note that Eqn. (III.34) locates m_f relative to the pericenter for path (T).

III.D.10. TIME OF FLIGHT SUMMARY

In the analysis discussed here the total time of flight involves motion along two arcs, at least. The first segment involves the motion from $\oplus$ to m_d; and, the second involves the arc from m_d to m_f. (In some instances, but not necessarily the case at hand, one may wish to account for the time within the sphere of influence. However, for present purposes this latter time increment will be ignored, and the mission, timewise, will be assumed to be composed solely of the two principal arcs for the transfer.)

The two arcs considered here may be segments of ellipses; or, the last arc may be a segment of a hyperbola. Thus, to account for the total "time of motion" it will be necessary to include both (possible) equations and to use them appropriately.

Since the equations for time, set down below, count time from pericenter passage it will be necessary to "manipulate" these expressions in order to obtain the desired results. One should note that these expressions involve the orbit parameters (a, ϵ) and the position coordinate (φ), only; thus, time is expressed as

$$t = t(a, \epsilon, \varphi);$$

or specifically:

(1) For an elliptic arc (from pericenter to some position φ);

$$t = \sqrt{\frac{a^3}{\mu_\odot}}\left[-\frac{\epsilon\sqrt{|\epsilon^2 - 1|}\sin\varphi}{1 + \epsilon\cos\varphi} + 2\tan^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan\frac{\varphi}{2}\right)\right]. \qquad \text{(III.35a)}$$

(2) For a hyperbolic arc (from pericenter to a position φ);

$$t = \sqrt{\frac{a^3}{\mu_\odot}}\left[\frac{\epsilon\sqrt{\epsilon^2 - 1}\sin\varphi}{1 + \epsilon\cos\varphi} - 2\tanh^{-1}\left(\sqrt{\frac{\epsilon-1}{\epsilon+1}}\tan\frac{\varphi}{2}\right)\right], \qquad \text{(III.35b)}$$

or

$$t = \sqrt{\frac{a^3}{\mu_\odot}}\left[\frac{\epsilon\sqrt{\epsilon^2 - 1}\sin\varphi}{1 + \epsilon\cos\varphi} - \ell n\left(\frac{\sqrt{\epsilon+1} + \sqrt{\epsilon-1}\tan(\varphi/2)}{\sqrt{\epsilon+1} - \sqrt{\epsilon-1}\tan(\varphi/2)}\right)\right]. \qquad \text{(III.35c)}$$

In computing the time of flight (neglecting the time passage when the vehicle inside the sphere of influence) the above equations are employed in the following manner:

(A) Time from $\oplus$ to m_d – use (III.35a) wherein

$$t = t\,(a_H, \varphi_{H_d}, \epsilon_H) \text{ – see eqn. (III.14), also.}$$

(B) Time from m_d to m_f – use (III.35a), or (III.35b), (III.35c), as follows:

(1) if $0 < \epsilon_T < 1.0$; then use (III.35a), calculating:

$$t_1 = t_1\,(a_T, \varphi_{T_f}, \epsilon_T),$$

$$t_2 = t_2\,(a_T, \varphi_{T_d}, \epsilon_T),$$

and then,

$$\Delta t = t_1 - t_2 = \text{time of flight from } m_d \text{ to } m_f; \qquad \text{(III.36a)}$$

(2) if $1.0 < \epsilon < \infty$; then use (III.35b) or (III.35c), calculating:

$$t_1 = t_1\,(a_T, \varphi_{T_f}, \epsilon_T),$$

$$t_2 = t_2\,(a_T, \varphi_{T_d}, \epsilon_T)$$

and then,

$$\Delta t = t_1 - t_2 = \text{time of flight from } m_d \text{ to } m_f. \qquad \text{(III.36b)}$$

III.D.11. ENERGY CHANGE DUE TO SWING-BY

Another of the trajectory parameters which can be significantly affected by the swing-by maneuver is the total specific energy, referred to heliocentric space. The change which occurs in the energy is caused by the vehicle's heliocentric velocity vector being altered (in direction), due to its passage closeby the disturbing body (which is, itself, in motion).

In order to describe the change in total energy (positive or negative) one should recall that the specific energy is defined by

$$E_1 \triangleq \frac{V^2}{2} - \frac{\mu}{r};$$

thus, in heliocentric space, V is the speed along the arc(s) (H and/or T), while $\mu = \mu_\odot$, and r defines position relative to $m_\odot$.

For this present case, the position at encounter has been assumed to be fixed; i.e. $r = r_d$, and consequently,

$$\Delta E \triangleq E_{1_T} - E_{1_H} = \left(\frac{V_{T_d}^2}{2} - \frac{\mu_\odot}{r_d}\right) - \left(\frac{V_{H_d}^2}{2} - \frac{\mu_\odot}{r_d}\right) ,$$

or

$$\Delta E = \frac{V_{T_d}^2}{2} - \frac{V_{H_d}^2}{2} . \tag{III.37}$$

This last expression will describe the energy gained ($\Delta E > 0$), or lost ($\Delta E < 0$), during the swing-by maneuver.

Essentially, now, the calculation procedures, by a Mode I analysis, are complete. In the next paragraphs, the more exacting scheme (Mode II) will be described.

III.E.1. MODE II TRANSFER, SWING-BY OPERATION

The extension to the earlier problem, introduced here, is such that a finite size for the sphere of influence is included in this next formulation, and is utilized throughout the computational procedure. That is, instead of having the two principal arcs end and begin at m_d, these are now carried only to the sphere of influence – where each arc intersects this imaginary boundary, and the motion through the sphere of influence becomes an integral part of the problem. The motion through this region becomes a real and essential piece of the overall analysis.

The assumption of r_{IN} as a finite (or non-zero) value adds a complication to the problem which was not previously encountered.

In describing the sphere of influence, its size will be assumed to be calculated from (see Eqn. (III.9))

$$\rho_I = r_i \left(\frac{m_i}{m_j}\right)^{2/5} ,$$

where i , j refer to the disturbing mass bodies being considered; or, specifically, for this problem where the encounter is with m_d, then

$$\rho_I = r_d \left(\frac{m_d}{m_\odot} \right)^{2/5} , \; \rho_I = \text{radius of sphere of influence, relative to } m_d \text{ as a center;} \tag{III.38}$$

and, wherein r_d locates m_d relative to $m_\odot$.

Now, the Mode II problem will be concerned with three (3) separate arcs of motion, namely;

(1) The Hohmann arc from $\oplus$ to $(\rho_I)_H$, the sphere of influence;

(2) the hyperbolic arc through the Sphere of Influence; and,

(3) the free trajectory from the Sphere of Influence to m_f !

III.E.2. THE PRE-ENCOUNTER TRAJECTORY

This path arc is (again) selected to be a portion of the Hohmann trajectory* connecting the orbits of $m_\oplus$ and m_f. As expressed in Eqns. (III.1) through (III.8) the characteristics of this arc are:

(a) $\epsilon_H = \dfrac{r_{apo} - r_{peri}}{r_{apo} + r_{peri}} \equiv \dfrac{r_f - r_\oplus}{r_f + r_\oplus}$ (eccentricity of the Hohmann arc)

(b) $V_H = \left[2 \dfrac{\mu_\odot}{r_H} \left(1 - \dfrac{r_H}{2a_H} \right) \right]^{1/2}$ (speed, at any point along the Hohmann arc)

(c) $V_{H_\oplus} = \left[2 V_{c_\oplus}^2 \left(1 - \dfrac{r_\oplus}{2a_H} \right) \right] = V_{c_\oplus} \left[\dfrac{2 r_f}{r_f + r_\oplus} \right]^{1/2}$ (speed, at $m_\oplus$ position on the Hohmann arc)

*As noted earlier, the selection of a Hohmann path is done mainly for convenience and familarity. In any case the initial arc may be selected as any conic the analyst would care to describe.

(d) $V_{H_f} = V_{c_f}\left[\dfrac{2r_\oplus}{r_f + r_\oplus}\right]^{1/2}$ (speed, at the m_f position on the Hohmann arc)

(e) $\Delta V_\oplus = V_{c_\oplus}\left(\sqrt{\dfrac{2r_f}{r_f + r_\oplus}} - 1\right)$ (increment in speed needed to leave the circle ($r_\oplus$) and get onto the Hohmann arc)

(f) $\Delta V_f = V_{c_f}\left(1 - \sqrt{\dfrac{2r_\oplus}{r_f + r_\oplus}}\right)$ (speed increment needed to get off the Hohmann arc, at m_f, and onto the circle (r_f))

(g) $p_H = a_H\,(1 - \epsilon_H^2)$
(orbital parameter)

(h) $b_H = a_H\sqrt{1 - \epsilon_H^2} = \sqrt{a_H\,p_H}$
(minor axis length)

(i) $\Delta t_H = \dfrac{P_H}{2} = \pi\sqrt{\dfrac{a_H^3}{\mu_\odot}}$ (time for motion, from $m_\oplus$ to m_f, on the Hohmann arc)

The quantities listed above describe the Hohmann ellipse (or, specifically, the semi-ellipse) on which the spacecraft is assumed to move initially. What has not been described here is the location and conditions of motion at the intercept between the vehicle's path and the "boundary" for the sphere of influence.

Because of the non-zero size for this boundary the "point" where intercept can occur is rather vague; actually, there is a fairly large length (of the major circle of this sphere) on which this point can lie. Assuming that the situation to be described (here) is fully one in a single plane, then the intercept can be described, and the trajectory characteristics at intercept can be determined.

III.E.3. SPEED AT ENTRY TO THE SPHERE OF INFLUENCE

According to Figure III.6, it is seen that the maximum magnitude for r_I, $r_{I_{max}} = r_d + \rho_I$; and, the least magnitude if $r_{I_{min}} = r_d - \rho_I$. In a like manner, the range of values for the angle φ_{H_I} is;

$$\left(\varphi_{H_d} - \delta\varphi_d\Big|_{max}\right) \leq \varphi_{H_I} \leq \left(\varphi_{H_d} - \delta\varphi_d\Big|_{min}\right) \qquad \text{(III.39)}$$

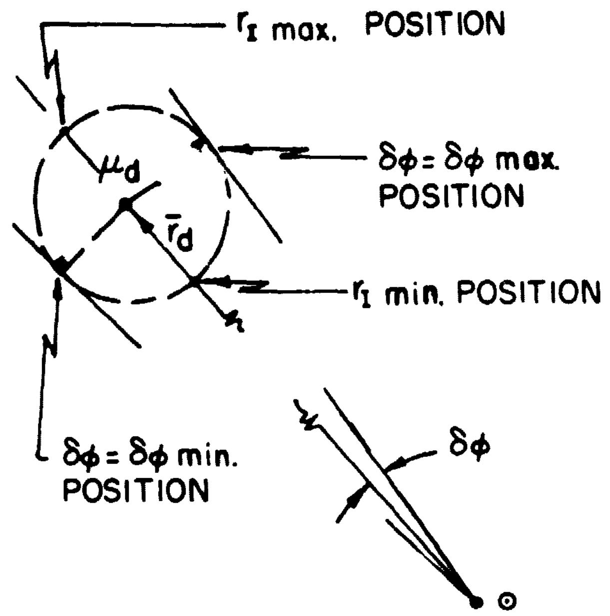


Figure III.6–Sketch Illustrating Extremes for the Sphere of Influence.

where

$$\delta\varphi_d = \varphi_{H_d} - \varphi_{H_I};$$

(these angles (φ_i) are measured relative to the pericenter of the Hohmann arc, with $m_\odot$ as a focus).

For the situation illustrated on the figure, it is evident that

$$\delta\varphi_d\Big|_{max} \geq 0, \ \delta\varphi_d\Big|_{min} \leq 0\,!$$

Note: At $\delta\varphi_d|_{max, min}$ the radius, $\bar{r}_I$, is tangent to the Sphere of Influence so that the intersection of the Hohmann trajectory with a major circle of the sphere is such that $|\bar{r}_I| < |\bar{r}_d|$, by some "small" amount in a real sense. It should be evident that at this condition $\bar{r}_I$ and $\bar{\rho}_I$ are orthogonal vectors (the scalar product vanishes); and, that, as such,

$$r_I = \sqrt{r_d^2 - \rho_I^2}\ ;$$

while, for this condition,

$$\delta\varphi_d = \sin^{-1}\left(\frac{\rho_I}{r_d}\right),$$

and the relationship between θ and $\delta\varphi$ is such that

$$\theta + \left| \delta\varphi \right|_{\substack{max,\\ min}} \cong \pi/2 .$$

This would indicate that for this condition

$$\cos\theta = \sin\delta\varphi$$

$(\delta\varphi = \left|\delta\varphi\right|_{max,min})$.

