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by Peter Crimi

Prepared by
ROCHESTER APPLIED SCIENCE ASSOCIATES, INC.
Rochester, N. Y.
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SUMMARY

A method has been developed which provides the exact solution to the perturbation equations of motion of a rotor in forward flight. Effects of compressibility, stall and reversed flow are taken into account, and there is no restriction on the number of degrees of freedom which can be analyzed.

The equations of motion of a rotor blade in forward flight were derived in terms of the normal modes of free vibration of the blade. Aerodynamic forces were expressed on the basis of quasi-steady flow, using strip theory. Perturbation equations were generated by expanding all aerodynamic forces in the perturbation variables about their steady-state values. The equations are of the form of a coupled set of linear, second-order differential equations with periodically varying coefficients.

A method was then developed for analyzing the stability of linear dynamic systems with periodically varying parameters. Stability is determined by calculation of all the characteristic values of the system. Thus, a solution provides rates of growth or decay of the motion following a disturbance, in addition to a determination of whether or not the system is stable.

A digital computer program was prepared to implement the general method for the case of a rotor blade with one, two or three degrees of freedom. Stability calculations were performed for comparison with results obtained by direct time integration of the nonlinear equations of motion of a rigid blade with flapping and lead-lag hinges. Limited calculations were also made of the aeroelastic stability of a model rotor blade for which experimental flutter data was available. Comparisons of the results indicate qualitative agreement in both cases.

INTRODUCTION

The aeroelastic stability of helicopter rotors is of concern for two reasons. First, at the high advance ratios associated with compound and stowable-rotor operation, the hostile aerodynamic environment could lead to dangerous instabilities. Secondly, in conventional operation, instabilities can occur which, while generally not of a catastrophic nature because of nonlinear effects, are nonetheless serious from a fatigue and control standpoint.

The analysis of rotor flutter for hovering flight is a relatively straightforward problem, the formulations being essentially the same as those for classical flutter of conventional aircraft. Considerable research, both theoretical and experimental, has dealt with the flutter of hovering rotors (see, for example, References 1 through 4). Agreement between theory and experiment has generally been good.

The problem for a rotor in forward flight is fundamentally different from that of the hovering case, due to the nature of the equations of motion. Because the relative flow imposed on a given blade section varies periodically as the blade rotates, the effective dynamic pressure, and hence the constant of proportionality of the aerodynamic forces, also varies periodically. As a consequence, the equations of motion of the rotor have periodically varying coefficients. Systems described by equations of this type, even though linear, display many unusual properties and in some respects resemble nonlinear systems (see Reference 5).

Linear differential equations with periodically varying coefficients have been the concern of applied mathematicians for over a century. The differential equation of second order bearing his name was discussed by Mathieu in 1868 in reference to the problem of a vibrating elliptic membrane. The more general second-order equation derived by Hill for determining the motion of the lunar perigee was presented by him in 1877. In 1883, Floquet determined the form of the solution for any linear differential equation with periodic coefficients.*

More recent analyses related to dynamic systems with periodic parameters are presented in References 7, 8 and 9. In Reference 7, the general $N^{\underline{th}}$ order problem is treated, while Reference 8 is concerned with spin-stabilized satellites and Reference 9 deals

*A discussion of the early history of Mathieu and related functions is given in Reference 5. Many papers of historical interest are also noted in Reference 6.

with mechanical instabilities of helicopter rotors. These analyses and similar ones treating related problems generally either rely on expansion in a small parameter to obtain a solution or are restricted to special forms of the differential equations.

Since a general method for analyzing linear systems with periodic parameters has not been available, it has been necessary in the past, in investigating rotor flutter, to rely primarily on direct integration of the equations of motion using an analog or digital computer. The results of analog studies of rotor stability are reported in References 10 and 11. Digital computers were used to obtain results discussed in References 12 and 13. Computer solutions are useful particularly in that nonlinear effects can readily be incorporated to give an accurate indication of system dynamics for a specific rotor and flight conditions. However, the results are often difficult to interpret and evaluate in terms of system stability, and, especially when a digital computer is used, their cost precludes conducting a thorough parametric study.

It has been possible to gain some insight into rotor aeroelastic characteristics by treating a single degree of freedom analytically (References 14 and 15). A single second-order differential equation represents the system. This equation can be reduced to the form of Hill's equation (Reference 6), for which there are techniques available to obtain solutions. The analyses have been confined to the flapping degree of freedom; it was found that this simple system is quite stable, but that flutter can occur at very high advance ratios (of the order of unity).

The study reported here was directed to obtaining the exact analytical solutions of the perturbation equations of motion of a rotor in forward flight. The values of system parameters or flight conditions were not restricted, nor were limitations placed on the number or types of degrees of freedom. Because the general problem was attacked, the solution obtained has application in areas other than rotor aeroelastic stability. The method presented can be used to determine the stability of any linear dynamic system with periodically varying parameters. Also, the computer program developed to implement the method has been segmented in such a way that it can be applied to the stability analysis of any linear dynamic system with three degrees of freedom.

SYMBOLS

a_{mn}, b_{mn}	dimensionless, periodic coefficients in the basic set of differential equations
c_{mn}, d_{mn}	dimensionless, periodic coefficients in the derived set of differential equations

c — blade chord, m

C_ℓ — blade section lift coefficient

C_d — blade section drag coefficient

C_m — blade section moment coefficient for moment about mid-chord, positive to increase incidence

d — drag per unit span of blade, N/m

F_v — aerodynamic force component per unit blade span in the y-direction, N/m

F_w — aerodynamic force component per unit blade span in the z-direction, N/m

f_{rj} — coefficients in the finite sum of trigonometric functions related to Δ

I_0 — mass moment of inertia about the elastic axis per unit span; kg-m

ℓ — lift per unit span of blade, N/m

ℓ_z — horizontal distance forward of the X_0axis of the elastic axis (x-axis), m

$M_{c/2}$ — aerodynamic moment per unit span about mid-chord, positive to increase incidence, N

M_ϕ — aerodynamic moment per unit span about the elastic axis, positive to increase incidence, N

m — blade mass per unit span, kg/m

N — number of degrees of freedom of the elasto-mechanical system

R — rotor radius, m

V — magnitude of resultant fluid velocity relative to a blade section, m/s

V_n — component of fluid velocity, relative to a blade section, normal to the X_0-Z_0 plane, m/s

V_t — component of fluid velocity, relative to a blade section, tangent to the X_0-Z_0 plane, m/s

V_f	magnitude of free-stream velocity (aircraft forward speed), m/s
v	displacement of blade section in the y-direction,m
w	displacement of blade section in the z-direction,m
w_i	wake-induced inflow, m/s
(X_0, Y_0, Z_0)	coordinates rotating with the blade, the Y_0 axis coincident with the shaft
(x, y, z)	coordinates fixed with respect to a local blade section, the x-axis coinciding with the elastic axis and the z-axis parallel to the chord
Z_a	distance of the elastic axis ahead of mid-chord,m
α	blade section angle of attack, rad
α_s	shaft tilt with respect to normal to free stream, positive for aft tilt of the negative Y_0-axis, rad
Δ	infinite determinant
ε	distance forward of the elastic axis of blade section mass center, m
ζ_n	displacement of the $n^{\underline{th}}$ coupled mode of free vibration of the elasto-mechanical system
θ	flapwise bending slope
λ_R	real part of system characteristic value $i\omega$
λ_I	imaginary part of system characteristic value $i\omega$
λ_{rj}	roots defining points at which Δ is singular
ρ	mass density of free stream, kg/m^3
Φ	angle between local blade chord and the X_0-Y_0 plane, rad
ϕ	torsional deflection about the elastic axis, positive to increase incidence, rad
ψ	chordwise bending slope
Ω	rotor rotational speed, rad/s

$i\omega$ characteristic value of the basic or derived systems, $i\omega = \lambda_R + i\lambda_I$

$\bar{\omega}_k$ natural frequency of the $k^{\underline{th}}$ coupled mode of free vibration of the rotating system, rad/s

DEVELOPMENT OF THE EQUATIONS OF MOTION OF A ROTOR IN FORWARD FLIGHT

The perturbation equations of motion for a rotor blade in forward flight are derived in this section. The set of coupled, linear differential equations with periodic coefficients which are obtained form the subject for analysis in the next section.

The elasto-mechanical segment of the aeroelastic system, assumed to have N degrees of freedom, is conveniently represented in terms of the N normal modes of free vibration. Normal modes and frequencies for a rotating beam can be obtained either from a continuous representation (Reference 16) or from a lumped-mass model (Reference 17).

Aerodynamic forces are derived here in accordance with the usual strip-theory assumptions. Quasi-steady flow is assumed as well. A perturbation analysis is then performed, with expansion of experimentally or otherwise determined aerodynamic coefficients in a Taylor series,about the nominal flow conditions,in the quantities defining the perturbations in the blade motions.

Consider a rotor blade, then, with angular speed Ω subjected to a uniform free stream of magnitude V_f and density ρ, as shown in Figure 1. The X_0-axis is taken coincident with the pitch, or feathering, axis, when the blade is in its undeformed position, as shown in the figure. The coordinates (x,y,z) also rotate with the blade and are fixed with respect to a local blade section of the undeformed blade, with origin at the elastic axis and with the z-axis parallel to the chordline. The horizontal offset of the elastic axis with respect to the X_0-axis is denoted ℓ_z. The angle which a section of the undeformed blade makes with the X_0-Z_0plane, combining built-in twist and pitch control settings, is denoted Φ.

Blade motions are defined by the five variables v, w, ϕ, θ and ψ. The deflections of the elastic axis in the y and z directions are v and w, respectively, the torsional deflection is ϕ, while θ and ψ are bending slopes:

$$\theta = \frac{\partial v}{\partial x} \quad , \quad \psi = - \frac{\partial w}{\partial x} \quad .$$

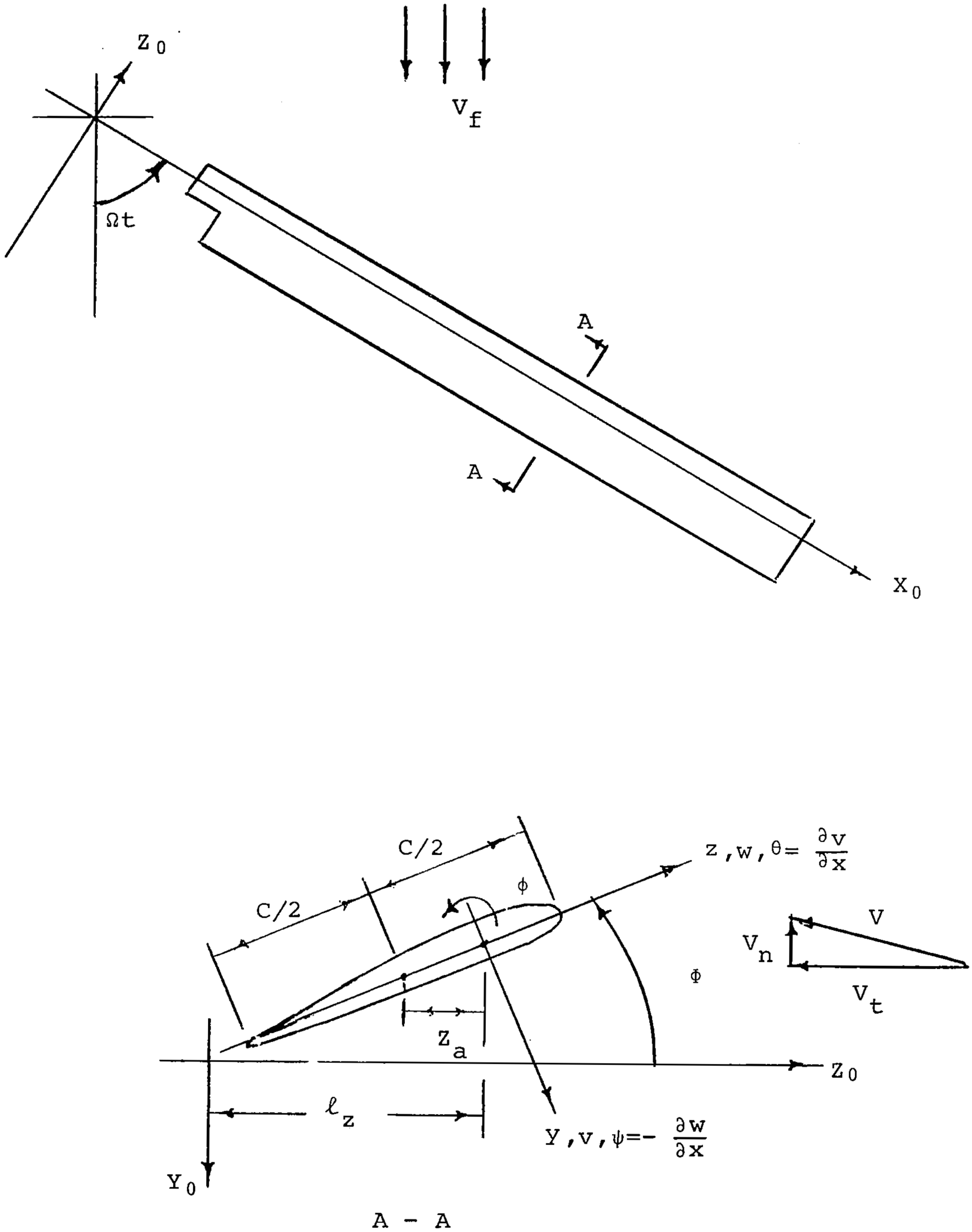

Figure 1. Coordinate systems for a rotor in forward flight.

These five variables are expressed in terms of the N coupled modes of free vibration selected to represent the rotating mechanical system. Specifically, these variables are expressible in the form:

$$\left.\begin{aligned}
v(x,t) &= \sum_{n=1}^{N} A_v^{(n)}(x)\,\zeta_n(t) \\
w(x,t) &= \sum_{n=1}^{N} A_w^{(n)}(x)\,\zeta_n(t) \\
\phi(x,t) &= \sum_{n=1}^{N} A_\phi^{(n)}(x)\,\zeta_n(t) \\
\theta(x,t) &= \sum_{n=1}^{N} A_\theta^{(n)}(x)\,\zeta_n(t) \\
\psi(x,t) &= \sum_{n=1}^{N} A_\psi^{(n)}(x)\,\zeta_n(t)
\end{aligned}\right\} \qquad (1)$$

where $A_v^{(n)}(x)$, $A_w^{(n)}(x)$, etc., denote the values of v, w, etc., obtained on excitation of the $n^{\underline{th}}$ coupled mode.

If Lagrange's equations are applied to the system, with the functions ζ_1, ζ_2, ..., ζ_N as generalized coordinates, the system equations of motion are given by

$$\ddot{\zeta}_k + \bar{\omega}_k^2\,\zeta_k = F_k(t) \quad , \quad k = 1,2,\ldots,N; \qquad (2)$$

where $\bar{\omega}_k$ is the natural frequency of the $k^{\underline{th}}$ mode and F_k is the generalized force applied to that mode. It is assumed that the mode shapes have been normalized so as to yield a generalized mass of unity.

The generalized forces are expressed by

$$F_k(t) = \int_0^R \{A_v^{(k)}(x)Q_v(x,t) + A_w^{(k)}(x)Q_w(x,t)$$

$$+ A_\phi^{(k)}(x)Q_\phi(x,t) + A_\psi^{(k)}(x)Q_\psi(x,t)\}dx \tag{3}$$

where R is rotor radius. Omitting the steady-state forcing terms, which are not of concern to a stability analysis, the force and moment distributions Q_v, Q_w, etc. are given by

$$\left.\begin{aligned}
Q_v(x,t) &= -\ 2m\Omega\varepsilon \sin \Phi\dot{\psi} + F_v(x,t) \\
Q_w(x,t) &= -\ 2m\Omega\varepsilon \cos \Phi\dot{\psi} + F_w(x,t) \\
Q_\phi(x,t) &= 2I_0\Omega \sin \Phi\dot{\psi} + M_\phi(x,t) \\
Q_\psi(x,t) &= -\ 2I_0\Omega\dot{\phi} + 2m\Omega\varepsilon[\dot{v} \sin \Phi + \dot{w} \cos \Phi]
\end{aligned}\right\} \tag{4}$$

where F_v and F_w are aerodynamic forces in the y and z directions, respectively, M_ϕ is the aerodynamic moment about the elastic axis, ε is the distance of the section mass center forward of the elastic axis, m is blade mass per unit span and I_0 is mass moment of inertia about the elastic axis per unit span. The terms containing $\dot{\psi}$ and $\dot{\phi}$ represent gyroscopic coupling terms. They are placed on the right-hand side because their inclusion in the vibration analysis to obtain mode shapes and frequencies would disrupt the orthogonality of the modes. No quantity Q_θ, similar to the quantity Q_ψ, appears in Eqs. (3) or (4) because the mass moment of inertia of a blade section about its chordline has been assumed to be negligible.

The next step is to obtain expressions for the aerodynamic forces and to perform the appropriate linearizations. To accomplish this define, first, an effective section incidence α,

$$\alpha = \Phi + \phi + \tan^{-1}\left(\frac{V_n}{V_t}\right)$$

where V_n and V_t are components of fluid velocity relative to the blade, at midchord, normal and tangential to the rotor plane, respectively. If α_s denotes shaft angle (positive for aft tilt of the Ω-vector), Z_a is distance of the elastic axis forward of mid-chord and Ωt is the blade azimuth relative to the downstream direction, then these velocity components are given by

$$\left.\begin{aligned} V_n &= [\dot{v} + Z_a\dot{\phi} - \Omega(Z_a - \ell_z)(\theta \cos \Phi + \psi \sin \Phi)] \cos \Phi \\ &\quad - \dot{w} \sin \Phi + V_f \cos \alpha_s \cos \Omega t(\theta \cos \Phi + \psi \sin \Phi) \\ &\quad + V_f \sin \alpha_s - w_i \\ V_t &= \Omega x + V_f \cos \alpha_s \sin \Omega t + \dot{w} \cos \Phi \\ &\quad + [\dot{v} + Z_a\dot{\phi} - \Omega(Z_a - \ell_z)(\theta \cos \Phi + \psi \sin \Phi)] \sin \Phi \end{aligned}\right\} \quad (5)$$

where w_i is the wake-induced inflow, assumed constant over the rotor plane. The factor $\Omega(\theta \cos \Phi + \psi \sin \Phi)$ appears in Eqs. (5) because the angular velocity directed along the Y_0 (shaft) axis, of magnitude Ω, has a component which is tangent to the blade when the blade is inclined to the $X_0 - Z_0$ plane, due to bending, the angle of inclination being $\theta \cos \Phi + \psi \sin \Phi$.

The aerodynamic lift ℓ, acting normal to the relative fluid velocity, and containing a term to account for the apparent camber due to rotation of the section about the x-axis, the drag d acting parallel to the relative velocity, and the moment $M_{c/2}$ acting about mid-chord, are given by

$$\left.\begin{aligned} \ell &= \tfrac{1}{2}\rho V^2 C C_\ell(\alpha) + \tfrac{\pi}{4}\rho V C^2 [\dot{\phi} - \Omega(\theta \cos \Phi + \psi \sin \Phi)] \\ d &= \tfrac{1}{2}\rho V^2 C C_d(\alpha) \\ M_{c/2} &= \tfrac{1}{2}\rho V^2 C^2 C_m(\alpha) \end{aligned}\right\} \quad (6)$$

where

$$V^2 = V_n^2 + V_t^2 \quad .$$

The blade chord is C and C_ℓ, C_d and C_m are experimentally determined two-dimensional force coefficients. The apparent-camber term arises due to the linear variation of quasi-steady downwash along the blade chord when the blade section is rotated about mid-chord. The aerodynamic forces appearing in Eqs. (4) are related to those in Eqs. (6) by the following expressions:

$$\left.\begin{aligned} F_v &= -\ell \cos\alpha - d \sin\alpha \\ F_w &= \ell \sin\alpha - d \cos\alpha \\ M_\phi &= M_{c/2} - Z_a \ell . \end{aligned}\right\} \qquad (7)$$

The next step is to perform a consistent linearization of Eqs. (7) by expanding the various functions appearing in those equations in Taylor series and discarding second-order quantities in v, w, ϕ, θ or ψ. The expansion is made about the nominal incidence α_0 and nominal relative speed V_0, where

$$\alpha_0 = \Phi + \xi ,$$

$$\xi = \tan^{-1}\left[\frac{V_f \sin\alpha_s - w_i}{\Omega x + V_f \cos\alpha_s \sin\Omega t}\right] ;$$

$$V_0 = [(V_f \sin\alpha_s - w_i)^2 + (\Omega x + V_f \cos\alpha_s \sin\Omega t)^2]^{1/2} ;$$

as obtained when all dependent variables vanish in the expressions for V and α. Specifically, the expansions of α and V yield

$$\alpha = \alpha_0 + \Delta\alpha + \ldots$$

$$= \alpha_0 + \phi + \frac{1}{V_0^2} [\Omega x + V_f \cos\alpha_s \sin\Omega t]\{[\dot{v} + Z_a\dot{\phi}$$

$$- \Omega(Z_a - \ell_z)(\theta \cos\Phi + \psi \sin\Phi)]\cos\Phi$$

$$- \dot{w} \sin\Phi + V_f \cos\alpha_s \cos\Omega t(\theta\cos\Phi + \psi \sin\Phi)\}$$

$$- \frac{1}{V_0^2} [V_f \sin \alpha_s - w_i]\{\dot{w} \cos \Phi + [\dot{v} + Z_a \dot{\phi}$$

$$- \Omega(Z_a - \ell_z)(\theta \cos \Phi + \psi \sin \Phi)] \sin \Phi\} + \ldots$$

$$V = V_0 + \Delta V + \ldots$$

$$= V_0 + \frac{1}{V_0} [V_f \sin \alpha_s - w_i]\{[\dot{v} + Z_a\dot{\phi}$$

$$- \Omega(Z_a - \ell_z)(\theta \cos \Phi + \psi \sin \Phi)]\cos \Phi$$

$$- \dot{w} \sin \Phi + V_f \cos \alpha_s \cos \Omega t(\theta \cos \Phi + \psi \sin \Phi)\}$$

$$+ \frac{1}{V_0} [\Omega x + V_f \cos \alpha_s \sin \Omega t]\{\dot{w} \cos \Phi + [\dot{v} + Z_a\dot{\phi}$$

$$- \Omega(Z_a - \ell_z)(\theta \cos \Phi + \psi \sin \Phi)]\sin \Phi\} + \ldots$$

Each term in Eqs. (7) is then expanded, using these expressions for α and V. For example, the first term in F_V is expanded by writing $\ell \cos \alpha$ as follows:

$$\ell \cos \alpha = \{\tfrac{1}{2}\rho V^2 C C_\ell(\alpha) + \tfrac{\pi}{4}\rho V C^2 [\dot{\phi} - \Omega(\theta \cos \Phi + \psi \sin \Phi)]\}\cos \alpha$$

$$= \{\tfrac{1}{2}\rho [V_0^2 + 2V_0\Delta V + \ldots]C[C_\ell(\alpha_0) + \left.\frac{dC_\ell}{d\alpha}\right|_{\alpha=\alpha_0} \Delta\alpha + \ldots]$$

$$+ \tfrac{\pi}{4}\rho [V_0 + \Delta V + \ldots]C^2[\dot{\phi} - \Omega(\theta \cos \Phi + \psi \sin \Phi)]\}[\cos \alpha_0$$

$$- \sin \alpha_0 \Delta\alpha + \ldots]$$

Thus, on discarding higher-order terms and the steady state term $\frac{1}{2}\rho V_0^2 C C_\ell(\alpha_0)\cos\alpha_0$, which is not of concern in a stability analysis, $\ell \cos \alpha$ is given by

$$
\begin{aligned}
\ell \cos\alpha = {} & \tfrac{1}{2}\rho C\left[\left.\frac{dC_\ell}{d\alpha}\right|_{\alpha=\alpha_0}\cos\alpha_0 - C_\ell(\alpha_0)\sin\alpha_0\right]\Big\{V_0^2\phi \\
& + [\Omega x + V_f\cos\alpha_s\sin\Omega t]\{[\dot v + Z_a\dot\phi - \Omega(Z_a - \ell_z)(\theta\cos\Phi \\
& \qquad + \psi\sin\Phi)]\cos\Phi \\
& \qquad - \dot w\sin\Phi + V_f\cos\alpha_s\cos\Omega t(\theta\cos\Phi + \psi\sin\Phi\} \\
& - [V_f\sin\alpha_s - w_i]\{\dot w\cos\Phi + [\dot v + Z_a\dot\phi - \Omega(Z_a - \ell_z)(\theta\cos\Phi \\
& \qquad + \psi\sin\Phi)]\sin\Phi\}\Big\} \\
& + \rho C C_\ell(\alpha_0)\cos\alpha_0\Big\{[V_f\sin\alpha_s - w_i]\{[\dot v + Z_a\dot\phi - \Omega(Z_a - \ell_z)(\theta\cos\Phi \\
& \qquad + \psi\sin\Phi)]\cos\Phi \\
& \qquad - \dot w\sin\Phi + V_f\cos\alpha_s\cos\Omega t(\theta\cos\Phi + \psi\sin\Phi)\} \\
& + [\Omega x + V_f\cos\alpha_s\sin\Omega t]\{\dot w\cos\Phi + [\dot v + Z_a\dot\phi - \Omega(Z_a - \ell_z)(\theta\cos\Phi \\
& \qquad + \psi\sin\Phi)]\sin\Phi\}\Big\} \\
& + \tfrac{\pi}{4}\rho V_0 C^2\cos\alpha_0[\dot\phi - \Omega(\theta\cos\Phi + \psi\sin\Phi)]
\end{aligned}
$$

It can be seen at this point that, by systematically and consistently expanding all terms in the manner outlined above, the effects of compressibility, stall and reversed flow have been retained in the formulation. The coefficients C_ℓ, C_d and C_m and their derivatives with respect to incidence, all evaluated at the nominal incidence $\alpha_0(x,t)$, appear in the formulations as periodically varying factors in the expressions for the coefficients in the equations of motion. Thus, provided the correct and complete variations of those coefficients with nominal incidence and nominal Mach number are retained, the effects of compressibility, stall and reversed flow on the linear stability of the system are properly taken into account.

