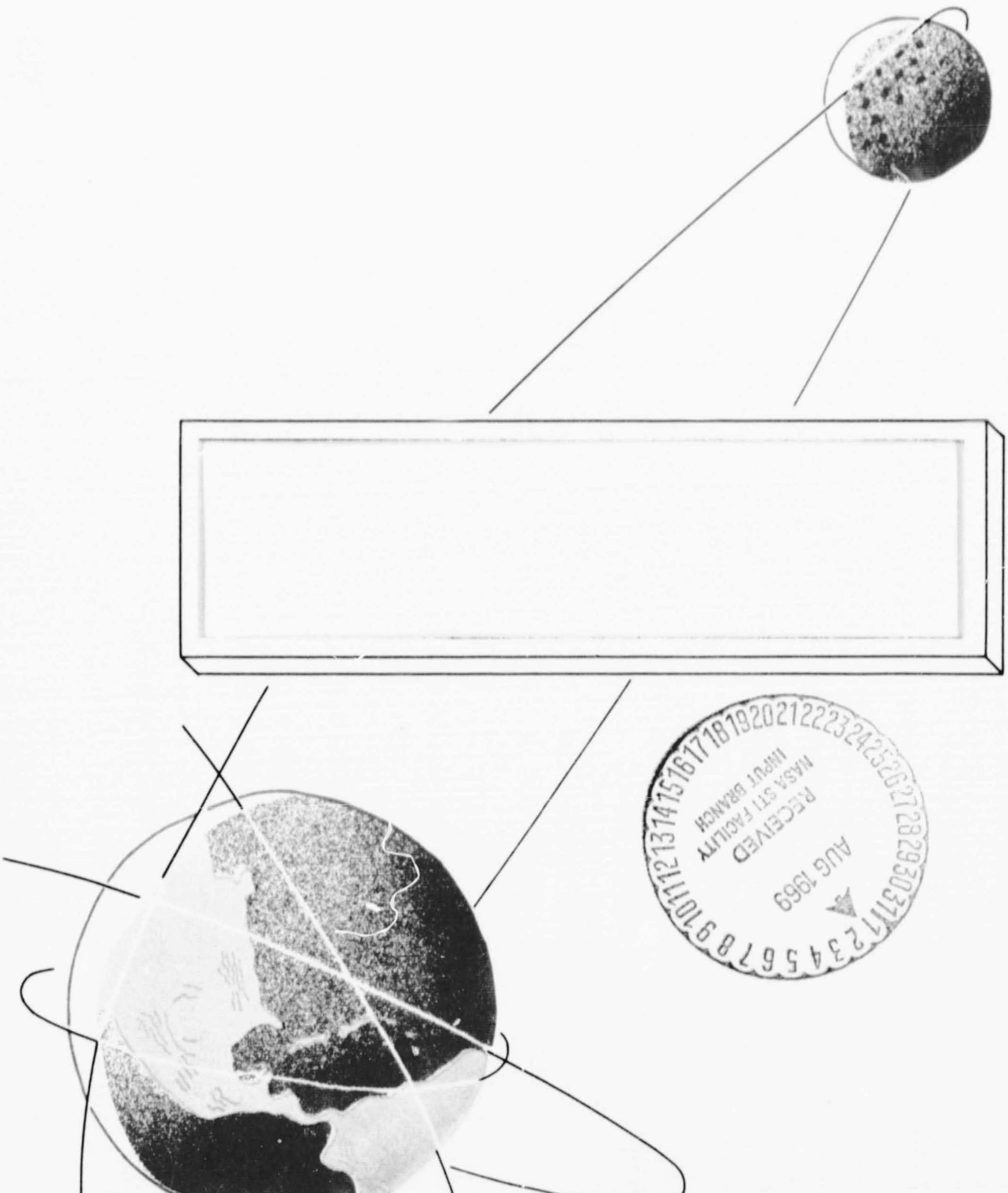

ENGINEERING MECHANICS RESEARCH LABORATORY

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SOME SUFFICIENT CONDITIONS FOR THE ASYMPTOTIC STABILITY OF LINEAR NONAUTONOMOUS SYSTEMS OF THE SECOND ORDER*

by

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Abstract

A quadratic Liapunov function with constant coefficients is used to obtain sufficient conditions for the asymptotic stability of linear systems of the second order. This Liapunov function is then optimized with respect to its coefficients to obtain simplified stability criteria for special classes of such systems.

Introduction. One of the more useful criteria for the stability of solutions of the equation

$$\ddot{x} + p(t)\,\dot{x} + q(t)x = 0 \tag{1}$$

is that obtained by Starzinski [1][1]. Using the same Liapunov approach which will utilized here he obtained sufficient conditions for asymptotic stability of the equilibrium as follows:[2]

$$p_1 > \sqrt{q_2} - \sqrt{q_1}\ ,\ p_2 < \frac{q_2 + 2\sqrt{q_1 q_2} + 5\,q_1}{\sqrt{q_2} - \sqrt{q_1}} \tag{2}$$

where p(t) and q(t) are continuous and satisfy

$$0 < p_1 \leq p(t) \leq p_2\ ,\ \ 0 < q_1 \leq q(t) \leq q_2 \tag{3}$$

for $t \geq t_0$. For the purpose of later comparison, define $\alpha = p_1/2$, $\omega = \sqrt{q_1} + p_1/2$. Inequalities (2) and (3) therefore require that the representative point parametrically defined by $\xi = p(t)$, $\eta = q(t)$, remain inside any one member of the

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[1] Numbers in brackets refer to list of references.

[2] The symbol $\sqrt{}$ will denote throughout the non-negative real square root, whose existence will always be assumed.

family of rectangular domains $R_{\alpha\omega}$ (see figure 1) bounded by

$$\begin{aligned} \xi &= 2\alpha \quad , \quad \alpha + \frac{1}{\alpha}(2\omega - \alpha)^2 \\ \eta &= (\omega - \alpha)^2 \; , \quad (\omega + \alpha)^2 \end{aligned} \tag{4}$$

for $t \geq t_o$, where $\omega > \alpha > o$.

In 1963 A. Ghizzetti [2], using an integral equation technique, proved that the equilibrium of (1) is asymptotically stable provided that the representative point parametrically defined by $\xi = p(t)$, $\eta = q(t)$, enters and remains inside any one member of the family of elliptical domains $E_{\alpha\omega}$ (see figure 2) defined by

$$\omega^2 (\xi - 2\alpha)^2 + (\eta - \alpha\xi + \alpha^2 - \omega^2)^2 - \alpha^2\omega^2 = 0 \tag{5}$$

for all $t \geq t_o$, where α and ω may be any positive numbers. Although this appears to be an excellent result, it will be shown to be always more restrictive than that obtained by utilizing a Liapunov function with constant coefficients.