III.E.4. A DESCRIPTION OF THE INTERCEPT POSITION (I)

The intercept to be considered here occurs as the vehicle flies along the initial transfer path. It is to occur in heliocentric space at the position denoted by φ_{H_I} and measured from the pericenter position for the Hohmann ellipse. Necessarily φ_{H_I} satisfies the constraint, noted above; namely, that

$$\varphi_{H_I} = \varphi_{H_d} \pm \delta\varphi_d , \qquad \text{(III.40)}$$

where $\pm\,\delta\varphi_d \leq \sin^{-1}(\rho_I/r_d)$. Following along with this condition, then, it is apparent that the radius to the point, I, can be expressed as

$$r_I = \frac{p_H}{1 + \epsilon_H \cos\varphi_{H_I}} , \qquad \text{(III.41a)}$$

from the conic equation.

As an alternate description for this radius, note that from Figure III.7 (for the condition shown)

$$r_I \sin\delta\varphi_d = \rho_I \sin\theta, \text{ and, } r_I \cos\delta\varphi_d + \rho_I \cos\theta = r_d$$

so that

$$r_I = \frac{\rho_I \sin\theta}{\sin\delta\varphi_d}; \text{ and/or, } r_I = \frac{r_d - \rho_I\cos\theta}{\cos\delta\varphi_d}. \qquad \text{(III.41b)}$$

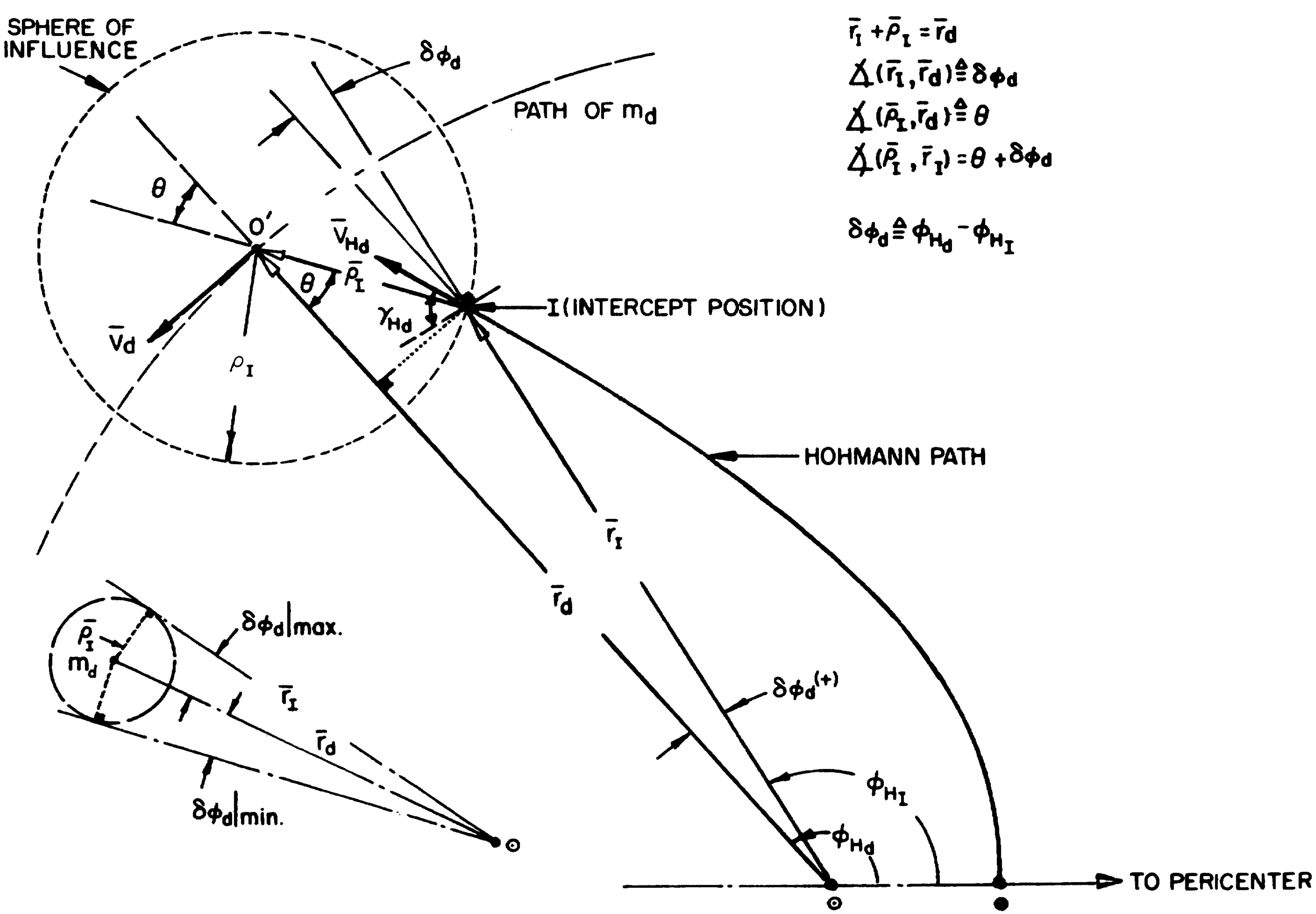


Figure III.7–The Geometric and Kinematic Conditions at Entry into the Sphere of Influence from the Hohmann Path

In Eqn. (III.41) the radius (r_I) was defined; consequently, now, one knows the coordinates locating the point where the vehicle penetrates the "sphere of influence." Even though this is a ficticious boundary, nevertheless it is essential that the intercept point (I) should be specified in order that the problem may proceed according to the assumptions set down earlier.

Knowing the position of (I), one can describe the spacecraft's heliocentric speed, at this point, from the specific energy equation. In this regard, it is easily verified that

$$V_{H_d} \triangleq |\overline{V}_{H_d}| = \left[2\frac{\mu_\odot}{r_I} - \frac{\mu_\odot}{a_H}\right]^{1/2} \tag{III.42}$$

(See Fig. III.7 for a graphical description of $\overline{V}_{H_d}$!)

Also, from the figure, it is evident that

$$\overline{r}_I + \overline{\rho}_I = \overline{r}_d;$$

thus

$$\overline{r}_I = \overline{r}_d - \overline{\rho}_I;$$

and, therefore,

$$\begin{aligned} r_I^2 &= r_d^2 + \rho_I^2 - 2\,\overline{r}_d \cdot \overline{\rho}_I \\ &= r_d^2 + \rho_I^2 - 2\,r_d\,\rho_I \cos(\theta + \delta\varphi_d). \end{aligned}$$

Including Eqn. (III.41) into the above relation, then,

$$\frac{p_H^2}{(1 + \epsilon_H \cos\varphi_{H_I})^2} = r_d^2 + \rho_I^2 - 2\,r_d\,\rho_I \cos(\theta + \delta\varphi_d),$$

wherein the unknowns (here) are the angles θ and $\delta\varphi_d$ (r_d is known, a priori; ρ_I is calculated from Eqn. (III.38); p_H and ϵ_H are calculated for the Hohmann path; and the angle $\varphi_{H_I} = \varphi_{H_d} - \delta\varphi_d$ is as described before).

A part of the present computational dilemma can be overcome by noting that θ and $\delta\varphi_d$ are connected through Eqn. (III.41b) – which can be used to express $\theta = \theta(\delta\varphi_d)$; that is,

$$\theta = \sin^{-1}\left[\frac{r_I \sin \delta\varphi_d}{\rho_I}\right] = \sin^{-1}\left[\frac{p_H}{1 + \epsilon_H \cos \varphi_{H_I}} \left(\frac{\sin \delta\varphi_d}{\rho_I}\right)\right] \tag{III.43}$$

Now, with θ defined, as shown in Eqn. (III.43), then

$$\frac{p_H^2}{[1 + \epsilon_H \cos(\varphi_{H_d} + \delta\varphi_d)]^2} = r_d^2 + \rho_I^2 - 2 r_d \rho_I \cos(\theta + \delta\varphi_d), \tag{III.44}$$

which is an explicit relation in the unknown $\delta\varphi_d$ (i.e. including Eqn. (III.43) to relate θ to $\delta\varphi_d$). Of course, it should be remembered that $\delta\varphi_d$ is constrained as indicated by the comment given with Eqn. (III.40).

An iteration and/or interpolation formula (or algorithm) can be developed to satisfy this expression, Eqn. (III.44). It should be evident that there are actually four values for $\bar{r}_I$ which could theoretically satisfy this relation (based on a given value of $|\delta\varphi_d|$); these represent the various quadrant intersections of the path (H) with the sphere of influence boundary at a value of ($\pm$) $\delta\varphi_d$. Necessarily, the physical problem and its constraints will dictate which of the points one should consider.

After having obtained $\delta\varphi_d$ from (III.44) – and, likewise, defining θ by Eqn. (III.43) – then the entry point is known, specifically and in toto, by the coordinates (r_I, φ_{H_I}). (This is tantemount to specifying a value for φ_{H_I} and then solving for r_I).

A last quantity needed to fully describe conditions at intercept with the sphere of influence is the velocity vector's elevation angle (γ_{H_d}) as noted on the figure. By definition this quantity is obtained from

$$\tan \gamma_{H_d} = \frac{\epsilon_H \sin \varphi_{H_I}}{1 + \epsilon_H \cos \varphi_{H_I}}, \tag{III.45}$$

which is used to "locate" the velocity vector relative to the local horizon, (which is normal to $\bar{r}_I$ here).

Occasionally one wishes to know the angularity of the velocity vector ($\bar{V}_{H_d}$) with respect to the radius vector ($\bar{\rho}_I$) – see Figure III.8! From the figure it is noted that (per the case shown)

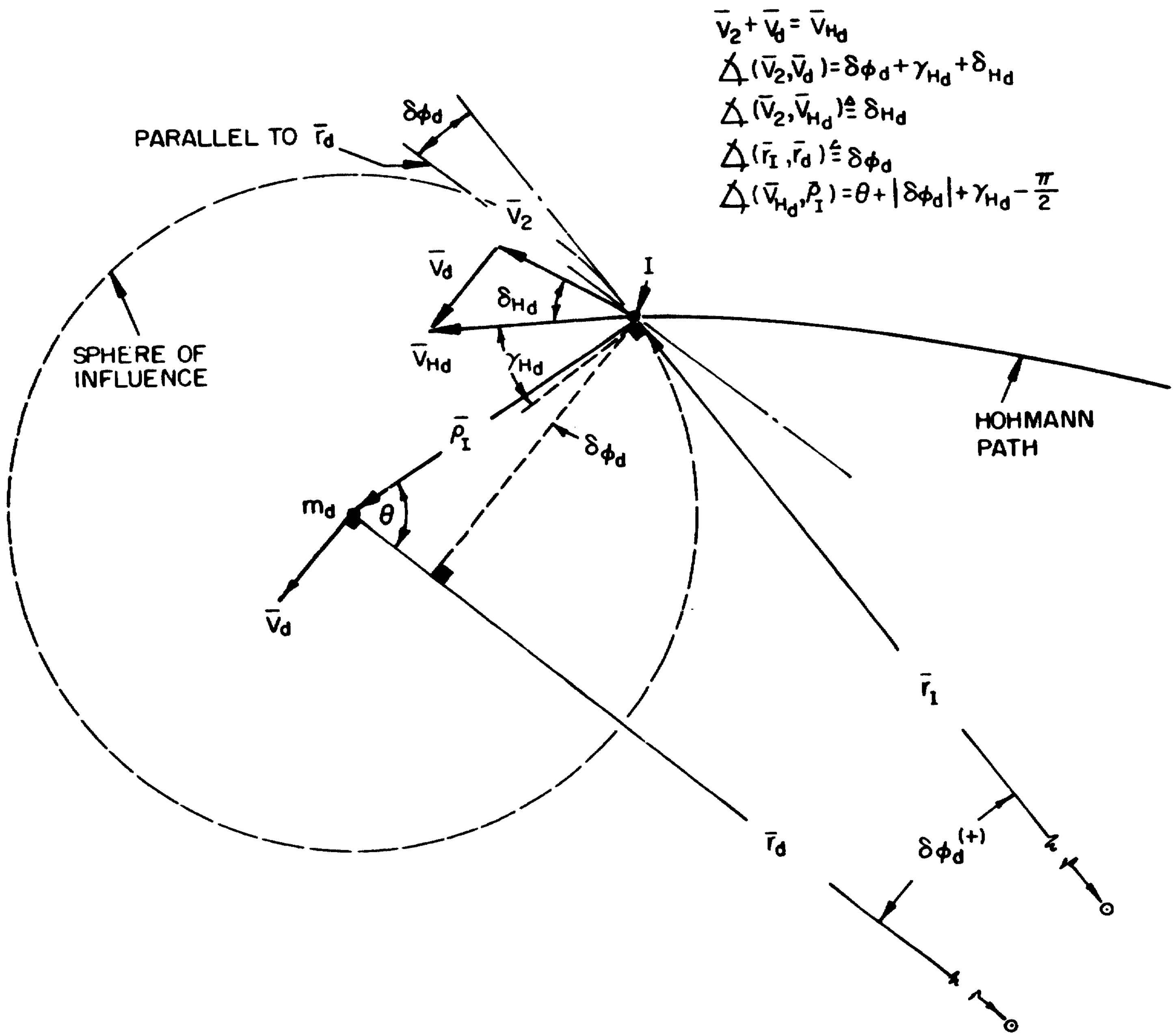


Figure III.8–Entry Conditions; Showing Planetocentric and Heliocentric Velocities, Angles of Reference and Coordinates