Once Eqs. (7) have been expanded as outlined above, those relations are substituted in Eqs. (4), which are in turn substituted in Eqs. (2). With the blade motions expressed in terms of the generalized coordinates, through Eqs. (1), the perturbation equations of motion are obtained. Specifically, those equations are:

$$\ddot{\zeta}_k + \bar{\omega}_k^2 \zeta_k - \sum_{n=1}^{N} [G_{kn}(t)\zeta_n + H_{kn}(t)\dot{\zeta}_n] = 0 \tag{8}$$

$$k = 1,2,\ldots,N$$

where

$$G_{kn}(t) = \int_0^R \{A_v^{(k)}(x)\mu_{vn}(x,t) + A_w^{(k)}(x)\mu_{wn}(x,t) + A_\phi^{(k)}(x)\mu_{\phi n}(x,t)\}\, dx$$

$$H_{kn}(t) = \int_0^R \{A_v^{(k)}(x)\lambda_{vn}(x,t) + A_w^{(k)}(x)\lambda_{wn}(x,t) + A_\phi^{(k)}(x)\lambda_{\phi n}(x,t) + A_\psi^{(k)}(x)\lambda_{\psi n}(x,t) \quad dx$$

$$\mu_{vn}(x,t) = \frac{\rho}{2} V_0^2 C \sigma_D A_\phi^{(n)}$$

$$+ \frac{\rho}{2} V_0 C [\cos \Phi A_\theta^{(n)} + \sin \Phi A_\psi^{(n)}]\{\sigma_D[V_f \cos \xi \cos \alpha_s \cos \Omega t$$

$$- \Omega(Z_a - \ell_z) \cos \alpha_0] + 2\sigma_L[\Omega(Z_a - \ell_z) \sin \alpha_0$$

$$- V_f \sin \xi \cos \alpha_s \cos \Omega t] + \frac{\pi}{2} C \Omega \cos \alpha_0\}$$

$$\lambda_{vn}(x,t) = - 2m\Omega\varepsilon \sin \Phi A_\psi^{(n)} - \frac{\pi}{4}\rho V_0 C^2 \cos \alpha_0 A_\phi^{(n)}$$

$$+ \frac{\rho}{2} V_0 C\{[\sigma_D \cos \alpha_0 - 2\sigma_L \sin \alpha_0][Z_a A_\phi^{(n)} + A_v^{(n)}]$$

$$- [\sigma_D \sin \alpha_0 + 2\sigma_L \cos \alpha_0] A_w^{(n)}\}$$

in which

$$\sigma_D = \left[C_\ell(\alpha_0) - \frac{dC_d}{d\alpha}\bigg|_{\alpha=\alpha_0}\right]\sin\alpha_0 - \left[C_d(\alpha_0) + \frac{dC_\ell}{d\alpha}\bigg|_{\alpha=\alpha_0}\right]\cos\alpha_0$$

$$\sigma_L = 2\left[C_\ell(\alpha_0)\cos\alpha_0 + C_d(\alpha_0)\sin\alpha_0\right]$$

$$\begin{aligned}\mu_{wn}(x,t) &= \frac{\rho}{2} V_0^2\, C\, \gamma_L\, \dot{A}_\phi^{(n)} \\ &+ \frac{\rho}{2} V_0\, C\left[A_\theta^{(n)}\cos\Phi + A_\psi^{(n)}\sin\Phi\right]\Big\{\gamma_L\Big[V_f\cos\alpha_s\cos\xi\cos\Omega t \\ &- \Omega(Z_a - \ell_z)\cos\alpha_0\Big] + \gamma_D\Big[V_f\cos\alpha_s\sin\xi\cos\Omega t \\ &- \Omega(Z_a - \ell_z)\sin\alpha_0\Big] - \frac{\pi}{2} C\,\Omega\sin\alpha_0\Big\}\end{aligned}$$

$$\begin{aligned}\lambda_{wn}(x,t) &= -\,2m\Omega\varepsilon\cos\Phi\, A_\psi^{(n)} + \frac{\pi}{4}\rho V_0 C^2\sin\alpha_0 A_\phi^{(n)} \\ &+ \frac{\rho}{2} V_0 C\Big\{\Big[\gamma_L\cos\alpha_0 + \gamma_D\sin\alpha_0\Big](Z_a A_\Phi^{(n)} + A_v^{(n)}) \\ &\qquad + \Big[-\gamma_L\sin\alpha_0 + \gamma_D\cos\alpha_0\Big]A_w^{(n)}\Big\}\end{aligned}$$

in which

$$\gamma_L = \left[C_\ell(\alpha_0) - \frac{dC_d}{d\alpha}\bigg|_{\alpha=\alpha_0}\right]\cos\alpha_0 + \left[C_d(\alpha_0) + \frac{dC_\ell}{d\alpha}\bigg|_{\alpha=\alpha_0}\right]\sin\alpha_0$$

$$\gamma_D = 2\left[C_\ell(\alpha_0)\sin\alpha_0 - C_d(\alpha_0)\cos\alpha_0\right]$$

$$\mu_{\phi n}(x,t) = \frac{\rho}{2} V_0^2 C \delta^1 A_\phi^{(n)}$$

$$+ \frac{\rho}{2} V_0 C \left\{ V_f \cos \alpha_s \cos \Omega t \left[\delta^1 \cos \xi + 2\delta \sin \xi \right] \right.$$

$$- \Omega (Z_a - \ell_z) \left[\delta^1 \cos \alpha_0 + 2\delta \sin \alpha_0 \right]$$

$$\left. + \frac{\pi}{2} \Omega C Z_a \right\} \left[A_\theta^{(n)} \cos \Phi + A_\psi^{(n)} \sin \Phi \right]$$

$$\lambda_{\phi n}(x,t) = 2 I_0 \Omega \sin \Phi A_\psi^{(n)} - \frac{\pi}{4} \rho V_0 C^2 Z_a A_\phi^{(n)}$$

$$+ \frac{\rho}{2} V_0 C \left\{ \left[\delta^1 \cos \alpha_0 + 2\delta \sin \alpha_0 \right] \left[Z_a A_\phi^{(n)} + A_v^{(n)} \right] \right.$$

$$\left. + \left[- \delta^1 \sin \alpha_0 + 2\delta \cos \alpha_0 \right] A_w^{(n)} \right\}$$

in which

$$\delta = C C_m(\alpha_0) - Z_a C_\ell(\alpha_0)$$

$$\delta^1 = C \left. \frac{dC_m}{d\alpha} \right|_{\alpha=\alpha_0} - Z_a \left. \frac{dC_\ell}{d\alpha} \right|_{\alpha=\alpha_0}$$

$$\lambda_{\psi n}(x,t) = - 2 I_0 \Omega A_\phi^{(n)} + 2 m \Omega \varepsilon \left[A_v^{(n)} \sin \Phi + A_w^{(n)} \cos \Phi \right]$$

DEVELOPMENT OF THE BASIC METHOD

The specific problem at hand here is to establish the means for determining the stability of a set of linear equations with periodic coefficients, such as those representing a helicopter rotor in forward flight, Eqs. (8). Those equations are put in a somewhat more convenient form by defining an independent variable z,

$$z = \tfrac{1}{2}\Omega t \quad ,$$

and changing the notation of the coefficients, according to

$$\left.\begin{aligned} a_{mn} &= -\frac{2}{\Omega} H_{mn} \\ b_{mm} &= \frac{4}{\Omega^2}\left[\bar{\omega}_m^2 - G_{mm}\right] \\ b_{mn} &= -\frac{4}{\Omega^2} G_{mn} \quad , \quad m \neq n \end{aligned}\right\} \quad m,n = 1,2,\ldots,N.$$

The equations to be analyzed then become

$$\frac{d^2\zeta_m}{dz^2} + \sum_{n=1}^{N}\left\{a_{mn}\frac{d\zeta_n}{dz} + b_{mn}\,\zeta_n\right\} = 0 \tag{9}$$

$$m = 1,2,\ldots,N;$$

where a_{mn} and b_{mn} are periodic functions:

$$a_{mn}(z + \pi) = a_{mn}(z)$$

$$b_{mn}(z + \pi) = b_{mn}(z).$$

As was noted in the introduction, equations of the form of Eqs. (9) have received considerable attention over the past century. Although attempts at general solutions have met with little success, the form of the solution in the general case has been developed. The theory of Floquet (see Reference 6) yields that

$$\zeta_n(z) = e^{i\omega z}\phi_n(z)$$

where $\phi_n(z)$ is periodic, with a period π, and $i\omega$ is a complex constant. Since the differential system (Eqs. (9)) is of order 2N, there are 2N values of $i\omega$ defining the solution for a given case. If any one of these characteristic values has a positive real part, the system is unstable.

In order to secure the theoretical basis for the solution and to avoid numerical difficulties, it is necessary to first operate on Eqs. (9) to obtain two related sets of equations. It is convenient for this purpose to use matrix notation. A column m-+rix X and square matrices A and B can be defined as

$$X = \begin{Bmatrix} \zeta_1 \\ \zeta_2 \\ \cdot \\ \cdot \\ \cdot \\ \zeta_N \end{Bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ b_{N1} & b_{N2} & \cdots & b_{NN} \end{bmatrix}$$

so that Eqs. (9) can be written

$$\frac{d^2X}{dz^2} + A\frac{dX}{dz} + BX = 0 \tag{10}$$

where the derivative of a matrix is the matrix formed by replacing each element by its derivative.

The first step in obtaining the two related sets of equations is to differentiate Eq. (10), yielding

$$\frac{d^3X}{dz^3} + A\frac{d^2X}{dz^2} + \left[\frac{dA}{dz} + B\right]\frac{dX}{dz} + \left(\frac{dB}{dz}\right)X = 0 \quad . \tag{11}$$

If Eq. (10) is solved for d^2X/dz^2 and the result substituted in Eq. (11), it is found that

$$\frac{d^3X}{dz^3} + C\frac{dX}{dz} + DX = 0 \tag{12}$$

where

$$C = \frac{dA}{dz} + B - A^2$$

$$D = \frac{dB}{dz} - AB \quad .$$

If, now, Eq. (12) is differentiated and the second derivative of X eliminated as before, it is found that

$$\frac{d^4X}{dz^4} + E\frac{dX}{dz} + FX = 0 \tag{13}$$

where

$$E = \frac{dC}{dz} + D - CA$$

$$F = \frac{dD}{dz} - CB \quad .$$

It can be shown that a set of functions is a solution of the original equation (Eq. (10)), if and only if it is a solution to both of the derived equations, Eq. (12) and Eq. (13). As a result, the solutions of Eq. (10) can be found by solving Eqs. (12) and (13) and identifying those solutions common to the latter two equations. The proof is straightforward, and is given in Appendix A.

Consider, first, the solution to Eq. (12). It is convenient at this point to abandon the matrix notation. Thus, if c_{mn} and d_{mn} denote the elements of matrices C and D, respectively, Eq. (12) gives that

$$\frac{d^3\zeta_m}{dz^3} + \sum_{n=1}^{N}\left(c_{mn}\frac{d\zeta_n}{dz} + d_{mn}\zeta_n\right) = 0 \ , \tag{14}$$

$$m = 1,2,\ldots,N \ .$$

From the theory of Floquet, the solution of Eqs. (14) must be of the form

$$\zeta_m(z) = e^{i\omega z}\phi_m(z)$$

$$= \sum_{k=-\infty}^{\infty} p_{mk}\, e^{i(2k+\omega)z}$$

$$m = 1,2,\ldots,N;$$

upon expansion of ϕ_m in a complex Fourier series. Also, the periodic coefficients in Eqs. (14) can be expanded in Fourier series:

$$\left.\begin{aligned} c_{mn} &= \sum_{k=-\infty}^{\infty} \xi_{mnk}\, e^{2ikz} \\ d_{mn} &= \sum_{k=-\infty}^{\infty} \eta_{mnk}\, e^{2ikz} \end{aligned}\right\} \quad m,n = 1,2,\ldots,N.$$

If the Fourier representations for the solution and for the coefficients are substituted in Eqs. (14) and the coefficients of e^{2ikz} are grouped and set equal to zero, a set of linear recursion relations in the unknown coefficients p_{mk} is obtained. Specifically, it is found that

$$0 = [i(2n + \omega)]^3 p_{rn}$$
$$+ \sum_{s=1}^{N} \left\{ \sum_{k=-\infty}^{\infty} \left[\eta_{rsk} + i(2n - 2k + \omega)\,\xi_{rsk} \right] p_{s,n-k} \right\} \tag{15}$$

$$n = \ldots,-2,-1,0,1,2,\ldots;$$

$$r = 1,2,\ldots,N$$

These relations constitute an infinite set of linear algebraic equations in the unknown Fourier coefficients p_{mk}. For this set of equations to have a nontrivial solution, the associated infinite determinant $\Delta(\omega)$ must vanish (a discussion of infinite determinants is given in Reference 6). The requirement that Δ vanish is the condition which determines ω, and hence the stability of the system.

In order that Δ be meaningfully defined, it is necessary to divide each of Eqs. (15) through by the coefficient of p_{rn}. With the unknowns then appropriately ordered, the diagonal elements of Δ are all unity and the off-diagonal elements all vanish in the limit as a row or column index of the determinant tends to either positive or negative infinity. Specifically, the elements $\sigma_{\mu\nu}$ of $\Delta(\omega)(\mu,\ \nu = 0,\ \pm 1,\ \pm 2 \ldots)$ are given by

$$\sigma_{Nn+r,\ Nk+s} = \alpha_{rs}(n,k;\omega)$$

$$n,k = \ldots,-2,-1,0,1,2,\ldots;$$

$$r,s = 1,2,\ldots,N,$$

where

$$\alpha_{rs}(n,k;\omega) = \frac{[i(2k+\omega)]^3\,\delta_{nk}\,\delta_{rs} + i(2k+\omega)\,\xi_{rs,n-k} + \eta_{rs,n-k}}{[i(2n+\omega)]^3 + i(2n+\omega)\,\xi_{rr_0} + \eta_{rr_0}}$$

in which

$$\delta_{ij} = 0\ ,\ i \neq j\ ;$$

$$= 1\ ,\ i = j.$$

Note that the diagonal elements $\sigma_{\mu\mu}$ are all unity. The construction of the determinant can perhaps be best visualized as made up of a collection of subarrays which are NxN in size. The location of any element within a subarray is determined from indices r and s. The location of each subarray is specified through indices n and k. The general arrangement of the α's in Δ is illustrated in Figure 2 for the case N=2.

The value of Δ is obtained, for a given ω, by evaluating the finite determinant formed by ranging n and k from, say, -L to L.

$$\Delta(\omega) = \begin{vmatrix}
 & & \vdots & & \vdots & & \vdots & \\
\cdots & 1 & \alpha_{12}(-1,-1) & \alpha_{11}(-1,0) & \alpha_{12}(-1,0) & \alpha_{11}(-1,1) & \alpha_{12}(-1,1) & \\
 & \alpha_{21}(-1,-1) & 1 & \alpha_{21}(-1,0) & \alpha_{22}(-1,0) & \alpha_{21}(-1,1) & \alpha_{22}(-1,1) & \cdots \\
\cdots & \alpha_{11}(0,-1) & \alpha_{12}(0,-1) & 1 & \alpha_{12}(0,0) & \alpha_{11}(0,1) & \alpha_{12}(0,1) & \\
 & \alpha_{21}(0,-1) & \alpha_{22}(0,-1) & \alpha_{21}(0,0) & 1 & \alpha_{21}(0,1) & \alpha_{22}(0,1) & \cdots \\
\cdots & \alpha_{11}(1,-1) & \alpha_{12}(1,-1) & \alpha_{11}(1,0) & \alpha_{12}(1,0) & 1 & \alpha_{12}(1,1) & \\
 & \alpha_{21}(1,-1) & \alpha_{22}(1,-1) & \alpha_{21}(1,0) & \alpha_{22}(1,0) & \alpha_{21}(1,1) & 1 & \cdots \\
 & & \vdots & & \vdots & & \vdots &
\end{vmatrix}$$

Figure 2. The arrangement of the determinant elements for the case N = 2

Successively larger values for L are then taken until the limit becomes apparent (see Reference 6).

The expression

$$\Delta(\omega) = 0 \tag{16}$$

by itself is not a particularly useful relation for determining ω, because of the limiting process involved in evaluating infinite determinants. It will be shown, however, that $\Delta(\omega)$ in fact constitutes a combined series-product expansion in ω of a finite sum of trigonometric functions. With $\Delta(\omega)$ expressed in the latter form, Eq. (16) can be solved explicitly for ω.

It should be noted here that Hill obtained such a relation for the second-order differential equation which bears his name (see Reference 6). The development which follows is a generalization of that result.

Two properties of the function $\Delta(\omega)$ must first be established. Specifically, it is asserted that

1. $\Delta(\omega)$ is an analytic function of ω, except for simple, identifiable poles;

2. $\Delta(\omega)$ is periodic in ω with a period of 2.

That $\Delta(\omega)$ is analytic can be concluded by noting that Δ is an absolutely convergent determinant--i.e., the product of the diagonal elements converges absolutely (in this case to unity) and the double sum of the off-diagonal elements converges absolutely. Uniform convergence and analyticity can then be established (see Reference 6).

Note that if the original equations (Eqs. (9)), rather than derived equations (Eqs. (14)), had been used to generate Δ, that determinant would not have been absolutely convergent since the double sum of off-diagonal elements generated from the original equations is only conditionally convergent. What follows hinges on the analyticity of Δ, hence the necessity for working with the differentiated sets of equations.

Clearly, too, the only singularities in Δ are simple poles at the points

$$\omega = -2n - i\lambda_{rj} \; ; \; n = 0, \pm 1, \pm 2, \ldots ;$$
$$r = 1, 2, \ldots N;$$
$$j = 1, 2, 3;$$

where the λ's are the 3N roots of the N cubic equations

$$\lambda^3 + \lambda\xi_{rr_0} + \eta_{rr_0} = 0 \ , \ r = 1,2,\ldots,N.$$

The periodicity of Δ is shown by substituting $\omega + 2$ for ω in the expression for α_{rs}, whereupon it is found that

$$\alpha_{rs}(n,k;\omega + 2) = \alpha_{rs}(n + 1,k + 1;\omega) \quad .$$

Thus, in the limit, the value of Δ is unchanged if ω is replaced by $\omega + 2$:

$$\Delta(\omega + 2) = \Delta(\omega).$$

Now, consider the function $D(\omega)$, defined by

$$D(\omega) = \Delta(\omega) + \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \cot[\tfrac{\pi}{2}(\omega + i\lambda_{rj})] \tag{17}$$

where the f_{rj}'s are constants. Observe that $D(\omega)$ is an analytic function and that

$$D(\omega + 2) = D(\omega).$$

Further, note that the term

$$f_{rj} \cot[\tfrac{\pi}{2}(\omega + i\lambda_{rj})]$$

has simple poles at $\omega = -2n - i\lambda_{rj}$ for $n = 0,\pm 1,\pm 2, \ldots,$ and has no other singularities. Thus, if the value for each constant f_{rj} is properly chosen, $D(\omega)$ will have no poles. Let the f_{rj}'s be so chosen, making $D(\omega)$ analytic throughout the finite part of the ω-plane. But D is clearly bounded at infinity; $\Delta(\omega)$ has a limit of one and the cotangents have limits of $\pm i$ for $\mathrm{Im}(\omega)\to\pm\infty$. Therefore, by Liouville's theorem, $D(\omega)$ is simply some constant, say c:

$$D(\omega) = c = \Delta(\omega) + \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \cot[\tfrac{\pi}{2}(\omega + i\lambda_{rj})] \quad . \tag{18}$$

It only remains to determine the values of the f_{rj}'s and of c. This is facilitated by first considering the limits of $D(\omega)$ with infinite ω:

$$\lim_{\mathrm{Im}(\omega)\to +\infty} D(\omega) = c = 1 - i \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \ ,$$

$$\lim_{\mathrm{Im}(\omega)\to -\infty} D(\omega) = c = 1 + i \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \ .$$

Clearly, then, c=1 and

$$\sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} = 0 \ .$$

Replacing c by unity and solving for Δ in Eq. (18) gives that

$$\Delta(\omega) = 1 - \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \cot[\tfrac{\pi}{2}(\omega + i\lambda_{rj})] \ .$$

Now, let 3N - 1 arbitrary (but finite) values of ω, say ω_1, $\omega_2, \ldots, \omega_{3N-1}$, be assigned in the above equation. The resulting 3N - 1 equations, together with the one obtained for infinite ω, provide sufficient relations to solve for the f_{rj}'s. More specifically, those constants are the solution of

$$\sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \cot[\tfrac{\pi}{2}(\omega_k + i\lambda_{rj})] = 1 - \Delta(\omega_k) \quad , \quad k = 1,2,\ldots,3N - 1;$$

$$\sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} = 0. \tag{19}$$

With the f_{rj}'s known, the determinantal equation, $\Delta(\omega) = 0$, can be replaced by

$$1 - \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \cot[\frac{\pi}{2} (\omega + i\lambda_{rj})] = 0 \quad . \tag{20}$$

A polynomial of degree 3N in $e^{i\pi\omega}$ can be readily derived from Eq. (20). The 3N roots of that polynomial determine the characteristic values for Eqs. (12).

