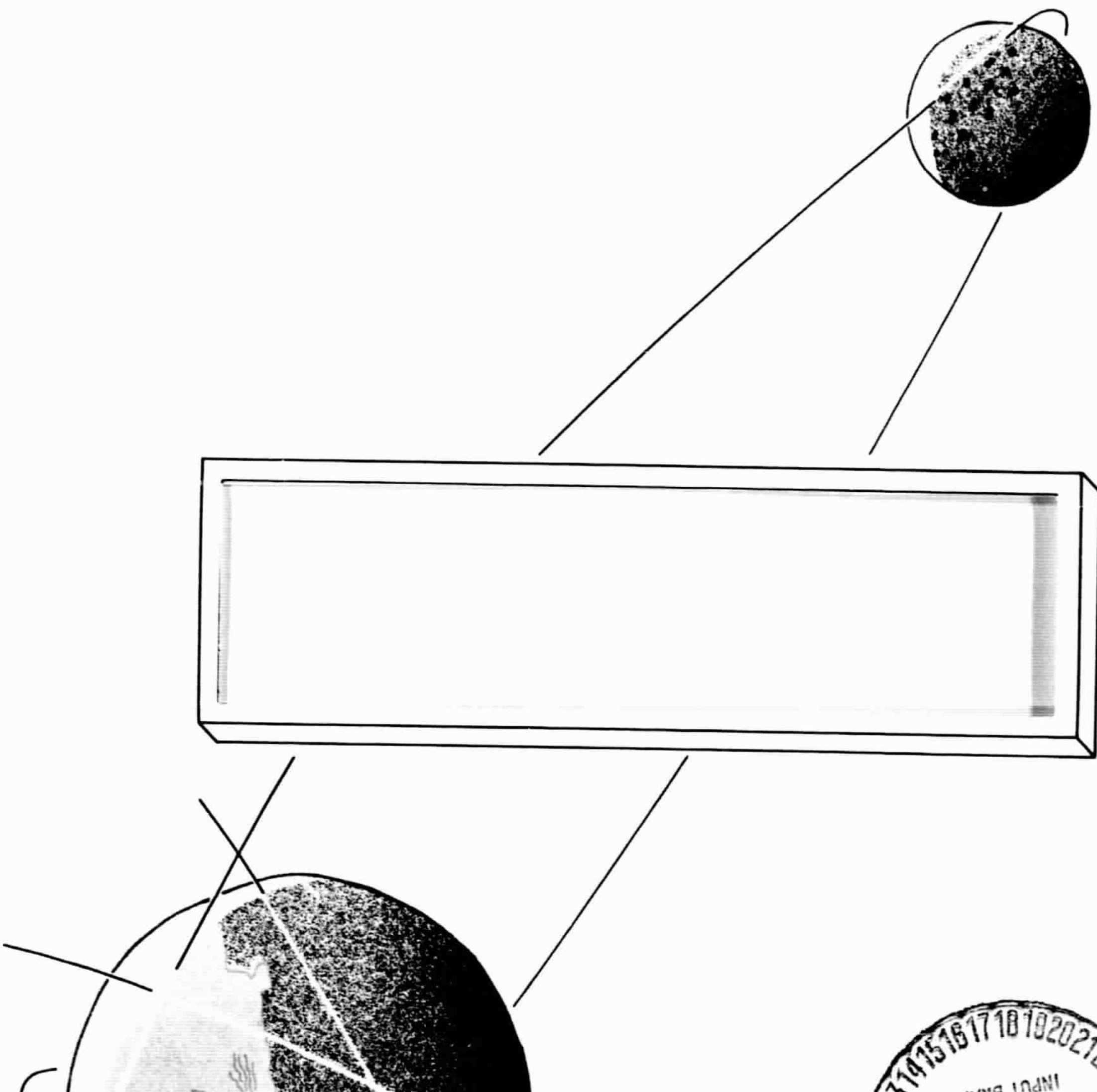

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ON THE SYNTHESIS OF SUBOPTIMAL, INERTIA-WHEEL ATTITUDE CONTROL SYSTEMS[1]

by

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EMRL 1065 October, 1968

[1] This work was partially supported by NASA Grant NAS8-18120 and NSF Grant GK-3273.

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Abstract

Two techniques are presented for the synthesis of sub-optimal systems using motor-driven inertia wheels as the source of torque for three-dimensional attitude control. These techniques approximately minimize the integral of a quadratic function of system error and control effort, and both procedures compensate for nonlinear inter-axis coupling. The techniques developed in this paper are applied to the design of an attitude control system for a typical artificial satellite. The resulting control laws are in feedback form. In a computer simulation, systems designed on the basis of the procedures developed are shown to respond faster and more accurately than those designed by optimization procedures based on linearized approximations of the equations of motion.