$$\sphericalangle (\overline{V}_{H_d}, \overline{r}_I) = \frac{\pi}{2} - \gamma_{H_d};$$

and, also, that

$$\sphericalangle (\overline{V}_{H_d}, \overline{\rho}_I) = (\theta + |\delta\varphi_d|) - \left(\frac{\pi}{2} - \gamma_{H_d}\right)$$

$$= \theta + |\delta\varphi_d| + \gamma_{H_d} - \frac{\pi}{2}; \qquad \text{(III.46)}$$

thus the angularity is defined in terms of known quantities.

III.E.5. TIME OF FLIGHT (FROM LAUNCH TO INTERCEPT, I).

Since the initial segment of the flight path is an arc of the Hohmann ellipse then one can calculate the time required to go from pericenter (at $m_\oplus$) to the entry point (I) by means of,

$$\Delta t = \sqrt{\frac{a_H^3}{\mu_\odot}}\left[-\sqrt{1 - \epsilon_H^2}\ \tan\gamma_{H_d} + 2\tan^{-1}\left(\sqrt{\frac{1-\epsilon_H}{1+\epsilon_H}}\ \tan\frac{\varphi_{H_I}}{2}\right)\right], \qquad \text{(III.47)}$$

wherein tan γ_{H_d} is defined above, Eqn. (III.45). This is only the first time increment for the overall flight. The remaining two (2) intervals, to be calculated, will be discussed and determined subsequently.

III.E.6. HYPERBOLIC TRAJECTORY, INSIDE THE SPHERE OF INFLUENCE

Within the sphere of influence the flight trajectory will (usually) be a hyperbola – thus the reference in the literature to a "hyperbolic encounter." In this section the nature and description of this part of the overall flight maneuver will be presented and discussed.

On Figure (III.8) the geometric and kinematic conditions at entry into the sphere of influence are depicted. The velocity vector for the vehicle, relative to the heliocentric (i.e. $\overline{V}_{H_d}$) and relative to the attracting mass (i.e. $\overline{V}_2$), are shown, along with the velocity of the disturbing mass (m_d) relative to the heliocenter (i.e. $\overline{V}_d$) – and these are related to one another by

$$\bar{V}_{H_d} = \bar{V}_2 + \bar{V}_d . \qquad \text{(III.48)}$$

Accordingly, this expression can be scalar expanded to yield

$$V_{H_d}^2 = V_2^2 + V_d^2 + 2\bar{V}_2 \cdot \bar{V}_d \equiv V_2^2 + V_d^2 + 2V_2V_d \cos(\bar{V}_2, \bar{V}_d), \qquad \text{(III.49)}$$

where

$$\measuredangle (\bar{V}_2, \bar{V}_d) = (\delta_{H_d} + \gamma_{H_d} + |\delta\varphi_d|), \text{ as shown.}$$

Rearranging Eqn. (III.48) to read

$$\bar{V}_2 = \bar{V}_{H_d} - \bar{V}_d \qquad \text{(III.50)}$$

then as expression for $|\bar{V}_2|$ is obtained as

$$V_2 = \sqrt{V_{H_d}^2 + V_d^2 - 2\bar{V}_{H_d} \cdot \bar{V}_d} \equiv \sqrt{V_{H_d}^2 + V_d^2 - 2V_{H_d}V_d \cos(\bar{V}_{H_d}, \bar{V}_d)} \qquad \text{(III.51)}$$

wherein $\measuredangle (\bar{V}_{H_d}, \bar{V}_d) = (\gamma_{H_d} + |\delta\varphi_d|)$, which is known! Consequently Eqns. (III.51) and (III.49) can be used to determine $|\bar{V}_2|$, in addition to the angle δ_{H_d} (= $\measuredangle (\bar{V}_2, \bar{V}_{H_d})$).

III.E.7. A MEASURE OF ASSUMPTION ACCURACY

One of the assumptions used here is that the hyperbolic trajectory through the sphere of influence is reasonably close to a two-body conic and its properties. One recalls that for such a geometry the asymptotes and the figure merge at large distances from the focus (as the radius approaches infinity). Certainly this situation does not exist here; and, even though the sphere of influence may be large there will always be some small (angular) separation between these two geometric lines. One measure of how accurate the present analysis is, compared to the assumption stated, would be the proximity of the radius (ρ_I) to the asymptote. That is, if the angle β (shown below) vanished, then the approximation becomes an "exact" hypothesis; and, as β becomes large (compared to zero) the degree of approximation is worsened. In this regard an expression for β is developed below; however, no numeric is assigned to the angle, as a means of ascertaining the degree of accuracy achieved by any particular analysis.

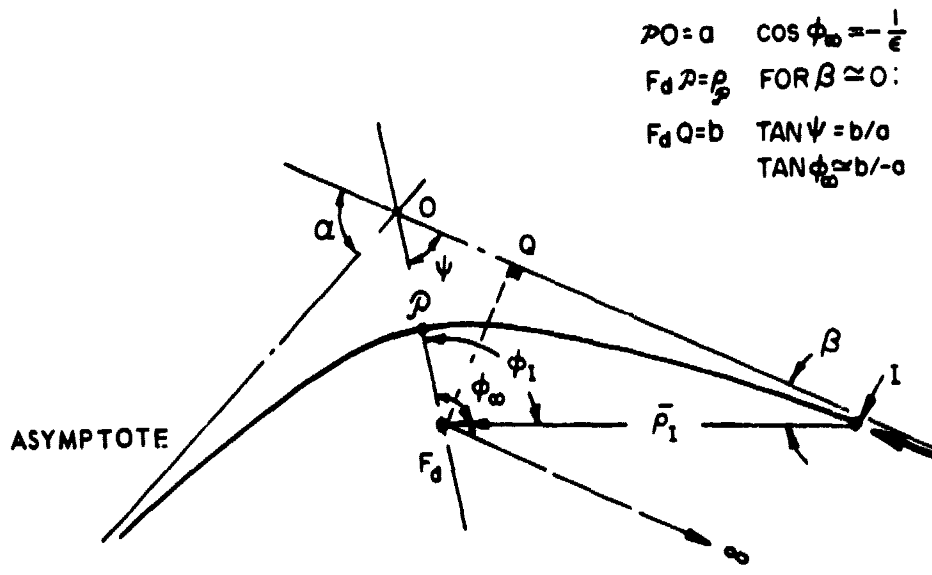


Figure III.9–Sketch Showing the Planet Centered Hyperbolic Path through the Sphere of Influence.

According to the sketch (above) the perpendicular to the asymptote (drawn through the Focus F_d) – i.e., the line $F_d Q$ ($\equiv$ b), – is related to ρ_I (the radius of the sphere of influence) by

$$F_d Q(\equiv b) = \rho_I \sin\beta \tag{III.52}$$

where $\beta = \measuredangle(\bar{V}_2, \bar{\rho}_I)$. From an inspection of Figure III.9 it is found that

$$\beta = (\delta_{H_d} + \gamma_{H_d} + |\delta\varphi_d|) - \left(\frac{\pi}{2} - \theta\right)$$

$$= \delta_{H_d} + \gamma_{H_d} + |\delta\varphi_d| + \theta - \frac{\pi}{2}\ , \tag{III.53}$$

wherein all quantities are (now) known.

An alternate evaluation of β could be achieved by determining the angle (ψ) of the asymptote, and the position angle (φ) corresponding to ρ_I, then describing β accordingly. This should be compared with the expression above to ascertain the deviation between the exact and approximation situations.

III.E.8. DETERMINATION OF PERICENTER

A quantity necessary to the successful study of an encounter problem (in a physical sense) is the expected planetocentric pericenter radius, ρ_p. In order to relate this quantity to "entry" conditions, and known terms, one might follow the procedure outlined below:

From conservation considerations (assuming, as noted before, the autonomy of the sphere of influence);

$$\rho_p V_p \cong (F_d Q) V_2 = b V_2 ,$$

and

$$V_p^2 - 2 \frac{\mu_d}{\rho_p} = V_2^2 - 2 \frac{\mu_d}{\rho_I} .$$

Combining these expressions, elimination V_p, then

$$V_2^2 - 2 \left(\frac{\mu_d}{\rho_I} - \frac{\mu_d}{\rho_p} \right) = \frac{b^2 V_2^2}{\rho_p^2}$$

leading to a definition of ρ_p as,

$$\rho_p \cong \frac{b}{\frac{b}{\mu_d} \left(V_2^2 - 2 \frac{\mu_d}{\rho_I} \right)} \left[\sqrt{1 + \left(\frac{bV_2}{\mu_d} \right)^2 \left(V_2^2 - 2 \frac{\mu_d}{\rho_I} \right)} - 1 \right] . \qquad \text{(III.54)}$$

III.E.9. ORBITAL PARAMETERS (HYPERBOLA)

The orbital parameters for the hyperbolic path are easily defined from the geometric, dynamic properties of the encounter (within the sphere of influence). These are evaluation below:

(1) The major length of the geometric figure (a) can be obtained from the planetocentric energy expression (E_1), since

$$(E_1)_d = \frac{V^2}{2} - \frac{\mu_d}{r} \equiv + \frac{\mu_d}{2a} \quad \text{(assuming a hyperbola)}$$

thus, for instance, for conditions corresponding to entry at the boundary of the sphere of influence,

$$a = \frac{\mu_d}{V_2^2 - \frac{\mu_d}{\rho_I}} = \frac{b}{\frac{b}{\mu_d}\left(V_2^2 - \frac{\mu_d}{\rho_I}\right)} . \qquad \text{(III.55)}$$

(2) Since the figure of the motion is a conic, then by definition, the radius to pericenter is

$$\rho_p \triangleq a(\epsilon - 1);$$

consequently, the eccentricity can be described from,

$$\epsilon = \frac{\rho_p}{a} + 1 . \qquad \text{(III.56a)}$$

As a matter of interest, comparing the expression above, for ρ_p, to that given in Eqn. (III.54) it is noted that (alternately)

$$\epsilon \cong \sqrt{1 + \left(\frac{bV_2}{\mu_d}\right)^2 \left(V_2^2 - 2\,\frac{\mu_d}{\rho_I}\right)} . \qquad \text{(III.56b)}$$

(3) The turn (or angular change in direction) for the vehicle and its velocity vector, due to the encounter, is indicated on the figure above by α. This angle can be defined from the following:

Note that

$$\alpha + 2\psi = \pi ,$$

wherein the angle ψ has been obtained from

$$\psi = \tan^{-1} \frac{b}{a} ;$$

or, alternately,

$$\psi \simeq \pi - \varphi_\infty ,$$

with

$$\varphi_{\infty} = \cos^{-1}\left(-\frac{1}{\epsilon}\right)$$

(recall that φ_{∞} is the <u>limit position angle</u> for the hyperbola). If one employs these approximations, then

$$\alpha = \pi - 2\psi \ [\text{where } \psi = \tan^{-1}(b/a)]; \qquad \text{(III.57a)}$$

or, alternately

$$\alpha \simeq -(\pi - \varphi_{\infty}) \cong -\psi. \qquad \text{(III.57b)}$$

This would indicate that $\alpha \simeq \pi/3$ for the encounter (regardless of the <u>initial</u> situation). Obviously this leads to an anomalistic condition which should be recognized as having arisen from the fact that, herein, the hyperbolic figure has been assumed to lie close to its own asymptotes; and, that the entry conditions to the sphere of influence <u>closely approximate</u> the limit conditions for a (planetocentric) hyperbolic trajectory. Necessarily, the degree of error which has been incurred in this analysis is directly linked to the extent which the initial entry state departs from the hyperbolic limit state. Some measure of this approximation can be linked to the angle, β.