In the same manner as is outlined above for Eqs. (12), the 4N characteristic values for Eqs. (13) can be obtained. The 2N values common to the two sets are those of the original equations, Eqs. (9).

OUTLINE OF BASIC COMPUTER PROGRAM

In order to implement the method derived for analyzing stability of dynamic systems with periodic parameters, a basic digital computer program was prepared for the analysis of systems with three degrees of freedom. The program accepts the pertinent data for a given system in the form of Fourier coefficients of the periodic coefficients in the equations of motion (Eqs. (9), with N=3), and then proceeds to calculate the six characteristic values of the system by the method derived in the previous section. For specific applications, a small subroutine can be prepared, if necessary, to generate the Fourier coefficients used as input by the basic computer program.

The overall flow of information guided by the formulations of the basic program are outlined below. The numbers in the outline correspond to those in the blocks on the flow diagram of Figure 3. The outline includes reference to the pertinent equations of the previous section. The relations guiding the more routine procedures, such as extraction of the roots of polynomials and solving of linear algebraic equations, have been omitted, but are embodied in the computer program, a listing of which is given in Appendix B.

1. Fourier Expansion of Coefficients

Given the coefficients of the equations of motion for a linear dynamic system with three degrees of freedom and periodically varying parameters with normal modes used as generalized coordinates, such as Eqs. (9), the coefficients are expanded

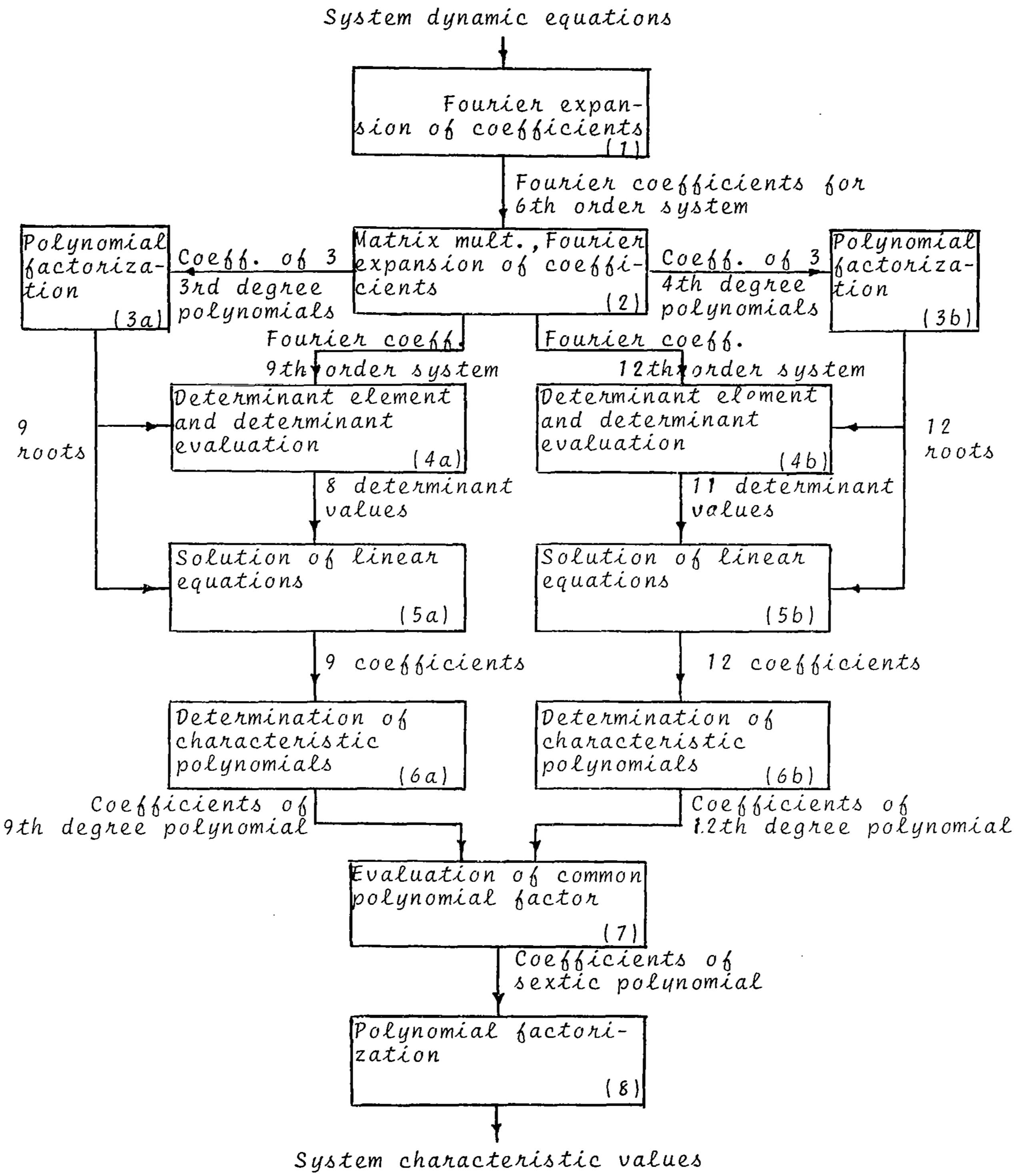

Figure 3. Procedure for calculating characteristic values of a dynamic system with three degrees of freedom and periodically varying parameters

in Fourier series. The specific output is a set of Fourier coefficients for the periodic coefficients a_{mn} and b_{mn} appearing in Eqs. (9) with N=3; m,n=1,2,3. This subroutine is associated with the particular dynamic system being analyzed. and can be described as preparing the input to the main stability-analysis program.

2. Matrix Multiplication and Fourier Expansion of Coefficients

Given the Fourier coefficients of the elements of the 3x3 matrices A and B appearing in Eq. (10), the Fourier coefficients of the elements of matrices C and D, appearing in Eq. (12), and of E and F, appearing in Eq. (13), are calculated. In this way, the coefficients of the higher-order, or differentiated, systems of equations are related to the original system of equations, Eqs. (9).

3. Polynomial Factorization

The three cubic polynomials associated with the determinant for the ninth-order system are factored to obtain the nine singularities of that determinant (subroutine 3a). The three quartic polynomials associated with the twelfth-order system are factored to obtain the twelve singularities of that determinant (subroutine 3b). The points ω_k at which the infinite determinants are to be evaluated are also selected, by specifying each of them to be a set distance δ from one of the singular points of the determinant. The accuracy of the solution was found to be quite sensitive to the value of δ, it generally being necessary to make δ as small as possible, without sacrificing numerical accuracy elsewhere in the program, in order to get satisfactory results. The reason for this behavior is not known, but is presumably connected in some way to the accuracy with which the infinite determinants are calculated, this being the most difficult, and hence least accurate, of the tasks performed by the program. A value for δ of 10^{-4} was used for most of the calculations reported here.

4. Determinant Element and Determinant Evaluation

For each value of ω_k, the determinant elements are computed and the infinite determinant is evaluated. It was anticipated that this step would be the most time-consuming one in the program, and so considerable care was taken to employ the most economical means for determinant evaluation. A process of triangularization was selected for evaluating finite determinants.

The limiting value of a determinant, for a given ω_k, as the number of elements increases without bound, is estimated as follows. Three determinant values, for three successively larger numbers of rows and columns, are first obtained. The determinant is then assumed to vary inversely as some unknown power of the number of rows and columns. The assumed form has three unknown constants, so the three determinant values, corresponding to three different determinant sizes, provide sufficient information to calculate the constants and hence an estimate of the limit as the number of elements becomes infinite. It has been found that three determinants, respectively 33x33, 39x39 and 45x45 in size, generally are sufficient to provide three-place accuracy in the result.

5. Set-up and Solution of the Linear Algebraic Equations

For the ninth-order system, the eight values of ω_k and the eight values of $\Delta(\omega_k)$ are used to compute the coefficients and inhomogeneous terms of Eqs. (19). The nine equations are then solved for the nine constants f_{rj}. A similar procedure is followed to obtain the twelve constants for the twelfth-order system. The equations are solved by triangularization.

6. Determination of the Characteristic Polynomials of the Higher-Order Systems

For the ninth-order system, the coefficients of the polynomial of the ninth degree in $e^{i\pi\omega}$ are derived from the determinantal equation, Eq. (20) (subroutine 6a). Similarly the coefficients of the characteristic polynomial of the twelfth-order system are obtained from the appropriate determinantal equation (subroutine 6b).

7. Evaluation of the Common Polynomial Factor

At this point, one could presumably compute the nine roots of the ninth-degree polynomial, the twelve roots of the twelfth-degree polynomial and identify the roots common to the two polynomials as being the characteristic values of the original system. However, this procedure requires evaluation of fifteen extraneous roots, followed by possible difficulties in identifying which roots are indeed common ones.

It was found, though, that the coefficients of the characteristic equation for the original system, which is actually a polynomial factor common to the two higher-degree polynomials, can be obtained in terms of the coefficients of these higher-degree polynomials. The steps taken to derive the necessary expressions are outlined in Appendix C. Subroutine 7 implements those expressions, yielding the coefficients of the sixth-degree polynomial characterizing the original system.

8. Polynomial Factorization

A standard library subroutine is used to obtain the six roots of the characteristic polynomial. The logarithm of each root is then evaluated (recall that the polynomial is formed of powers of $e^{i\pi\omega}$) to obtain the six characteristic values, and hence the stability, of the system.

During the check-out of the computer program, it was found necessary to extract the roots of the higher-degree polynomials. Since little additional running time is consumed by those calculations, determination of the roots and characteristic values for the ninth and twelfth-order systems has been retained in the program for purposes of comparison.

APPLICATION OF THE METHOD

The computer program implementing the general method was first checked out by means of a test case, the characteristic values of which could be derived in advance. The program was used next to analyze a rotor system with two degrees of freedom for which solutions by direct time integration had been obtained previously. Lastly, calculations were conducted for a model rotor system having three degrees of freedom, for which flutter boundaries had been obtained experimentally. These applications are discussed in detail below.

Test Case Calculations

It was found necessary in the course of the check-out of the computer program to have available a test case for which the characteristic values of the original sixth-order system as well as of the derived ninth-order system were known in advance. Such a case was generated by appropriately transforming Mathieu's equation. The sixth-order system was then made up from three of these equations, treated as a coupled system, the program not being able to distinguish whether the equations are coupled or not.

The derivation of the test case was carried out as follows. Consider Mathieu's equation, which is generally written in the form

$$\frac{d^2u}{dz^2} + (a - 2q \cos 2z)u = 0 \tag{21}$$

where a and q are specified constants. The characteristic values $\pm\mu$ of Mathieu's equation are a function of a and q (see Reference 5).

Now, let a new dependent variable y be defined by

$$y = ue^{-\frac{1}{6}f(z)}$$

where f(z) is periodic with a period π, but otherwise may be regarded as arbitrary. If u is substituted in Eq. (21) in terms of y and f the following differential equation for y is obtained:

$$\frac{d^2y}{dz^2} + A\frac{dy}{dz} + By = 0 \tag{22}$$

where

$$A = \frac{1}{3} f(z)$$

$$B = a - 2q \cos 2z + \frac{1}{36} f^2 + \frac{1}{6}\frac{df}{dz}$$

Suppose, now, f is given by

$$f(z) = \sum_{n=0}^{\infty} (A_n \cos 2nz + B_n \sin 2nz).$$

It then follows that the characteristic values of y, as a solution of Eq. (22), must be $\mu - A_0/6$ and $-\mu - A_0/6$. Thus, we have derived a second-order differential equation with periodic coefficients, of a quite general form, for which the characteristic values are known.

Further, suppose a third-order equation is derived by the method described previously which eliminates the second derivative, namely

$$\frac{d^3y}{dz^3} + C\frac{dy}{dz} + Dy = 0 \tag{23}$$

where

$$C = B + \frac{dA}{dz} - A^2$$

$$D = \frac{dB}{dz} - AB \quad .$$

It is possible, through further transformation and utilization of the relations given on p. 134 of Reference 5, to show that the additional characteristic value of Eq. (23) is the negative of the sum of the other two values, i.e., $A_0/3$.

The three equations of the test case were all derived from the Mathieu's equation having a = 1, q = 0.32. For those values of a and q, it follows that μ = .15813 + i (see p. 105 of Reference 5). Only the first three terms of the series for f(z) were retained for each equation of the test case. The values of A_0, A_1 and A_2 chosen and the associated characteristic values for the three degrees of freedom are as follows:

A_0	A_1	A_2	$\mu - \frac{A_0}{6}$	$-\mu - \frac{A_0}{6}$	$A_0/3$
- 4.5	2.0	0.6	0.90813 + i	0.59187 - i	- 1.5
- 0.6	2.0	- 0.6	0.25813 + i	- 0.05813 - i	- 0.2
3.0	- 2.0	0.6	- 0.34187 + i	- 0.65813 - i	1.0

The Fourier coefficients of the functions A and B appearing in Eq. (22) were calculated for the three degrees of freedom and were supplied as inputs to the computer program. The characteristic values which were calculated for the ninth and twelfth order systems are tabulated below, together with the independently derived exact (anticipated) values. The quantities λ_R and λ_I given are related to ω by

$$i\omega = \lambda_R + i\lambda_I .$$

CHARACTERISTIC VALUES OF THE 9TH ORDER SYSTEM

Calculated		Anticipated	
λ_R	λ_I	λ_R	λ_I
0.90813	1.0	0.90813	1.0
0.59185	- 1.0	0.59187	- 1.0
0.25814	1.0	0.25813	1.0
- 0.05816	- 1.0	- 0.05813	- 1.0
- 0.34184	1.0	- 0.34187	1.0
- 0.65813	- 1.0	- 0.65813	- 1.0
- 1.50000	0.0	- 1.50000	0.0
- 0.19998	0.0	- 0.20000	0.0
1.00000	0.0	1.00000	0.0

CHARACTERISTIC VALUES OF THE 12TH ORDER SYSTEM

Calculated		Anticipated	
λ_R	λ_I	λ_R	λ_I
0.90803	1.0	0.90813	1.0
0.59187	- 1.0	0.59187	- 1.0
0.25802	1.0	0.25813	1.0
- 0.05960	- 1.0	- 0.05813	- 1.0
- 0.34188	1.0	- 0.34187	1.0
- 0.65813	- 1.0	- 0.65813	- 1.0
0.97699	1.0	-	-
0.02369	- 1.0	-	-
- 0.10970	1.0	-	-
- 0.41425	0.0	-	-
0.21431	0.0	-	-
- 1.38935	- 1.0	-	-

As can be seen from the tabulated results, the characteristic values common to the ninth and twelfth order systems are those of the sixth-order system. Note, too, that the program for the most part predicts the characteristic values for this case to four significant figures.

The characteristic values were also calculated from the sixth-degree polynomial which was derived from the two higher-degree polynomials. The resulting roots were unacceptably different from the correct values, apparently because the polynomial as so derived is sensitive to slight inaccuracies in the coefficients of the higher-degree polynomials for this case. Thus, it may be necessary to rely on direct comparison of ninth-order and twelfth-order solutions to determine the correct sixth-order solutions in some instances.

Comparison With Direct Time Integration

The analysis of stability by direct time integration on a digital computer of a rigid rotor blade with flapping and lead-lag hinges is reported in Reference 12. The nonlinear representations of both inertial and aerodynamic forces are utilized in the equations of motion. The basic blade for which numerical results were obtained had zero offset of the flapping hinge and 0.05R offset of the lead-lag hinge. The blade had a constant chord except for a cut-out from the axis of rotation to 0.2 R. In the calculations reported in Reference 12, the rotor was unloaded and at zero shaft angle. Further details can be found in Reference 12.

For two of the cases analyzed in Reference 12, the variations of flapping and lead-lag angles with time are presented. These cases had values of advance ratio μ of 0.6 and 1.4, respectively. The mass constant γ' for both cases was 1.6 ($\gamma' = \rho CR^4/I_h$, where I_h is the mass moment of inertia about the lead-lag hinge). The case for the lower advance ratio is reported to be very stable and the case with $\mu = 1.4$ is indicated to be stable but near a boundary of neutral stability. The time histories of the blade displacements, from Reference 12, are reproduced in Figures 4 and 5.

The appropriate parameter values for these two cases were inserted in the linearized equations of motion of a rotor blade developed previously in this report. Series representations of the coefficients in the equations, including the first eleven harmonics of rotor rotational speed, were generated and supplied to the main stability-analysis program. The characteristic values calculated are as follows:

DEGREE OF FREEDOM	CHARACTERISTIC VALUES			
	$\mu = 0.6$		$\mu = 1.4$	
	λ_R	λ_I	λ_R	λ_I
Flapping	- 1.13400	0.60206	- 0.73379	- 1.0
	- 1.13400	- 0.60206	- 2.56010	1.0
Lead-Lag	- 0.00353	0.56195	- 0.00534	0.56206
	- 0.00353	- 0.56195	- 0.00534	- 0.56206

Examination of these results indicates qualitative agreement with the results of the analysis by direct time integration but there is evidence of some differences in quantity. At $\mu = 0.6$, the flapping motion should damp by a factor $e^{-2} = 0.135$ in $4/(-\lambda_R) = 3.527$ radians of azimuth change, by the result obtained here, whereas Figure 4 indicates a much more rapid decrease. On the other hand, at $\mu = 1.4$, the factor e^{-2} should apply to the flapping motion for a change of $4/(-\lambda_R) = 5.452$ radians, but Figure 5 indicates that the flapping motion is considerably less stable than that.

From the qualitative viewpoint, the comparison is more favorable. The decrease in stability of the flapping degree of freedom with increasing μ is in evidence in the result obtained here, since λ_R is less negative at $\mu = 1.4$ than at $\mu = 0.6$. The stability of a given system is determined, of course, by the least negative, or most positive, value of λ_R. Also, in agreement with the indications of Figures 4 and 5, the lead-lag motion is only slightly damped, with a factor e^{-2} decrease occurring in 1,130 radians at $\mu = 0.6$ and in 750 radians at $\mu = 1.4$.

The quantitative differences in the predictions of the flapping motion can be attributed to the nonlinear effects which are included in the direct-integration solution, but which are absent from the formulations analyzed here. This can be seen as follows. With the rotor unloaded, as is the case here, the equations for rigid-body flapping and lead-lag motion become decoupled when linearized. Thus, any coupling of the motion which is detected can be attributed, in this case, to nonlinear dynamic-coupling effects. Since the motions plotted in Figures 4 and 5 were initiated by a disturbance in flapping, the considerable lead-lag motion must then all be due to nonlinear effects. Furthermore,

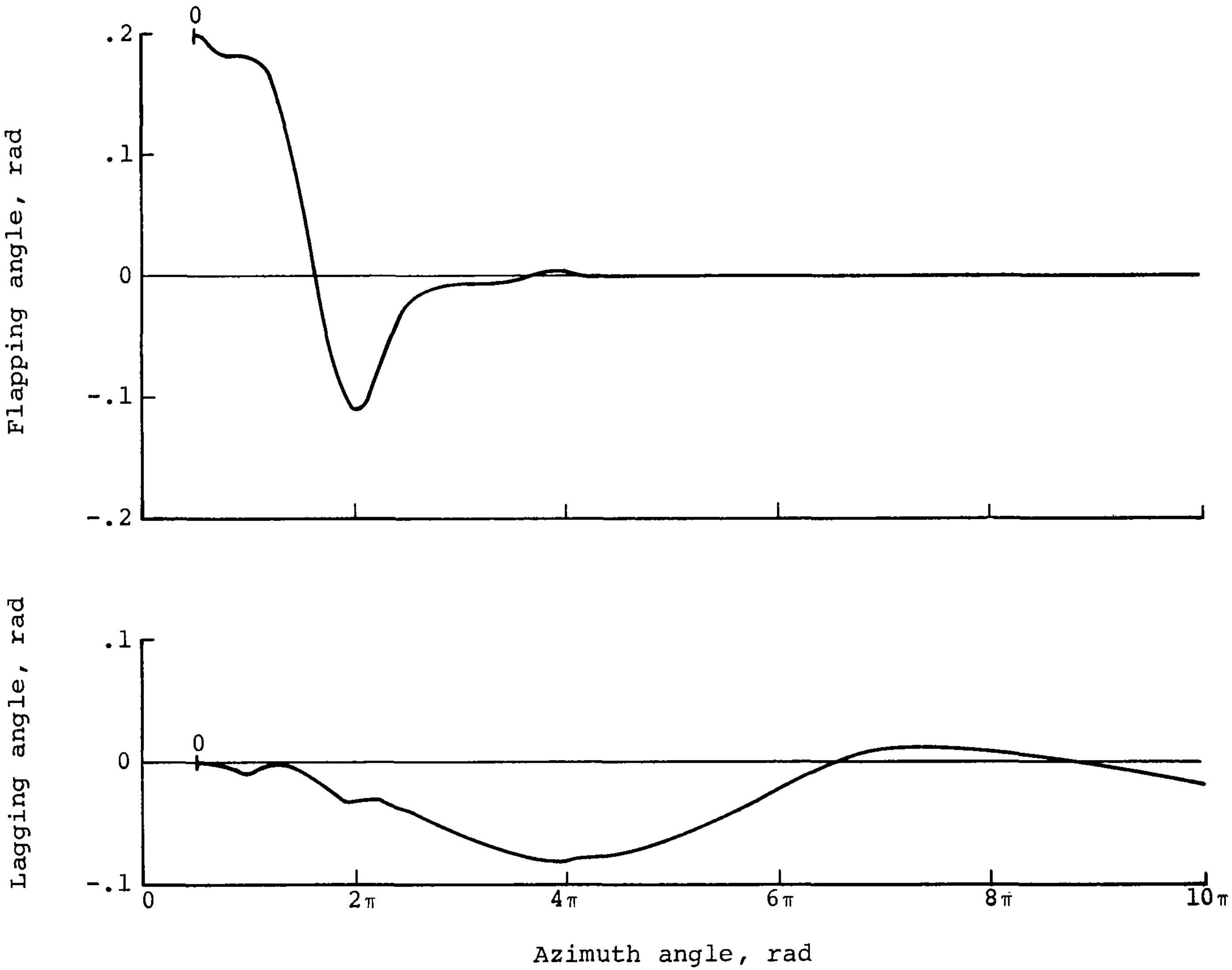

Figure 4. Blade motions for advance ratio μ = 0.6 (from Reference 12).