In the following section a general criterion is obtained for equation (1) by use of a Liapunov method. This result, although identical to the general result of Starzinski [1], Razumichin [3], and others, is presented in a form more suitable for direct application. Succeeding sections will develop simple specialized results for several particular cases.

The General Criterion. Defining $x_1 = x$, $x_2 = \dot{x}$, (1) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= - p(t)x_2 - q(t)x_1 \end{aligned} \tag{6}$$

Consider the time-independent positive definite Liapunov function

$$V = (\alpha x_1 + x_2)^2 + \omega^2 x_1^2 \tag{7}$$

The time derivative of V, taken according to (6),

$$\dot{V} = -(x_1,x_2) \begin{bmatrix} 2\alpha q(t) & \alpha p(t)+q(t)-\alpha^2-\omega^2 \\ \alpha p(t) + q(t)-\alpha^2-\omega^2 & 2(p(t)-\alpha) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{8}$$

is negative definite provided the Sylvester inequalities

$$p(t) - \alpha \geq \delta \tag{9}$$

$$4\alpha q(t)[p(t)-\alpha]-[\alpha p(t)+q(t)-\alpha^2-\omega^2]^2 \geq \delta \tag{10}$$

are satisfied for all $t \geq t_o$, and some $\delta > 0$.

Sufficient conditons for asymptotic stability of all solutions of the system (1) are thus that there exist three positive numbers, α, ω, and δ, such that the representative point ($p(t)$, $q(t)$) remains inside, and bounded away from, the skewed parablola shown in figure 3 which is defined by either

$$\eta^2 - 2\,(\alpha\xi - \alpha^2 + \omega^2)\eta + (\alpha\xi - \alpha^2 - \omega^2)^2 = 0 \qquad (11a)$$

or

$$x^2 = \omega^2(2y - 2\alpha^2 - \omega^2) \qquad (11b)$$

for some positive numbers α and ω. The domains defined by (4) and (5) are compared with that of (11) in figure 4. Although the parabolic result is well known in less convenient forms, the simplicity of figure 3 and equations (11a,b) indicates that an appropriate choice of Liapunov function parameters can be made with considerably more ease than previously when considering a given physical system.

As an illustration of the procedure for choosing α and ω, consider a system of the form (1) where

$$p(t) = c(q(t) + d) \quad , \; c>0 \qquad (12)$$

Noting that $dq/dp = 1/c$, figure 3 clearly suggests that an appropriate choice for α is

$$\alpha = 1/c \tag{13}$$

since the slope of the upper (lower) branch of the parabola is always larger (less) than α. Further, figure 3 suggests that a minimum lower bound on $p(t)$ will be determined if the point (α,ω^2) lies on the line $\xi = c(\eta + d)$, leading to the choice

$$\omega^2 = 1/c^2 - d \tag{14}$$

Thus asymptotic stability is assured if

$$\begin{aligned} &c>0 \\ &dc^2<1 \\ &q(t) - \tfrac{1}{c}2 + d \geq \varepsilon > 0 \end{aligned} \tag{15}$$

Alternatively, a minimum lower bound on $q(t)$ should be determined if the point $(\frac{\omega^2+\alpha^2}{\alpha}, 0)$ lies on the line $\xi = c(\eta + d)$, implying

$$\omega^2 = d - 1/c^2 \tag{16}$$

and another set of sufficient conditions for asymptotic stability is

$$c>0$$
$$dc^2>1 \tag{17}$$
$$q(t) \geq \varepsilon > 0$$

Here q(t) may be unknown and not necessarily bounded. The validity of either (15) or (17) is sufficient for asymptotic stability.

Particular Cases. Special results derivable from the preceding will now be obtained for system (1), where p(t) and q(t) satisfy one or more of the following sets of conditions for $t \geq t_o$:

Case I. $0<p_1 \leq p(t) \leq p_2 \;,\; 0<q_1 \leq q(t) \leq q_2$ (18)

Case II. $0<p_1 \leq p(t) \;,\; b_1 \leq q(t)-ap(t) \leq b_2$ (19)

Case III. $0<p_1 \leq p(t) \;,\; b_1 \leq q(t)-ap(t) \leq b_2$ (20)

$$ap_1 + b_1 \leq q_1 \leq q(t)$$

The object is to determine sufficient conditions for asymptotic stability of (1) in terms of the parameters p_1, p_2, q_1, q_2, a, b_1, b_2, for each of the three cases. It is noted that by (11a) the lower branch of the parabolic boundary of figure 3 is defined by

$$\eta = (\sqrt{\alpha\xi - \alpha^2} - \omega)^2 \tag{21}$$

and the upper by

$$\eta = (\sqrt{\alpha\xi - \alpha^2} + \omega)^2 \tag{22}$$

where ω is assumed positive.

<u>Case I</u>. Reference to (18), (21), (22), and figure 5 indicates that (1) is asymptotically stable provided there exist positive numbers α , ω , δ_1 , δ_2 , p_2^* , such that

$$q_1^* = (\sqrt{\alpha p_1 - \alpha^2} - \omega)^2 \tag{23a}$$

$$q_1^* = (\sqrt{\alpha p_2^* - \alpha^2} - \omega)^2 \tag{23b}$$

$$q_2^* = (\sqrt{\alpha p_1 - \alpha^2} + \omega)^2 \tag{23c}$$

$$\delta_1 = \sqrt{q_1} - \sqrt{q_1^*} \tag{23d}$$

$$\delta_2 = \sqrt{q_2^*} - \sqrt{q_2} \tag{23e}$$

$$p_2^* \geq p_2 \tag{23f}$$

(23a,b,c) indicate

$$\sqrt{\alpha p_2^* - \alpha^2} - \sqrt{\alpha p_1 - \alpha^2} = 2\sqrt{q_1^*} \tag{24a}$$

$$\omega = \sqrt{\alpha p_1 - \alpha^2} + \sqrt{q_1^*} \tag{24b}$$

$$\sqrt{q_2^*} - \sqrt{q_1^*} = 2\sqrt{\alpha p_1 - \alpha^2} \tag{24c}$$

which, by (23d,e), become

$$\alpha(p_2^* - p_1) = 4(\sqrt{q_1} - \delta_1)^2 + 4(\sqrt{q_1} - \delta_1)\sqrt{\alpha p_1 - \alpha^2} \tag{25a}$$