ON THE SYNTHESIS OF SUBOPTIMAL, INERTIA-WHEEL ATTITUDE CONTROL SYSTEMS

1. Introduction

Many studies concerning the application of mathematical optimization techniques to the design of spacecraft attitude control systems have appeared in recent years. Most research in this area has been focused on time- and fuel-optimal, gas-jet control about a single axis.[1-4] Optimization of attitude control systems using inertia wheels as the source of control torque has been treated in considerably less detail. Flugge-Lotz and Marbach[1] and Schwartz[5] have proposed minimum-time and minimum-energy systems for inertia-wheel control about a single axis. Kalman, Englar, and Bucy[6] have presented a three-dimensional, inertia-wheel, attitude control system which approximately minimizes the integral of a quadratic function of system error and control effort, but an exceedingly large response time limits the usefulness of this system.

In the present study, two procedures are presented for the synthesis of suboptimal systems using motor-driven inertia wheels as the source of control torque for three-dimensional attitude control. These techniques approximately minimize the integral of a quadratic function of system error and control effort, and contrary to other treatments[6-9], both procedures compensate for nonlinear, inter-axis coupling. The methods proposed are applicable to the design of attitude control systems which use inertia-wheels for the correction of small errors.

The procedures developed in this paper are applied to the design of an attitude control system for a typical artificial satellite, the Nimbus. In a computer simulation, systems designed on the basis of the procedures outlined in this study are shown to respond faster and more accurately than those designed by optimization procedures based on linearized approximations of the equations of motion.

2. Preliminary Considerations

Attitude control consists of applying torque to a spacecraft in such a way as to place and hold it in a specific angular orientation with respect to a three-dimensional frame of reference. In this study, the angular velocity and acceleration of the reference frame is assumed to be of an order of magnitude smaller than the expected angular position and velocity errors. The spacecraft is considered rigid, and control torque is available about the three principal axes. Since the control system is to be used to correct small attitude errors, the equations of motion may be simplified by use of standard small angle assumptions, that is, terms of second order and higher involving the angular position and velocity errors may be neglected. Also it is assumed that the angular velocity of the inertia wheels relative to the spacecraft may be much larger than that of the spacecraft. Furthermore, it is not uncommon for the moments of inertia of the spacecraft to be more than a thousand times as large as those of the control wheels[6,7]; consequently, the moments of inertia of the control wheels may be ignored when summed with the moments of inertia of the spacecraft. Many important attitude control situations fall within the context of the above restrictions. The simplified equations of motion are

$$\dot{x} = Ax + C(y)x + Bu + Bg + Bh$$
$$\dot{y} = -u \tag{1}$$

where x is a vector representing the spacecraft attitude error (angular position and velocity); u is a vector representing the control torque available from the inertia wheels; y is a vector representing the angular momentum of the inertia wheels relative to the spacecraft; g and h are vectors representing the measurable and unmeasurable disturbance torques respectively; the elements of the matrix C(y) are the angular momenta of the inertia wheels; and A and B are constant matrices. Typical forms for A, B and C(y) are presented in Section 4; however, it should be noted that the functional form of the elements of C(y) in terms of x and t is unknown.

The value of the integral of a quadratic function of the system error plus a quadratic function of the control effort has been widely used as a measure of control system performance. Such a performance index is often analytically attractive, and for inertia-wheel attitude control, a quadratic cost functional also makes sense from a physical standpoint.

The suboptimal systems developed in this study reduce the angular position and velocity errors to zero rapidly, and also approximately minimize,

$$J(x_o, y_o, u, t_o) = \int_{t_o}^{\infty} (x'Qx + u'Ru)dt \qquad (2)$$

where Q and R are positive-definite, diagonal, constant matrices.* A small value of this integral indicates that both the error and control effort are kept small during most of the control interval.

3. Development of Suboptimal Techniques

The optimization problem considered is the determination of the control, u, which transfers any initial state, x_o, to the origin for the system governed by (1) and also minimizes the integral performance index J.