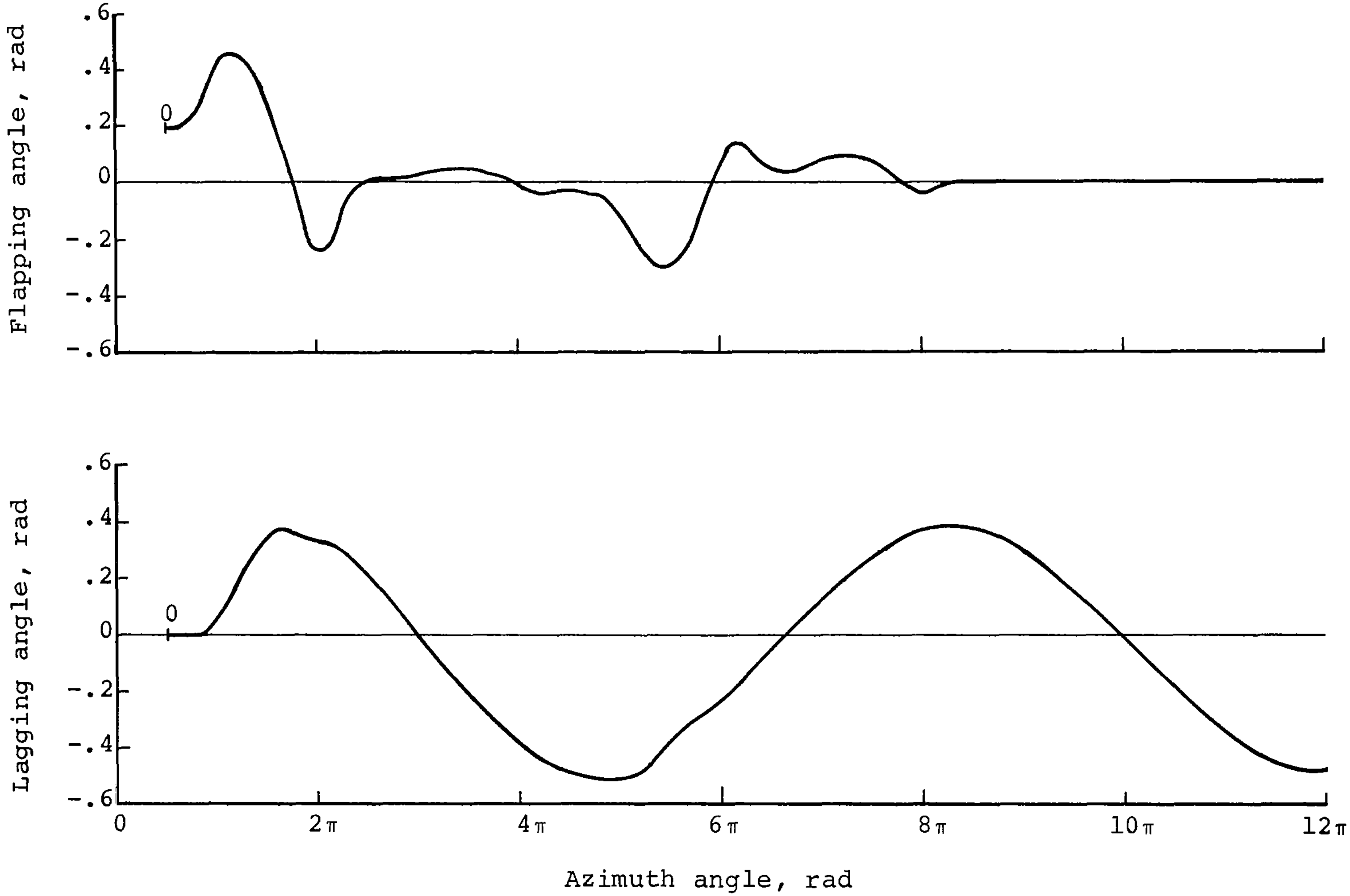

Figure 5. Blade motions for advance ratio $\mu = 1.4$ (from Reference 12).

the flapping motion for $\mu = 1.4$ at the higher azimuth angles appears to have been excited by the persisting lead-lag motion, resulting in an apparent damping of the motion which is much less than would otherwise be the case.

Comparison With Experimental Data

Reference 10 reports the results of an experimental investigation of the flutter of a model helicopter rotor in forward flight. The model rotor had a single blade with a radius of four feet and with a flapping hinge through the axis of rotation. The blade had a constant chord of 3.5 inches and a root cutout of 6 inches. Inertial and elastic properties of the blade are given in Reference 10. The blade was relatively stiff in torsion, but the control system was made flexible, so the primary contributions to blade motions derived from rigid-body pitch, flapping motions and deflections in the first flapwise bending mode. The main parameters of the study were advance ratio, control system stiffness and chordwise mass center location.

The case selected for comparison had a ratio of first flapwise bending frequency $\bar{\omega}_{\phi_1}$ to control frequency $\bar{\omega}_{\theta_0}$ (nonrotating) of 1.31 and a chordwise mass center location of 42.5% of chord aft of the leading edge, giving a value of 0.139 for equivalent mass center location as defined in Reference 10. This case was chosen because the data as plotted in Figure 8e of Reference 10 indicated there should be a relatively large change in stability with advance ratio. Natural frequencies and mode shapes for the first three coupled modes of the blade were calculated for a value of the ratio $\bar{\omega}_{\theta_0}/\Omega = 4.37$, and Fourier coefficients of the coefficients of the equations of motion were calculated for values of advance ratio μ of 0.0, 0.09, 0.175, 0.24 and 0.30. The Fourier coefficients for each value of μ were supplied to the main computer program and the system characteristic values computed.

Subsequent to these calculations, it was determined through communications with one of the authors of Reference 10 that the data of Figure 8e of that reference were mislabelled. The symbols for values of $\bar{\omega}_{\phi_1}/\bar{\omega}_{\theta_0}$ of 1.31 and 0.63 were reversed. The experimentally determined flutter boundary actually corresponding to the calculations performed, consisting of a plot of $\bar{\omega}_{\theta_0}/\Omega$ versus μ, taken from Figure 8e of Reference 10, is reproduced in Figure 6. The points at which characteristic values were calculated are indicated by asterisks on the line drawn at $\bar{\omega}_{\theta_0}/\Omega = 4.37$.

As can be seen from Figure 6, the mislabelling of the data has led to a rather unsatisfactory comparison of theory with experiment. The experimental data shows very little change with μ. Hence, for a comparison with this case, the characteristic values should have been determined at several different values of ω_{θ_0}/Ω, for fixed μ, rather than at fixed $\bar{\omega}_{\theta_0}/\Omega$ for various values of μ. Unfortunately, time limitations prevented carrying out more extensive calculations. However, some information can still be derived from the calculations which were performed.

The characteristic values obtained for each advance ratio are as follows:

$\mu = 0.0$		$\mu = 0.09$		$\mu = 0.175$	
λ_R	λ_I	λ_R	λ_I	λ_R	λ_I
- 0.06273	0.02028	- 0.06271	0.02800	- 0.06266	0.02871
- 0.06273	- 0.02028	- 0.06271	- 0.02800	- 0.06266	- 0.02871
- 0.33052	0.20163	- 0.33055	0.20185	- 0.33068	0.20267
- 0.33052	- 0.20163	- 0.33055	- 0.20185	- 0.33068	- 0.20267
- 0.47015	0.08874	- 0.47016	0.08954	- 0.47012	0.09152
- 0.47015	- 0.08874	- 0.47016	- 0.08954	- 0.47012	- 0.09152

$\mu = 0.24$		$\mu = 0.30$	
λ_R	λ_I	λ_R	λ_I
- 0.06268	0.02957	- 0.06283	0.03060
- 0.06268	- 0.02957	- 0.06283	- 0.03060
- 0.33120	0.20366	- 0.33227	0.20489
- 0.33120	- 0.20366	- 0.33227	- 0.20489
- 0.47010	0.09342	- 0.47013	0.09510
- 0.47010	- 0.09342	- 0.47013	- 0.09510

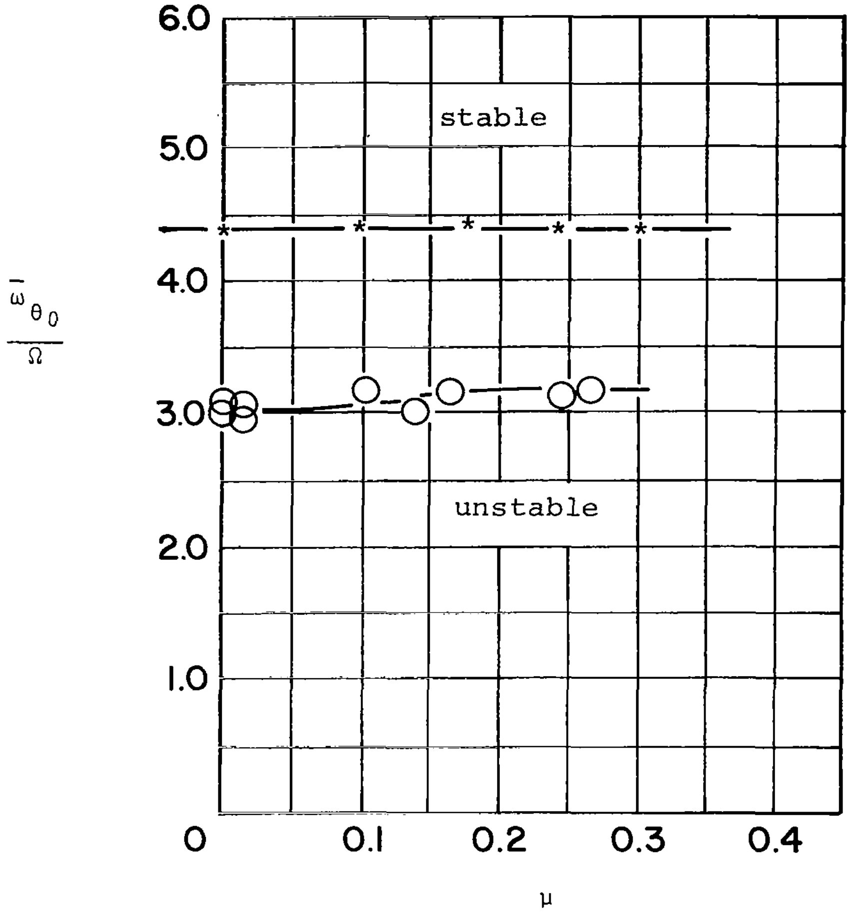

Figure 6. Experimental flutter boundary for a model rotor blade. Ratio of nonrotating pitch natural frequency $\bar{\omega}_{\theta_0}$ to rotational speed Ω versus advance ratio μ, for $\bar{\omega}_{\phi_1}/\bar{\omega}_{\theta_0} = 1.31$ and $\bar{x}_e/C = .139$ (from Reference 10). Points at which characteristic values were calculated are indicated by asterisks.

The results of the calculations indicate, first, that the stability of the rotor for the control stiffness selected is essentially independent of advance ratio over the range of μ considered. The relative insensitivity to advance ratio changes is certainly in agreement with the plot of Figure 6. Secondly, it should be noted that the rotor is predicted to be very nearly unstable. The value for λ_R of -0.063 indicates that about ten rotor revolutions are required to damp the motion by a factor e^{-2}. The limited calculations which were performed, then, are at least in qualitative agreement with the experimental results. A definitive quantitative comparison must await more extensive calculations.

CONCLUDING REMARKS

A method has been developed for analyzing the aeroelastic stability of helicopter rotors in forward flight. The method employs formulations for calculating the characteristic values of the perturbation equations of motion, the latter being a coupled set of second-order, linear differential equations with periodic coefficients. The characteristic values, which are the zeros of an infinite determinant, are calculated from an equivalent analytic form for the infinite determinant consisting of a finite sum of trigonometric functions.

A digital computer program was prepared which implements the method for three degrees of freedom. Calculations of characteristic values carried out for a test case demonstrated the practicality of the method, with anticipated results generally obtained to four significant figures.

Calculations were carried out for comparison with results of a direct time integration on a digital computer of the equations for a rigid rotor blade with flapping and lead-lag hinges. The results were in qualitative agreement. Quantitative differences were attributable to the nonlinear effects which were included in the direct time integration.

Lastly, limited calculations were performed for comparison with experimentally derived flutter boundaries for a model rotor blade with three degrees of freedom. The calculations indicate that the rotor is only marginally stable at the control stiffness selected and that the stability is relatively insensitive to advance ratio, in agreement with the experimental results.

APPENDIX A

Relationship Among the Solutions of Original and Differentiated Systems

Let X_0 be a solution of

$$\frac{d^3X_0}{dz^3} + C\frac{dX_0}{dz} + DX_0 = 0 \tag{A-1}$$

as well as of

$$\frac{d^4X_0}{dz^4} + E\frac{dX_0}{dz} + FX_0 = 0 \quad . \tag{A-2}$$

From Eq. (A-2),

$$\frac{d^4X_0}{dz^4} + \left[\frac{d^2A}{dz^2} + \frac{2dB}{dz} - AB - \frac{d}{dz}(A^2) - \left(\frac{dA}{dz} + B - A^2\right)A\right]\frac{dX_0}{dz}$$
$$+ \left[\frac{d^2B}{dz^2} - \frac{d}{dz}(AB) - \left(\frac{dA}{dz} + B - A^2\right)B\right]X_0 = 0 \; . \tag{A-3}$$

But Eq. (A-1), differentiated once, gives

$$\frac{d^4X_0}{dz^4} = -\left[\frac{dA}{dz} + B - A^2\right]\frac{d^2X_0}{dz^2} - \left[\frac{d^2A}{dz^2} + \frac{dB}{dz} - \frac{d}{dz}(A^2) + \frac{dB}{dz} - AB\right]\frac{dX_0}{dz}$$
$$- \left[\frac{d^2B}{dz^2} - \frac{d}{dz}(AB)\right]X_0 \quad . \tag{A-4}$$

If (A-4) is substituted in (A-3), a number of terms cancel and a common factor can be extracted, with the result that

$$- \left[\frac{dA}{dz} + B - A^2\right] \left[\frac{d^2X_0}{dz^2} + A\frac{dX_0}{dz} + BX_0\right] = 0$$

Thus, X_0 must be a solution of the original equation, Eq. (10). It is, of course, also true that every solution of Eq. (10) is a solution of Eq. (A-1) and of Eq. (A-2). Thus, X_0 is a solution of Eq. (10) if and only if it is a solution of both Eq. (A-1) and Eq. (A-2).