$$\omega = \sqrt{\alpha p_1 - \alpha^2} + \sqrt{q_1} - \delta_1 \tag{25b}$$

$$\sqrt{q_2} - \sqrt{q_1} = 2\sqrt{\alpha p_1 - \alpha^2} - \delta_1 - \delta_2 \tag{25c}$$

and thus

$$\alpha = \frac{1}{2}(p_1 \pm \varepsilon) \ , \ \varepsilon \triangleq \sqrt{p_1^2 - (\sqrt{q_2} - \sqrt{q_1} + \delta_1 + \delta_2)^2} \tag{26a}$$

$$\omega = \frac{1}{2}(\delta_2 - \delta_1) + \frac{1}{2}(\sqrt{q_2} + \sqrt{q_1}) \tag{26b}$$

$$(p_1 \pm \varepsilon)(p_2^* - p_1) = 8(\sqrt{q_1} - \delta_1)^2 + 4(\sqrt{q_1} - \delta_1)(\sqrt{q_2} - \sqrt{q_1} + \delta_1 + \delta_2) \tag{26c}$$

The least restrictive choice for α is clearly

$$\alpha = \tfrac{1}{2}(p_1-\varepsilon) \tag{27}$$

for which (26c) becomes

$$p_2^* = p_1 + 4 \; \frac{2(\sqrt{q_1}-\delta_1)^2+(\sqrt{q_1}-\delta_1)(\sqrt{q_2}-\sqrt{q_1}+\delta_1+\delta_2)}{p_1 - \sqrt{p_1^2-(\sqrt{q_2}-\sqrt{q_1}+\delta_1+\delta_2)^2}} \tag{28}$$

The existence of $\alpha > o$, $\omega > o$, $p_2^* \geq p_2$, is assured by (26a,b), (28), provided

$$p_1 \geq \sqrt{q_2} - \sqrt{q_1} + \delta_1 + \delta_2 \tag{29a}$$

$$\sqrt{q_2} + \sqrt{q_1} > \delta_1 - \delta_2 \tag{29b}$$

$$p_2 \leq p_1 + 4 \; \frac{2(\sqrt{q_1}-\delta_1)^2+(\sqrt{q_1}-\delta_1)(\sqrt{q_2}-\sqrt{q_1}+\delta_1+\delta_2)}{p_1 - \sqrt{p_1^2-(\sqrt{q_2}-\sqrt{q_1}+\delta_1+\delta_2)^2}} \tag{29c}$$

for some $\delta_1 > o, \delta_2 > o$. Letting $\delta_2 = \delta_1$ and choosing δ_1 as the smaller of the two positive values obtained by solving (29a) and(29c), respectively, as equalities, sufficient conditions for asymptotic stability of (1) satisfying (18)(Case I) are found as follows[3]:

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[3]See footnote 2.

$$(p_1 - \sqrt{p_1^2 - (\sqrt{q_2} - \sqrt{q_1})^2})(p_2 - p_1) < 4\, q_1 + 4\sqrt{q_1\, q_2} \tag{30a}$$

$$p_1 > \sqrt{q_2} - \sqrt{q_1} \tag{30b}$$

A different procedure for obtaining a similar result was outlined by Starzinski [1], but the much more restrictive conditions (2) are considered to be his practical result [4].

<u>Case</u> <u>II</u> Reference to (19), (21), (22), and figure 6 indicates that (1) is asymptotically stable provided there exist positive numbers ω , δ_1 , δ_2 , such that

$$q_1^* = (\sqrt{ap_1 - a^2} - \omega)^2 \tag{31a}$$

$$q_2^* = (\sqrt{ap_1 - a^2} + \omega)^2 \tag{31b}$$

$$\delta_1 = \sqrt{ap_1 + b_1} - \sqrt{q_1^*} \tag{31c}$$

$$\delta_2 = \sqrt{q_2^*} - \sqrt{ap_1 + b_2} \tag{31d}$$

where α has been chosen as $\alpha = a$, implying $p_1 > a > 0$. (31a,b) imply

$$\omega = \sqrt{ap_1 - a^2} \pm \sqrt{q_1^*} \tag{32a}$$

$$\omega = -\sqrt{ap_1 - a^2} + \sqrt{q_2^*} \tag{32b}$$

where the upper sign will be used in (32a) since this choice leads to the least restrictive criteria. By (31c,d) and (32a,b)

$$ap_1 + b_1 > 0 \tag{33a}$$

$$\omega = \sqrt{ap_1 - a^2} + \sqrt{ap_1 + b_1} - \delta_1 \tag{33b}$$

$$\sqrt{ap_1 + b_2} - \sqrt{ap_1 + b_1} = 2\sqrt{ap_1 - a^2} - (\delta_1 + \delta_2) \tag{33c}$$

which can be satisfied by positive δ_1 , δ_2 , and ω , if and only if

$$\sqrt{ap_1 + b_2} - \sqrt{ap_1 + b_1} < 2\sqrt{ap_1 - a^2} \tag{34a}$$

$$ap_1 + b_1 > 0 \tag{34b}$$

Restrictions (34a,b) therefore assure asymptotic stability of (1) under conditions (19) (Case II).[4]

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[4]See footnote 2.

This case is applicable to the problem previously considered, where $p(t) = c(q(t) + d)$, $c>0$. Parameters a, b_1, and b_2 of (19) are thus

$$
\begin{aligned}
b_1 &= b_2 = -d \\
a &= 1/c
\end{aligned}
\tag{35}
$$

and conditions (34a,b) become

$$
\begin{aligned}
&\sqrt{p_1/c - 1/c^2} > 0 \\
&p_1/c - d > 0
\end{aligned}
\tag{36}
$$

Consequently asymptotic stability is assured for all $p(t)$ such that

$$p(t) \geq p_1 \tag{37}$$

where

$$p_1 > \max\begin{cases} cd \\ 1/c \end{cases} \qquad (c>o) \tag{38}$$

or, equivalently, for all

$$q(t) \geq q_1 \tag{39}$$

where

$$q_1 > \max \begin{cases} 0 \\ 1/c^2 - d \end{cases} \qquad (c>o) \tag{40}$$

Conditions (15) and (17) are contained in (39) and (40).