Three techniques for the solution of the above problem are the calculus of variations[10], Pontryagin's maximum

*A prime denotes the transpose of a vector or matrix.

principle[11], and the Hamilton-Jacobi theory[12]. Theoretically, each of these approaches gives enough information to determine the mathematically optimal control for the problem defined by (1) and (2); however, the Hamilton-Jacobi theory gives the most direct approach to the determination of both optimal and suboptimal control laws in feedback form. This approach depends upon the minimization with respect to u of a scalar function

$$H(x,y,u,t) = x'Qx+u'Ru+p'Ax+p'C(y)x+p'Bg(t)+p'Bh(t)+p'Bu \tag{3}$$

where p(t) is a vector of the same dimensionality as x. The optimal control $u^o(t) = -\frac{1}{2}R^{-1}B'p$ minimizes H. A scalar function, called the Hamiltonian is obtained by substitution of u^o into (3), and is given by

$$H^o = x'Qx - \tfrac{1}{4}p'BR^{-1}B'p + p'Ax + p'C(y)x+p'Bg(t) +p'Bh(t). \tag{4}$$

In the Hamilton-Jacobi approach, the vector p(t) is set equal to the gradient of a scalar function V(x,y,t), (i.e. p(t) = $V_x(x,y,t)$), where V is a twice-continuously differentiable function satisfying the partial differential equation

$$V_t + H^o(x,y,V_x,t) = 0 , \quad V(0,y,t) = 0 . \tag{5}$$

Equation (5) is known as the Hamilton-Jacobi equation, and its solution, V(x,y,t) evaluated at x_o, y_o, and t_o is the minimum value of the integral performance index J.

Method I. In the attitude control problem, the analytical solution of (5) appears impossible. Thus it is necessary to develop procedures for generating control laws which are suboptimal (approximately optimal). The first procedure for suboptimal control consists of using the control system to eliminate the most substantial nonlinear terms and the measurable disturbance torque. The resulting system is then treated as linear for purposes of optimization. The control system must compensate for the unmeasurable disturbance torque indirectly by controlling the attitude errors created by this torque; consequently, in the sequel h(t) is omitted from the equations of motion from which the control laws are derived.

A portion of the control, denoted by u_c is used to eliminate all nonlinear terms and the measurable distrubance torque in (1). From (1), $Bu_c = - C(y)x - g(t)$. The remainder of the control is denoted by u_L. Applying u_c to (1) yields $\dot{x} = Ax + Bu_L$.

The control u_L is to be selected in such a way as to minimize $J = \int_{t_o}^{\infty} (x'Qx + u_L'Ru_L)dt$. Using u_c, the Hamilton-Jacobi equation is

$$- \frac{1}{4} V_x{}'BR^{-1}B'V_x + V_x{}'Ax + x'Qx = 0 \qquad (6)$$

and $u_L = -\frac{1}{2} R^{-1}B'V_x$. Kalman[12] has demonstrated that the quadratic function, $V(x) = x'Px$, is a solution to (6) provided P is a symmetric, positive definite matrix such that

$$PA + A'P - PB'R^{-1}BP + Q = 0. \qquad (7)$$

The control, u, is the sum of u_L and u_c.

The transient response of the system is determined by the values selected for the weighting matrices Q and R. In the remainder of this work it is assumed that identical response is desired about each axis of the spacecraft. The above assumption considerably simplifies algebraic manipulations, although it is not a condition which must be satisfied before the suboptimal procedures developed in this study can be applied.

Method II. The second method for suboptimal control is an extension of a technique developed by Garrard, McClamroch, and Clark[13]. The control, u, is divided into two components, u_d and u_s, where $Bu_d = - g(t)$. Using u_d, (1) reduces to $\dot{x} = Ax + C(y)x + Bu_s$.

The control u_s is chosen to approximately minimize

$J = \int_{t_o}^{\infty} (x'Qx + u_s{}'Ru_s)dt$. The Hamilton-Jacobi equation is

$$V_x'[Ax + C(y)x] - \tfrac{1}{4} V_x'BR^{-1}B'V_x + x'Qx = 0 \;, \; V(0,y) = 0. \tag{8}$$

An approximate solution to the above equation may be obtained by assuming V(x,y) to be represented by a power series in ε of the form

$$V(x,y) = \sum_{n=2}^{\infty} \varepsilon^{(n-2)} V_n(x,y). \tag{9}$$

Substituting (9) into (8) and equating coefficients of powers of ε to zero gives

$$\frac{\partial V_2'}{\partial x} Ax - \tfrac{1}{4} \frac{\partial V_2'}{\partial x} BR^{-1}B' \frac{\partial V_2}{\partial x} + x'Qx = 0 \;,$$

$$\frac{\partial V_3'}{\partial x} Ax + \frac{\partial V_2'}{\partial x} Cx - \tfrac{1}{2} \frac{\partial V_2'}{\partial x} BR^{-1}B' \frac{\partial V_3'}{\partial x} = 0 \;, \tag{10}$$

$$\vdots$$

$$\frac{\partial V_n'}{\partial x} Ax + \frac{\partial V_{n-1}'}{\partial x} Cx - \tfrac{1}{4} \sum_{k>2}^{n} \frac{\partial V_k'}{\partial x} BR^{-1}B' \frac{\partial V_{n+2-k}}{\partial x} = 0.$$