APPENDIX B

Listing of Basic Computer Program - See errata sheet

The program was coded in FORTRAN IV. A CDC 6400 digital computer was employed for all calculations.

```
      PROGRAM Q(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      DIMENSION L2(5)
      DIMENSION ER(350),EI(350),FR(350),FI(350),FFR(150),FFI(150)
      DIMENSION DELT(150),ALPHA(150),BETA(150),GAMMA(150)
      DIMENSION GR(5000),GI(5000)
      DIMENSION CI(250),DR(250),DI(250),DUM(810)
      DIMENSION YR9(9),YI9(9),YR12(12),YI12(12),BIGA(9),BIGAH(13),YY(13)
      DIMENSION AK9(9),AK12(13),DYR9(11),DYI9(11),DYR12(12),DYI12(12)
      DIMENSION SIGMA(7)
      DIMENSION XI9(10),ET9(10),XI12(13),ET12(13),XI6(7),ETA6(7)
      DIMENSION R12TR(13),R12TI(13)
      DIMENSION AUM(1,1),AY(15),AR(125),AI(125),BR(125),BI(125),CR(250)
      DIMENSION PSI9(9),R06TR(6),R06TI(6),NDR9(9),NDI9(9),NDR12(12)
      DIMENSION NDI12(12),DETR(12),DETI(12),PIRNEW(12),PIROLD(12)
      DIMENSION PIINEW(12),PIIOLD(12),AUMR9(8,14),AUMI9(8,14)
      DIMENSION AUMR12(11,14),AUMI12(11,14)
      DIMENSION ROLY(9),PSI12(13),ETA9(10),ETA12(13)
      DIMENSION RR(20),RRI(20),ROOTI(20),ROOTR( 20),Z(20),Y(20),POLY(20)
      DIMENSION XR(12),XI(12)
      DIMENSION CCH(150),CC(150),PSI6(7)
      DIMENSION RR12(13),RRI12(13)
      DIMENSION A9(9),B9(9),C9(9),AH(12),BH(12),CH(12)
      DIMENSION ICHG(15)
C
C      THE GR AND GI ARRAYS ARE DETERMINANT ELEMENT STORAGE NOT NEEDED
C     THROUGHOUT THE PROGRAM AND ARE AVAILABLE FOR FOR OTHER USES BEFORE
C     AND AFTER THE DETERMINANT ROUTINE
C
      EQUIVALENCE(ALPHA(1,1),AR(1)),(BETA(1,1),AI(1)),(GAMMA(1,1),BR(1))
      EQUIVALENCE (DELT(1,1),BI(1))
      EQUIVALENCE(R12TR(1),DUM(1)),(R12TI(1),DUM(15))(AY(1),DUM(30))
      EQUIVALENCE(PSI9(1),DUM(50)),(R06TR(1),DUM(60)),(R06TI(1),DUM(70))
      EQUIVALENCE(NDR9(1),DUM(80)),(NDI9(1),DUM(90)),(NDR12(1),DUM(100))
      EQUIVALENCE (DYR9(1),DUM(115)),(DYI9(1),DUM(130)),(RR(1),DUM(145))
      EQUIVALENCE(DYR12(1),DUM(166)),(NDI12(1),DUM(180))
      EQUIVALENCE(DETR(1),DUM(193)),(DETI(1),DUM(206))
      EQUIVALENCE(PIRNEW(1),DUM(220)),(PIROLD(1),DUM(233))
      EQUIVALENCE(PIINEW(1),DUM(246)),(PIIOLD(1),DUM(260))
      EQUIVALENCE  (AUMR9(1,1),DUM(273)),(AUMI9(1,1),DUM(386))
      EQUIVALENCE(AUMR12(1,1),DUM(500)),(AUMI12(1,1),DUM(655))
      EQUIVALENCE(AK9(1),GR(1)),(AK12(1),GR(10)),(SIGMA(1),GR(25))
      EQUIVALENCE(GR(35),XI9(1)),(ET9(1),GR(45)),(XI12(1),GR(60))
```

```
C
C     FORMULA FOR GR,GI,ALPHA,BETA,GAMMA,DELT    IS  3(ND+1)**2
C      MAINFRAME PROGRAM FOR THE NASA FLUTTER
C
      REAL MU
      NUMMA=1
      INP = 0
      NO= 9
C     LK = NF  A SPECIIFIED INTEGER    USUALLY LESS THAN 3
C
      NIN=5
      NP=6
      NOUT=NP
 1001 FORMAT(1H0*START*)
      WRITE(NP,1001)
      WRITE(NP,1001)
      LI=3
      LJ=3
      MU=.9
      GAMMAP=2.
      NF=3
      LK=NF
      N=NF
      K=1
      LM = 3
      LJ1 = LJ+1
 7731 FORMAT(8F10.4)
 7345 FORMAT(4E13.4)
 7734 FORMAT(1X,I3,*,*I3*,*I3,  4E20.8)
      L=3LLGO
      L=3LMKK
      DO 1575  MM=1,250
      DUM(MM) = 0
 1575 CONTINUE
      KEN=3LKEN
      L2(1)=2LSA
      L2(2)=4LCLAD
      L2(3)=3LOSF
      L2(4)=0
      CALL SEGMENT(KEN,1,_2,1,NUMMA)
C
C     FOR A FLUTTER RUN CALL SA FOR THE AR, AI, BR, BI
C
      CALL  SA(AR,AI,BR,BI,LI,LJ,NF,GAMMAP,MU,DEG)
      L2(1)=3LS1B
```

```
      L2(2)=0
      CALL SEGMENT(KEN,1,L2,1,0)
C
C    GET CR(IJK) AND CI(IJK) BY THIS CALL TO GET INPUT FOR SECT2A
C
      CALL S1B(OUTPUT,CR,CI,INPUT,AR,AR,AI,AI,BR,BI,LI,LJ,LK,NF,NP)
C
C       GET THE DR(IJK), AND DI(IJK) BY THIS CALL
C   WHERE DUM IS A DUMMY VECTOR OF LENGTH BR(IJK) BUT ZERO EVERYWHERE
C
      CALL S1B(OUT,DR,DI,IN,AR,BR,AI,BI,DUM,DUM,LI,LJ,LK,NF,NP)
      L2(1)=3LS1C
      L2(2)=0
      CALL SEGMENT(KEN,1,L2,1,0)
C
C       GET THE ER(IJK), AND EI(IJK) BY THIS CALL
C
      CALL S1C(AR,AI,CR,CI,CR,CI,DR,DI,ER,EI,LI,LJ,LK,NF)
C
C       GET THE FR(IJK), AND FI(IJK) BY THIS CALL
C
      CALL S1C(BR,BI,CR,CI,DR,DI,DUM,DUM,FR,FI,LI,LJ,LK,NF)
      PI=3.1415926
      DELTA= .0001/PI
      L2(1)=2LS3
      L2(2)=4LPROD
      L2(3)=0
      KEN=3LLGO
C     GET THE EVALUATION PONTS FOR THE 9TH AND 12TH ORDERS
      CALL SEGMENT(KEN,1,L2,1,NUMMA)
      CALL S3(1,PSI9,ETA9,YR9,YI9,K9,IN,DELTA,CR,DR,NF,9,NP)
      CALL S3(1,PSI12,ETA12,YR12,YI12,K12,IN,DELTA,ER,FR,NF,12,NP)
        OUT=0
      EPS=.000000000001
        DELTAX= .0000000001
      IN= 0
      L=3LMKK
      L2(2)=2LUP
      L2(3)=3LPRE
      L2(1)=3LS2A
      L2(4)=0
```

```
      KEN=3LKEN
      CALL SEGMENT (KEN,1,L2,1,NUMMA)
C
C     D DOES SETUP AND FINDS ALL CONVERGED DETERMINANTS
C
      CALL D(FFR,FFI,GR,GI,ALPHA,BETA,GAMMA,DELT,CR,CI,DR,DI,AUMR9,AUMI9,
     1,AUMR12,AUMI12,DYR9,DYI9,PIROLD,PIRNEW,YR9,YI9,NDR9,NDI9,9,K9,1,N)
      CALL D(FFR,FFI,GR,GI,ALPHA,BETA,GAMMA,DELT,ER,EI,FR,FI,AUMR9,AUMI9
     1,AUMR12,AUMI12,DYR12,DYI12,PIIOLD,PIINEW,YR12,YI12,NDR12,NDI12,
     212,K12,1,N)
      L2(1)=2LS4
      L=3LMKK
      L2(2)=0
C
C       SET UP THE 9TH ORDER SIMULTANEOUS EQUATIONS
C      TAKING ADVANTAGE OF COMPLEX -DETERMINANT PAIRS
C
      CALL SEGMENT(KEN,1,L2,1,NUMMA)
      CALL S4(CC,BIGA,A9,B9,C9,XR,XI,U,PSI9,ETA9,9,K9,PI,DYR9,DYI9,YR9,
     1 YI9)
C
C       SOLVE THE SIMULTANEOUS EQUATIONS
C
      CALL SIMQ(CC,9,BIGA)
C
C       SET UP THE 12TH ORDER  SIMULTANEOUS EQUATIONS
C
      CALL S4(CCH,BIGAH,AH,BH,CH,XRH,XIH,U,PIS12,ETA12,12,K12,PI,
     1 DYR12,DYI12,YR12,YI12)
      L2(1)=4LSIMQ
      L2(2)=0
      CALL SEGMENT(KEN,1,L2,1,NUMMA)
C
C       SOLVE THE SIMULTANEOUS EQUATIONS
C
      CALL SIMQ(CCH,12,BIGAH)
 1186 L2(2)=0
      L2(1)=3LS5A
      L2(2)= 3LTEA
      L2(3)=4LPRQD
      L2(4)=0
      CALL SEGMENT(KEN,1,L2,1,NUMMA)
```

```
C
C        SET UP THE POLYNOMIAL COEFFICIENTS FOR THE 9TH ORDER AND GET
C
       CALL S5A(OUT,AK9,IN,K9,4,BIGA,B9,C9,IERR)
 1177  AY(1) = AK9(9)
       AY(2) = AK9(8)
       AY(3) = AK9(7)
       AY(4) = AK9(6)
       AY(5) = AK9(5)
       AY(6) = AK9(4)
       AY(7) = AK9(3)
       AY(8) = AK9(2)
          AY(9) = AK9(1)
       AY(10) = 1.
C
C        GET THE POLYNOMIAL ROOTS
C
       CALL  PROD(AY,9,RR,RRI,YY,NUM,IERR)
C
C       THE PSI AND ETA FROM SUBROUTINE TEA
C
C
 1175  CALL TEA(9,6,RR,RRI,XI9,ET9)
       K12 = KTWELV
C     SET UP THE 12TH ORDER POLYNOMIAL COEFFICIENTS
C
       CALL S5A(OUT,AK12,IN,K12, 6,BIGAH,BH,AH,CH,IERR)
 1171  AY(1) = AK12(12)
       AY(2) = AK12(11)
       AY(3) = AK12(10)
       AY(4) = AK12(9)
       AY(5) = AK12(8)
       AY(6) = AK12(7)
       AY(7) = AK12(6)
       AY(8) = AK12(5)
```

```
      AY(9) = AK12(4)
      AY(10) = AK12(3)
      AY(11) = AK12(2)
      AY(12) = AK12(1)
      AY(13) = 1.
C
C      GET THE ROOTS FROM PROD
C
      CALL  PROD(AY,12,R12TR,R12TI,YY,NUM,IERR)
C
C   GET THE 12TH ORDER PSI AND ETA FROM TEA
C
 1169 CALL TEA(12,6,R12TR,R12TI,XI12,ET12)
C    ON TO SECT6
      L2(1)=2LS6
      L2(2)=0
      L2(3)=0
      KEN= 3LLGO
      CALL SEGMENT(KEN,L2,1,NUMMA)
C
C      GET THE COMMON POLYNOMIAL COEFFICIENTS
C
      CALL S6(OUT,SIGMA,IN,AK9,AK12,NP)
 1167  Z(7) = 1
      Z(6) = SIGMA(1)
      Z(5) = SIGMA(2)
      Z(4) = SIGMA(3)
      Z(3) = SIGMA(4)
      Z(2) = SIGMA(5)
      Z(1) = SIGMA(6)
      KEN=3LLGO
      L2(1)=4LPROD
      L2(2)=36TEA
      L2(3)=3LPAT
      L2(4)=0
      CALL SEGMENT(KEN,1,L2,1,NUMMA)
C
C      GET THE 6TH ORDER ROOTS
C
```

```
      CALL  PRQD(Z,6,R06TR,R06TI,YY,NUM,IERR)
C
C      GET THE 6TH ORDER PSI AND ETA FROM TEA
C
      CALL TEA(6,6,R06TR,R06TI,XI6,ETA6)
      GA=GAMMAP
C
C     GET THE FINAL OUTPUT SHEETS
C
      CALL PAT(AUMR9,AUMR12,PSI9,PSI12,XI9,XI12,XI6,ETA9,ETA12,ET9,ET12
     1,ETA6,    GA,MU,NF,RR,RRI,R12TR,R12TI,R06TR,R06TI,NDR9,NDR12,YR9,YR
     212,PIROLD,PIRNEW,PIIOLD,PIINEW,AUMI9,AUMI12,NDI9,NDI12,YI9,YI12)
      STOP
      END
      NOLIST
*END
```

```
      SUBROUTINE SA(AR,AI,BR,BI,LI,LJ,NF,GAMMAP,MU,DEG)
      DIMENSION Y( 30),YY( 30),W( 30),WW( 30),ZZ( 30),Z1( 30),AMO(30),
     1 CLA(30),APHI1(30),BPHI1(30) ,AR(125),AI(125),BR(125),BI(125)
      DIMENSION APHI2(30),BPHI2(30)
C     SUBROUTINE FOR GENERATING THE  AR, AI,BR,BI, FO THE NASA FLUTTER
C     EQUATIONS.
      REAL MU,MU1,MUSIPH,MUCOPH, MZZERO,MUZERO,MOXPHI,MOZPHI
C     SET UP EXTERNALS       CLA(I), AMO(I), GAMMAP, MU
      LK =NF
      NF2 = 2.*NF
      ANF2 = NF2
      DEG= 360./ANF2
      NP = 6
      LJ1 = LJ+1
      AMO(1) = 0
      AMO(2) = .3
      AMO(3) = .4
      AMO(4) = .5
      AMO(5) = .6
      AMO(6) = .65
      AMO(7) = .7
      AMO(8) = .75
      AMO(9) =1.
      CLA(1) = 6.398
      CLA(2) = 6.398
      CLA(3) =6.517
      CLA(4)= 6.876
      CLA(5) = 7.792
      CLA(6) = 8.594
      CLA(7) = 9.072
      CLA(8) = 10.027
      CLA(9) = 10.027
      DO 50 I = 1,LI
      DO 50 J = 1,LJ
```

```
      DO 50 K = 1,NF
      IJK = (I*LI-LJ1+J)*LK+K
      AR(IJK) = 0
      AI(IJK) = 0
      BR(IJK) = 0
      BI(IJK) = 0
   50 CONTINUE
      NINT = 20
      ANINT = NINT
      G = (.97-.2)/ANINT
      H = (1,-,2)/ANINT
      PI = 3,1415926536
      PI2 = PI/2,
      NF2 = NF*2
      ANF = NF
      RNF = 1./ANF
      PHI =(PI*ANU)/ANF
      MU1= 1, + MU
      PI180     = PI/180,
      NDEG = 360,/DEG
      NINT=NINT+1
      DO 1 NPHI = 1,NDEG
      ANPHI = NPHI-1
      PHEYE= ANPHI*DEG
      PHI = ANPHI*PI180*DEG
      COSPHI = COS(PHI)
      SINPHI = SIN(PHI)
      MUSIPH = MU*SINPHI
      MUCOPH = MU*COSPHI
      NINT1 = NINT+1
```

```
      DO 2 J= 1,NINT1
      CHAY = J-1
      L = J+1
      Z = .2+CHAY*G
      X = .2+ CHAY*H
      ZMU = Z+ MUSIPH
      EMU = X+  MUSIPH
      MZZERO= ABS(ZMU)
      MUZERO = ABS(EMU)
      IF(EMU) 52,53,53
   53 A0XPHI = 0
      GO TO 55
   52 A0XPHI = PI
   55 M0XPHI = (.8*MUZERO)/MU1
      IF(ZMU) 56,57,57
   57 A0ZPHI = 0
      GO TO 59
56     A0ZPHI = PI
   59 M0ZPHI = (.8*MZZERO)/MU1
      CALL CLAD(PI,AMO,CLA,A0XPHI,M0XPHI,CLX,CDX,NP,IER)
      CALL CLAD(PI,AMO,CLA,A0ZPHI,M0ZPHI,CLZ,CDZ,NP,IERR)
       YY(J) = MUZERO*(X-.05)*(X-.05)*CDX
      Y(J) = MUZERO*(X-.05)*CDX
      WW(J) = MZZERO*Z*Z*CLZ
      W(J) = MZZERO*Z*CLZ
      ZZ(J) = MUZERO*X*X*CDX
      Z1(J) = MUZERO*X*CDX
    2 CONTINUE
```

```
      DO 2 J= 1,NINT1
      CHAY = J-1
      L = J+1
      Z = .2+CHAY*G
      X = .2+ CHAY*H
      ZMU = Z+ MUSIPH
      EMU = X+  MUSIPH
      MZZERO= ABS(ZMU)
      MUZERO = ABS(EMU)
      IF(EMU) 52,53,53
   53 A0XPHI = 0
      GO TO 55
   52 A0XPHI = PI
   55 M0XPHI = (.8*MUZERO)/MU1
      IF(ZMU) 56,57,57
   57 A0ZPHI = 0
      GO TO 59
  56   A0ZPHI = PI
   59 M0ZPHI = (.8*MZZERO)/MU1
      CALL CLAD(PI,AMO,CLA,A0XPHI,M0XPHI,CLX,CDX,NP,IER)
      CALL CLAD(PI,AMO,CLA,A0ZPHI,M0ZPHI,CLZ,CDZ,NP,IERR)
       YY(J) = MUZERO*(X-.05)*(X-.05)*CDX
      Y(J) = MUZERO*(X-.05)*CDX
      WW(J) = MZZERO*Z*Z*CLZ
      W(J) = MZZERO*Z*CLZ
      ZZ(J) = MUZERO*X*X*CDX
      Z1(J) = MUZERO*X*CDX
    2 CONTINUE
```

```
      CALL QSF(H,YY,YY,NINT)
      CALL QSF(H,Y,Y,NINT)
      CALL QSF( G,WW,WW,NINT)
      CALL QSF(G,ZZ,ZZ,NINT)
      CALL QSF(G,W,W,NINT)
      CALL QSF(H,Z1,Z1,NINT)
      APHI1(NPHI) = YY(NINT)
      BPHI1(NPHI) = MUCOPH*Y(NINT)
      APHI2(NPHI) = 0.4286874*(WW(NINT) + ZZ(NINT))
      BPHI2(NPHI) = 0.4286874*( W(NINT) + Z1(NINT))  *MUCOPH
    1 CONTINUE
      DO 15 K = 1,NF
      K11 = (LI-LJ1+1)*LK+K
      K22= (LI*2-LJ1+2)*LK+K
      AK1 = K-1
      PIAK1 = PI*AK1
      VALUE1 = 0
      VALUE2 = 0
      VALUE3 = 0
      VALUE3 = 0
      VALUE4 = 0
      VALUE5 = 0
      VALUE6 = 0
      VALUE7 = 0
      VALUE8 = 0
      DO 10 L = 1,NF2
      ANU = L-1
      PIK1NU = (PIAK1*ANU)/ANF
      COSPI=COS(PIK1NU)
      SINPI = SIN(PIK1NU)
      VALUE1 = VALUE1 +APHI1(L)*COSPI
```

```
      VALUE3 = VALUE3 +BPHI1(L) • COSPI
      VALUE4 = VALUE4 +BPHI1(L)*SINPI
      VALUE5 = VALUE5+APHI2(L)*COSPI
      VALUE6 = VALUE6+APHI2(L)*SINPI
      VALUE7 = VALUE7 +BPHI2(L) *COSPI
      VALUE8 = VALUE8 + BPHI2(L)* SINPI
   10 CONTINUE
      AR(K11) = VALUE1*RNF*GAMMAP
      AR(K22) = VALUE5*RNF *GAMMAP
      AI(K11) = -VALUE2*RNF  *GAMMAP
      AI(K22) = -VALUE6*RNF*GAMMAP
      BR(K11) = 2.*VALUE3*RNF  *GAMMAP
      BR(K22) = 2.*VALUE7*RNF *GAMMAP
      BI(K11) = -2.*VALUE4*RNF      *GAMMAP
      BI(K22) = -2.*VALUE8*RNF       *GAMMAP
   15 CONTINUE
      K221 =(2*LI-LJ1+2)*LK+1
      BR(K221) = BR(K221)+4.
      K111 =(LI-LJ1+1)*LK+1
      BR(K111) = BR(K111) + .3157896
      K133     = (3*LI-LJ1+3)*LK+1
      K233     = (3*LI-LJ1+3)*LK+2
      AR(K133)=2.
      BR(K133) = 2.
      AR(K233) = 1.0
      RETURN
      END
```

```
      SUBROUTINE CLAD(PI,AM,CL,A,AMO,CLX,CDX,NP,IERR)
      DIMENSION AM(1) , CL(1)
      IERR=0
       IF (A-PI)1,2,1
    2 CLX= 5.73
      CDX= 0.02
      RETURN
    1 IF(A) 999,4,999
    4 CDX=0.01
    3 MA = AMO*10. + 1.
      IF(MA-11) 40,40,41
   41 IERR=2
      RETURN
   40 GO TO (100,101,101,101,104,105,106,107,108,108,108),MA
  100 CLA =  CL(1)
      GO TO 111
  101 CLA= (AMO-AM( 2))*( CL(2)-CL(1))+CL(2)
      GO TO 111
  104 CLA = (AMO-AM(3)) *(CL(4)-CL(3))+ CL(3)
      GO TO 111
  105 CLA = (AMO-AM(4))*(CL(5)-CL(4))+ CL(4)
      GO TO 111
  106 IF(AMO-AM(6))60,60,61
   60 CLA = (AMO-AM(5)) *(CL(6)-CL(5))+ CL(5)
      GO TO 111
   61 CLA = (AMO-AM(6))* (CL(7)-CL(6))+ CL(6)
      GO TO 111
  107 IF(AMO-AM(8)) 63,108,108
   63 CLA =(AMO-AM(7)) * (CL(8)-CL(7))+ CL(7)
      GO TO 111
  108 CLA = CL(8)
  111 CLX=CLA
      RETURN
  999 IN=1
      RETURN
      END
```

```
      SUBROUTINE QSF(H,Y,Z,NDIM)
      DIMENSION Y(1),Z(1)
      HT=.3333333*H
      INTEGRATION BY SIMPSONS RULE
      IF(NDIM-5)7,8,1
    1 SUM1=Y(2)+Y(2)
      SUM1=SUM1+SUM1
      SUM1=HT*(Y(1)+SUM1+Y(3))
      AUX1=Y(4)+Y(4)
      AUX1=AUX1+AUX1
      AUX1=SUM1+HT*(Y(3)+AUX1+Y(5))
      AUX2=HT*(Y(1)+3.875*(Y(2)+Y(5))+2.625*(Y(3)+Y(4))+Y(6))
      SUM2=Y(5)+Y(5)
      SUM2=SUM2+SUM2
      SUM2=AUX2-HT*(Y(4)+SUM2+Y(6))
      Z(1)=0.
      AUX=Y(3)+Y(3)
      AUX=AUX+AUX
      Z(2)=SUM2-HT*(Y(2)+AUX+Y(4))
      Z(3)=SUM1
      Z(4)=SUM2
      IF(NDIM-6)5,5,2

      INTEGRATION LOOP
    2 DO 4 I=7,NDIM,2
      SUM1=AUX1
      SUM2=AUX2
      AUX1=Y(I-1)+Y(I-1)
      AUX1=AUX1+AUX1
      AUX1=SUM1+HT*(Y(I-2)+AUX1+Y(I))
      Z(I-2)=SUM1
      IF(I-NDIM)3,6,6
    3 AUX2=Y(I)+Y(I)
      AUX2=AUX2+AUX2
      AUX2=SUM2+HT*(Y(I-1)+AUX2+Y(I+1))
    4 Z(I-1)=SUM2
    5 Z(NDIM-1)=AUX1
```

```
      Z(NDIM)=AUX2
      RETURN
    6 Z(NDIM-1)=SUM2
      Z(NDIM)=AUX1
      RETURN
C     END OF INTEGRATION LOOP
C
    7 IF(NDIM-3)12,11,8
C
C     NDIM IS EQUAL TO 4 OR 5
    8 SUM2=1.125*HT*(Y(1)+Y(2)+Y(2)+Y(2)+Y(3)+Y(3)+Y(3)+Y(4))
      SUM1=Y(2)+Y(2)
      SUM1=SUM1+SUM1
      SUM1=HT*(Y(1)+SUM1+Y(3))
      Z(1)=0.
      AUX1=Y(3)+Y(3)
      AUX1=AUX1+AUX1
      Z(2)=SUM2-HT*(Y(2)+AUX1+Y(4))
      IF(NDIM-5)10,9,9
    9 AUX1=Y(4)+Y(4)
      AUX1=AUX1+AUX1
      Z(5)=SUM1+HT*(Y(3)+AUX1+Y(5))
   10 Z(3)=SUM1
      Z(4)=SUM2
      RETURN
C
C     NDIM IS EQUAL TO 3
   11 SUM1=HT*(1.25*Y(1)+Y(2)+Y(2)-.25*Y(3))
      SUM2=Y(2)+Y(2)
      SUM2=SUM2+SUM2
      Z(3)=HT*(Y(1)+SUM2+Y(3))
      Z(1)=0.
      Z(2)=SUM1
   12 RETURN
      END
```

```
      SUBROUTINE S1B(OUT,CR,CI,IN,AR,BAR,AI,BAI,BR,BI,LI,LJ,LK,NF,NP)
      DIMENSION CI(1), CR(1), AR(1), BAR(1), BAI(1), AI(1), BR(1), BI(1)
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
C   CALL  SECT1(CR,CI,AR,AR,AI,AI,BR,BI,LI,LJ,LK,NF)
C   FOR THE CALCULATION OF THE CR(I,J,K) AND THE CI(I,J,K) USE...
C   FOR THE CALCULATION OF THE DR(I,J,K) AND THE DI(I,J,K) USE...
C     CALL  SECT1(DR,DI,AR,BR,AI,BI,DUM,DUM,LI,LJ,LK,NF)
C    WHERE DUM IS A   VECTOR AS LONG AS BI AND BR BUT = TO ZERO EVERYWHERE
C
C      THE IMPUT TO THIS PROGRAM CONSISTS OF THE COMPLEX FOURIER COEFFICIENTS
C      OF 18 PERIODIC FUNCTIONS.  SPECIFICALLY, THE PROGRAM IS TO BE SUPPLIED
C      WITH...
C        AREAL(I,J,K)                      AIMAG(I,J,K)
C
C        BREAL(I,J,K)                      BIMAG(I,J,K)
C
C          FOR I = 1, 2, 3
C          FOR J = 1, 2, 3
C          FOR K = 1,2, , , , , NF
C
C          WHERE NF IS A SPECIFIED INTEGER
C
C    THIS SUBROUTINE IS A BRANCH  THAT PROCEEDS FROM THE CALCULATION
C    OF COMPLEX FOURIER COEFFICIENTS FOR 18 FUNCTIONS RELATED TO THE ABOVE,
C
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
C      EQUATIONS FOR COMPUTATIONS
C
      LJ=LJ+1
      LJ1 = LJ-1
      LK=NF
      N2F1 = 2 * NF -1
      NF1 = NF + 1
      KL = 2*NF-1
C
      DO 5 I=1, LI
C
      DO 5 J = 1, LJ1
```

```
C
      IKL = (LI*I-LJ+J)*KL
            IJ=(LI * I -LJ+J)*LK
C     FOR K = 1, 2, ... , NF
C
                  DO  55  K = 1 , NF
      AK = K-1
      IJKL = IKL+K
      IJK = IJ + K
      CI(IJKL) = 2.* AK * BAR(IJK) + BI(IJK)
      CR(IJKL)= -2.*AK*BAI(IJK) + BR(IJK)
C   PERFORM THE SUMMATIONS
      CONSTR =0
      CONSTI = 0
C
      DO  7 L= 1, 3
      IL = (LI*I- LJ + L)*LK
       JL= (LI*L -LJ+J)*LK
      CONST1 =0
      CONST4 =0.
C
      DO 15 N = 1 , K
      ILN = IL+N
      LJK1N = JL + K +1 -N
      ARILN = AR(ILN)
      BALJK1=BAR(LJK1N)
       AIILN=AI(ILN)
      BILJK1=BAI(LJK1N)
       APPLE= BALJK1*ARILN
      PEAR = AIILN*BILJK1
      CONST1 = CONST1+APPLE-PEAR
C     CONST1=CONST1+ARILN*BALJK1-AIILN*BILJK1
      APPLE=ARILN*BILJK1
C     CONST1 = CONST1 + AR(ILN) * BAR(LJK1N) - AI(ILN) * BAI(LJK1N)
C     CONST4 = CONST4 + AR(ILN) + BAI(LJK1N) + AI(ILN) * BAR(LJK1N)
      PEAR = AIILN*BALJK1
      CONST4=CONST4+APPLE+PEAR
   15 CONTINUE
```

```
C
C        NOTE ...,  THE LAST SUM ON N IS OMITTED FOR K = NF
      CONST2 = 0
      CONST5 = 0
      IF(NF-K) 17, 17, 16
   16 NFK=NF-K
C
      DO 20 N=1, NFK
      ILN1 = IL + N + 1
      LJNK=JL+N+K
      ILNK = IL + N + K
      LJN1 = JL + N +1
      ARILN1=AR(ILN1)
      BRLJNK=BAR(LJNK)
      AIILN1=AI(ILN1)
      BILJNK=BAI(LJNK)
      ARILNK=AR(ILNK)
      BRLJN1=BAR(LJN1)
      AIILNK=AI(ILNK)
      BILJN1=BAI(LJN1)
      APPLE=ARILN1*BRLJNK
      PEAR = AIILN1*BILJNK
      PEACH=ARILNK*BRLJN1
      PLUM = AIILNK*BILJN1
      CONST2 = CONST2+APPLE+PEAR+PEACH+PLUM
C    1  BILJN1
C     CONST2=CONST2+ARILN1*BRLJNK+AIILN1*BILJNK+ARILNK*BRLJN1+AIILNK*
C     CONST2 = CONST2 + AR(ILN1) * BAR(LJNK) + AI(ILN1) * BAI(LJNK) +
C    1  AR(ILNK) * BAR(LJN1) + AI(ILNK) * BAI(LJN1)
      APPLE=ARILN1*BILJNK
      PEAR  = AIILN1*BRLJNK
      PEACH = ARILNK*BILJN1
```

```
