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OPTIMAL POINTWISE FEEDBACK CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

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ABSTRACT

Systems described by parabolic partial differential equations are formulated as ordinary differential equations in a Sobolev space of a given order. Quadratic cost criteria are then formulated in terms of inner products on this Sobolev space. Existence of an optimal control is proved both in the case where the system operator is coercive and in the case where the system operator is the infinitesimal generator of a semigroup of operators. The optimal control is given by a linear state feedback law. The feedback operator is shown to be the bounded, positive, self-adjoint solution of a nonlinear operator equation of the Riccati type. This operator can also be represented by an integral operator whose kernel satisfies a Riccati-like integro-differential equation.

These results are specialized, in a straightforward manner, to the case of pointwise control. The optimal pointwise control is given by a simplified linear control law which depends on the control point location. The general results are also specialized to obtain the modal approximation to the pointwise control problem and to demonstrate the optimality of output feedback for a particular class of output transformations. The pointwise feedback control laws, in these cases, are characterized, structurally, by a measurement operation which is independent of control point location and a gain operation which is directly dependent on control point location. Several examples relating to the scalar heat equation are solved.

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CHAPTER I

INTRODUCTION

A great deal of research in recent years has been devoted to the optimal control of distributed parameter systems. With few exceptions this work has focused on the use of distributed, rather than lumped, controls. In many cases of practical interest, however, it is actually desired to control distributed parameter systems by means of finite-dimensional controls. Examples of such cases would be the control of the flexure of a launch booster using only rocket thrust and control of wing and fuselage flexure in aircraft by means of rudder, flap, aileron, and spoiler manipulation. In both of these examples the controls are a finite number of pointwise controls, that is, controls applied at isolated points within the spatial domain of definition of the distributed parameter system. Other examples of systems in which pointwise control might be applied are heat diffusion systems, systems described by wave equations such as longitudinally vibrating beams and transmission lines, transversely deflecting flexible beams, and mechanically vibrating systems.

Traditional approaches to solving this type of problem include solving for an optimal distributed control and then approximating the distributed control by a finite number of lumped controls, or the modal approach, as used by Johnson,[1] for example, in which the system is assumed to be adequately described by a finite number of modes and the resulting finite-dimensional optimization problem is solved for the optimal modal control. The former approach becomes inadequate if we wish to determine feedback control laws rather than open-loop controls, and the latter fails when the system cannot be described by a countable number of modes, when the number of modes necessary for adequate description of the system is prohibitively large, when it is computationally difficult to determine the eigenvalues and eigenfunctions, or when it is difficult to judge which modes are indeed the dominant ones.

The purpose of this research is to formulate the pointwise control problem as a distributed parameter control problem and to present a unified approach to solving this problem within the framework of existing distributed parameter control theory. The distributed parameter systems we shall consider are described by parabolic and hyperbolic partial differential equations. Examples of parabolic partial differential equations are the scalar heat diffusion equation and the heat diffusion equation in the plane. Wave equations and the equation for transverse deflection of a flexible beam fall within the category of hyperbolic partial differential equations. The cost functional is quadratic in the deviation of the state distribution from a desired distribution and in the control energy. The choice of such a cost functional is motivated by the desire to derive, in the distributed parameter case, results of comparable elegance to those of finite-dimensional control problems with quadratic cost--namely, linear feedback control laws and simply expressed quadratic optimal cost functions.

There are many approaches to the solution of general distributed parameter control problems. One of the earliest systematic approaches was that of Butkovskii's.[2] He presents a maximum principle for distributed parameter optimal control problems analogous to Pontryagin's maximum principle for lumped parameter control problems. The distributed parameter systems which Butkovskii considers are those described by systems of integral equations and the necessary conditions for optimality which he obtains by variational techniques are also in the form of integral equations. Since we shall consider systems described by parabolic and hyperbolic partial differential equations, our control problem is not in a form in which the Butkovskii maximum principle is immediately applicable. There are methods, namely Green's function techniques, whereby the partial differential equation description of a distributed parameter system may be transformed to an equivalent integral equation description, but these techniques tend to be difficult to apply to the general classes of spatial differential operators we shall consider.