In order to determine V_2, V_3, ... V_n, the above equations must be solved successively. The first equation of (10) is identical to (6). The remaining equations in (10) are linear, partial differential equations; solutions can be obtained by assuming $V_n = x'M_n(y)x$ for $n \geq 3$, where $M_n(y)$ is treated as a symmetric matrix. Substituting into (10) yields the following set of linear algebraic equations

$$M_3(A-BR^{-1}B'P) + (A' - PB'R^{-1}B)M_3 = PC - C'P$$
$$\vdots \tag{11}$$
$$M_n(A-BR^{-1}B'P) + (A'-PB'R^{-1}B)M_n = \sum_{k>3}^{n} M_kBR^{-1}B'M_{n+2-k}-M_{n-1}C-C'M_{n-1}$$

These equations may be solved for $M_n(y)$, and the control, u_s, is given as

$$u_s = -R^{-1}B' [P + \sum_{n=3}^{\infty} \varepsilon^{(n-2)}M_n(y)]x. \tag{12}$$

The complete control, u, is the summation of its two components u_s and u_d.

If the nonlinear coupling terms, $C(y)x$, are ignored, minimization of J yields a control law, u, which is nearly identical to u_L. This procedure is called linearization in the sequel, and as demonstrated in the next section, a control designed on the basis of such a procedure may prove inadequate.

4. Suboptimal Control of a Non-Symmetric Satellite

The suboptimal control laws developed in the previous section are applied to the synthesis of an inertia-wheel control system for an artificial earth satellite. The nominal distance from the surface of the earth to the center of mass of the satellite is 500 nautical miles, and the orbital eccentricity is 50 nautical miles. The earth is assumed to be spherical. In Fig. 1, the set of orthogonal axes R_1, R_2, and R_3 are the reference axes with which the spacecraft is to be aligned. The R_3 axis is assumed to point toward the center of the earth, and the R_2 axis is perpendicular to the orbital plane; consequently, the angular velocity of the reference axes is $\Omega_R' = [0, \Omega_{R_2}, 0]$, where Ω_{R_2} is the angular velocity of a line connecting the center of mass of the satellite with the center of the earth. (nominally $\Omega_{R_2} = 0.85x10^{-3}$ rad/sec).

The set of orthogonal axes B_1, B_2, and B_3 are the principal axes of the spacecraft and are denoted as the roll, pitch, and yaw axes respectively. The components along the body axes of the angular velocity of the spacecraft relative to the reference axes are called the roll, pitch, and yaw rates, and the angular errors between the body and reference axes are the roll, pitch, and yaw angles.

The maximum expected values of the roll, pitch, and yaw angles are 0.175 rad, and the maximum expected angular velocity is 0.01 rad/sec about each axis. The control system must reduce the roll, pitch, and yaw angels to 0.0175 rad within four minutes. The moments of inertia of the spacecraft are 200 slug-ft^2 about the roll axis, 150 slug-ft^2 about the pitch axis, and 100 slug-ft^2 about the yaw axis. The maximum allowable inertia-wheel angular momentum is 10 slug-ft^2/sec about the roll axis, 7.5 slug-ft^2/sec about the pitch axis, and 5 slug-ft^2/sec about the yaw axis. The system characteristics and performance specifications outlined abover are nearly identical to those for the Nimbus satellite[6].

If x_1 = roll angle, x_2 = roll rate, x_3 = pitch angle, x_4 = pitch rate, x_5 = yaw angle, x_6 = yaw rate, and y_i = angular momentum relative to the spacecraft of the i-th inertia wheel, the elements of A are zero except $a_{12} = a_{34} = a_{56} = 1$, the elements of B are zero except $b_{21} = 1/I_1$, $b_{42} = 1/I_2$, $b_{63} = 1/I_3$ (I_i is the moment of inertia of the spacecraft about the i-th principal axis),

$$C(y) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y_3/I_1 & 0 & y_2/I_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_3/I_2 & 0 & 0 & 0 & y_1/I_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y_2/I_3 & 0 & -y_1/I_3 & 0 & 0 \end{bmatrix},$$

and $g(t) = [y_3\Omega_{R_2}, I_2\dot{\Omega}_{R_2}, -y_1\Omega_{R_2}]$.