Case III. Reference to (20), (21), (22), and figure 7 indicates that (1) is asymptotically stable provided there exist numbers $\omega>o$, $\delta_1>o$, $\delta_2>o$, and $\delta_3\geq o$, such that

$$q_1^* = (\sqrt{ap_1-a^2} - \omega)^2 \tag{41a}$$

$$q_2^* = (\sqrt{ap_1-a^2} +\omega)^2 \tag{41b}$$

$$q_1^* = (\sqrt{ap_2^* -a^2} - \omega)^2 \tag{41c}$$

$$\delta_1 = \sqrt{q_1} - \sqrt{q_1^*} \tag{41d}$$

$$\delta_2 = \sqrt{q_2^*} - \sqrt{ap_1+b_2} \tag{41e}$$

$$\delta_3 = ap_2^* +b_1 - q_1^* \tag{41f}$$

$$p_2^* \geq p_1 \tag{41g}$$

where α has been chosen as $\alpha = a$, implying $p_1 > a > 0$. Relations (41a,b,c,g) are satisfied for

$$\omega = \sqrt{ap_1 - a^2} + \sqrt{q_1^*} \tag{42a}$$

$$\omega = -\sqrt{ap_1 - a^2} + \sqrt{q_2^*} \tag{42b}$$

$$\omega = \sqrt{ap_2^* - a^2} - \sqrt{q_1^*} \tag{42c}$$

which imply

$$\omega = \sqrt{ap_1 - a^2} + \sqrt{q_1} - \delta_1 \tag{43a}$$

$$\sqrt{ap_1 + b_2} - \sqrt{q_1} = 2\sqrt{ap_1 - a^2} - (\delta_1 + \delta_2) \tag{43b}$$

$$\delta_3 = ap_1 + b_1 + 4(\sqrt{q_1} - \delta_1)\sqrt{ap_1 - a^2} + 3(\sqrt{q_1} - \delta_1)^2 \tag{43c}$$

If the restrictions

$$\sqrt{ap_1 + b_2} - \sqrt{q_1} < 2\sqrt{ap_1 - a^2} \tag{44a}$$

$$ap_1 + b_1 + 4\sqrt{q_1}\sqrt{ap_1 - a^2} + 3q_1 > 0 \tag{44b}$$

are satisfied, there do exist a $(\delta_1 + \delta_2) > 0$ which satisfies (43b) and a sufficiently small positive $\delta_1 < (\delta_1 + \delta_2)$ such that both ω and δ_3 are positive by (43a,c). Therefore system (1) with conditions (20) (Case III) is asymptotically stable provided (44a,b) are satisfied.[5] Case II is seen to be included in Case III when $q_1 = ap_1 + b_1$ in conditions (20).

Summary. Sufficient conditions have been found for the asymptotic stability of all solutions of the system (1). In particular, the following theorems have been proven:[6]

A. If there exist positive α , ω , and δ , such that the functions p(t) and q(t) satisfy

$$4\alpha q(t)[p(t)-\alpha] - [\alpha p(t)+q(t)-\alpha^2-\omega^2]^2 \geq \delta$$

for all $t \geq t_o$, the equilibrium of (1) is asymptotically stable.

B. If the functions p(t) and q(t) satisfy

$$0 < p_1 \leq p(t) \leq p_2$$

$$0 < q_1 \leq q(t) \leq q_2$$

where

$$p_1 > \sqrt{q_2} - \sqrt{q_1}$$

$$(p_1 - \sqrt{p_1^2 - (\sqrt{q_2} - \sqrt{q_1})^2})(p_2-p_1) < 4 q_1 + 4\sqrt{q_1 q_2}$$

[5,6] See footnote 2.

for all $t \geq t_o$, the equilibrium of (1) is asymptotically stable.

C. If the functions $p(t)$ and $q(t)$ satisfy

$$0 < p_1 \leq p(t)$$

$$b_1 \leq q(t) - a\, p(t) \leq b_2$$

$$ap_1 + b_1 \leq q_1 \leq q(t)$$

where

$$\sqrt{ap_1 + b_2} - \sqrt{q_1} < 2\sqrt{ap_1 - a^2}$$

$$a_1 p_1 + b_1 + 4\sqrt{q_1}\sqrt{ap_1 - a^2} + 3q_1 > 0$$

for all $t \geq t_o$, the equilibrium of (1) is asymptotically stable.