Wang[3] derives a maximum principle for distributed parameter systems described by partial differential equations by using a dynamic programming procedure. The necessary conditions he obtains are in the form of partial differential equations. An unfortunate aspect of Wang's maximum principle is that, although it is systematic in principle, there is no systematic way of treating boundary conditions. Moreover, in a strict mathematical sense, it is impossible to prove existence and uniqueness of optimal solutions in the function space in which Wang formulates his control problems.

A step in the direction of formulating distributed parameter control problems in a form more amenable to the application of well-known system theoretic concepts is taken by Balakrishnan,[4] who considers the state distribution in the distributed system to be a point in some Banach space and then regards the partial differential equation describing the time evolution of the state distribution to be an ordinary differential equation in the strong topology of the Banach space. He uses the assumption that the system spatial differential operator is the infinitesimal generator of a semigroup of operators, the infinite dimensional analogy of the transition matrix in finite dimensional systems, and proceeds to solve final value problems and time-optimal problems by means of well-known functional analytic methods. Fattorini[5] works along these same lines in considering the controllability of distributed parameter systems containing both distributed control and boundary control. Unfortunately for the problem we wish to consider, or, more precisely, for quadratic cost functionals, the Banach spaces used by Balakrishnan and Fattorini are much too general.

There have been several applications of the above techniques. Egorov[6, 7] attacks a problem with both interior and boundary control. He considers the system partial differential equations and boundary condition equations as dynamics and introduces appropriate adjoint variables to obtain a maximum principle separated into an interior inequality and a boundary inequality.

Sakawa[8] considers linear one-dimensional distributed parameter systems, with boundary control, as represented by integral equations, and, using variational techniques, derives integral equation necessary conditions which are simpler in form, but less general in application, than Butkovskii's maximum principle.

Yeh and Tou[9] treat systems in which the controlled object moves continuously through the plant with a constant velocity. With the control assumed to be constrained in magnitude, the authors minimize a quadratic criterion via Butkovskii's maximum principle. The optimal control is shown to be the solution of a Fredholm integral equation of the second kind.

Kim and Erzberger[10] also consider the minimization of a quadratic cost functional, using a dynamic programming approach to obtain a set of functional equations analogous to the matrix Riccati equation for lumped systems. They solve these equations by a method based on the eigenfunction representation of the Green's function.

Axelband[11] solves the problem of minimizing the norm of the difference of a distributed parameter system output from a desired output by the use of a functional analytic formulation similar to Balakrishnan's. He obtains an optimal solution by a convex programming algorithm.

Sirazetdinov[12,13] considers a quadratic cost functional and, using stability theory and dynamic programming arguments, proves the optimality of a distributed control law which is linear in the state of the system and derives integro-differential equations for the coefficients of the optimal cost function. He applies this to the problem of controlling aerodynamic and elastic deformation of an airframe.

Yavin and Sivan[14] treat the optimal control of longitudinal vibrations in a flexible rod held fixed at one end. From a partial differential equation formulation they obtain the proper Green's function for transformation to an integral equation. Using a quadratic criterion and a control applied at the force end, they obtain necessary conditions in the form of a Fredholm equation of the second kind. An approximate open-loop control is obtained by approximating the kernel by a sequence of degenerate kernels.

In a recent book, Lions[15] formulates quadratic distributed parameter control problems in Hilbert spaces in which the terms of the quadratic cost functional may be written as inner products. He shows, for systems described by spatial differential operators satisfying a certain definiteness condition, that solutions to the system equation exist and are continuous with respect to the control in the topology of the Sobolev space of order equivalent to the order of the system spatial differential operator. Using these Sobolev spaces, he is able to prove the existence and uniqueness of an optimal control and to determine the necessary conditions for the optimality of this control. Moreover, he shows that the optimal control is specified by a linear feedback control law and that the feedback operator satisfies a differential equation similar to the matrix Riccati equation obtained for finite-dimensional systems.