The inertia-wheel control logic is synthesized by linearization and the two suboptimal procedures developed in the previous section. The control obtained by linearization is

$$\begin{aligned} u_1 &= -I_1p_{22}x_2 - I_2p_{12}x_1 - y_3\Omega_{R_2} \\ u_2 &= -I_2p_{22}x_4 - I_2p_{12}x_3 - I_2\dot{\Omega}_{R_2} \\ u_3 &= -I_3p_{22}x_6 - I_3p_{12}x_5 + y_1\Omega_{R_2} \end{aligned} \qquad (13)$$

If nonlinear coupling, $C(y)x$, were indeed negligible, values of $p_{12} = 3.5\text{x}10^{-2}$rad/sec and $p_{22} = 5.75\text{x}10^{-4}$rad/sec^2 would yield adequate transient response. However, this is not the case as is demonstrated later.

Method I for suboptimal control yields

$$\begin{aligned} u_1 &= -I_1p_{22}x_2 - I_1p_{12}x_1 - y_3\Omega_{R_2} + y_3x_4 - y_2x_6. \\ u_2 &= -I_2p_{22}x_4 - I_2p_{12}x_3 - I_2\dot{\Omega}_{R_2} - y_3x_2 - y_1x_6 \\ u_3 &= -I_3p_{22}x_6 - I_3p_{12}x_5 + y_1\Omega_{R_2} + y_2x_2 + y_1x_4 \end{aligned} \qquad (14)$$

Using Method II, a two-term approximation of (12) is

$$u_{1_s} = - I_1 p_{22} x_2 - I_1 p_{12} x_1 - \frac{(I_1 - I_2)}{2I_2} y_3 x_4 - \frac{(I_1 - I_3)}{2I_3} y_2 x_6$$

$$- \frac{p_{12}}{2p_{22}} \frac{(I_1 + I_2)}{I_2} y_3 x_3 - \frac{p_{12}}{2p_{22}} \frac{(I_1 + I_3)}{I_3} y_2 x_5$$

$$u_{2_s} = - I_2 p_{22} x_4 - I_2 p_{12} x_3 + \frac{(I_1 - I_2)}{2I_1} y_3 x_2 - \frac{(I_3 - I_2)}{2I_3} y_1 x_6 \qquad (15)$$

$$+ \frac{p_{12}}{2p_{22}} \frac{(I_1 + I_2)}{I_1} y_3 x_1 + \frac{p_{12}}{2p_{22}} \frac{(I_2 + I_3)}{I_3} y_1 x_5$$

$$u_{3_s} = - I_3 p_{22} x_6 - I_3 p_{12} x_5 + \frac{(I_1 - I_2)}{2I_1} y_2 x_2 + \frac{(I_3 - I_2)}{2I_2} y_1 x_4$$

$$- \frac{p_{12}}{2p_{22}} \frac{I_2 + I_3)}{I_2} y_1 x_3 + \frac{p_{12}}{2p_{22}} \frac{(I_1 + I_3)}{I_1} y_2 x_1$$

Stability of (1) under the action of control laws of the general form of (15) has been demonstrated by Garrard and Walker[14]. A functional block diagram of the spacecraft and control system is illustrated in Fig. 2.

In simulating the characteristics of the spacecraft and the control system on a digital computer, control laws

TABLE I

Initial Conditions - Pitch, Yaw, and Roll Angels = 0.175 radians Pitch, Yaw, and Roll Rates = 0.1 radians/sec						
Method of control System Design	Linearization		Suboptimal: Method I		Suboptimal: Method II	
Initial Inertia-Wheel Angular Momentum (% of Maximum)	0	60	0	60	0	60
Response Time (Minutes)	4.0	15.0	3.0	3.2	3.7	3.2
Peak Torque (ft-lbs x 10^{-4})	4.501	5.083	4.501	6.583	4.501	13.952
Peak Power (ft-lbs/sec x 10^{-3})	92.69	611.42	91.32	789.9	126.0	3345.4
Energy Consumed (ft-lbs x 10^{-3})	107.7	1194.5	112.5	859.1	105.0	857.6
Quadratic Performance Index	9.170	13.060	8.930	12.190	10.310	9.641

given by linearization and suboptimal Methods I and II were employed for several sets of initial conditions. Values of zero and 60 per cent of the maximum for the initial, inertia-wheel angular momentum were used for each set of initial conditions. The exact equations of motion were used in the simulation, and disturbance torques due to the gravity gradient were included. The results of a typical simulation are given in Table 1.