References

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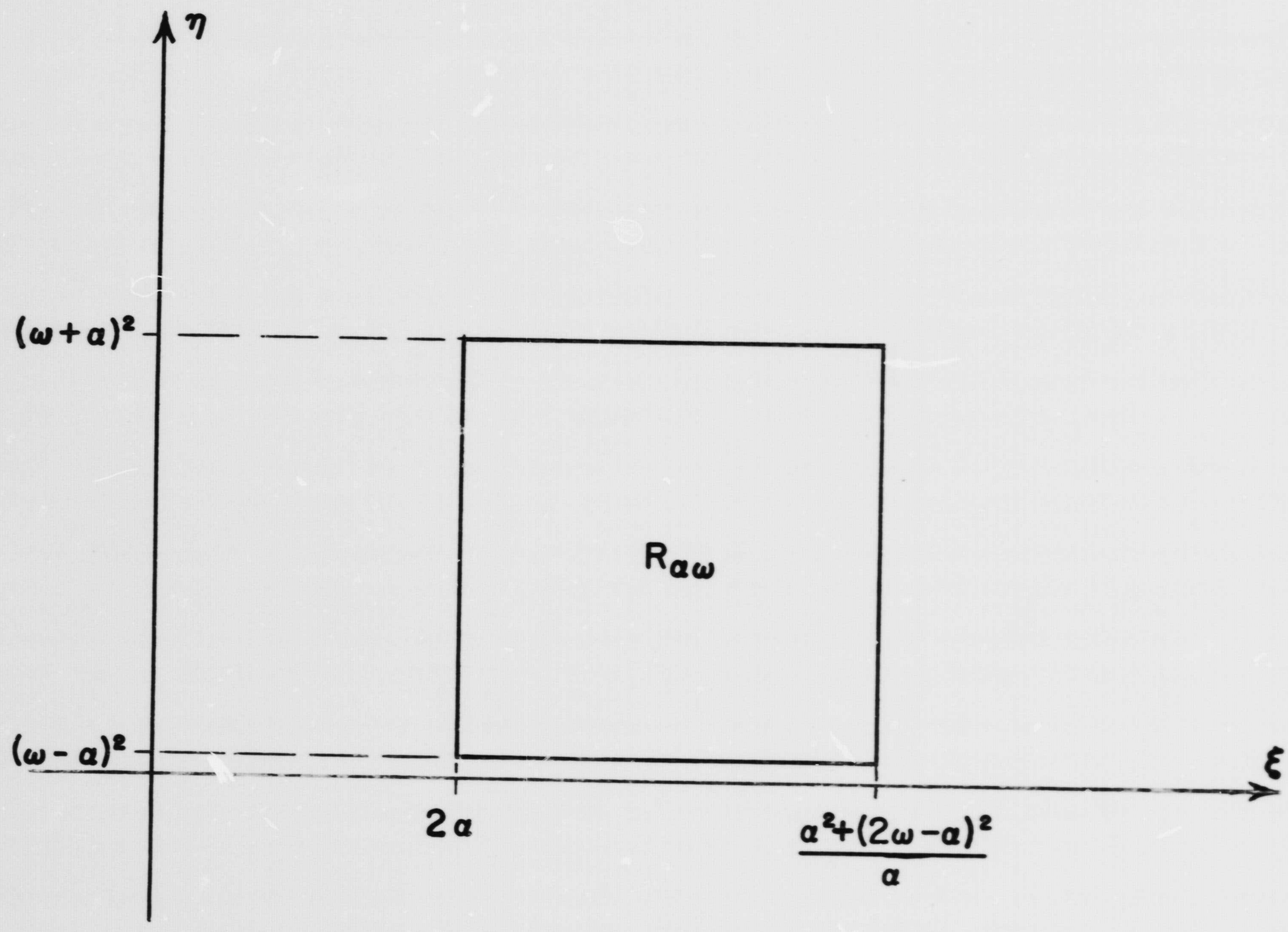
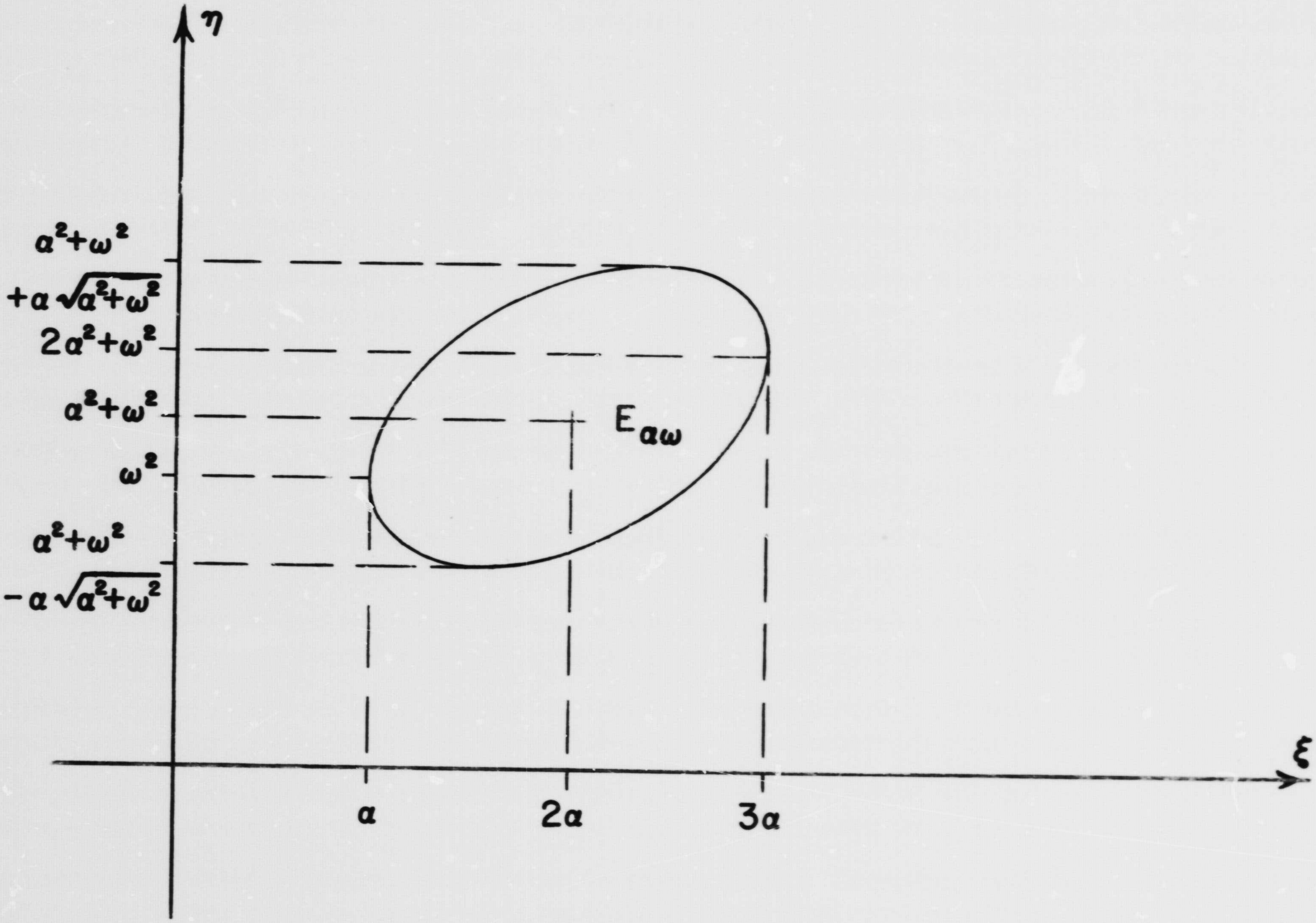


FIGURE 1



η
$\alpha^2+\omega^2$
$+\alpha\sqrt{\alpha^2+\omega^2}$
$2\alpha^2+\omega^2$
$\alpha^2+\omega^2$
ω^2
$\alpha^2+\omega^2$
$-\alpha\sqrt{\alpha^2+\omega^2}$
$E_{\alpha\omega}$
ξ
α
2α
3α