Lions' results are the foundations upon which this research is built. We shall extend the class of system spatial differential operators considered by Lions to include those which are infinitesimal generators of semigroups of operators and will show that the results obtained by Lions for his more restrictive class also hold in the more general case. A fact of key importance which we shall use is that differential operators defined on a Sobolev space are closed operators in the topology of that Sobolev space. This is one of the requirements for an operator to be the infinitesimal generator of a semigroup of operators. Another useful feature of Sobolev spaces is that boundary conditions become easy to handle when the state space for the system is a Sobolev space. We shall also show that distributed systems driven by finite dimensional controls (the pointwise control problem) fall within the framework of this formulation and the results obtained for a general class of controls are specialized to the case of pointwise control in a straightforward manner. It should be noted that Russell[16] attacks the problem of constrained pointwise control with a minimum system energy cost functional.

He does not develop a Hilbert space formulation of the problem and he circumvents the unboundedness of the system spatial differential operator by assuming that his initial states have bounded spatial derivatives. By treating our pointwise control problem as a special case of a general quadratic optimization problem in a Sobolev space, we need not consider any such confining assumptions on the initial conditions.

The organization of the thesis is as follows: Chapter II provides the mathematical background necessary for the formulation of parabolic and hyperbolic optimal control problems. The material is presented in such a form as to point out continually the relationships between infinite dimensional and finite dimensional system theoretic concepts. Sobolev spaces of finite order are defined by means of distribution theory. Elliptic differential operators of the coercive and, more general, strongly elliptic type are defined on these Sobolev spaces. It is then shown that parabolic and hyperbolic partial differential equations may be written as ordinary differential equations in the Sobolev space. The remainder of the chapter is devoted to semigroups of operators--their definition, the concept of infinitesimal generator, and the presentation of a formula analogous to the variation of constants formula in finite dimensional systems.

In Chapter III we present the precise mathematical formulation of the parabolic and hyperbolic optimal control problems. The parabolic control problem is then specialized to the case of pointwise control.

Chapter IV is concerned with the solution of the parabolic optimal control problem in both the case where the system operator is assumed to be coercive and in the case where the system operator is assumed to be the infinitesimal generator of a semigroup of operators. The path to a solution first involves proving that a unique solution indeed exists.

In the coercive system operator case it will be shown that because of continuity of the optimal state in the optimal control, the optimal control is given by a linear state feedback law in which the feedback operator is the solution of a Riccati operator differential equation. Under the assumption that the system operator is the infinitesimal generator of a semigroup of operators this continuity relation is not easily demonstrable, but it is shown that if a solution of the Riccati operator equation exists, then the optimal control is given by a linear feedback control law. It will then be shown that such a solution does exist. The remainder of the chapter contains a discussion of the behavior of optimal solutions when the terminal time approaches infinity and an alternative formulation of bounded operators on a Sobolev space as integral operators and the subsequent modification of the Riccati operator equation.

With optimal solutions to the parabolic control problem having been determined for general control spaces, we specialize the results to the case of pointwise-control in Chapter V and show that the optimal feedback operator in the pointwise control case is of a simpler form from a computational point of view. The second part of this chapter is concerned with the infinite terminal time pointwise control problem. It will be shown that by a judicious choice of the quadratic cost functional the modal analytic formulation of the pointwise control problem is obtained. This approach will enable us to make conclusions about the optimality of modal analytic solutions which we are unable to make by the straightforward techniques of modal analysis alone. We then consider the case where the entire state is not available to be fed back, but only the outputs of a finite number of measuring devices. It will be shown that if the measuring devices are of a certain class, then the optimal control law will consist of feeding back only the outputs of these devices.

The concluding Chapter VI contains a summary of the results obtained and recommendations for further research.

It should be stressed that throughout the thesis a general class of distributed parameter optimal control problems will be solved, and the results will be specialized so as to obtain results in the pointwise optimal control problem and to obtain insight into the modal analytic and measurement problems.