      PLUM = AIILNK*BRLJN1
       CONST5 = CONST5 + APPLE-PEAR-PEACH+PLUM
C     CONST5 = CONST5+ARILN1*BILJNK-AIILN1*BRLJNK-ARILNK*BILJN1+AIILNK*
C    1 BRLJN1
C     CONST5 = CONST5 + AR(ILN1) * BAI(LJNK) - AI(ILN1)* BAR(LJNK) -
C    1  AR(ILNK) * BAI(LJN1) + AI(ILNK)  * BAR(LJN1)
   20 CONTINUE
C
   17 CONSTR= CONSTR + CONST1 + CONST2
      CONSTI = CONSTI + CONST4 + CONST5
    7 CONTINUE
      CR(IJKL)= CR(IJKL)- CONSTR
      CI(IJKL)= CI(IJKL)- CONSTI
   55 CONTINUE
C
C     FOR  K = NF+1, NF+2, ...., 2NF-1
C
      DO 9 K = NF1,N2F1
      CONSTR=0.0
      CONSTI = 0.0
      IJKL = IKL + K
C
         DO 8 L= 1,3
      IL= (LI*I -LJ+L)*LK
      JL = ( LI*L -LJ+J)*LK
      CONST3 = 0.0
      CONST6 = 0.0
      K1N = K + 1 - NF
C
   22 DO 30 N = K1N ,NF
      ILN = IL+N
      LJK1N = JL + K + 1 -N
      ARILN=AR(ILN)
```

```
      BRLJK1=BAR(LJK1N)
      AIILN=AI(ILN)
      BILJK1=BAI(LJK1N)
      CONST3 = CONST3+ARILN*BRLJK1-AIILN*BILJK1
      CONST6 = CONST6+ARILN*BILJK1+AIILN*BRLJK1
C     CONST3 = CONST3 + AR(ILN) * BAR(LJK1N) - AI(ILN) * BAI(LJK1N)
C     CONST6 = CONST6 + AR(ILN) * BAI(LJK1N) + AI(ILN) * BAR(LJK1N)
   30 CONTINUE
C
   21 CONSTR = CONSTR + CONST3
    8 CONSTI=CONSTI+CONST6
      CI(IJKL)=-CONSTI
      CR(IJKL)=     - CONSTR
    9 CONTINUE
C
    5 CONTINUE
C
      LJ= LJ-1
       RETURN
      END
```

```
      SUBROUTINE  PROD(XCOF,M,ROOTR,ROOTI,COF, NUM,IER)
      DIMENSION XCOF(1),COF(1),ROOTR(1),ROOTI(1)
      DOUBLE PRECISION X0,XO,Y0,YO,X,Y,XPR,YPR,UX,UY,V,YT,XT,U
      DOUBLE PRECISION XT2,YT2,SUMSQ,DX,DY,TEMP,ALPHA
C    COMPUTES THE REAL AND COMPLEX ROOTS OF A POLYNOMIAL
C   USING THE NEWTON RAPHSON ITERATION TECHNIQUE
C    PARAMETERS
C   XCOF  VECTOR OF M+1 COEFFICIENTS OF THE POLYNOMIAL
C                  ORDERED FROM SMALLEST TO LARGEST POWER
C      COF A WORKING VECTOR OF SIZE M+1
C      M       THE ORDER OF THE POLYNOMIAL
C      ROOTI      RESULTANT VECTOR OF LENGTH M OF IMAGINARY PARTS
C     ROOTI(1) IS THE INITIAL VALUE OF THE Y GUESS IMAGINARY PART
C      ROOTR      RESULTANT VECTOR OF LENGTH M OF REAL ROOTS
C     ROOTR(1) IS THE INITIAL VALUE OF THE Y GUESS REAL PART
C      IER    ERROR CODE
C                 IER = 0    NO ERROR
C     SUBROUTINE POLRT(XCOF,COF,M,ROOTR,ROOTI,IER)
C                 IER = 1       M IS LESS THAN 1
C                 IER = 2       M IS GREATER THAN 36
C                 IER = 3   UNABLE TO DETERMINE ROOTS IN 50 ITERATIONS
C                 IER = 4   HIGH  ORDER COEFFICIENT IS ZERO
C             PROGRAMMED BY     K G BLEMEL
C                               R.A.S.A.   INC.
C                     ROCHESTER N.Y.    716 I
C                 ROCHESTER N. Y.     716 271 3450
C     SUBROUTINE POLRT(XCOF,COF,M,ROOTR,ROOTI,IER)
      IFIT = 0
      N=M
      IER =0
      IF(XCOF(N+1))10,25,10
   10 IF(N)15,15,32
   15 IER=1
   20 RETURN
   25 IER=4
      GO TO 20
   30 IER=2
      GO TO 20
```

```
32 IF(N-36) 35,35,30
35 NX=N
   NXX = N+1
   N2 = 1
   KJ1 = N+1
   DO 40 L=1,KJ1
   MT = KJ1-L+1
40 COF(MT) = XCOF(L)
45 XO = .00500101
   YO = 0.01000101
   IN = 0
50 X = XO
   XO = -10.*YO
   YO = -10.*X
   X=XO
   Y=YO
   IN = IN +1
   GO TO 59
55 IFIT =1
   XPR = X
   YPR = Y
59 ICT = 0
60 UX = 0.0
   UY = 0.0
   V = 0.0
   YT = 0.0
   XT =  1.0
   U = COF(N+1)
   IF(U) 65,130,65
65 DO 70 I = 1,N
   L = N - I +1
   TEMP = COF(L)
   XT2 = X*XT-Y*YT
   YT2 = X*YT +Y*XT
   V = V + TEMP *YT2
   U = U + TEMP*XT2
   FI=I
   UX=UX+FI*XT*TEMP
   UY=UY-FI*YT*TEMP
   XT = XT2
```

```
 70 YT = YT2
    SUMSQ = UX*UX+UY*UY
    IF(SUMSQ) 75,110, 75
 75 DX=(V*UY-U*UX)/SUMSQ
    X = X + DX
    DY = - (U*UY + V*UX)/SUMSQ
    Y  = Y +  DY
 78 IF(DABS(DY)+DABS(DX)-1.D-12) 100,80,80
 80 ICT = ICT +1
    IF(ICT-500) 60,85,85
 85 IF(IFIT) 100,90,100
 90 IF(IN-5) 50,95,95
 95 IER = 3
    RETURN
100 DO 105 L = 1,NXX
    MT = KJ1 -L+1
    TEMP = XCOF(MT)
    XCOF(MT) = COF(L)
105 COF(L) = TEMP
    ITEMP = N
    N=  NX
    NX =  ITEMP
    IF(IFIT) 120 , 55,120
110 IF(IFIT) 115,50,115
115 X = XPR
    Y = YPR
120 IFIT = 0
    IF(X) 122,125,122
122 IF(DABS(Y/X)-1.D-10) 135,125,125
```

```
125 ALPHA = X+X
    SUMSQ = X*X + Y*Y
    N = N-2
    GO TO 140
130  X = 0.0
    NX = NX  -1
    NXX = NXX -1
135 Y = 0.0
    SUMSQ = 0.0
    ALPHA = X
    N = N-1
140 COF(2) = COF(2) + ALPHA* COF(1)
    K=L
145 DO 150 L = 2,N
    K=K
150 COF(L+1) = COF(L+1)+ALPHA*COF(L) - SUMSQ*COF(L-1)
155 ROOTI(N2) = Y
    ROOTR(N2) = X
    N2=N2+1
    IF(SUMSQ)160,165,160
160 Y=-Y
    SUMSQ=0.
    GO TO 155
165 IF(N) 20,20,45
    END
```

```
      SUBROUTINE  S1C(AR,AI,CR,CI,ACR,ACI,DR,DI,ER,EI,LI,LJ,LK,NF)
      DIMENSION CR(1),CI(1),DR(1),DI(1),ER(1),EI(1)
      DIMENSION AR(1), AI(1), ACR(1),ACI(1)
C     ROCHESTER APPLIED SCIENCE ASSOCIATES
C      100 ALLENS CREEK ROAD
C     ROCHESTER, N. Y.                      716 271-3450
C    THIS SUBROUTINE IS FOR THE COMPUTATION OF  ER(I,J,K) AND EI(I,J,K)
C     TO COMPUTE ER(I,J,K) AND EI(I,J,K) USE...
C      CALL SECT1C(AR,AI,CR,CI,CR,CI,DR,DI,ER,EI,LI,LJ,LK,NF)
C      OR THE CALCULATION OF FR(I,J,K) AND FI(I,J,K)
C      CALL SECT1C(BR,BI,CR,CI,DR,DI,DUM,DUM,FR,FI,LI,LJ,LK,NF)
C     LK IS THE LENGTH OF K
C
      LK=3*NF-2
      LK2NF = 2*NF-1
      LKNF = NF
      LJ=LJ+1
      LJ3=LJ+1
      LJ1= LJ-1
      DO 5 I=1,LI
        DO 5 J= 1,LJ1
      IJ = (LI*I-LJ+J)*LK
      IJ2 = (LI*I-LJ+J)*LK2NF
      DO 45 K=1,NF
      AK= K+K-2
      IJK=IJ+K
      IJK2 = IJ2+K
      ER(IJK)  = -AK*ACI(IJK2)+DR(IJK2)
      EI(IJK) = AK*ACR(IJK2)+DI(IJK2)
C     ER(IJK)=-AK*ACI(IJK)+DR(IJK)
C     EI(IJK) = AK*ACR(IJK)+DI(IJK)
      CONSTR=0
      CONSTI=0
      DO 55 L=1,3
      IL = (LI*I-LJ+L)*LK2NF
      JL = (LI*L-LJ+J)*LKNF
C      JL =(LI*L-LJ+J)*LK
C      IL = (LI*I-LJ+L)*LK
      CONST1=0
      CONST7=0
      NFK=NF-K
      IF(NFK) 3,3,4
    4 DO 15 N = 1,NFK
```

```
      ILN1=IL+N+1
      LJKN=JL+K+N
      CONST1=CONST1+CR(ILN1)*AR(LJKN) +CI(ILN1)*AI(LJKN)
      CONST7= CR(ILN1) * AI(LJKN) - CI(ILN1)*AR(LJKN) + CONST7
   15 CONTINUE
    3 CONST2=0.0
      CONST8=0
      NF1=NF-1
      DO 20 N= 1, NF1
      ILKN=IL+K+N
      LJN1=JL+N+1
      CONST2=CR(ILKN) * AR(LJN1) + CI(ILKN)* AI(LJN1) + CONST2
      CONST8 = CONST8 - CR(ILKN)* AI(LJN1) + CI(ILKN)*AR(LJN1)
   20 CONTINUE
      CONST3    =0
      CONST9 =0
      DO 25 N=1,K
      ILK1N = IL+K+1-N
      LJN= JL+N
      CONST3 = CONST3 + CR(ILK1N) * AR(LJN) - CI(ILK1N) * AI(LJN)
      CONST9= CONST9 + CR(ILK1N) * AI(LJN) + CI(ILK1N) * AR(LJN)
   25 CONTINUE
      CONSTR=CONSTR+ CONST1+CONST2+CONST3
      CONSTI=CONST7 + CONST8 + CONST9  +CONSTI
   55 CONTINUE
      ER(IJK) = ER(IJK)- CONSTR
      EI(IJK)= EI(IJK)- CONSTI
   45 CONTINUE
C
      NF1= NF+1
      NF21 = 2*NF-1
      DO 60 K = NF1,NF21
      IJK=IJ+K
      IJK2 = IJ2+K
      AK = K + K -2
```

```
      EI(IJK) = AK* ACR(IJK2)+DI(IJK2)
      ER(IJK) = -AK*ACI(IJK2) + DR(IJK2)
C     EI(IJK) = AK*ACR(IJK)+DI(IJK)
C     ER(IJK) = -AK*ACI(IJK)+DR(IJK)
      CONSTR=0
      CONSTI=0
      DO 65 L=1,3
      NN=2*NF-1-K
      IL = (LI*I-LJ+L)*LK2NF
      JL = (LI*L-LJ+J)*LKNF
C     IL=(LI*I-LJ+L)*LK
C     JL = (LI*L-LJ+J)*LK
      CONST4 =0.0
      CONST10=0.0
      IF(NN)66,66,67
   67 DO 70 N = 1,NN
       ILKN=IL+K+N
      LJN1 = JL+N+1
      CONST10=-CR(ILKN)* AI(LJN1) + CI(ILKN) * AR(LJN1) + CONST10
      CONST4 = CR(ILKN) * AR(LJN1) + CI(ILKN) * AI(LJN1)  + CONST4
   70 CONTINUE
   66 CONST5=0.0
      CONST11 = 0
C
      DO 80 N=1,NF
      ILK1N=IL+K+1-N
      LJN=JL+N
      CONST11 = CONST11 + CR(ILK1N)*AI(LJN)+CI(ILK1N)*AR(LJN)
      CONST5=CONST5+CR(ILK1N)*AR(LJN)-CI(ILK1N)*AI(LJN)
   80 CONTINUE
C
      CONSTR=CONSTR+CONST4+CONST5
      CONSTI=CONSTI+CONST10+ CONST11
   65 CONTINUE
      ER(IJK) = ER(IJK) - CONSTR
      EI(IJK) = EI(IJK)  - CONSTI
   60 CONTINUE
```

```
C
      NF2 = 2*NF
      NF32= 3*NF-2
      DO 100 K=NF2,NF32

      IJK=IJ+K
      IJK2 = IJ2+K
      CONSTR = 0
      CONSTI=0
C
      DO 105 L=1,3
C     IL= (I*LI-LJ+L)*LK
      IL = (I*LI-LJ+L)*LK2NF
C     JL =(LI*L-LJ+J)*LK
      JL = (LI*L-LJ+J)*LKNF
      CONST6= 0
      CONST12 = 0
      NN = K-2*NF * 2
      IF(NF- NN) 101, 102, 102
C
  102 DO 110 N = NN , NF
      ILK1N= IL+K+1-N
      LJN=JL+N
      CONST6 = CR(ILK1N) * AR(LJN) - CI(ILK1N) * AI(LJN)
      CONST12 = CR(ILK1N) * AI(LJN) + CI(ILK1N) * AR( LJN)
  110 CONTINUE
C
  101 CONSTR= CONSTR + CONST6
      CONSTI = CONSTI + CONST12
  105 CONTINUE
      ER(IJK)=  - CONSTR
      EI(IJK) = - CONSTI
  100 CONTINUE
C
    5 CONTINUE
C
      LJ=LJ-1
      RETURN
      END
*END
```

```
*DECK,S4
      SUBROUTINE S4(CC,BIGA,A,B,C,XR,XI,U,SI,E,NO,K,PI,DYR,DYI,YR,YI)
      DIMENSION XR(13),XI(13)
      DIMENSION A(1),B(1),C(1),CC(1),YR(1),YI(1),DYR(1),DYI(1),X(1),E(1)
      DIMENSION BIGA(1),U(1),SI(1)
      WRITE(6,1445)
 1445 FORMAT( * ENTRY INTO S4*)
C    SETTING UP OF THE LINEAR ALGEBRAIC EQUATIONS 9TH AND 12TH ORDER SYSTEMS
C     PROGRAMMED FOR DR, P, CRIMI
C     ROCHESTER APPLIED SCIENCE
C     ROCHESTER NEW YORK
C      AC716 271 3450
C     PROGRAM FOR SETTING UP THE  SYSTEMS OF SIMULTANEOUS EQUATIONS
C     FOR THE 9TH AND 12TH ORDER SYSTEMS
C     INPUT PARAMETERS
C     A(1) IS DUMMY STORAGE
C     B(1) IS DUMMY STORAGE
C     C(1) IS DUMMY STORAGE
C    MAXIMUM NECESSARY LENGTH OF A,B,C IS 12
C   DELTY IS THE PARAMTER TO BE USED IF THE DENOMINATOR BECOMES SMALL

C     CC(1) IS THE RETURNED SET FO SIMULTANEOUS EQUATION MATRICES
C    C(M,N) IS OF ORDER C(9,9) OR C(12,12)      THEREFORE C(81) OR C(144)
C     Y ARE THE INPUT VALUES OF THE DETERMINANT
C     DY   ARE THE DELTA Y
C     X(1) IS DUMMY STORAGE  MAX IS X(12)
C    E(1) IS ETA(J)  MAX IS ETA(12)
C     U (1) IS DUMMY STORAGE  MAX DIMENSION IS U(12)
C     A PROGRAM FOR SETTING UP THE SIMULTANEOUS EQUATIONS FOR 9TH AND 12TH ORDE
      NO1 = NO-1
      NK=K*2
      DO 5 N=2,NK,2
      A(N)=PI*SI(N)
      B(N) = PI*E(N)
    5 CONTINUE
      N1 = NK+1
      NK1 = NK-1
      DO 10 N=N1,NO
      C(N)=PI*SI(N)
   10 CONTINUE
      DO 25 MM=1,NK1,2
      XR(MM)=-PI*YR(MM)
      XI(MM)=-PI*YI(MM)
      XXR=XR(MM)
      XXI=XI(MM)
      DO 15 N=1,NK1 ,2
      NN=N+1
```

```
      BNN=B(NN)+XXI
      XIB=B(NN)-XXI
      AA=A(NN)
      XAA=XXR+AA
      EXAA=EXP(XAA)
      EMXAA=EXP(-XAA)
      COSHY=(EXAA+EMXAA)*.5
      U(N)=((SIN(BNN)/(COSHY -COS(BNN)))+(SIN(XIB)/(COSHY-COS(XIB))))*.5
   15 CONTINUE
      DO20 N = 2,NK,2
      AA=A(N)
      XXAA=XXR+AA
      BN=B(N)
      XIBN=XXI+BN
      BNXI=BN-XXI
      EXXAA=EXP(XXAA)
      EMXXAA=EXP(-XXAA)
      SINHY=(EXXAA-EMXXAA)*.5
      COSHY=(EXXAA+EMXXAA)*.5
 7022 FORMAT(* NOP = * I5)
      U(N)=((SINHY/(COSHY-COS(XIBN)))+(SINHY/(COSHY-COS(BNXI))))*.5
   20 CONTINUE
      DO 21 N =  N1,NO
      XXCN=XXR+C(N)
      EXCN=EXP(XXCN)
      EMXCN=EXP(-XXCN)
      SINHY=(EXCN-EMXCN)*.5
      COSHY=(EXCN+EMXCN)*.5
      U(N) = SINHY/(COSHY-COS(XXI))
   21 CONTINUE
       DO 30 N=1,NO
      MN=(MM-1)*NO+N
   30 CC(MN) = U(N)
   25 CONTINUE
      DO 125 MM=2,NK,2
      XXR=-PI*YR(MM)
      XXI=-PI*YI(MM)
      DO 130 N=1,NK1,2
      XXRA=XXR+A(N+1)
      EXXRA=EXP(XXRA)
      EMXXRA=EXP(-XXRA)
      SINHY=(EXXRA-EMXXRA)*.5
      COSHY=(EXXRA+EMXXRA)*.5
      BNXI=B( N+1)-XXI
```

```
      XIBN=B(N+1)+XXI
      U(N)=(SINHY/(COSHY-COS(BNXI))-(SINHY/(COSHY-COS(XIBN))))*.5
  130 CONTINUE
      DO 135 N=2,NK,2
      XXAN=XXR+A(N)
      BNXI=B(N)-XXI
      XIBN=B(N)+XXI
      COSHY=(EXP(XXAN)+EXP(-XXAN))*.5
      U(N)=(-SIN(BNXI)/(COSHY-COS(BNXI))+SIN(XIBN)/(COSHY-COS(XIBN)))*.5
  135 CONTINUE
      DO 140 N=N1,NO,1
      XRCN=XXR+C(N)
      COSHY=(EXP(XRCN)+EXP(-XRCN))*.5
      U(N)=SIN(XXI)/(COSHY-COS(XXI))
  140 CONTINUE
      DO 145 N=1,NO
      MN=(MM-1)*NO+N
  145 CC(MN)=U(N)
  125 CONTINUE
       DO 150 MM=N1,NO1
      M=(MM-1)*NO
      XXR= -PI*YR(MM)
      DO 155 N=1,NK1,2
      XRAN=XXR+A(N+1)
      COSHY=(EXP(XRAN)+EXP(-XRAN))*.5
      MN=M+N
      BN1= B(N+1)
      CC(MN)=SIN(BN1)/(COSHY-COS(BN1))
  155 CONTINUE
      DO 160 N=2,NK,2
      MN=M+N
      XRAN=XXR+A(N)
      EXRAN=EXP(XRAN)
      EMXRAN=EXP(-XRAN)
      SINHY=(EXRAN-EMXRAN)*.5
```

```
      COSHY=(EXRAN+EMXRAN)*.5
  160 CC(MN)=SINHY/(COSHY-COS(B(N)))
      DO 165 N=N1,NO
      MN=M+N
      XRCN=(XXR+C(N))*,5
      EXRN=EXP(XRCN)
      EMXRN=EXP(-XRCN)
      SINHY=EXRN-EMXRN
      COSHY=EXRN+EMXRN
      CC(MN)= COSHY/SINHY
  165 CONTINUE
  150 CONTINUE
      DO 40 N = 1,NK1,2
      MN=NO*(NO-1)+N
      MN1 = MN+1
      CC(MN1) = 1.
      CC(MN) = 0.0
   40 CONTINUE
      DO 70 N=NK ,NO
      MN=NO*NO1+N
      CC(MN)=1.0
   70 CONTINUE
      DO 50 M=1,NK1,2
      BIGA(M)=DYR(M)-1.
   50 BIGA(M+1)=-DYI(M+1)
      DO 60 M=N1,NO1
   60 BIGA(M)=DYR(M)-1.
      BIGA(NO)=0
      RETURN
      END
*END
```

```
      SUBROUTINE S2A(OUT,FFR,FFI,GR,GI,ALPH,BET,BET,GAMM,DELT,PIR,PII,
     1  IN,CR,CI,DR,DI,YR,YI,N,ND,LI,LJ,NF,M,NP,NO,EPS,DELTAX)
C    TO GET THE REAL AND IMAGINARY PART OF A COMPLEX DETERMINANT
C     SUBROUTINE SECT2B(OJT,FFR,FFI,GR,GI,ALPH,BET,GAMM,DELT,PIR,PII,
      DIMENSION CR(1), CI(1),DR(1),DI(1),ER(1),EI(1)
      DIMENSION FFR(1), FFI(1)
       DIMENSION FR(1),FI(1),GR(1),GI(1)
      ALF(FFRI,GRIJ,FFII,GIIJ,AB)      =(FFRI*GRIJ +FFII*GIIJ)*AB
      BETF(FFRI,GIIJ,FFII,GRIJ,AB)=(FFRI*GIIJ-FFII*GRIJ)*AB
      FRF(YR,YI,R,CRMM1,DRMM)=YR**3-3,*YR*((2.*R+YI)**2)+YR*CRMM1+DRMM
      FIF(YR,YI,R,CRMM1) =(2,*R+YI)*(3.*YR*YR-((2*R+YI)**2)+CRMM1   )
      GRF(YR,YI,S,CRM,CIM,DRM)=YR*CRM-(2,*S+YI)*CIM+DRM
      GIF(YR,YI,S,CRM,CIM,DIM)=YR*CIM+(2,*S+YI)*CRM+DIM
      FHI(YR,YI,R,CRMM1)=(2,*R+YI)*(4,*YR*(YR*YR-((2,*R+YI)**2))+CRMM1)
      FHR(YR,YI,R,CRMM1,DRMM1)=((YR*YR-(2,*R+YI)**2)**2)+YR*CRMM1*
     1(4,*R*YR+2,*YR*YI)**2 +DRMM1
      NIN=5
      NP=6
      ORDER=NO
      IF(ORDER-9,)1,2,1
    1  IF(ORDER-12,)4,3,4
 1005 FORMAT(41H1IN DELY NEITHER A 9TH OR 12TH ORDER CALL)
    3 NOR=2
      LK=3*NF-2
      GO TO 8
    2 NOR=1
      LK = 2*NF-1
    8 LJ1 =3*(2*ND+1)
C    LJ1 IS USED FOR 2 DIMENSION INDICES
      LM=LI
      LN=LJ
      LJ=LJ+1
      LK33 = LI*LJ*LK
      MD=-ND
      MZ=0
C
```

```
      MM1=(LI*M-LJ+M)*LK+1
      CRMM1 = CR(MM1)
      DRMM1 = DR(MM1)
      DO 89 MR=MD,ND
      R=MR
      I = 3*(MR+ND)+M
      NIRM=3*(ND-MR)+M
   23 GO TO (66,67),NOR
C     DO 9 TH ORDER EQUATIONS
   66 FFII=FIF(YR,YI,R,CRMM1)
      FFRI=FRF(YR,YI,R,CRMM1,DRMM1)
      ALPH(I)=1./(FFRI*FFRI+FFII*FFII    )
      FRN= FRF(YR,-YI,R,CRMM1,DRMM1)
      FIN=FIF(YR,-YI,R,CRMM1)
      BET(NIRM)=1./(FRN*FRN+FIN*FIN)
      GAMM(NIRM)=FRN
      DELT(NIRM)=FIN
      FFI(I)=FFII
      FFR(I)=FFRI
      GO TO 89
   67   FFRI=FHR(YR,YI,R,CRMM1,DRMM1   )
      FFII=FHI(YR,YI,R,CRMM1)
      FFI(I)=FFII
      FFR(I)=FFRI
      ALPH(I)=1./(FFII*FFII+FFRI*FFRI)
      FRN=FHR(YR,-YI,R,CRMM1,DRMM1)
      FIN=FHI(YR,-YI,R,CRMM1)
      BET(NIRM)=1./(FRN*FRN+FIN*FIN)
      GAMM(NIRM)=FRN
      DELT(NIRM)=FIN
   89 CONTINUE
      DO 5 MS=MD,ND
      S=MS
      S2 = S+S
      MSND = ND-MS
      MSND3 = MSND+MSND+MSND
      NDMS=ND+MS
      NDMS3 = NDMS+NDMS+NDMS
```

```
      DO 5 MR = MS,ND
      R=MR
      NDMR = ND-MR
      NDMR3 = NDMR+NDMR+NDMR
      MRND= MR+ND
      MRND3 = MRND+MRND+MRND
      MRMS1 = MR-MS+1
      DO 5 M=1,LM
      I = MRND3+M
C     I = 3*(MR+ND)+M
C     THE I INDEX REQUIRES ONLY MR AND M
C    INDICES ARE IN THREE DIMENSIONS
C     NIRM= 3*(-MR+ND)+M
      NIRM = NDMR3+M
      BA=BET(NIRM)
      FRNEGIJ=GAMM(NIRM)
      FINEGIJ=DELT(NIRM)
      NIRM1=(NIRM-1)*LJ1
C     IRM= 3*(MR+ND)+M
      FFRI=FFR(I)
      FFII = FFI(I)
      AB=ALPH(I)
      IRM = I
        IRM1 = (IRM-1)*LJ1
      DO 5 N=1,LN
      MN=(LI*M+LJ+N)*LK
C
C     MNRS1=MN+MR-MS+1
      MNRS1= MN+MRMS1
      NJSN = MSND3+N
C     NJSN= 3*(-MS+ND)+N
C     NEGIJ=(NIRM-1)*LJ1 +NJSN
      NEGIJ = NIRM1+NJSN
      JSN = NDMS3+N
C     JSN= 3*(MS+ND)+N
```

```
      IJ= IRM1+JSN
C     IJ=(IRM-1)*LJ1 +JSN
      IF(MRMS1-LK) 6177,6177,6117
 6117 GR(IJ) = 0.
      GI(IJ) = 0
      GR(NEGIJ)=0.
      GI(NEGIJ)=0.
      GO TO 5
 6177 IF(M-N)2111,617,2111
  617 IF(MR-MS)  2111,2112,2111
 2112 GR(IJ) = 1.
      GI(IJ) = 0
      GR(NEGIJ)=1.
      GI(NEGIJ)=0.
      GO TO 5
 2111 CRMNRS= CR(MNRS1)
      CIMNRS= CI(MNRS1)
      DRMNRS= DR(MNRS1)
      DIMNRS= DI(MNRS1)
      GRIJ=GRF(YR,YI,S,CRMNRS,CIMNRS,DRMNRS)
      GIIJ=GIF(YR,YI,S,CRMNRS,CIMNRS,DIMNRS)
      TEMPA =(FFRI*GRIJ + FFII*GIIJ)*AB
      GI(IJ)= (FFRI*GIIJ-FFII*GRIJ)*AB
      GR(IJ) = TEMPA
 3111 GRNEGIJ=GRF(YR,-YI,S,CRMNRS,CIMNRS,DRMNRS)
      GINEGIJ=GIF(YR,-YI,S,CRMNRS,CIMNRS,DIMNRS)
      GI(NEGIJ)=-BETF(FRNEGIJ,GINEGIJ,FINEGIJ,GRNEGIJ,BA)
      GR(NEGIJ)=ALF(FRNEGIJ,GRNEGIJ,FINEGIJ,GINEGIJ,BA)
    5 CONTINUE
 8887 K=K
 9999 K=K
      M=6*ND+3
C
      M1 = M-1
      DO 200 KM = 1,M1
      NST = 3
      M2J = KM
      MKM=M-KM
      M2=MKM+1
      M21M = (M2-1)*M
      MM=(M2-1)*M+M2
C
```

```
      GRMM= GR(MM)
      GIMM= GI(MM)
      D =  GRMM*GRMM + GIMM*GIMM
      DI= 1./D
      IF(D-EPS)71,72,72
   72 MONEY=0
      DO 2200 I=1,MKM
      I1M= (I-1)*M
      IM = I1M+M2
      IM=(I-1)*M+M2
      GRIM = GR(IM)
      GIIM = GI(IM)
      GAM = GRIM*GRMM + GIMM*GIIM
      GA= DI*GAM
      DE=DI*(GIIM*GRMM-GRIM*GIMM)
C      FFI(I)=GAM/D
C     FFR(I) = (GIIM*GRMM - GRIM*GIMM)/D
C     DE= FFR(I)
C     GA = FFI(I)
C     DO 200 J = 1, MKM
      DO2200 J = 1, MKM
      IJ = I1M+J
C     IJ=(I-1)*M+J
      MJ= M21M+J
C     MJ=(M2-1)*M+J
      BE = GI(MJ)
      AL = GR(MJ)
      GR(IJ) = GR(IJ)-GA*AL+DE*BE
      GI(IJ) = GI(IJ) - GA*BE-DE*AL
 2200 CONTINUE
  200 CONTINUE
C     REPLACE BETA(IJ)
C
      PIIN = GI(1)
      PIRN = GR(1)
      M1 = M-1
C
      DO 103 K = 2,M1
```

```
      KK=(K-1)*M+K
      GRKK = GR(KK)
      GIKK= GI(KK)
      PIR = GRKK*PIRN-GIKK*PIIN
      PII = PIIN*GRKK + GIKK*PIRN
      PIRN=PIR
  103 PIIN=PII
C
      LJ=LJ-1
      RETURN
C     UNLIKELY EVENT     A(MM)**2    + B(MM)**2    TO SMALL
   71 NA=M-KM
C
      DO 73 LL=1,NA
          LUVM=(LL-1)*M+J
      IF(GR(LUVM) -DELTAX) 91,91,92
   91 IF(GI(LUVM)-DELTAX) 733,733,92
   73 CONTINUE
C
  733 WRITE(NP,1002)DELTAX,D,M2J
 1002 FORMAT(54H ****UNABLE TO FIND AN ALPHA OR BETA LARGER THAN DELTA /
     1 21H IN SUBROUTINE DELY       /
     2 10H DELTAX=    , E20.8,17H  OLD VALUE USED= E20.8,10H  COLUMN
     1  I5)
      IF(D)   72,999,72