A check on the accuracy of ψ may be afforded (in part) by the following manipulations:

(a) Knowing ρ_I, then a value for φ_{∞} is obtained from the conic equation (exactly) as,

$$\varphi_{\ell im} \left(\stackrel{\Delta}{=} \varphi_{\infty}\right) = \cos^{-1}\left[\frac{a(\epsilon^2 - 1)}{\epsilon \rho_I} - \frac{1}{\epsilon}\right];$$

then according to Fig. III.10,

$$\psi = \pi - (\varphi_{\ell im} + \beta),$$

where β has been given by Eqn. (III.45)! (b) Using the result above, calculate

$$\alpha = \pi - 2\psi ,$$

and compare this to Eqn. (III.57a) – or to (III.57b)!

With the information which has been obtained to this point in the analysis one is ready to proceed, next, to a description of the exit conditions from the sphere of influence.

The flight through the sphere of influence has occured over a figure which has dynamic and geometric symmetry; thus the speed at entry (V_2) is the same as the speed at exit (V_3) – in agreement with the assumptions made earlier. That is,

$$|\overline{V}_3| \equiv |\overline{V}_2| . \tag{III.58}$$

Also, in agreement with the assumptions employed herein, the vector $\overline{V}_d$ (describing the velocity of m_d relative to $m_\odot$) is a constant; and, by the simplifications introduced, the vectors $\overline{V}_2$ and $\overline{V}_3$ are (essentially) directed along the hyperbola's asymptotes. In this regard, then,

$$\measuredangle (\overline{V}_d, \overline{V}_2) + \alpha = \measuredangle (\overline{V}_3, \overline{V}_d), \tag{III.59}$$

where α is defined by Eqn. (III.57). Also, since $\overline{V}_3 + \overline{V}_d = \overline{V}_{T_D}$ (where $\overline{V}_{T_D}$ is the heliocentric velocity of the space vehicle at exit from the sphere of influence) then,

$$V_{T_D} = \sqrt{V_3^2 + V_d^2 + 2\,\overline{V}_3 \cdot \overline{V}_d}$$

or

$$V_{T_D} = \sqrt{V_3^2 + V_d^2 + 2V_3\,V_d \cos\,(\overline{V}_3, \overline{V}_d)} \tag{III.60}$$

wherein the cos $(\overline{V}_3, \overline{V}_d)$ can be obtained from Eqn. (III.59); and, the angle $\measuredangle (\overline{V}_d, \overline{V}_2)$ used above, has been noted to satisfy the relation

$$\measuredangle (\overline{V}_d, \overline{V}_2) \equiv \delta_{H_d} + \gamma_{H_d} + |\delta\varphi_d| ,$$

with $\delta\varphi_d = \varphi_{H_d} - \varphi_{H_I}$.

The magnitude of $\overline{V}_{T_D}$ is known, now; however, to complete a description of the vector its direction must be specified also. This can be done by locating the vector (in angular position) relative to (say) $\overline{V}_3$. In this regard note that (See Fig. III.10).

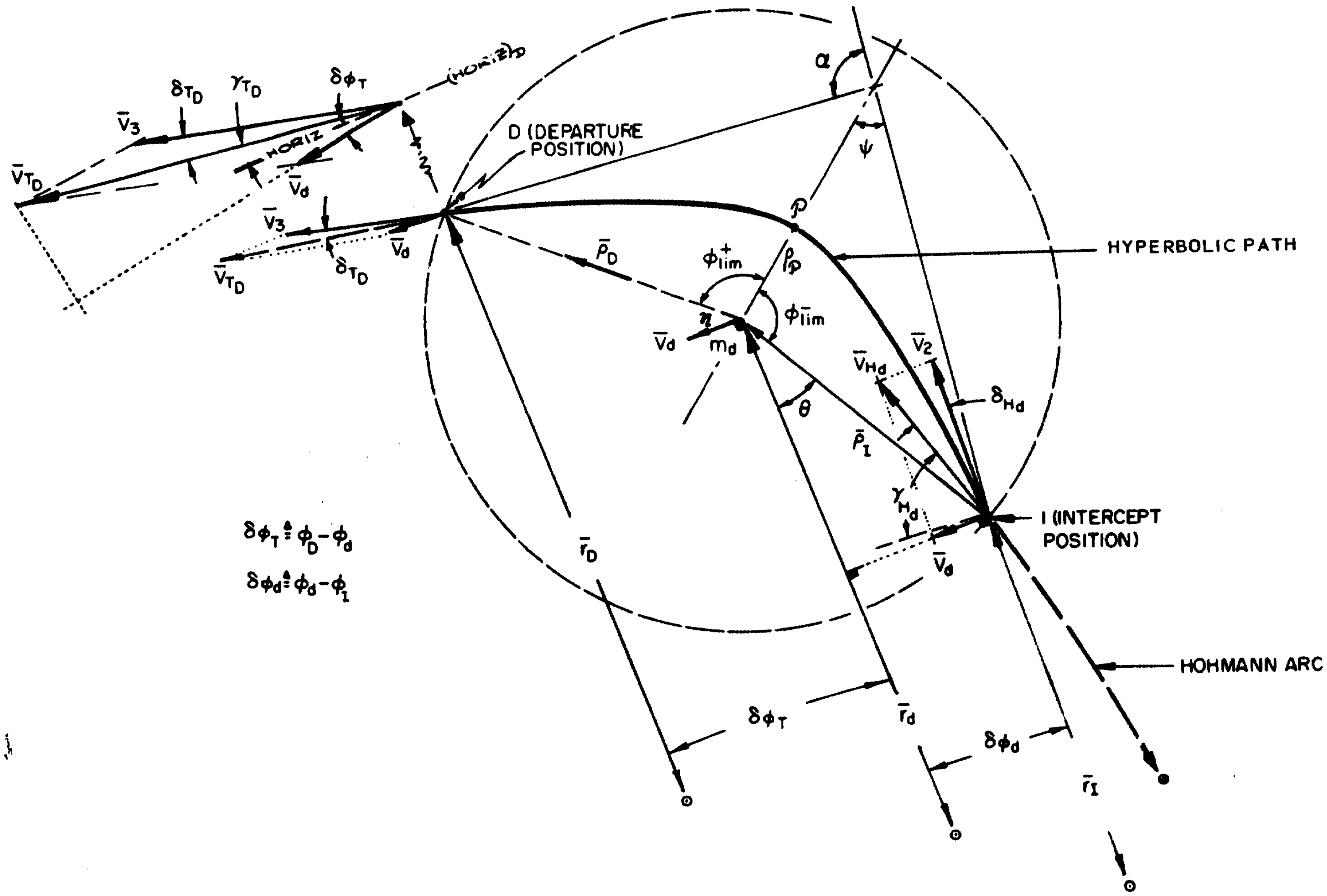


Figure III.10—Sketch of the Flight Path Inside the Sphere of Influence, with Conditions at Entry and Exit Described

$$|\overline{V}_{T_D}| \sin(\overline{V}_{T_d}, \overline{V}_d) = |\overline{V}_3| \sin(\overline{V}_3, \overline{V}_d)$$

or

$$\measuredangle (\overline{V}_{T_D}, \overline{V}_d) = \sin^{-1} \left[\frac{|\overline{V}_3| \sin(\overline{V}_3, \overline{V}_d)}{|\overline{V}_{T_D}|} \right]. \qquad \text{(III.61a)}$$

where $\measuredangle (\overline{V}_3, \overline{V}_d)$ is described by Eqn. (III.59). Also, as a check, this angle is described by

$$\measuredangle (\overline{V}_{T_D}, \overline{V}_d) = \cos^{-1} \left[\frac{|\overline{V}_d| + |\overline{V}_3| \cos(\overline{V}_3, \overline{V}_d)}{|\overline{V}_{T_D}|} \right]. \qquad \text{(III.61b)}$$

According to the construction shown on the figure there is an angle equivalence statement which is useful here; namely,

$$\measuredangle (\overline{V}_{T_D}, \overline{V}_d) = \gamma_{T_D} + \delta_{\varphi_T}, \qquad \text{(III.61c)}$$

where Eqns. (III.61) describe the angle $(\overline{V}_{T_D}, \overline{V}_d)$. The angles γ_{T_D} and δ_{φ_T} define, (first) the elevation angle for $\overline{V}_{T_D}$ (as it relates to the "final" flight path (T)); and, second, (δ_{φ_T}) describes the absolute angular separation between the two position vectors $\overline{r}_d$ and $\overline{r}_D$.

In describing these angles it is necessary to obtain the vector magnitude $|\overline{r}_D|$. A definition of $\overline{r}_D$ is obtained (see the figure produced here) from

$$\overline{r}_D = \overline{r}_d + \overline{\rho}_D;$$

leading directly to

$$r_D = [r_d^2 + \rho_D^2 + 2\overline{r}_d \cdot \overline{\rho}_D]^{1/2},$$

hence

$$r_D = \sqrt{r_d^2 + \rho_D^2 + 2r_d \rho_D \cos(\overline{r}_d, \overline{\rho}_D)} \qquad \text{(III.62)}$$

wherein $\measuredangle (r_d, \rho_D) = \pi/2 - \eta$ (See Fig. III.10). However, η is defined from a description of the total angle about m_d; i.e.,

$$2\pi = 2|\varphi_{\ell_{im}}| + |\theta| + \pi/2 + \eta,$$

so

$$\eta \cong \frac{3\pi}{2} - (2|\varphi_{\ell_{im}}| + |\theta|). \tag{III.63}$$

As a consequence of (III.63) one notes that

$$\sphericalangle(\bar{r}_d, \bar{\rho}_D) = \frac{\pi}{2} - \eta = 2|\varphi_{\ell_{im}}| + |\theta| - \pi. \tag{III.64}$$

Next, as a means of describing the angle $\delta\varphi_T$, one can make use of the following geometric identity; that is,

$$|\bar{\rho}_D| \cos \eta = |\bar{r}_D| \sin \delta\varphi_T,$$

thus

$$\delta\varphi_T = \sin^{-1}\left[\frac{|\bar{\rho}_D| \cos \eta}{|\bar{r}_D|}\right]. \tag{III.65}$$

(Note that the sign of $\delta\varphi_T$ has been arbitrarily set by $0 \le \delta\varphi_T \le \pi/2$, then $\delta\varphi_T = |\varphi_D - \varphi_d|$, where the $\sphericalangle\varphi_i$ locate "D" and "m_d" relative to some consistent pericentric origin).

Knowing η (Eqn. (III.63)) and $\delta\varphi_T$ (Eqn. (III.65)) then the elevation angle (γ_{T_D}) can be obtained from Eqn. (III.61c). This angle locates the velocity vector ($\bar{V}_{T_D}$) relative to the local horizon as it is described for the final trajectory arc, T. Recall that this final arc extends from "D" to the terminus (which is located at m_f, and/or its time position in heliocentric space).

The one quantity left to be described – for the trajectory through the sphere of influence – is the time of flight. This will be deferred for the moment, but will be discussed subsequently.

III.E.10. THE TERMINAL TRAJECTORY

The terminal trajectory extends from point "D", on the imaginary boundary of the sphere of influence, to the orbit of the terminal body (m_f). Because of the encounter with the massive body (m_d) this final arc is not a continuation of the pre-encounter arc (the Hohmann ellipse). Actually the terminal path has a different eccentricity and size, hence the latter orbital parameters (ϵ_T, a_T) must be determined.