FIGURE 2

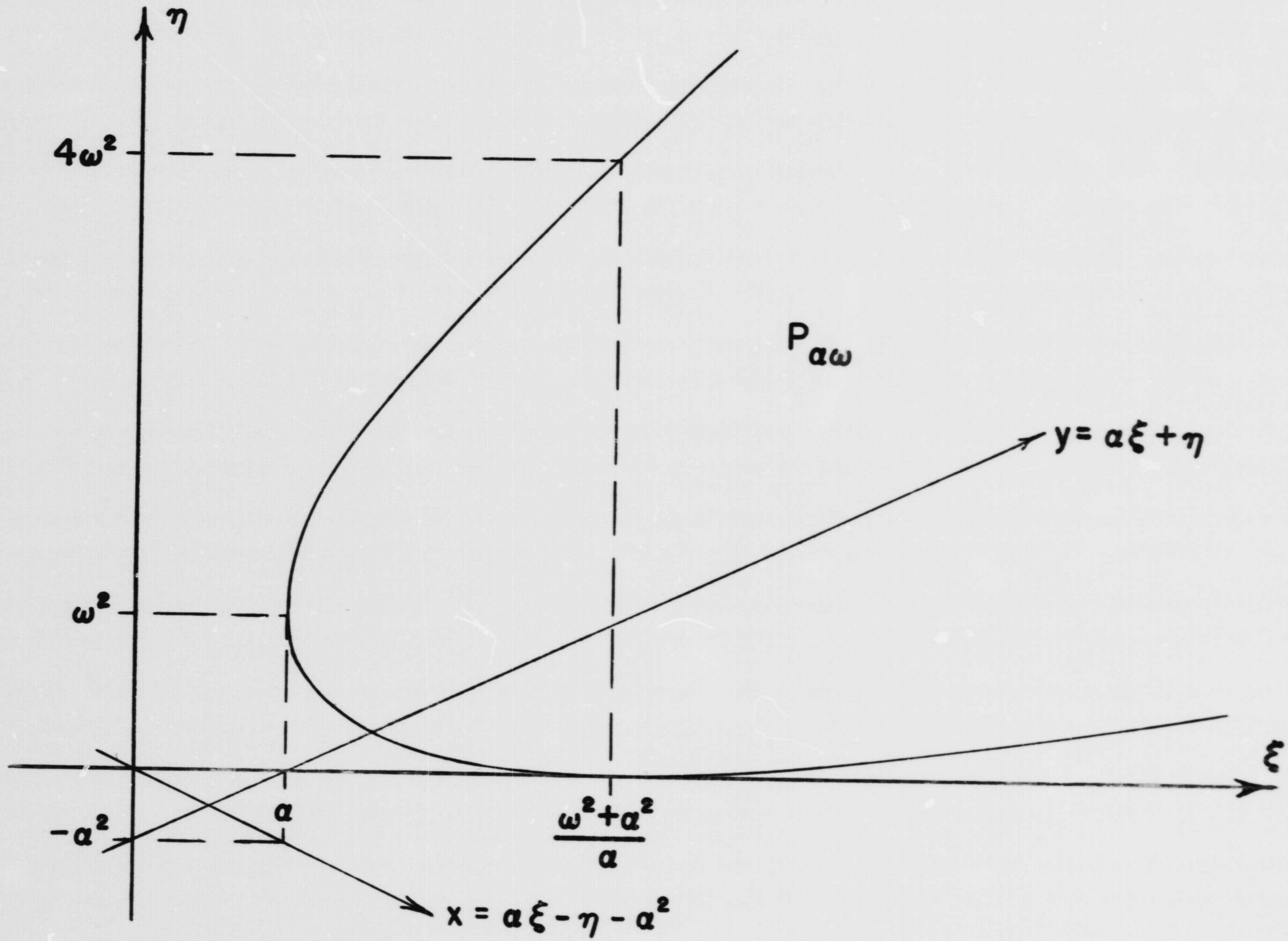


FIGURE 3

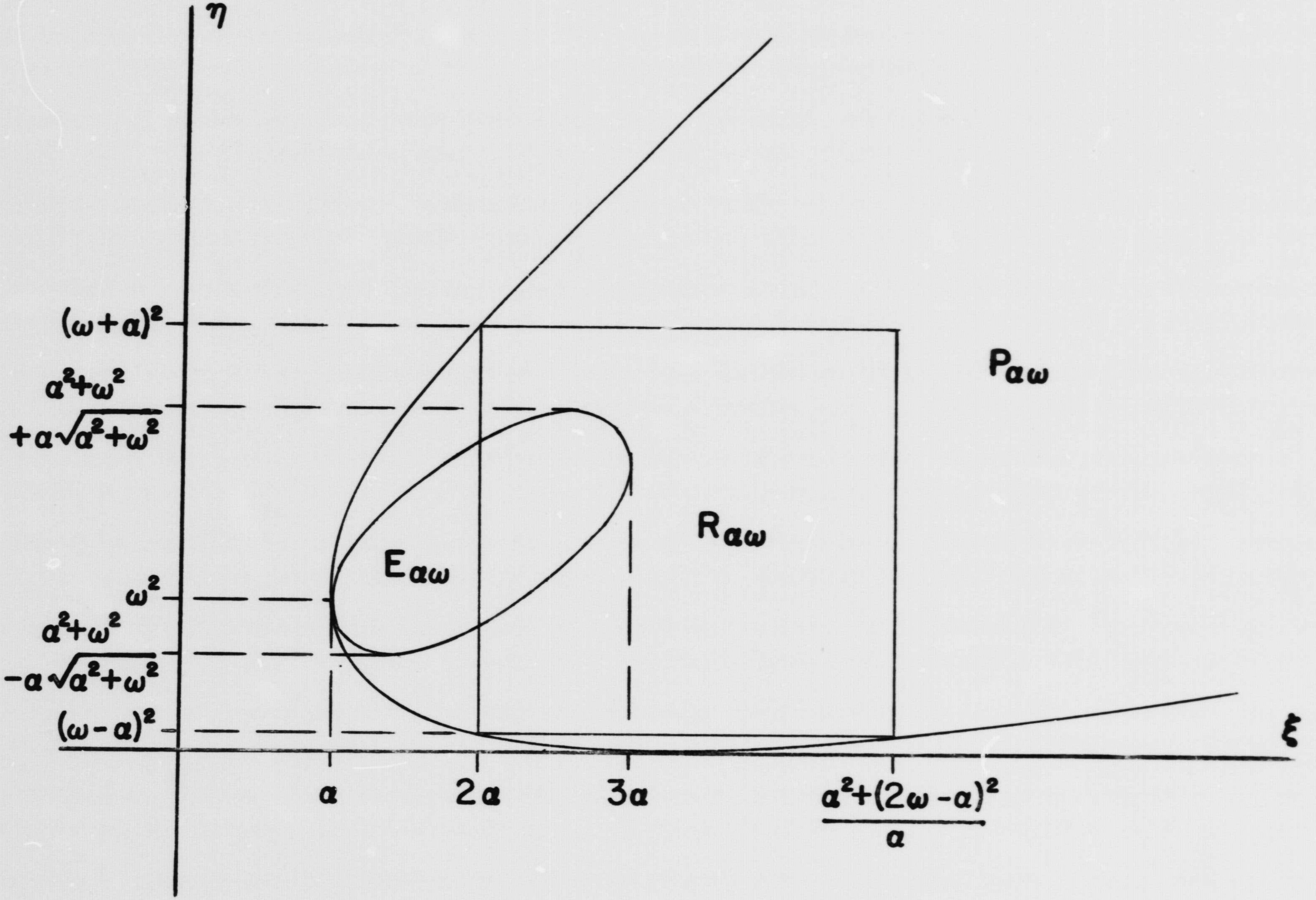

η
$(\omega+\alpha)^2$
$\alpha^2+\omega^2$
$+\alpha\sqrt{\alpha^2+\omega^2}$
ω^2
$\alpha^2+\omega^2$
$-\alpha\sqrt{\alpha^2+\omega^2}$
$(\omega-\alpha)^2$
$P_{\alpha\omega}$
$R_{\alpha\omega}$
$E_{\alpha\omega}$
ξ
α
2α
3α
$\frac{\alpha^2+(2\omega-\alpha)^2}{\alpha}$