   92 DO 77 J=1,NA
      NAJ=(NA-1)*M+J
      KJ=(LL-1)*M+J
      GR(NAJ) = GR(NAJ) + GR(KJ)
      GI(NAJ) = GI(NAJ) + GI(KJ)
      NANA = (NA-1)*M + NA
      D = GR(NANA)*GR(NANA)+GI(NANA)*GI(NANA)
   77 CONTINUE
C
 4007 FORMAT(21H ADJUSTMENT ON COLUMN   I5)
      WRITE(NP,4007) M
      GO TO  72
  999 WRITE(NP,1009)
 1009 FORMAT(1H050HNECESSARY TO ABORT DUE TO SINGULARITY                )
      STOP
      DELTAX = 0.0
      LJ=LJ-1
    4 WRITE(NP,1005)
      RETURN
      END
```

```
      SUBROUTINE UP(PC,ND,NSTART,NP)
      DIMENSION PC(18)
C     SUBROUTINE TO ESTABLISH CRITERION FOR CONVERGENCE
C         THE LAST THREE DETERMINANTS MUST BE MONOTONIC, CONVERGING
      EPSIL = .075
      EP2 = 2.*EPSIL
   51 ND1 = ND-1
      ND2 = ND-2
      PA   = PC(ND1)-PC(ND2)
       PD  = PC(ND)-PC(ND1)
      PA65=PA
      PA76=PD
      PA1 = ABS(PA)
      PD1=ABS(PD)
      IF(PD1- PA1)8,55,55
    8 K=K
      IF(PA) 1,2,2
    1 PA=-1.
      GO TO 3
    2 PA=1.
    3 IF(PD) 4,5,5
    4 PD=-1.
      GO TO 6
    5 PD=1.
    6 IF(PA+PD)7,55,7
    7 PD=PD1/PC(ND1)
      PA=PA1/PC(ND2)
      PA=ABS(PA)
      PD=ABS(PD)
      IF(PA-EPSIL) 17,17,55
   17 IF(PD-EPSIL) 18,18,55
   18 K=K
      RATIO=PA76/PA65
      IF(RATIO)55,55,19
   19 IF(RATIO-.8)54,55,55
   54 CALL PRE(ND,PC,NP)
      NSTART = 0
      RETURN
   55 CONTINUE
      NSTART = 1
      RETURN
      END
```

```
      SUBROUTINE PRE(ND,PC,NP)
      DIMENSION PC(18)
C        PREDICT THE CONVERGEED VALUES BASED ON THE LAST THREE
C     DETERMINANTS
      REAL K2K1,K2K3,MU,K1,K2,K3
      DELTA=.00001
      C1 = PC(ND-2)
      C2 = PC(ND-1)
      C3 = PC(ND)
      K1=ND-2
      K2 = ND-1
      K3 = ND
      K2K3= K2/K3
      K2K1=K2/K1
C   WRITTEN FOR THE INFINITE DETERMINANT  SUBROUTINE WITH ASYMPTOTIC
C     LIMITS.
      PC(2) = 0
      MU = (C1-C2)/(C2-C3)
      P=.00001
      FP = MU*(1.-K2K3**P)-K2K1**P+1
      A = ABS(FP)/FP
C   THIS GETS NEGATIVE OR POSITIVE ONE
      AM = 1
      AP = 1
      M20=1
      M100=100
      M1 = 1
   13 K=K
      DO 1 M = M20,M100,M1
      AP = K2K1*AP
      AM = K2K3*AM
      FP = MU-MU*AM -AP+1
```

```
      B = ABS(FP)/FP
      IF(A+B) 1,10,1
C    ZERO IS CHANG E OF SIGN
    1 CONTINUE
      PC(2) = 3
      RETURN
   11 K=K
   10 M50 = M+50
   18 PN=M
   19 FA = MU*(1,-K2K3**PN)-K2K1**PN+1
      DO 15 L = 1,50
      PNK2K3 = K2K3**PN
      PNK2K1= K2K1**PN
      PN1 = PN +(MU*(1,-PNK2K3)-PNK2K1 +1)/(PNK2K1*ALOG(K2K1)+MU*PNK2K3*
     1ALOG(K2K3))
      PN1FP=(PN1-PN)/PN1
      PN = PN1
      B = MU*(1.-K2K3**PN1)-K2K1**PN1 +1
      PC(3)=PN1
      IF(ABS(PN1FP)-DELTA)16,16,15
   15 FA = B
   14 PC(2)=2
      RETURN
   16 PC(1) =(C2*K2K1**PN1-C1)/( K2K1**PN1-1. )
      PC(2) = 0
      RETURN
      END
```

```
      SUBROUTINE S5A(OUT,AK,IN,K,MORDER,F,PB,PA,PC,IERR)
      INTEGER R21
      INTEGER R1
      INTEGER R,R2,R22
      DIMENSION BIGA(13,13),A(13,13),B(13,13),C(13,13)
      DIMENSION BIGAP(13,13,6),BP(13,13,6),CP(13,13,6),AP(13,13,6)
      DIMENSION F(13),AC(13),AK(13)
      DIMENSION AR(13),AS(13),P(13),Q(13),PC(13),PA(13),PB(13)
C
C
C    THE POLYNOMIAL COEFFICIENTS OF THE HIGHER ORDER SYSTEMS.
C     THE 9 THE AND 12TH ORDER SYSTEMS HAVE ASSOCIATED
C      WITH THEM CHARACTERISTIC POLYNOMIALS OF DEGREE 9 AND 12
C     RESPECTIVELY.  THE COEFFICIENTS OF THESE POLYNOMIALS ARE NEEDED TO
C      DEFINE THE SHARACTERISTIC EQUATIONS OF THE SYSTEMS8 AND
C      ARE COMPUTED AS FOLLOWS,,.
C    THE COEFFICIENTS DESIRED ARE K1,,,,K9
C
      NP=6
      DO 45 M=1,K
      M2= 2*M
      M21=M2-1
      PBM2 = PB(M2)
      COSB2M = COS(PBM2)
      PAM2 = PA(M2)
      EA2M = EXP(-PAM2)
      F2M = F(M2)
      E2A2M = EXP(-2.*PAM2)
      F2M1=F(M21)
      AR(M) = 2.*EA2M*(F2M1*SIN(PBM2)+F2M*COSB2M)
      AS(M) = -2.*F2M*E2A2M
      P(M) = -2.*EA2M*COSB2M
      Q(M) = E2A2M
      WRITE(NP,2000)P(M),Q(M),AR(M),AS(M)
   45 CONTINUE
      K1=K+1
      DO 10 M=K1,MORDER
      M2 = 2*M
      M21 = M2-1
      PCM2 = -PC(M2)
      EC2M = EXP(PCM2)
      PCM21 =-PC(M21)
      EC2M1 = EXP(PCM21)
      F2M = F(M2)
```

```
      F2M1 = F(M21)
      AR(M) = 2.*(F2M1*EC2M1+F2M*EC2M)
      CC9 =  PCM21+PCM2
C     CC9 = -PCM21-PCM2
      ECC9 = EXP(CC9)
      AS(M) = -2.*ECC9*(F2M1+F2M)
        P(M) = -(EC2M1+EC2M)
      Q(M) = ECC9
      WRITE(NP,20000)M
20000 FORMAT(5H M =  ,I5/)
      WRITE(NP,2000)P(M),Q(M),AR(M),AS(M)
 2000 FORMAT(1H0,/15X,4HP(M),15X,4HQ(M),15X,4HR(M),15X,4HS(M),/
     1 /3X,4E19.6)
   10 CONTINUE
C    NOW WE DEFINE A SUCCESSION OF TRIANGULAR ARRAYS,
C      A(I,2R),   ...  I= 1,2,...,2R          R= 1,2,3,4
C       ACCORDING TO    A1,2) = P(1)
C                       A(2,2) = Q(1)
C                       A(I,2R) = BIGA(I,R),*Q(R) + B(I,R)*P(R)  + C(I,R)
C     WHERE BIGA(I,R) = A(I-2,2R-2)
C              B(I,R)= A(I-1,2R-2)
C              C(I,R) = A(I,2R-2)
C    WHILE IN COMPUTATION TAKE,..
C      A(I,2R) = 0 FOR  I NEGATIVE
C       AND TAKE   A(0,2R)=1
C   IN ORDER TO HAVE  A(0,2R)  WE SHALL ADD 1 TO ALL I INDICES TO AVOID ZERO
      NORDER=MORDER*2 +1
      NL1= NORDER +1
      DO 25 I = 1,NL1
      DO 25 R  = 1,NL1
   25 A(I,R) = 0.
C     ADJUSTED TO AVOID ZERO INDICES   A,1,2) AND A(2,2)
      A(2,2) = Q(1)
      A(1,2) = P(1)
      R=MORDER
      MR2 = 2.*MORDER
      MR21= MR2+1
      DO 5 I = 1,MR2
      KI=I
      KI1 = KI-1
      KI2= KI-2
      DO 5 R = 1,MORDER
      R2 = 2*R
C     R22 = 2*R-2+1
C     ADJUSTED TO AVOID ZERO INDICES
      R22=R2-1
      IF(KI2) 1,2,3
    1 A(KI2,R22) = 0
```

```
      GO TO 7
    2 A(KI2,R22) = 1.
    7 IF(KI1) 4,6,3
    4 A(KI1,R22) = 0
      GO TO 3
    6 A(KI1,R22) = 1.
    3 BIGA(I,R) = A(KI2,R22)
      B(I,R) = A(KI1,R22)
      C(I,R) = A(KI,R22)
      R21 = R2+1
      A(KI,R21) = BIGA(I,R)*Q(R) + B(I,R)*P(R) + C(I,R)
    5 CONTINUE
      A(3,3) = 0.0
      A(2,3) = 0.0
      A(1,3) = 0
      DO 21 R = 4,R2,2
      R1 = R+1
      DO 21 I = 1,R2
      A(I,R) = A(I,R1)
   21 A(I,R1) = 0
C     WRITE(NP,300)(((I,J,A(I,J)),I=1,13),J=1,13)
  300 FORMAT(1X,2HA(,I5,1H,I5,2H)=E20,6)
C    NEXT, DEFINE THE SUCCESSION OF MODIFIED TRAINGULAR ARRAYS
C     AP(I,2R,M)    I= 1,2,...,2R   R=1,2,...,MORDER-1 M = 1,2,...MORDER
C     NOTE THE SUCCESSION HALTS AT R=MORDER-1
C     THE ARRAYS OF AP  ARE FORMED FOR A GIVEN M EXACTLY AS THE A(I,R)
C   EXCEPT BEFORE STARTING REPLACE THE VALUE OF P(M) BY P(4) AND Q(M) BY Q(4)
      NORDER=13
      DO 30 I = 1, NORDER
      DO 30 R = 1,NORDER
      DO 30 M = 1,MORDER
   30 AP(I,R,M) = 0.0
      DO 58 M = 1,MORDER
      TMPP=P(M)
      TMPQ=Q(M)
      Q(M)=Q(MORDER)
      P(M) = P(MORDER)
      AP(2,2,M) = Q(1)
      AP(1,2,M) = P(1)
      ML1 = MORDER -1
      MR2 = 2*MORDER
CEXCEPT BEFORE STARTING REPLACE    P(M) BY P(MORDER) AND SAME FOR Q(M)
      MR21 = MR2+1
      DO 50 I = 1,MR2
      KI=I
      KI1=KI-1
      KI2=KI-2
```

```
C     ADJUSTED TO AVOID ZERO INDICES
      DO 50 R = 1,ML1
      R2 = 2*R
      R22 = R2-1
      IF(KI2)  51,52,53
   51 AP(KI2,R22,M) = 0
      GO TO 57
   52 AP(KI2,R22,M) = 1
   57 IF(KI1) 54,56,53
   54 AP(KI1,R22,M) = 0
      GO TO 53
   56 AP(KI1,R22,M) = 1.
   53 BIGAP(I,R,M) = AP(KI2,R22,M)
      BP(I,R,M) = AP(KI1,R22,M)
      CP(I,R,M) = AP( KI,R22,M)
      R21 = R2+1
      AP(I,R21,M) = BIGAP(I,R,M)*Q(R)+ BP(I,R,M)*P(R) + CP(I,R,M)
   50 CONTINUE
      Q(M) = TMPQ
      P(M) = TMPP
   58 CONTINUE
C     NOW BACK SHIFT
      R= MORDER-1
      R2 = 2*R
      DO 121 I = 1,NORDER
      DO 121 M=1,MORDER
      DO 121 R = 4,R2,2
      R1 = R+1
      AP(I,R,M) = AP(I,R1,M)
  121 AP(I,R1,M) = 0
        DO 122 M=1,MORDER
      AP(1,3,M)=0
      AP(2,3,M)=0
  122 CONTINUE
  400 FORMAT(2X,3HAP(,I5,1H,I5,1H,I5,3H)= E20.6)
C     IT IS THEN POSSIBLE TO COMPUTE THE COEFFICIENTS C(S)  S=1,2,.,.8
C     ACCORDING TO ...
      CONST1 = 0
          IF(MORDER-4) 199,200,201
  201 IF(MORDER-6)199,202,199
  200 DO 100 M = 1,MORDER
  100 CONST1= CONST1+AR(M)
      AC(1) = A(1,8) + CONST1
      CONST1 = 0
```

```
      DO 110 M = 1,4
  110 CONST1 = CONST1 + AR(M) * AP(1,6,M) + AS(M)
      AC(2) = A(2,8) + CONST1
      DO 1200 MS=3,7
      CONST1 = 0
      DO 120 M=1,4
       MS1 = MS-1
      MS2 = MS-2
      CONST1 = CONST1 + AR(M)*AP(MS1,6,M)+AS(M)*AP(MS2,6,M)
  120 CONTINUE
      AC(MS)   = A(MS,8) + CONST1
 1200 CONTINUE
C     WRITE(NP,500)((I,AC(I)),I=1,8)
  500 FORMAT(1H0, 1X,3HAC(,I5, 3H)= ,E20.6)
      CONST1 = 0
      DO 130 M = 1,4
      CONST1 = CONST1 + AS(M)*AP(6,6,M)
  130 CONTINUE
      AC(8) = A(8,8) + CONST1
      AK1 = EXP(-PC(9))
      AK0 = 2.*F(9)*AK1
      AK(1) = AC(1) -AK1+AK0
      DO 150 MS= 2,8
      MS1 = MS-1
      AK(MS) =AC(MS) -AK1*AC(MS1) + AK0*A(MS1,8)
  150 CONTINUE
      AK(9) = AK0 * A(8,8) - AK1 * AC(8)
  550 FORMAT(3X,//,3X4HAK0=,E20.6,5X,4HAK1=,E20.6/)
  650 FORMAT(1H ,1X,/,2X,4HAK( ,I5,3H)= E20.6)
       WRITE(NP,650)((I,AK(I)),I=1,9)
      IERR=0
      RETURN
  202 CONST1 = 0
      DO 205 M=1,MORDER
      CONST1 = CONST1 + AR(M)
  205 CONTINUE
      AK(1) = A(1,12) + CONST1
      CONST1 = 0
      DO 210 M = 1, MORDER
      CONST1 = CONST1 + AR(M)*AP(1,10,M)+AS(M)
```

```
210 CONTINUE
    AK(2) = A(2,12) + CONST1
    DO 215 MS=3,11
    MS1 = MS-1
    MS2 = MS-2
    CONST1 = 0
    DO 212 M= 1,MORDER
    CONST1 = CONST1 + AR(M)*AP(MS1,10,M)+AS(M)*AP(MS2,10,M)
212 CONTINUE
    AK(MS)   = A(MS,12) + CONST1
215 CONTINUE
    CONST1 = 0
    DO 220 M=1,MORDER
    CONST1 = CONST1 + AS(M) * AP(10,10,M)
220 CONTINUE
    AK(12) = A(12,12) + CONST1
    WRITE(NP,650)((I,AK(I)),I=1,12)
    IERR= 0
    RETURN
199 IERR=MORDER
    RETURN
    END
```

```
      SUBROUTINE SIMQ(A,N,Y)
      DIMENSION A(1),Y(1),ICHG(15),SV(15)
C     SUBROUTINE FOR SOLVING SIMULTANEOUS EQUATIONS USING KROUTS METHOD
      DO 1000 I=1,N
      II = (I-1)*N+I
      SV(I) = A(II)
C     SV(I)=A(I,I)
      IF(SV(I))5,46,5
    5 Y(I)=Y(I)/SV(I)
      DO 1000 J=1,N
       IJ = (I-1)*N+J
      A(IJ) = A(IJ)/SV(I)
 1000 CONTINUE
C1000 A(I,J)=A(I,J)/SV(I)
      DO 101 K=1,N
      KK = (K-1)*N+K
C     AMX=ABS(A(K,K))
      AMX=ABS(A(KK))
      IMX=K
      DO 15 I=K,N
      IK = (I -1)*N+K
      IF(ABS(A(IK))-AMX) 15,15,14
C     IF (ABS(A(I,K))-AMX) 15,15,14
C  14 AMX=ABS(A(I,K))
   14 AMX = ABS(A(IK) )
      IMX=I
   15 CONTINUE
      IF(AMX)27,46,27
   46 N=-N
      RETURN
   27 IF(IMX-K)8,9,8
    8 DO 22 J=1,N
      KJ = (K-1)*N+J
C     TEMP=A(K,J)
      TEMP=A(KJ)
      IMXJ =(IMX-1)*N+J
      A(KJ) = A(IMXJ)
C     A(K,J)=A(IMX,J)
```

```
C  22 A(IMX,J)=TEMP
   22 A(IMXJ )=TEMP
      ICHG(K)=IMX
      TEMP=Y(K)
      Y(K)= Y(IMX)
      Y(IMX)= TEMP
      GO TO 10
    9 ICHG(K)=K
C  10 A(K,K)=1,/A(K,K)
   10 A(KK) = 1./A(KK)
      DO 33 J=1,N
      IF (J-K) 6,33,6
    6 KJ = (K-1)*N+J
      A(KJ) = A(KJ)*A(KK)
C   6 A(K,J)=(A(K,J))*(A(K,K))
   33 CONTINUE
      Y(K) = Y(K)*A(KK)
C     Y(K) = Y(K)*A(K,K)
      DO 44 I=1,N
      IK = (I-1)*N+K
    7 DO 45 J=1,N
      IF(I-K)17,44,17
   17 IF(K-J)18,45,18
   18 IJ = (I-1)*N+J
      KJ = (K-1)*N+J
      A(IJ) = A(IJ) -(A(IK)*A(KJ))
C  18 A(I,J)=A(I,J)-(A(I,K))*(A(K,J))
   45 CONTINUE
C     Y(I) = Y(I)-A(I,K)*Y(K)
      Y(I) = Y(I)-A(IK)*Y(K)
   44 CONTINUE
      DO 99 I=1,N
      IF(I-K)26,99,26
   26 IK = (I-1)*N+K
      A(IK) = -A(IK)*A(KK)
```

```
C   26 A(I,K)=-(A(I,K))*(A(K,K))
    99 CONTINUE
   101 CONTINUE
       DO 70 K=1,N
       L=N+1-K
       KI=ICHG(L)
       IF (L-KI) 68,70,68
    68 DO 69 I=1,N
       IL = (I-1)*N+L
C      TEMP = A(I,L)
       TEMP = A(IL)
       IKI =(I-1)*N+KI
C      A(I,L) = A(I,KI)
       A(IL) = A(IKI)
       A(IKI) = TEMP
C      A(I,KI) = TEMP
    69 CONTINUE
    70 CONTINUE
       DO 1001 I=1,N
       DO 1001 J=1,N
       IJ = (I-1)*N+J
C1001 A(I,J)=A(I,J)/SV(J)
       A(IJ) = A(IJ)/SV(J)
  1001 CONTINUE
       RETURN
       END
```

```
      SUBROUTINE S6(OUT,SS,IN,K,KHAT,NP)
      DIMENSION K(12),KHAT(12),S(12),SS(12)
C     SUBROUTINE FOR DETERMINING THE COMMON POLYNOMIAL COEFFICIENTS
      REAL  KHAT,K
      REAL         KHAT1,KHAT2,KHAT3,KHAT4,KHAT5,KHAT6,KHAT7,KHAT8
     1, KHAT9,KHAT10,KHAT11,KHAT12,
     1 K1,K2,K3,K4,K5,K6,K7,K8,K9,K10,K11,MU1,MU1SQ,MUMU,
     2 L4,L3,L5,MU3,MU2,MJU,MJUU,K12,MU2SQ
      NP=6
      KHAT1= KHAT(1)
      KHAT2=KHAT(2)
      KHAT3 = KHAT(3)
      KHAT4 = KHAT(4)
      KHAT5=KHAT(5)
      KHAT6=KHAT(6)
      KHAT7=KHAT(7)
      KHAT8=KHAT(8)
      KHAT9=KHAT(9)
      KHAT10=KHAT(10)
      KHAT11= KHAT(11)
      KHAT12=KHAT(12)
      K1= K(1)
      K2= K(2)
      K3 = K(3)
      K4 = K(4)
      K5 = K(5)
       K6 = K(6)
      K7 = K(7)
      K8=K(8)
      K9 = K(9)
      MU1 = K9 / KHAT12
      MU2 = (K8-KHAT11*MU1)/KHAT12
      MU2SQ=MU2*MU2
      MU3 = (K7-KHAT10*MU1-KHAT11*MU2)/KHAT12
        MU1SQ = MU1 * MU1
      MUMU = MU2/MU1SQ
      MUUU = MU2SQ/MU1SQ - MU3/MU1
      H11 = KHAT1 - K1
      H10 = KHAT2 - K2
       L4 = H10 -K1*H11
```

```
      A12           = K7*H11 + K8 + K5/MU1 - K6*MUMU - KHAT8
      L5 = H11
      L3 = KHAT3 - K3 -K1*H10 + H11*(K1*K1 - K2)
      A13           = K7*(H10-K1*H11) + K8*H11 + K9 + K6/MU1 - KHAT9
      A21         =  K5 + K2/MU1 - K3*MUMU      +K4*MUUU/MU1 -KHAT5
      A22           = K5*H11 + K6 + K3/MU1 - K4*MUMU     - KHAT6
      A23 = K5*(H10-K1*H11)+K6*H11+K7+K4/MU1-KHAT7
      A31           = K3 + 1/MU1 - K1*MUMU+(K2*MUUU)/MU1 - KHAT3
      A32           = K3*H11 + K4 + K1/MU1 - K2*MUMU- KHAT4
      A33   =        K3*(H10-K1*H11) + K4*H11 + K5 + K2/MU1 - KHAT5
      B1     = KHAT10  -K7*L3 - K8 * L4 - K9*L5
        B2 = KHAT8-K8-K5*L3-K6*L4-K7*L5
      B3   =   KHAT6 - K6 - K3*L3 - K4*L4 - K5*L5
C     FROM CRAMERS RULE...
C     D1 = DELTA1/D,    D2 = DELTA2/D,    D3 = DELTA3/D
      A2233 = A22*A33 - A32 * A23
      A1233 = A12*A33 - A32*A13
      A1223 = A12*A23 - A22 *A13
      D = A11*A2233 - A21*A1233 + A31*A1223
       IF(D) 15,5,15
 1002 FORMAT(1H1,46H***DENOMINATOR D FOR CRAMERS RULE IS ZERO EOJ
    5 WRITE(NP,1002)
      STOP
   15 CONTINUE
      DELTA1 = B1*A2233 - B2*A1233 + B3 *A1223
      A2133 = A21*A33 - A31*A23
      A1133 = A11*A33 - A31*A13
      A1123 = A11*A23 - A21*A13
      DELTA2 = -B1*A2133 + B2 * A1133 - B3 *A1123
      A2132 = A21*A32 - A31*A22
      A1132 = A11 * A32 - A31*A12
      A1122 = A11*A22 - A21*A12
      DELTA3 = B1 *A2132  - B2*A1132 + B3 *A1122
      D1 = DELTA1/D
      D2 = DELTA2/D
      D3 = DELTA3/D
C   THE  COEFFICIENTS  THEN ARE GIVEN BY ...
      SS(1)=K1-D3
      SS(2) = K2-D2-D3*SS(1)
      SS(3) = K3-D1-D2*SS(1)-D3*SS(2)
      SS(4)=K4 - D1*SS(1) - D2 *SS(2)-D3*SS(3)
      SS(5) = K5 - D1*SS(2) - D2*SS(3)-D3 *SS(4)
      SS(6) = K6 -D1*SS(3)-D2*SS(4)-D3*SS(5)
      RETURN
      END
```

```
*DECK,TEA
      SUBROUTINE TEA(NO,NP,ROOTR,ROOTI,XI,ETA)
      DIMENSION ROOTR(1),ROOTI(2),XI(1),ETA(1)
C     SUBROUTINE FOR THE GENERATION FO THE PSI AND ETA VECTORS FOR EACH
C     SET OF POLYNOMIAL ROOTS
      PI = 3.1415926
      PITWO = PI/2.
      PI2 = 2.*PI
      PIONE = 1./PI
      PINV=1./PI2
      DO 15 I=1,NO
      SQS = ROOTR(I)*ROOTR(I)+ROOTI(I)*ROOTI(I)
      XI(I) = -PINV*ALOG(SQS)
      IF(ROOTR(I))14,50,14
   14 GAMMA=ATAN(ROOTI(I)/ROOTR(I))
      ETA(I) = GAMMA
      IF(ROOTR(I)) 10,50,15
   50 IF(ROOTI(I)) 51,52,53
   51 ETA(I) =-PITWO
      GO TO 15
   52 K=K
   53 ETA(I) = PITWO
      GO TO 15
   10 IF(ROOTI(I)) 30,40,35
   40 ETA(I) = 3.1415926
      GO TO 15
   35 ETA(I) = ETA(I) + PI
      GO TO 15
   30 ETA(I)=ETA(I)-PI
   15 ETA(I) = -PIONE*ETA(I)
      RETURN
      END
*END
```

```
      SUBROUTINE PAT(A,B,X,Z,X9,X12,X6,D,E,E9,E12,E6,G,MU,NF,RR9,RI9,
     1 RR12,RI12,RR6,RI6,ND9,ND12,Y9,Y12,PIROLD,PIRNEW,PIIOLD,PIINEW,
     2                  S,T,NDI9,NDI12,YI9,YI12)
      DIMENSION PIIOLD(1),PIINEW(1),     S(8,14),      T(11,14),NDI9(1)
C    SUBROUTINE FOR THE GENERATION OF THE SUMMARY TABLES FOR THE NASA FLUTTER
      DIMENSION YI9(1),YI12(1)
      DIMENSION PIROLD(1),PIRNEW(1),A(8,14),B(11,14),X9(1),X12(1),X6(1)
       DIMENSION D(1),E9(1),E12(1),E6(1),RR9(1),RI9(1),RR12(1),RI12(1)
      DIMENSION RR6(1),RI6(1),Y9(1),E(1),X(1),Z(1),ND9(1),ND12(1),Y12(1)
      REAL MU
      GAMMA=G
      ND=6
      NP=6
      WRITE(NP,50)GAMMA,MU,NF,((X(J),D(J),Y9(J),YI9(J)),J=1,9),((Z(J),
     1 E(J),Y12(J),YI12(J)),J=1,12)
      WRITE(NP,54)
      DO 25 J = 1,8
      L = ND9(J)
      WRITE(ND,51) ((Y9(J),A(J,L+1),A(J,L-2),A(J,L-1),A(J,L),ND9(J)),
     1 PIRNEW(J))
      WRITE(ND,551)((YI9(J),S(J,L+1),S(J,L-2),S(J,L-1),S(J,L),ND9(J)),
     1 PIINEW(J) )
   25 CONTINUE
   54 FORMAT(55X,24HNINTH ORDER SYSTEM            /)
   55 FORMAT(55X,24HTWELFTH ORDER SYSTEM           /)
      WRITE(NP,55)
      DO 20 M=1,11
      L=ND12(M)
      JM8= M
      WRITE(ND,51)Y12(JM8),B(JM8,L+1),B(JM8,L-2),B(JM8,L-1),
     1 B(JM8,L),ND12(M) ,PIROLD(JM8)
      WRITE(ND,551)YI12(JM8),T(JM8,L+1),T(JM8,L-2),T(JM8,L-1),
     1T(JM8,L),ND12(M) ,PIIOLD(JM8)
   20 CONTINUE
      WRITE(NP,53)((X9(J),E9(J),RR9(J),RI9(J)),J=1,9),((X12(J),E12(J),
     1RR12(J),RI12(J)),J=1,12),((X6(J),E6(J),RR6(J),RI6(J)),J=1,6)
```

```
 50 FORMAT(1H1,38X,55HSTABILITY OF DYNAMIC SYSTEMS WITH PERIODIC PARAM
   1METERS                    /
   2   /49X,40HROTOR WITH FLAP AND LEAD-LAG HINGES
   3/55X,7HGAMMA = G12.4/55X,7H    MU = G12.4/55X,7H    NF = I5/
   452X,40HSINGULARITIES AND EVALUATION POINTS                 /
   5 30X, *H            XI                        ETA                 YR
   2                  YI*                               /             55X*NI
   6NTH ORDER SYSTEM* /9(30X,4E20.8/)/55X,*TWELFTH ORDER SYSTEM*
   4/12(30X,4E20,8/)/60X,18HDETERMINANT VALUES
   5/15X ,1HY,18X,7HD. EST,14X,7HDELTA 1,14X,7HDELTA 2
   6 14X,7HDELTA 3,10X,6HND MAX  10H      N              /)
 51 FORMAT(1X,*REAL*5E20,8,I12,F10.5)
551 FORMAT(1X,*IMAG*5E20,8,2X,I10,F10.5)
 52 FORMAT(1X,5E20.8,2X,I15,F10.5)
 53 FORMAT(*1*/55X,21HCHARACTERISTIC VALUES//55X,24HNINTH ORDER SYSTEM
   2               //19X,2HXI,19X,3HETA,9X,18HREAL PART OF ROOT   ,3X,
   4 19HIMAG. PART OF ROOT          /, 9(10X,4E20.8/),
   3/,55X,24HTWELFTH ORDER SYSTEM
   4//19X,2HXI,19X,3HETA9X,18HREAL PART OF ROOT ,3X,
   519HIMAG. PART OF ROOT               /12(10X,4E20.8/),
   6/55X,24HSIXTH ORDER SYSTEM
   7//19X,2HXI,19X,3HETA9X,18HREAL PART OF ROOT ,3X,
   819HIMAG. PART OF ROOT               /6(10X,4E20.8/),/1H1)
    RETURN
    END
*END
```