In regard to these parameters it should be apparent that there is (now) enough information known about the terminal arc to specify values for each. For instance, employing the expression for eccentricity

$$\epsilon = \sqrt{1 + \frac{2E_1 h^2}{\mu^2}},$$

and recalling that this flight path is described as having constant E_1 and h – in heliocentric space – then it can be shown that

$$\epsilon_T = \left[\left(\frac{r_D V_{T_D}^2}{\mu_\odot} - 1\right)^2 \cos^2\gamma_{T_D} + \sin^2\gamma_{T_D}\right]^{1/2} . \qquad \text{(III.66)}$$

Also, from the energy equation it is easy to show that the trajectory's characteristic length is described by

$$a_T = \frac{r_D}{\dfrac{V_{T_D}^2}{\mu_\odot / r_D} - 2} \qquad \text{(III.67)}$$

III.E.11 ANGULARITY OF THE LINE OF APSIDES

As a next task, to find the amount which the terminal apsidal axis has been shifted, relative to the initial (Hohmann) figure, the following method is suggested.

By means of the conic equation, written for the terminal arc, the appropriate position angle (φ_T) for the radius $\bar{r}_D$ is obtained as:

$$\varphi_{T_D} = \cos^{-1}\left[\frac{a_T \,|\, \epsilon_T^2 - 1 \,|}{r_D\, \epsilon_T} - \frac{1}{\epsilon_T}\right]. \qquad \text{(III.68)}$$

Now, define the angularity (change) for the apsidal axes as, $\Delta\varphi$; which is the angular separation between the pericenter locations for the two trajectories $(\ )_H$ and $(\ )_T$! A sketch illustrating this shift in position(s) is shown below.

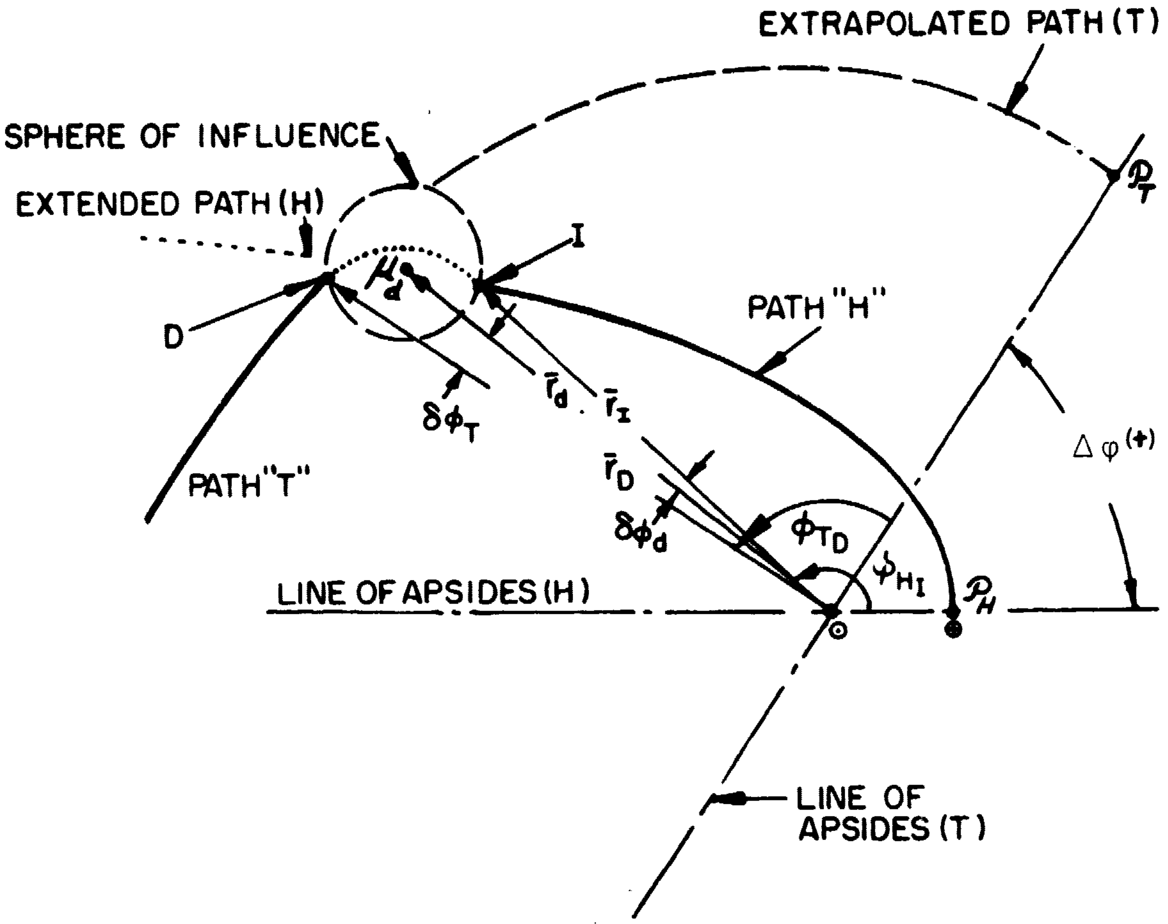


Figure III.11–Sketch Illustrating the Angular Positioning of Pericenter ($\mathbb{P}$) on the Original (Hohmann) Path, and the Final, Post-Encounter Trajectory.

In agreement with the figure shown above the angle φ_D locates r_D relative to the pericenter ($\mathbb{P}_T$). This, in turn, is separated from the original pericenter location ($\mathbb{P}_I$) by the angle $\Delta\varphi$; and, in addition, one notes that

$$\varphi_{T_D} + \Delta\varphi = \varphi_{H_I} + \delta\varphi_d + \delta\varphi_T,$$

hence

$$\Delta\varphi = (\varphi_{H_I} + \delta\varphi_d + \delta\varphi_T) - \varphi_{T_D}, \tag{III.69}$$

where these various angles have been determined previously. One should note that $\Delta\varphi \geq 0$ (dependent on the angle summation noted above). What is shown on the sketch (above) is a 'supposed' positive valued $\Delta\varphi$!

The last parameter needed to complete the trajectory analysis is a description of the position on the terminal path "T" where the spacecraft will intersect the orbit of m_f ! Knowing the orbital radius, (r_f), of this planet (or other celestial body), as an a priori, then by means of the conic equation it is easy to define the appropriate position angle as

$$\varphi_{T_f} = \cos^{-1}\left[\frac{a_T\,|\,\epsilon_T^2 - 1\,|}{r_f\,\epsilon_T} - \frac{1}{\epsilon_T}\right], \qquad \text{(III.70)}$$

where the absolute value is employed in the event that the final trajectory may be either hyperbolic or elliptic!

III.E.12. TOTAL TIME OF FLIGHT

In calculating the total time of flight it should be recognized that there are three (3) arcs to be considered; hence, three equations for time must be included in this evaluation. Since one or more of these arcs may be for either an elliptic or a hyperbolic arc then the two appropriate general expressions for time are:

(a) For an elliptic arc (time measured from pericenter passage);

$$t = \sqrt{\frac{a^3}{\mu}}\left[\frac{-\,\epsilon\sqrt{|\,\epsilon^2 - 1\,|}\,\sin\varphi}{1 + \epsilon\cos\varphi} + 2\tan^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\,\tan\frac{\varphi}{2}\right)\right]. \qquad \text{(III.71a)}$$

(b) For the hyperbolic arc (time measured from pericenter passage);

$$t = \sqrt{\frac{a^3}{\mu}}\left[\frac{\epsilon\sqrt{|\,\epsilon^2 - 1\,|}\,\sin\varphi}{1 + \epsilon\cos\varphi} - 2\tanh^{-1}\left(\sqrt{\frac{\epsilon-1}{1+\epsilon}}\,\tan\frac{\varphi}{2}\right)\right]; \qquad \text{(III.71b)}$$

or

$$t = \sqrt{\frac{a^3}{\mu}}\left[\frac{\epsilon\sqrt{|\,\epsilon^2 - 1\,|}\,\sin\varphi}{1 + \epsilon\cos\varphi} - \ln\left(\frac{\sqrt{\epsilon+1} + \sqrt{\epsilon-1}\cdot\tan(\varphi/2)}{\sqrt{\epsilon+1} - \sqrt{\epsilon-1}\cdot\tan(\varphi/2)}\right)\right]. \qquad \text{(III.71c)}$$

It is noted that, symbolically, these formulas define time as a function of the orbital parameters and other quantities, as

$$t = t\ (\mu,\ a,\ \varphi,\ \epsilon). \tag{III.72}$$

The three time intervals, making up a complete description of the mission flight time, are obtained as the time required to fly the three separate arcs making up the complete mission path. Of course, this statement is in agreement with the assumptions set forth earlier. These three arcs are:

(1) The Hohmann path length, from launch at $m_\oplus$ to intercept with the sphere of influence (at I); this considers an arc from a heliocentric position $\varphi = 0$ to one denoted as $\varphi = \varphi_{H_I}$!

(2) The hyperbolic arc, within the sphere of influence, is that arc whose angular extent is from a planetocentric position of $-\varphi_{\ell_{im}}$ to one denoted by $+\varphi_{\ell_{im}}$; and,

(3) the arc along the terminal trajectory, extending from the exit heliocentric position, $\varphi = \varphi_{T_D}$, to the terminus position, $\varphi = \varphi_{T_f}$.

In order to describe these time intervals from the expressions above the computations are carried out in the following manner:

(a) For the first arc (along the Hohmann path) the time is described via Eqn. (III.71a) where the quantities employed in that expression are noted to be

$$t = t\ (\mu_\odot\ a_H,\ \varphi_{HI},\ \epsilon_H)\ . \tag{III.73a}$$

(b) Within the sphere of influence the arc is a hyperbolic section, thus use is made of either Eqn. (III.71b) or (III.71c). Also since this arc is symmetric about the pericenter (and the figure's axis) then the time of flight would be <u>twice</u> the value calculated from either of the expressions noted above. Thus,

$$t = 2t(\mu_d,\ a,\ \varphi_{\ell_{im}},\ \epsilon); \tag{III.73b}$$

wherein, in reality, $\varphi_{\ell_{im}}$ is the angle corresponding to a radius ρ_I <u>in</u> planetocentric space.

(c) For the final arc the trajectory may be <u>either</u> an ellipse or hyperbola – this can be ascertained from Eqn. (III.66), where the eccentricity is described. As a consequence, this time must be calculated from the appropriate expression given by Eqns. (III.71). <u>Also</u>, since the arc covered by the spacecraft on this leg of the mission is described by

$$\varphi_{T_D} \leq \varphi \leq \varphi_{T_f}$$

then it will be necessary to compute the flight time by the scheme noted below:

The time increment of interest here is,

$$t \triangleq \Delta t = t_f - t_D , \tag{III.73c}$$

wherein

$$t_f = t\,(\mu_\odot,\ a_T,\ \varphi_{T_f},\ \epsilon_T),$$

and

$$t_D = t(\mu_\odot,\ a_T,\ \varphi_{T_D},\ \epsilon_T).$$

III.E.13. ENERGY CHANGE DUE TO THE SWING-BY MANEUVER

Calculations carried out in the sections prior to this one have been concerned, primarily, with the physical and geometric aspects of this overall maneuver. However, one of the more important parameters associated with this operation is still to be determined – that is, the energy change which has been brought about by the "swing-by" itself.

For a determination of the energy change one should recognize, first, that the description is concerned with the energy change referred to heliocentric space; and, also, that the considerations are in regard to both the kinetic and potential energies referred to this space.

Defining the change in energy, symbolically, as

$$\Delta E_1 = E_{1_T} - E_{1_H} \tag{III.74}$$

where, typically, for example,

$$E_{1_T} = \frac{V_{T_D}^2}{2} - \frac{\mu_\odot}{r_D} \equiv \pm \frac{\mu_\odot}{2a_T} \tag{III.75a}$$

(again, the ± have been written to account for the possibility of either an elliptic or hyperbolic terminal path); and where, for example,

$$E_{1_H} = \frac{V_{H_d}^2}{2} - \frac{\mu_\odot}{r_I} \equiv - \frac{\mu_\odot}{2a_H} \qquad \text{(III.75b)}$$

(written, as such, since the initial path was described as a Hohmann segment). From Eqns (III.75) one can define the change in both kinetic energy and potential energy (per unit of spacecraft mass); or, alternately, one can define the change in total energy, per se, by simply describing the change in a_i ($i = T, H$). For either scheme chosen a consistent definition of ΔE_1 would be obtained.