FIGURE 4

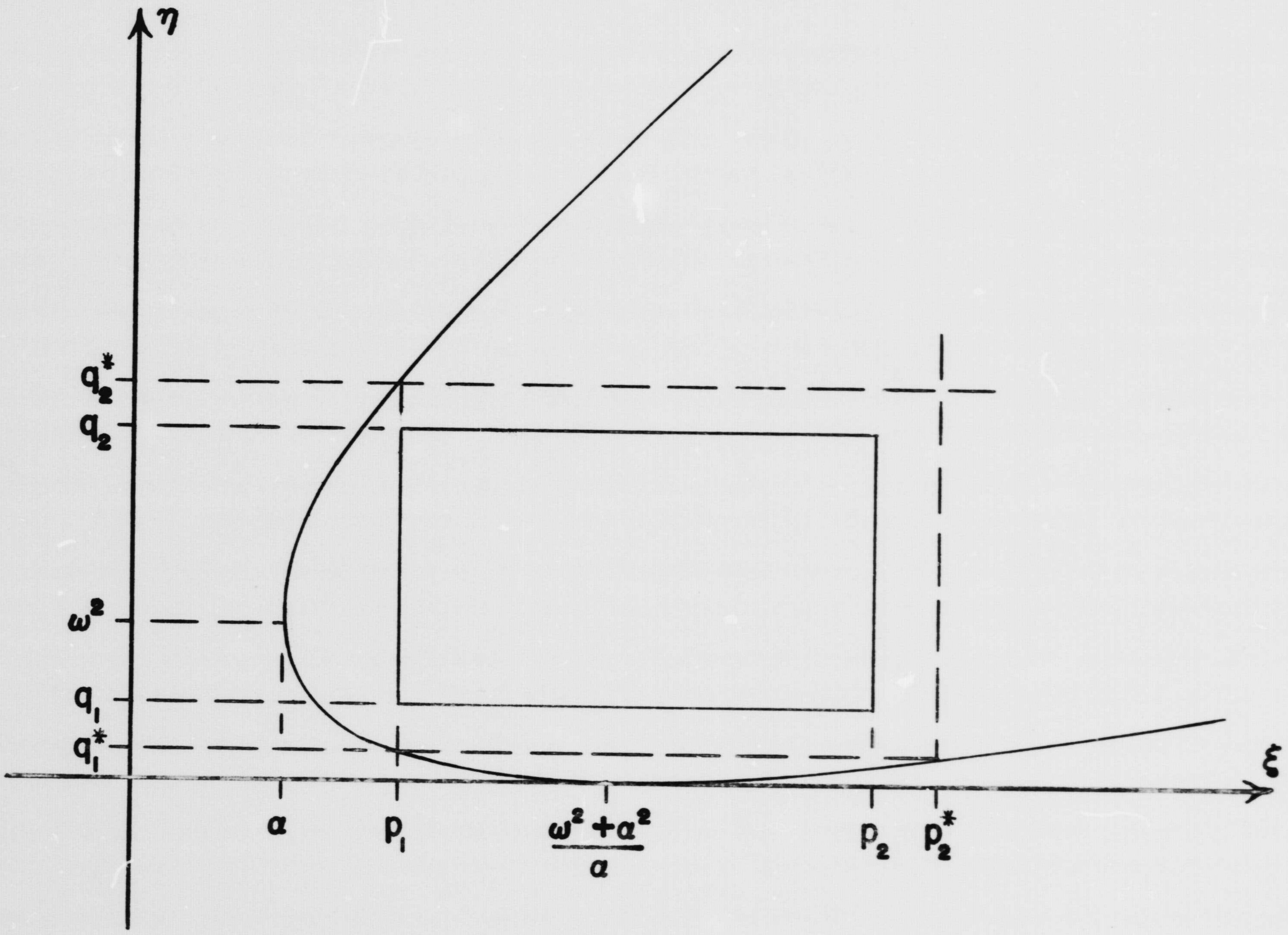

η
q_2^*
q_2
ω^2
q_1
q_1^*
ξ
α
p_1
$\frac{\omega^2+\alpha^2}{\alpha}$