APPENDIX C

Method for Extraction of the Common Polynomial Factor From the Two Higher-Degree Polynomials

Given the two polynomials

$$N(z) = z^9 + k_1 z^8 + k_2 z^7 + \ldots + k_8 z + k_9$$

$$T(z) = z^{12} + \hat{k}_1 z^{11} + \hat{k}_2 z^{10} + \ldots + \hat{k}_{11} z + \hat{k}_{12}$$

where it is known a priori that N and T both contain a sixth-degree factor S,

$$S(z) = z^6 + \sigma_1 z^5 + \sigma_2 z^4 + \ldots + \sigma_5 z + \sigma_6,$$

it is required to determine $\sigma_1, \sigma_2, .., \sigma_6$ in terms of the coefficients k_n and $\hat{k}_n$.

Note, first, that the ratio of T to N must reduce to the ratio of a sixth-degree polynomial to a cubic, the common sextic factor cancelling off:

$$\frac{T(z)}{N(z)} = \frac{z^6 + \lambda_1 z^5 + \lambda_2 z^4 + \ldots + \lambda_5 z + \lambda_6}{z^3 + \mu_1 z^2 + \mu_2 z + \mu_3} \tag{C-1}$$

Now, a set of nine linear algebraic equations can be formed, the solution of which gives the values of the λ's and μ's. This set is obtained by multiplying Eq. (C-1) through by $N(z)[z^3 + \mu_1 z^2 + \mu_2 z + \mu_3]$ and grouping the coefficients of like powers of z. There is considerable redundancy, with a total of fourteen equations in the nine unknowns. A convenient set of nine are (with the associated power of z from which each was obtained indicated on the left)

z^{14}: $$\hat{k}_1 + \mu_1 = k_1 + \lambda_1$$

z^{13}: $$\hat{k}_2 + \hat{k}_1\mu_1 + \mu_2 = k_2 + k_1\lambda_1 + \lambda_2$$

z^{12}: $$\hat{k}_3 + \hat{k}_2\mu_1 + \hat{k}_1\mu_2 + \mu_3 = k_3 + k_2\lambda_1 + k_1\lambda_2 + \lambda_3$$

z^5 : $$\hat{k}_{10} + \hat{k}_9\mu_1 + \hat{k}_8\mu_2 + \hat{k}_7\mu_3 = k_9\lambda_1 + k_8\lambda_2 + k_7\lambda_3 + k_6\lambda_4 + k_5\lambda_5 + k_4\lambda_6$$

z^7 : $$\hat{k}_8 + \hat{k}_7\mu_1 + \hat{k}_6\mu_2 + \hat{k}_5\mu_3 = k_8 + k_7\lambda_1 + k_6\lambda_2 + k_5\lambda_3 + k_4\lambda_4 + k_3\lambda_5 + k_2\lambda_6$$

z^9 : $$\hat{k}_6 + \hat{k}_5\mu_1 + \hat{k}_4\mu_2 + \hat{k}_3\mu_3 = k_6 + k_5\lambda_1 + k_4\lambda_2 + k_3\lambda_3 + k_2\lambda_4 + k_1\lambda_5 + \lambda_6$$

z^0 : $$\hat{k}_{12}\mu_3 = k_9\lambda_6$$

z^1 : $$\hat{k}_{11}\mu_3 + \hat{k}_{12}\mu_2 = k_8\lambda_6 + k_9\lambda_5$$

z^2 : $$\hat{k}_{10}\mu_3 + \hat{k}_{11}\mu_2 + \hat{k}_{12}\mu_1 = k_7\lambda_6 + k_8\lambda_5 + k_9\lambda_4$$

If the first three and last three of these equations are properly combined and substituted in the fourth, fifth and sixth equations, then a set of three equations for the coefficients μ_1, μ_2 and μ_3 is obtained:

$$\sum_{j=1}^{3} \alpha_{ij}\mu_{4-j} = \beta_i \qquad i = 1,2,3 \quad ; \qquad \text{(C-2)}$$

where

$$\alpha_{11} = k_7 + \frac{k_4}{\xi_1} - k_5 \frac{\xi_2}{\xi_1^2} + \frac{k_6}{\xi_1}\left[\left(\frac{\xi_2}{\xi_1}\right)^2 - \frac{\xi_3}{\xi_1}\right] - \hat{k}_7$$

$$\alpha_{12} = k_7 h_{11} + k_8 + \frac{k_5}{\xi_1} - k_6 \frac{\xi_2}{\xi_1^2} - \hat{k}_8$$

$$\alpha_{13} = k_7[h_{10} - k_1 h_{11}] + k_8 h_{11} + k_9 + \frac{k_6}{\xi_1} - \hat{k}_9$$

$$\alpha_{21} = k_5 + \frac{k_2}{\xi_1} - k_3 \frac{\xi_2}{\xi_1^2} + \frac{k_4}{\xi_1}\left[\left(\frac{\xi_2}{\xi_1}\right)^2 - \frac{\xi_3}{\xi_1}\right] - \hat{k}_5$$

$$\alpha_{22} = k_5 h_{11} + k_6 + \frac{k_3}{\xi_1} - k_4 \frac{\xi_2}{\xi_1^2} - \hat{k}_6$$

$$\alpha_{23} = k_5[h_{10} - k_1 h_{11}] + k_6 h_{11} + k_7 + \frac{k_4}{\xi_1} - \hat{k}_7$$

$$\alpha_{31} = k_3 + \frac{1}{\xi_1} - k_1 \frac{\xi_2}{\xi_1^2} + \frac{k_2}{\xi_1}\left[\left(\frac{\xi_2}{\xi_1}\right)^2 - \frac{\xi_3}{\xi_1}\right] - \hat{k}_3$$

$$\alpha_{32} = k_3 h_{11} + k_4 + \frac{k_1}{\xi_1} - k_2 \frac{\xi_2}{\xi_1^2} - \hat{k}_4$$

$$\alpha_{33} = k_3[h_{10} - k_1 h_{11}] + k_4 h_{11} + k_5 + \frac{k_2}{\xi_1} - \hat{k}_5$$

$$\beta_1 = \hat{k}_{10} - k_7\ell_3 - k_8\ell_4 - k_9\ell_5$$

$$\beta_2 = \hat{k}_8 - k_8 - k_5\ell_3 - k_6\ell_4 - k_7\ell_5$$

$$\beta_3 = \hat{k}_6 - k_6 - k_3\ell_3 - k_4\ell_4 - k_5\ell_5$$

while

$$h_{11} = \hat{k}_1 - k_1$$

$$h_{10} = \hat{k}_2 - k_2$$

$$\ell_3 = \hat{k}_3 - k_3 - k_1(\hat{k}_2 - k_2) + (\hat{k}_1 - k_1)(k_1{}^2 - k_2)$$

$$\ell_4 = \hat{k}_2 - k_2 - k_1(\hat{k}_1 - k_1)$$

$$\ell_5 = \hat{k}_1 - k_1$$

and

$$\xi_1 = \frac{k_9}{\hat{k}_{12}}$$

$$\xi_2 = \frac{1}{\hat{k}_{12}} (k_8 - \hat{k}_{11}\xi_1)$$

$$\xi_3 = \frac{1}{\hat{k}_{12}} \left[k_7 - \hat{k}_{10}\xi_1 - \frac{\hat{k}_{11}}{\hat{k}_{10}} \xi_2\right]$$

Eqs. (C-2) are readily solved for μ_1, μ_2 and μ_3, using Cramer's rule. The coefficients of the sextic now follow immediately, since

$$S(z) = \frac{N(z)}{z^3 + \mu_1 z^2 + \mu_2 z + \mu_3}$$

from which

$$\sigma_1 = k_1 - \mu_1$$

$$\sigma_2 = k_2 - \mu_2 - \mu_1\sigma_1$$

$$\sigma_3 = k_3 - \mu_3 - \mu_2\sigma_1 - \mu_1\sigma_2$$

$$\sigma_4 = k_4 - \mu_3\sigma_1 - \mu_2\sigma_2 - \mu_1\sigma_3$$

$$\sigma_5 = k_5 - \mu_3\sigma_2 - \mu_2\sigma_3 - \mu_1\sigma_4$$

$$\sigma_6 = k_6 - \mu_3\sigma_3 - \mu_2\sigma_4 - \mu_1\sigma_5$$

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