To some degree the effectiveness of the hyperbolic encounter can be measured in terms of the heliocentric speed gained for a trajectory passage close to the massive disturbing body. In this regard, an indication of such is given by comparing the quantities V_{H_d} (pre-encounter speed) and V_{T_D} (post-encounter speed). Also, for that case where the final path is highly energetic ($E_{1_T} > 0$) it is of some value to note the magnitude of the hyperbolic excess speed along the terminal trajectory. This topic will be briefly mentioned below.

Recalling that hyperbolic excess speed is defined as

$$V_\infty \triangleq \sqrt{\frac{\mu}{a}} \,,$$

then for the terminal path this quantity would be

$$V_{\infty_T} = \sqrt{\frac{\mu_\odot}{a_T}} \,; \qquad \text{(III.76a)}$$

or, as related to escape speed for this trajectory (referred to some convenient position),

$$V_{\infty_T}^2 = V_{T_D}^2 - 2\frac{\mu_\odot}{r_D} \equiv V_{T_D}^2 - V_{esc}^2 \qquad \text{(III.76b)}$$

since $V_{esc}^2 = 2\mu_\odot/r_D$ (at point "D", in heliocentric space). Necessarily if, from Eqn. (III.76b), the resultant is negative then the vehicle did not attain hyperbolic speed. On the other hand, if the resultant is positive then the escape, and hyperbolic speed, was reached by the vehicle during the encounter, and as a consequence of it.

IV. REMARKS

In the foregoing sections two approximate schemes, which allow the analyst to study a Swing-By operation, have been outlined. Of these, the Mode I method is simplest to use and manipulate; while Mode II is a more exact procedure insofar as the basic assumptions used herein are concerned.

Needless to say these methods are not "exact" in a physical and mathematical sense; and, as such, both schemes will suffer some loss in accuracy under certain conditions. To illustrate that the methods used herein are not in full agreement with the physical situation; recall that the assumption has been made whereby m_d is presumed to be fixed in a spatial position when the Swing-By occurs. In many instances such an approximation may not seriously affect the result; yet, in other cases this assumption might lead to unacceptable errors. One means of correcting for this (one) objection would be to properly account for the rotation of the m_d velocity vector, $\bar{V}_d$, at exit from the sphere of influence. Knowing the time of motion in planetocentric space it is immediately evident what the magnitude of this angularity would be – and, of course, its direction is known. Such a correction may (or may not) be of consequence; this will be a matter for the analyst to decide.

In general, the methods presented here have been developed to aid in the evaluation of planetary swing-by maneuvers, as a means of performing long distance-and-time missions. It is not suggested that these schemes will provide the accuracy needed for (say) guidance and precise performance calculations; however, the various influences which do arise as a consequence of the "Swing-By" can be determined, and evaluated in a not-too-rough manner.

Some of the more subtle aspects of this maneuver (type) can be found in the bibliographical listings; the interested reader is urged to consult these for this information.

V. BIBLIOGRAPHY

1. Luidens, Roger W., Miller, Brent A., Kappraff, Jay M., Jupiter High-Thrust Round-Trip Trajectories, NASA TN D-3739, Washington, D.C., 1966.

2. Karner, Robert J., An Analysis of Two Approximate Methods for the Determination of the Trajectory of a Planetary Flyby, VPI Thesis, Blacksburg, Va., 1968.

3. Behuncik, J. A., Characteristics of Planetary Fly-by Trajectories, GSFC X-643-67-501, Greenbelt, Md., 1967.

VI. SYMBOLS

a	semi-major length of a conic (trajectory)
b	semi-major length of a conic (trajectory)
E_1	specific energy of a particle in motion about a primary
ΔE_1	a change in specific energy
F	location of a conic's focus
G	universal gravitational constant
h	specific moment of momentum ($\triangleq \lvert \bar{r} \times \bar{V} \rvert$)
i, j, k	indices
m	mass
m_d	mass of the disturbing planet (about which "swing-by" occurs)
$m_{\odot}$	solar mass [See Eqn. (III.9)]
p	orbital parameter (semi-latus recturn)
P	period of motion; corresponding to given values of a, μ
r	radial distance (radius)
R	a radius [See Eqn. (III.9)]
r_{IN}	radius of the sphere of influence
t	time
V	speed
$\bar{V}_2, (V_2)$;	heliocentric velocity (speed) at entry to; and exit from;
$\bar{V}_3, (V_3)$	the sphere of influence
V_{H_d}, V_{T_d}	heliocentric speed, at m_d, on the Hohmann (H) and Terminal (T) trajectory arcs
α	relative inclination between both asymptotes of a hyperbolic path
β	angular position of conic's asymptote <u>relative</u> to limit radius (ρ_I)
γ	elevation angle for velocity
Γ	speed ratio (See Eqn. A.9, Appendix A)
$\Delta(\ ), \delta(\ )$	increment values ( )
δ_{jd}	angularity of $\bar{V}_i$ ($i = 2, 3$) relative to $\bar{V}_{jd}$ ($j = H, T$)

2

ϵ eccentricity

θ angle between radii ($\bar{\rho}$, $\bar{r}_d$)

μ gravitational parameter ($\overset{\Delta}{=}$ Gm; $\mu_\odot$ = $Gm_\odot$, $\mu_\oplus$ = $Gm_\oplus$, μ_d = Gm_d)

ξ asymptote positioning angle (See Appendix B)

$\bar{\rho}$,(ρ) planetocentric radius, measured relative to m_d as primary

φ polar positioning angle (measured from pericenter)

ψ angularity of asymptotes, referred to a conic's axis; see Fig. III.3.

ω a positioning angle (Appendix A)

<u>Subscripts</u>

apo corresponding to apocenter

D refers to <u>exit</u> point, leaving sphere of influence

d refers to disturbing mass (m_d).

f corresponds to final position (orbit, mass, etc.)

H refers to the Hohmann arc

h a heliocentric referenced quantity

I refers to the initial (entry) point to sphere of influence

peri, $\wp$ corresponding to pericenter

∞ refers to extreme (limit) position on a hyperbolic path

$(\ )_{2,3}$ quantities associated with $\bar{V}_2$, $\bar{V}_3$

$\oplus$ refers to earth quantities

$\odot$ refers to sun (solar) quantities

APPENDIX A

Geometry for the Planetary Swing-By

The planetary swing-by, per se, is considered to be confined to that region in space which lies within the planet's sphere of influence. In a simple definition of this region, the radius of this sphere is defined by

$$r_{IN} = R\left(\frac{m_d}{m_\odot}\right)^{2/5}$$

(See Eqn. (III.9)), wherein:

$r_{IN} \triangleq$ radius of the sphere measured from m_d

$R \triangleq$ heliocentric radius locating m_d (wrt $m_\odot$)

$m_d, m_\odot \triangleq$ mass of the planet, and sun, respectively.

The presumption made here is that the swing-by trajectory is a hyperbola; and, consequent to the definition above, the conic will have geometric and dynamic symmetry; with the planet, m_d, (hence $\mu = Gm_d/r^2$) serving as the focus.

For the geometric and kinematic conditions to be described next, concern is centered within the sphere of influence; though conditions at the boundary are also of consequence. In this regard the velocity vectors (see Fig. A.1) at the entry (I) to the sphere, and exit (D) from it are of interest. Also, these vectors, as referred to heliocentric space and planetocentric space, are of equal importance. For these purposes then, the vectors

$$\overline{V}_{i_h}, \overline{V}_{i_d} \; (i = I, D; \text{ etc})$$

are defined, wherein $(\;)_h$ refers to heliocentric space, and $(\;)_d$ implies quantities referred to the disturbing mass (m_d).

Since the planet (m_d) is in motion, in heliocentric space, then its velocity $\overline{V}_d$ is described and used here. For purposes of this analysis $|\overline{V}_d|$ is constant; and one of the assumptions to be employed here will suggest that $\overline{V}_d$ is also constant, but during the time of the planetary encounter only.

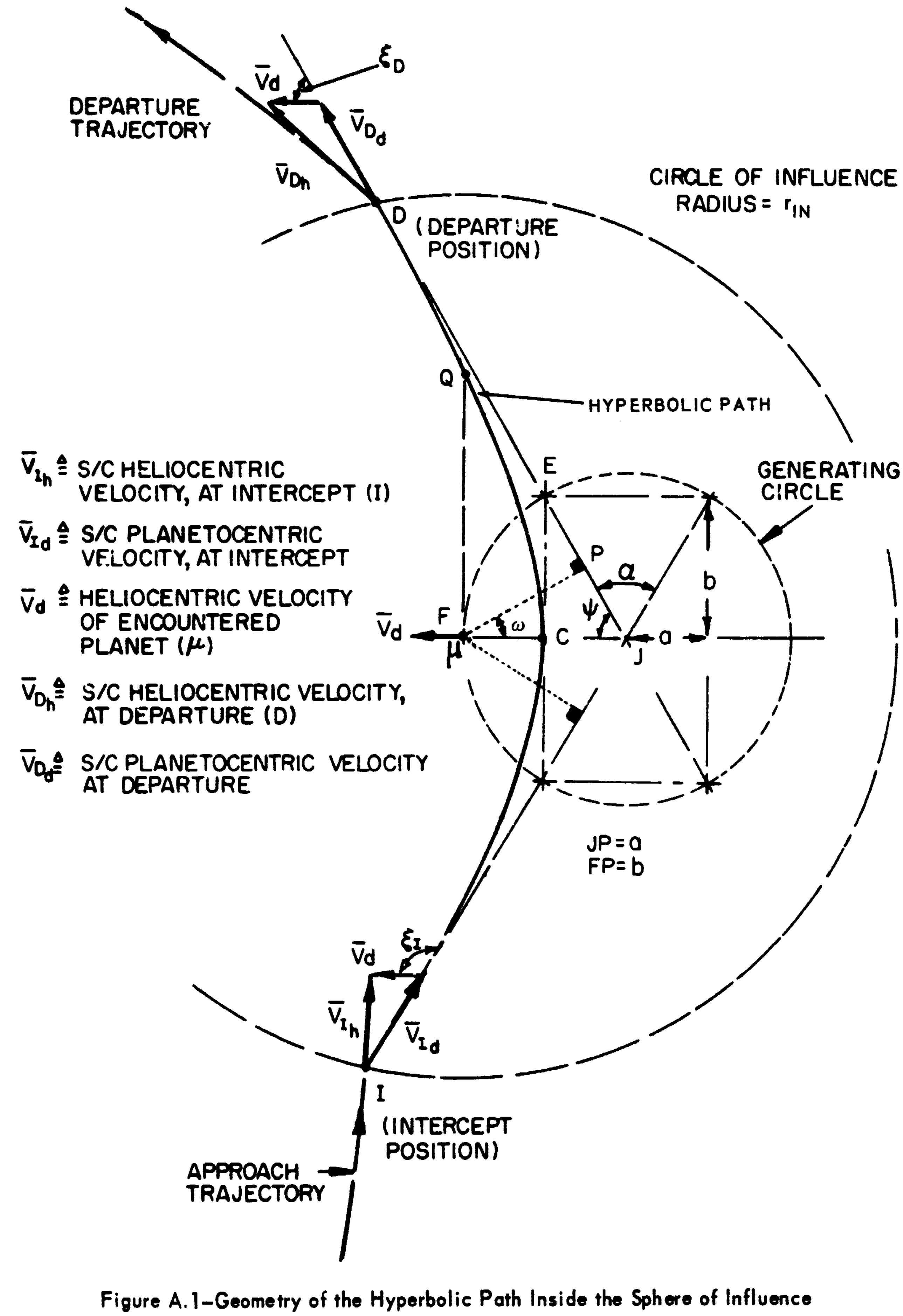


Figure A.1–Geometry of the Hyperbolic Path Inside the Sphere of Influence

Certain angles, seen in Fig. A.1, are of interest here; those of primary concern are listed and described below:

$\alpha \triangleq$ turning angle for the velocity vector during passage through the sphere of influence;

ψ = inclination of the asymptotes relative to the axis of the planet centered hyperbolic trajectory;

ω = the angle of inclination for the perpendicular to the asymptote, measured from the figure's axis of symmetry. (Note that $\omega + \psi = \pi/2$.)