p_2
p_2^*

FIGURE 5

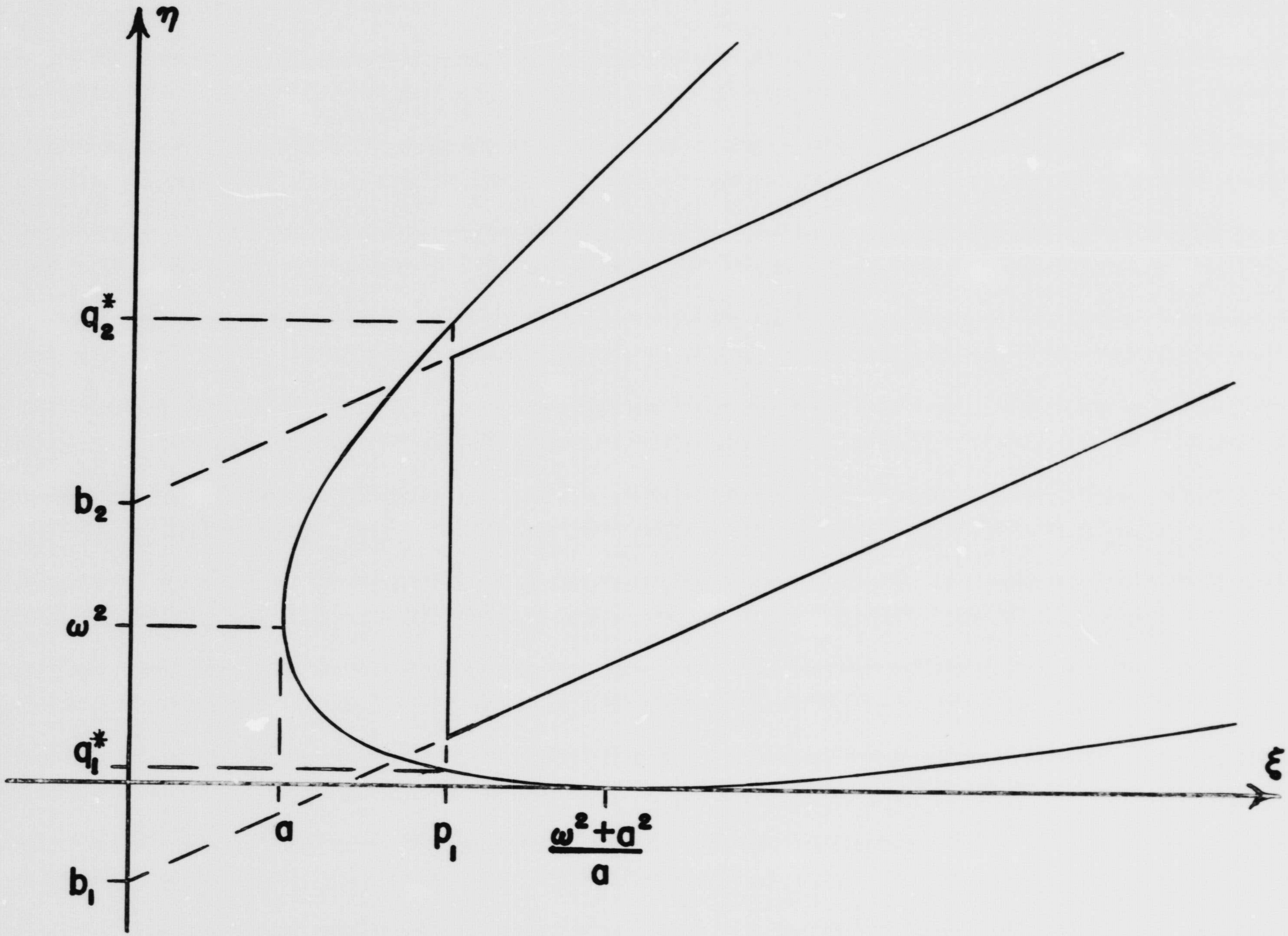


FIGURE 6

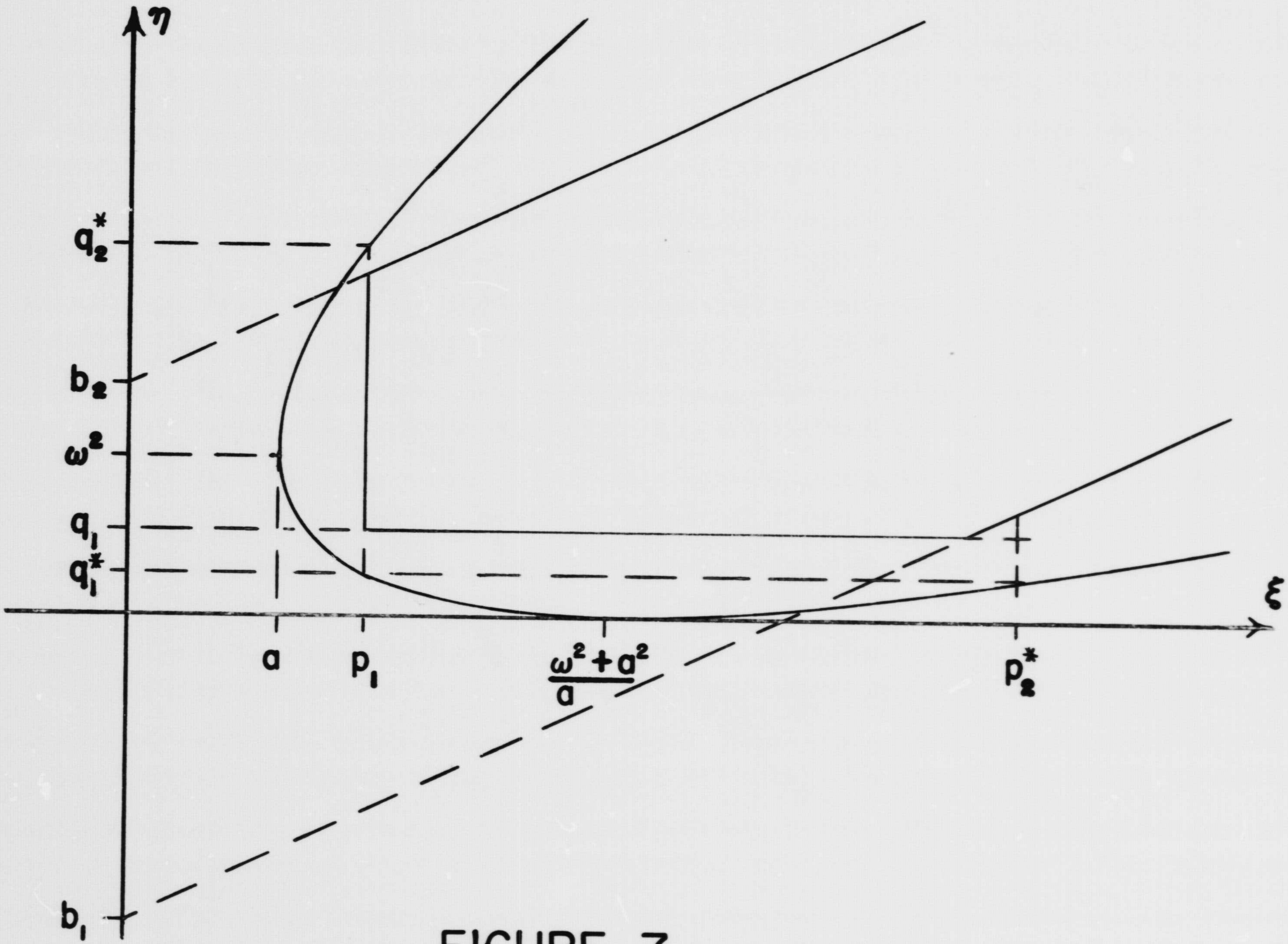


FIGURE 7