In all cases, the system designed on the basis of linearization failed to sufficiently reduce the angular error within the required time for an initial, inertia-wheel angular momentum of 60 per cent of the maximum expected value. Both suboptimal systems had adequate response times for all sets of initial conditions tested; however, the system designed on the basis of Method I appears to have smaller torque and power requirements than does the system designed on the basis of Method II.

The performance of the system designed by linearization approached the performance of the suboptimal systems for zero initial, inertia-wheel angular momentum. This was to be expected since all three control laws are nearly the same for small values of inertia-wheel angular momentum.

The response of the spacecraft is illustrated in Figs. 3 and 4 for a value of initial, inertia-wheel angular momentum of 60 per cent of the maximum, and the unacceptable response of the system designed on the basis of linearization is evident. Fig. 3 shows that the system synthesized by use of Method II gives less overshoot than the system designed by use of Method I. However, the suboptimal system based on Method II yields extremely oscillatory response; this is illustrated in both figures.

Kalman et al.[6] synthesized a control system for the satellite considered in this example. In this design all linear, time-invariant terms in the equations of motion were retained, but all nonlinear terms were neglected. Optimization was performed with respect to a quadratic performance index; however, the best response time obtained was over two hours.

5. Conclusions

The suboptimal control techniques developed in this work appear to provide effective methods for synthesizing inertia-wheel attitude control systems. Both procedures take nonlinear inter-axis coupling into account. As demonstrated in the example, unacceptable system response may result if such coupling is ignored. Both techniques yield control laws in feedback form, and the suboptimal systems developed in this study give considerably more accurate control that is provided by systems based on linearized approximations of the equations of motion.

Results obtained from application of the two techniques of for suboptimal, inertia-wheel control indicate that Method I is slightly superior in the following ways:

(1) The control system designed by use of Method I has lower torque and power requirements.

(2) Method I yields more accurate response.

(3) Method I is computationally easier to use and gives simpler control laws.

Better results might be obtained from Method II if more terms were used in the approximate solution of the Hamilton-Jacobi equation.

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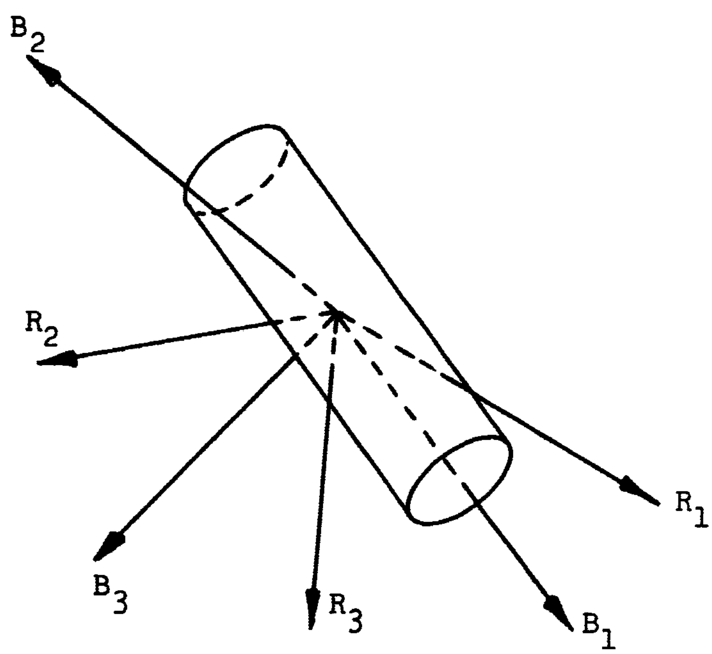