A satellite approaches the sphere of influence, along its heliocentric flight path (approach trajectory), and intersects this imaginary boundary at I. The heliocentric velocity at this point is $\overline{V}_{I_h}$, while the planetocentric velocity is $\overline{V}_{I_d}$. These velocities are related according to

$$\overline{V}_{I_h} = \overline{V}_{I_d} + \overline{V}_d. \tag{A.1}$$

Looking at Figure A.1 it is noted that several of the angles which are of interest here are related by the following

$$\alpha + 2\Psi = \pi,$$

and

$$\Psi = \frac{\pi}{2} - \omega,$$

hence

$$\alpha = 2\omega. \tag{A.2}$$

Also, it is seen that

$$\tan\Psi = b/a,$$

while

$$\tan\omega = a/b, \tag{A.3}$$

hence

$$\tan \omega = \cot \Psi,$$

which is also evident from the second relation given in Equation (A.2).

Since $\alpha = 2\omega$, then from Equation (A.2) and the definition of b it follows that

$$\tan \alpha = \tan 2\omega = \frac{2\tan \omega}{1 - \tan^2 \omega} = \frac{2\,\frac{a}{b}}{1 - \frac{a^2}{b^2}} \equiv \frac{2\sqrt{\epsilon^2 - 1}}{\epsilon^2 - 2} \;; \tag{A.4}$$

also, with

$$\tan \omega = \frac{a}{b} = \frac{1}{\sqrt{\epsilon^2 - 1}}\,, \tag{A.5}$$

then

$$\tan \Psi = \frac{b}{a} = \sqrt{\epsilon^2 - 1}\,. \tag{A.6}$$

From the conic equation, it is noted that the radius to pericenter, defined as $r(\varphi = 0)$ is,

$$r_p = \frac{p}{1 + \epsilon} \equiv a(\epsilon - 1);$$

similarly, since the semi-minor length $b = a\sqrt{\epsilon^2 - 1}$, and the eccentricity can be determined from $\epsilon^2 = 1 + b^2/a^2$, then it follows that

$$\frac{b^2}{a^2} + 1 = \epsilon^2 = \left(\frac{r_p}{a} + 1\right)^2 ;$$

or, after manipulation

$$b = r_p \sqrt{1 + 2\,\frac{a}{r_p}}\,. \tag{A.7}$$

Equation (A.7) is important to these problems since it is (both) the semi-minor axis length as well as the perpendicular distance to the asymptotes from the focus, F, (see Fig. A.1).

The semi-major axis length (a) is also significant in that according to previous assumptions the vectors $\bar{V}_{I_d}$ and $\bar{V}_{D_d}$ are essentially parallel to the asymptotes. Consequently these play the roles (approximately) of the so-called "hyperbolic excess velocity" vectors ($\bar{V}_\infty$), which are defined as being parallel to the asymptotes at the extremities of the hyperbola (as $r \to \infty$).

One should recall that this velocity vector has a magnitude defined by

$$V_\infty = \sqrt{\frac{\mu}{a}} . \tag{A.8}$$

Next, a parameter (Γ), describing a speed ratio, will be introduced: this quantity will be defined (here) as

$$\Gamma^2 \triangleq \frac{V^2}{\mu / r_p} , \tag{A.9}$$

where $\sqrt{\mu / r_p}$ is the circular speed for an orbit of radius r_p (the pericentric radius). As a consequence of this definition it can be shown that the following expressions occur:

(1) Using Eqns. (A.8, A.9),

$$\frac{a}{r_p} \simeq \Gamma_{I_d}^{-2} ; \text{ where } \Gamma_{I_d}^2 = \frac{V_{I_d}^2}{\mu / r_p} . \tag{A.10a)*}$$

(2) From Eqns. (A.7, A.9),

$$\frac{\mu}{b} = \frac{V_{I_d}^2}{\Gamma_{I_d} \sqrt{2 + \Gamma_{I_d}^2}} ; \text{ or, } b = r_p \sqrt{1 + \frac{2}{\Gamma_{I_d}^2}} . \tag{A.10b}$$

*One should note that $\Gamma_{I_d} \equiv \Gamma_{D_d}$, since $|\bar{V}_{I_d}| \equiv |\bar{V}_{D_d}|$.

(3) From Eqns. (A.5, A.9),

$$\tan \Psi = \cot \omega = \Gamma_{I_d} \sqrt{2 + \Gamma_{I_d}^2} . \tag{A.10c}$$

(4) Combining Eqns. (A.8, A.9, and A.4),

$$\tan \alpha = \frac{2 \Gamma_{I_d} \sqrt{2 + \Gamma_{I_d}^2}}{\Gamma_{I_d}^2 (2 + \Gamma_{I_d}^2) - 1} . \tag{A.10d}$$

and

(5) Utilizing the Eqns. for (r_p) and (A.8),

$$\epsilon = 1 + \Gamma_{I_d}^2 . \tag{A.10e}$$

Note: these expressions apply only for the hyperbolic encounter.

From these expressions it is apparent that for a fixed value of r_p, as V_{I_d} is increased the angle α decreases. That is, the "turning" of the velocity vector is reduced; this can be attributed to a reduction in time during which the spacecraft can be influenced by the attracting (or fly-by) mass. Therefore, to obtain a maximum of turning (α) it is essential that a slowest approach velocity ($\bar{V}_{I_d}$) should be available, at a fixed value of r_p. Conversely, for $|\bar{V}_{I_d}|$ fixed the value of α increases as r_p is reduced; hence, a close passage to the attracting planet is desired if α is to be maximized (for a given value of $|\bar{V}_{I_d}|$).

In substance, then, the maximum turning (α) occurs when r_p and $|\bar{V}_{I_d}|$ are smallest, compatible with the assumptions and constraints of the mathematical model assumed here.

It should be recognized that there are some least values for quantities connected with this study. For instance, the smallest value of r_p which can be used here is that corresponding to the dimensions of the attracting planet (and its atmosphere, in those cases where atmospheres are significant). Secondly, the entire problem has pre-supposed that the trajectory, during swing-by, is hyperbolic; therefore $|\bar{V}_{I_d}|$ – and $|\bar{V}_{D_d}|$ – must be at least as large as the parabolic speed relative to the attracting planet. This assures the geometric form of the conic assumed earlier.

One final remark before leaving this section; this is in regard to the distance FP (= B) shown on Fig. A.1. It has been stated (without proof) that this length is equal to the semi-minor axis (b) of the geometric figure; actually this is easily demonstrated, and will not be developed here.

Since the hyperbolic trajectory is assumed to be generated as a two-body flight path, it should be evident that the usual two-body relations are applicable and can be used to describe this geometry. Also, the state of motion – as well as time of flight – relative to the planetary mass (μ), can be described from two-body results.

Velocity and Energy Change from the "Swing-By" Maneuver

Figure (A.1) shows the heliocentric exit and entrance velocity vectors as $\bar{V}_{D_h}$ and $\bar{V}_{I_h}$, respectively. Consequently the change in velocity (δV), due to an encounter with an attracting mass can be defined as

$$\delta\bar{V} \triangleq \bar{V}_{D_h} - \bar{V}_{I_h} . \tag{A.11}$$

Making use of (A.1) it is evident that

$$\delta\bar{V} = \bar{V}_{D_h} - \bar{V}_{I_h} \equiv \bar{V}_{D_d} - \bar{V}_{I_d} , \tag{A.12}$$

as seen from the sketch here, (Fig. A.2).

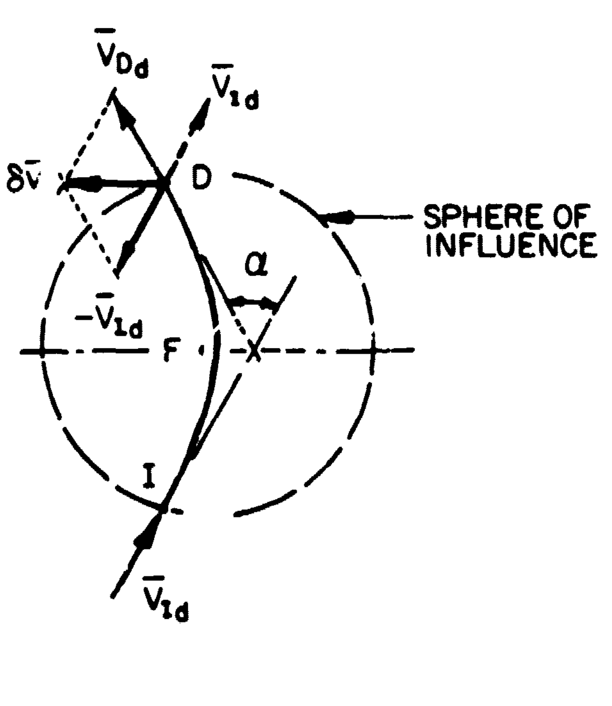


Figure A.2–Sketch illustrating the velocity change, $\delta\bar{V} = \bar{V}_{D_d} - \bar{V}_{I_d}$. Here the Circle of Influence is a cross-section of the Sphere of Influence.

The magnitude of this change ($|\delta\bar{V}|$) is obtained as

$$\Delta V \triangleq |\delta\bar{V}| = [(\bar{V}_{D_d} - \bar{V}_{I_d}) \cdot (\bar{V}_{D_d} - \bar{V}_{I_d})]^{1/2} ;$$

or, expanding and recognizing that $V_{D_d} \equiv V_{I_d}$, then

$$\Delta V = V_{I_d} \sqrt{2(1 - \cos \alpha)} \equiv V_{D_d} \sqrt{2(1 - \cos \alpha)}. \qquad \text{(A.13a)}$$

An equivalent expression, more compactly written,* is

$$\Delta V = 2V_{I_d} \sin \frac{\alpha}{2} . \qquad \text{(A.13b)*}$$

The total energy, relative to the heliocenter, at entry to the sphere of influence (at I) and exit from it (at D), can be written, generally, as

$$E_{1_i} = \frac{V_{i_h}^2}{2} - \frac{\mu_\odot}{r_{i_h}} = - \frac{\mu_\odot}{2a_i} \quad (i = I, D)$$

where $\mu_\odot$ is the gravitational constant for the sun; r_{i_h} is the radius to the points (i) from the sun; and, "a_i" is the semi-major length describing the heliocentric trajectories (before and/or after encounter). In general the energy is <u>not</u> the same before and after the swing-by – this is normally one of the reasons for attempting this maneuver – therefore, a change in total specific energy, $\delta E_i >< 0$, is expected.

Under the assumptions made for this analysis, it will be presumed that in heliocentric space, $r_{I_h} \simeq r_{D_h}$; therefore, the change in energy may be expressed (approximately) by

$$\delta E_1 \simeq E_{1_D} - E_{1_I} = \frac{V_{D_h}^2}{2} - \frac{V_{I_h}^2}{2} \qquad \text{(A.14a)}$$

(since the r_i are assumed equal).

*It was noted earlier that $\alpha/2 = \omega$, and that $V_{I_d} \simeq \sqrt{\mu/a}$, hence an approximate evaluation of ΔV is obtained as $\Delta V = 2V_{I_d} \sin \alpha/2 \simeq 2\sqrt{\mu/a} \sin \omega = 2\sqrt{\mu/a} \cos \psi$.