Fig. 1 Coordinate Axes

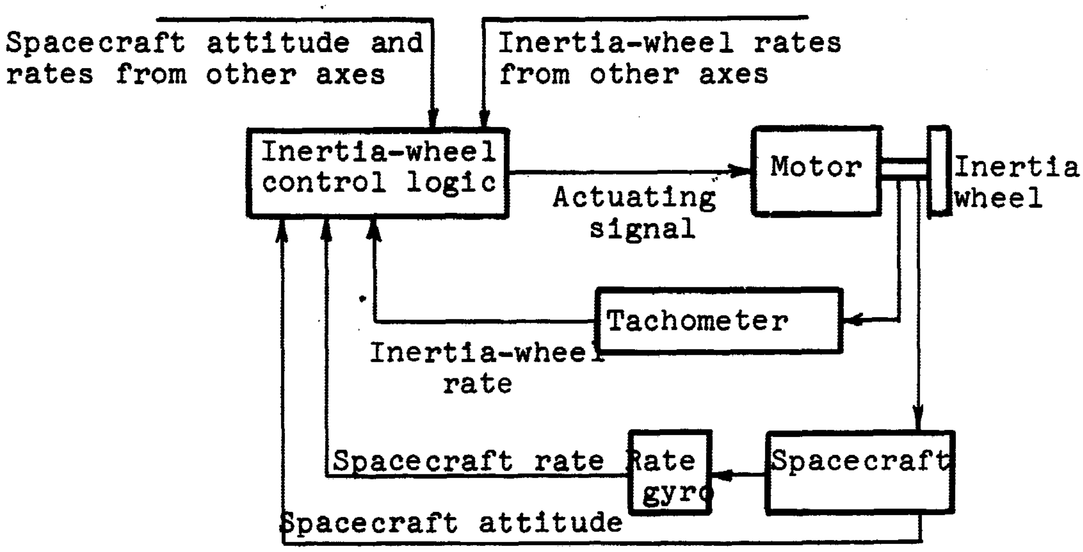

Fig. 2 Functional Block Diagram of Spacecraft and Inertia-Wheel Control System

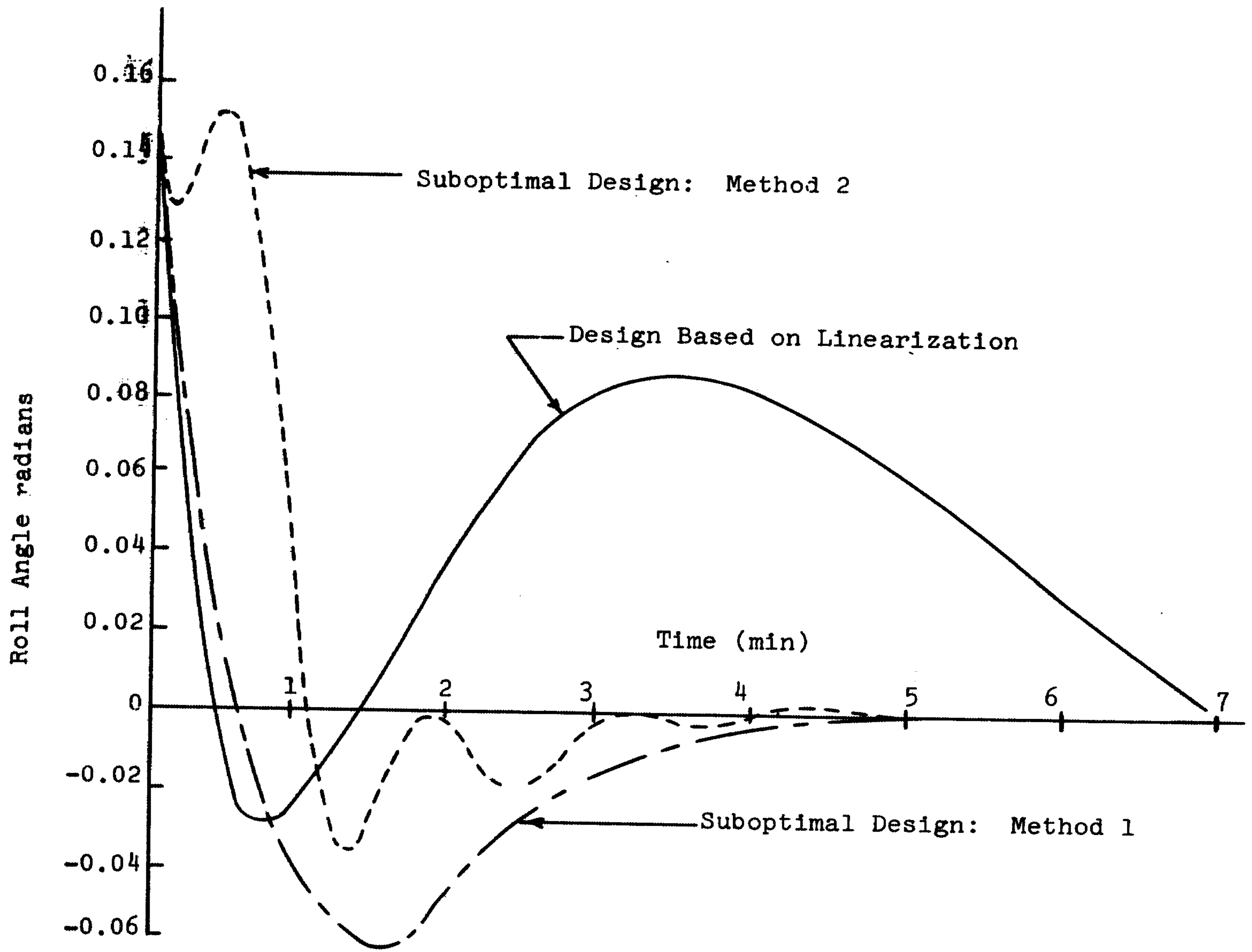

Fig. 3 Roll Angle vs Time; Initial, Inertia-Wheel Angular Momentum = 60% of Maximum

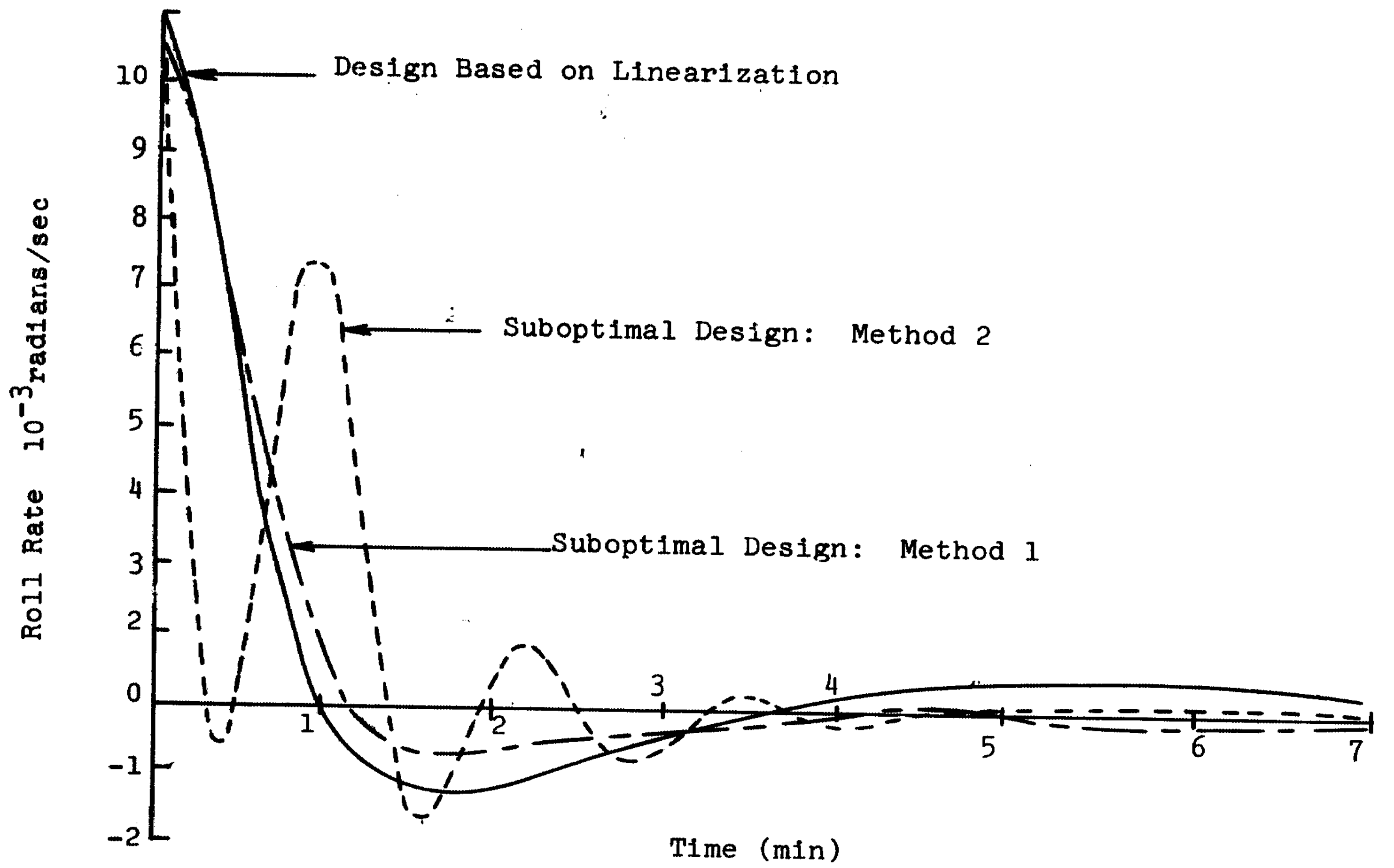

Fig. 4 Roll Rate vs Time; Initial, Inertia-Wheel Angular Momentum = 60% of Maximum