In this expression,

$$V_{D_h}^2 = |\overline{V}_{D_d} + \overline{V}_d|^2 = V_{D_d}^2 + V_d^2 + 2\,\overline{V}_{D_d} \cdot \overline{V}_d$$

and

$$V_{I_h}^2 = |\overline{V}_{I_d} + \overline{V}_d|^2 = V_{I_d}^2 + V_d^2 + 2\,\overline{V}_{I_D} \cdot \overline{V}_d;$$

so that, accounting for the angles ξ_j (j = I , D) on Figure A.2; also recalling that $V_{D_d} \doteq V_{I_d}$ then,

$$V_{D_h}^2 = V_{I_d}^2 + 2\,V_d\,V_{I_d}\cos\xi_D + V_d^2 \equiv V_{D_d}^2 + 2\,V_{D_d}\,V_d\cos\xi_D + V_d^2;$$

also

$$V_{I_h}^2 = V_{I_d}^2 + 2\,V_{I_d}\,V_d\cos\xi_I + V_d^2 \equiv V_{D_d}^2 + 2\,V_{D_d}\,V_d\cos\xi_I + V_d^2;$$

and, therefore,

$$\delta E_1 \cong V_d\,V_{I_d}(\cos\xi_D - \cos\xi_I) \tag{A.14b}$$

which is equivalent to

$$\delta E_I \simeq \overline{V}_d \cdot (\overline{V}_{D_d} - \overline{V}_{I_d}) \equiv \overline{V}_d \cdot \delta\overline{V} \tag{A.14c}$$

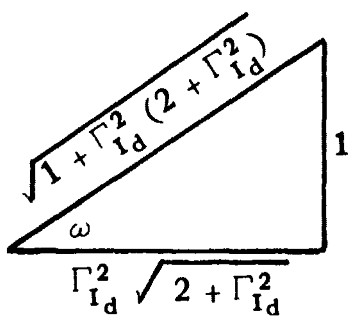
(See Eqn. (A.12)).

It should be evident from these expressions that $\delta E_1 \gtrless 0$ depending on the orientations of $\overline{V}_{I_h}$ and $\overline{V}_{D_h}$. Another means of stating this is to note that the sign of δE_1 is compatible with the direction cosine of $\delta\overline{V}$ relative to $\overline{V}_d$ (See Eqn. (A.14c)); or, it depends on the sign and/or magnitude of $\cos\xi_D$ and $\cos\xi_I$ (See Eqn. (A.14b)). In any regard, it is seen that the energy change, relative to the heliocenter, can be described as a <u>gain</u> in energy ($\delta E_1 > 0$) or a loss of energy ($\delta E_1 < 0$), for the satellite, depending <u>on the</u> approach to <u>and</u> retreat from the sphere of influence. If $\delta\overline{V}$ is generally in the direction of $\overline{V}_d$ then energy is <u>gained</u>; however, if the direction of $\delta\overline{V}$ is in opposition to $\overline{V}_d$, then the vehicle's passage will represent a <u>giving up</u> of <u>energy</u> to the attracting body, or a "loss" of energy relative to the heliocenter.

Before leaving this section, it is desirable to express some of these results in terms of the defined speed ratio (Γ_{I_d}). In this regard, it should be recalled that

$$\tan\omega = \frac{a}{b} \simeq \frac{1}{\Gamma_{I_d}\sqrt{2+\Gamma_{I_d}^2}}$$

then according to the sketch (below)



$$\sin\omega = \frac{1}{\sqrt{1+\Gamma_{I_d}^2\left(2+\Gamma_{I_d}^2\right)}} \;;$$

and, according to the footnote with Eqn. (A.13b).

$$\Delta V \simeq 2\sqrt{\frac{\mu}{a}}\,\sin\omega = 2\sqrt{\frac{\frac{\mu}{r_p}\Gamma_{I_d}^2}{1+\Gamma_{I_D}^2\left(2+\Gamma_{I_d}^2\right)}} \;. \tag{A.15}$$

Extending this to encompass the statement given as Eqn. (A.14c) it is evident that

$$\delta E_1 \simeq \bar{V}_d \cdot \frac{\delta\bar{V}}{|\delta\bar{V}|}\left[2\sqrt{\frac{\frac{\mu}{r_p}\Gamma_{I_d}^2}{1+\Gamma_{I_d}^2\left(2+\Gamma_{I_d}^2\right)}}\right] , \tag{A.16}$$

where (now) the magnitude of the change in velocity, and the change in energy, due to the encounter with an attracting mass, have been expressed in terms of known (or selected) quantities, and the speed ratio (Γ_{I_d}).

APPENDIX B

Extremals for the Change in Velocity and Specific Total Energy

Having evaluated the changes produced by a "swing-by", the next logical step is to investigate these to ascertain the conditions and magnitudes for the extremals of each. Taking these in logical order, the change in velocity (magnitude) will be determined first.

In order to describe an extremium for the change in velocity the magnitude of $\delta\bar{V}$ will be studied first since the direction of $\delta\bar{V}$ depends on the vector difference $\bar{V}_{D_d} - \bar{V}_{I_d}$, wherein each of these vectors has the same magnitude. Generally the extremal will be dependent on V_{I_d} ($\equiv V_{D_d}$) since the pericenter distance ($r_{\wp}$) is dependent on the energy (hence the entry speed) for the hyperbolic path. In this regard, then, the extremal for ΔV will be determined in terms of V_{I_d} (or, equivalently, in terms of Γ_{I_d} – and, in particular, in terms of $\Gamma_{I_d}^2$ since this is the order of this quantity as it appears in Equation (A.15)).

Therefore, differentiating Eqn. (A.15) with respect to $\Gamma_{I_d}^2$ and setting the result to zero leads to the expression

$$\sqrt{\frac{\mu/r_{\wp}}{\Gamma_{I_d}^2\ [1 + \Gamma_{I_d}^2\,(2 + \Gamma_{I_d}^2)]^3}}\ (1 - \Gamma_{I_d}^2) = 0;$$

from which the condition for the extremal is

$$\Gamma_{I_d}^2 = 1, \tag{B.1}$$

and/or

$$V_{I_d}^2 = \frac{\mu}{r_{\wp}}.$$

Necessarily this suggests a condition for a maximum since the minimum for ΔV is obviously zero.

Equation (B.1) implies that if ΔV is to become $(\Delta V)_{max}$, then the speed(s) V_{I_d} (and V_{D_d}) should be equivalent to the circular speed corresponding to an orbit of radius $r_{\wp}$!

Using Equation (B.1) in Equation (A.15) one finds the maximum of $\delta\bar{V}$ to be

$$\Delta V_{max} = \sqrt{\frac{\mu}{r_p}}, \tag{B.2}$$

hence the magnitude of the maximum gain in velocity is equivalent to this (circular) speed also. The direction of $\delta\bar{V}$ has been determined as the vector sum of $\bar{V}_{D_d}$ and $-\bar{V}_{I_d}$; these (in turn) depend on the entry and exit positions at the sphere of influence. Note that the gain (which is depicted here) is independent of the planet's velocity ($\bar{V}_d$) and is solely determined by the closeness of the passage to the planet (r_p), as well as the size of the attracting mass (μ).

The extremal for the change in specific energy is more involved since (see Eqns. (A.14)) there is an implied conditionality for the orientation of vectors involved here. Certainly the extremal for this quantity depends on both the direction and magnitude of $\delta\bar{V}$. For the moment, neglecting the magnitude of $\delta\bar{V}$, it is evident that δE_1 has its largest value when the angle between $\bar{V}_d$ and $\delta\bar{V}$ is zero; and, in contrast, δE_1 is a least value when these vectors are directed opposite to one another. Thus, the extremals of δE_1 are dependent on the statement

$$\cos(\bar{V}_d, \delta\bar{V}) = \pm 1.$$

Now, having defined a magnitude, $(\Delta V)_{max}$, then it would appear, from Eqn. (A.14b), that the extremals for δE_1 are defined according to the conditions

$$(1)\ \Delta V \equiv (\Delta V)_{max} \tag{B.3}$$

and

$$(2)\ \cos\xi_D - \cos\xi_I = \pm 1.$$

Condition (1) has been described, see Eqn. (B.2); condition (2) can be established, however, it should be noted that in agreement with Figure A.1,

$$\xi_I - \xi_D = \alpha;$$

therefore, condition (2) is better stated as

$$\cos\xi_D - \cos(\xi_D + \alpha) = \pm 1; \tag{B.4}$$

or, expanding,

$$\cos \xi_D(1 - \cos \alpha) + \sin \xi_D \sin \alpha = \pm 1.$$

After a value for α is decided upon, then it becomes a relatively simple matter to evaluate Eqn. (B.4) for the angle ξ_D ; and, to use Eqn. (B.3) to ascertain a proper value for ξ_I .

It should be apparent that the angle α is not arbitrary; as a matter of fact Eqns. (A.13b), (B.1) through (B.3) lead immediately to the result

$$(\Delta V)_{max} = \left(2 V_{I_d} \sin \frac{\alpha}{2}\right)_{max}$$

hence,

$$\sqrt{\frac{\mu}{r_p}} \equiv 2 \sqrt{\frac{\mu}{r_p}} \sin \frac{\alpha}{2} \quad ;$$

or, the requirement that is implied here is

$$\alpha \Big|_{(\Delta v)_{max}} = \frac{\pi}{3} \qquad \text{(B.5)}$$

Now, solving Eqn. (B.4), accounting for the proper value of α, it is found that

$$\frac{\cos \xi_D}{2} + \frac{\sqrt{3}}{2} \sin \xi_D = \pm 1$$

and, therefore,

$$\cos \xi_D = \pm \frac{1}{2}$$

which requires that

$$\xi_D = \begin{cases} \dfrac{\pi}{3}, & \text{for } \cos \xi_D = \dfrac{1}{2} \\[2ex] \dfrac{2\pi}{3}, & \text{for } \cos \xi_D = -\dfrac{1}{2} \text{ !} \end{cases} \qquad \text{(B.6)}$$

Now, using Equation (B.3), it is observed that the corresponding values of ξ_1, needed to extremize δE_1, are obtained from

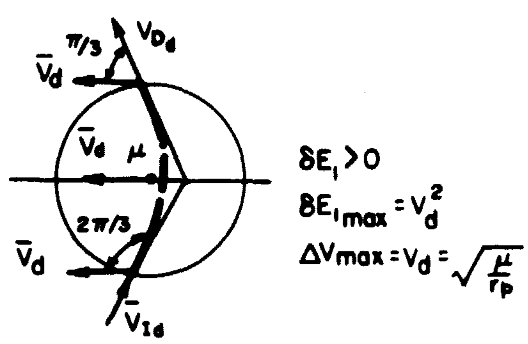
$$\cos \xi_I = \mp \frac{1}{2}$$

and, as a summary, the following table is presented.

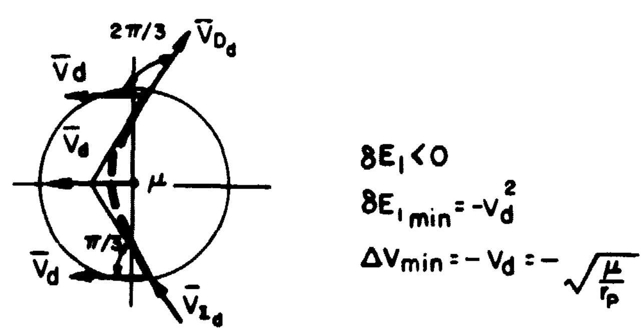
Table B.1
Conditions for Extremizing δE_1 Due to a Planetary Swing-By Maneuver

δE_1	$\cos(\overline{V}_d, \delta\overline{V})$			Comment
$(\delta E_1)_{max}$	0	$\pi/3$	$2\pi/3$	Energy is added for the satellite's heliocentric path
$(\delta E_1)_{min}$	π	$2\pi/3$	$\pi/3$	Energy is extracted from the satellite's heliocentric path

Graphically, the solutions noted in the above tabulation correspond to the following fly-by conditions, in the sphere of influence:



(energy is added to the heliocentric path)



(energy is extracted from the heliocentric path)

The information above indicates that the extremals for δE_1 are obtained by: (1) a gain in energy by passing behind the attracting mass; and doing so by keeping the vectors $\overline{V}_d$ and $\delta\overline{V}$ parallel (directly), also having $\Delta V = (\Delta V)_{max}$. The second extremal, $(\delta E_1)_{min}$, occurs when the satellite vehicle (or attracted mass) passes in front of the attracting mass; this occurs when $\delta\bar{V}$ and $\bar{V}_d$ are oppositely parallel, but while ΔV is retained at the level of $(\Delta V)_{max}$. All of this is, of course, based on the assumption that r_p has been pre-selected, a priori, and that this parameter does not otherwise influence the extremals.

Apparently conditions relative to the direction of $\bar{V}_{I_h}$, $\bar{V}_{D_h}$, etc., the size of the trajectory flown during the passage through the sphere of influence, and the magnitude of $\bar{V}_d$ (other than $\sqrt{\mu/r_p}$) will produce changes in $\delta\bar{V}$ and δE_1 but not the values corresponding to the extremals unless the necessary conditions for the extremals are satisfied.