CHAPTER II

MATHEMATICAL BACKGROUND

2.1 INTRODUCTION

The purpose of this chapter is to lay the mathematical foundation for the discussions and derivations in succeeding chapters. The various results stated in this chapter do not exhaustively cover the field of differential operators and partial differential equations, but serve to form a relatively complete set of tools to be applied to the problems of interest. The guiding philosophy for both choice of results to be discussed in this chapter and direction of theory in the sequel is the attempt to provide results for distributed parameter systems which are roughly parallel to known results in lumped parameter theory. In order to achieve this parallelism, related concepts in distributed parameter theory must be provided for such lumped parameter system concepts as state and state space, matrix operators, equations of state, transition matrices, and variation of constants formulae.

Section 2.2 is concerned with the concept of state in distributed parameter systems and the discussion of particular spaces of (generalized) functions which serve as state spaces for systems described by partial differential equations.

The reason for the choice of the spaces in Section 2.2 is made more clear when spatial differential operators are discussed in Section 2.3 and it is seen that elements of these spaces have sufficient smoothness to qualify as elements of the domain of differential operators. The properties of coercivity and strong ellipticity of differential operators are treated in this section. The distinction between these two concepts will not be apparent until necessary conditions for optimality are discussed in Chapter IV.

Parabolic and hyperbolic partial differential equations and their boundary conditions are introduced in Sections 2.4 and 2.5. Emphasis is placed throughout these two sections on the fact that these equations serve as equations of state exactly as ordinary differential equations describe the evolution of finite-dimensional state variables.

In Section 6 the concept of a semigroup of operators, the analog of the transition matrix in the finite dimensional case, is defined and explored. In addition to the properties of these semigroups the manner in which an operator may generate a semigroup of operators is discussed. This is further elaborated on in Section 2.7 where strong ellipticity is shown to be a sufficient condition for a differential operator to be the infinitesimal generator of a semigroup of operators.

The final section of the chapter contains the relation of the semigroup of operators generated by the system operator of a forced (controlled) system to solutions of this system. This expression for solutions of the forced system corresponds directly to the variation of constants formula for the state of a finite dimensional forced system.

2.2 DISTRIBUTION THEORETIC CONCEPTS AND SOBOLEV SPACES

The state of a finite dimensional system can be identified as a point in a finite dimensional Euclidean vector space. In distributed parameter systems the state is a function, at each instant of time, defined on the given spatial region, or, alternatively, the state is a point in an infinite dimensional (function) space. For the purpose of preparation for our subsequent study of quadratic performance criteria, attention will be focused on the Hilbert space of square integrable functions on the spatial region of definition. As will be shown, this space is not quite suitable for distributed parameter applications, but certain subspaces, namely the Sobolev spaces of finite order, are. As a preliminary to the definition of Sobolev spaces, a brief discussion of distribution theory is required.

Let us denote by D and ∂D the spatial region of definition and its boundary. The variable z is used to denote a point in D. Further let $C_0^\infty(D)$ be the space of infinitely differentiable functions of compact support on D. The space of bounded linear functionals on $C_0^\infty(D)$ (i.e., the dual of $C_0^\infty(D)$) is called the space of distributions on D and is denoted by $\mathcal{D}'(D)$. An element F of $\mathcal{D}'(D)$ has the form

$$F(\phi) = \int_D f(z)\phi(z)dz \quad \forall \quad \phi \in C_0^\infty(D)$$

where $f(\cdot)$ is some Lebesque integrable function on D. The most familiar example of a distribution is the Dirac δ-function or impulse, $\delta(z-z')$, which is the linear functional

$$\Delta(\phi) = \int_D \delta(z-z')\phi(z)dz = \phi(z')$$

There are several properties of the space of distributions which we shall exploit. First, the space of square integrable functions on D, $L^2(D)$, is a subset of the space of distributions. This is easily seen by noting the fact that $C_0^\infty(D) \subset L^2 D)$ (any infinitely differentiable function with compact support in D must be square integrable on D) and, therefore, the dual space of $L^2(D)$ must be contained in the dual space of $C_0^\infty(D)$, namely $\mathcal{D}'(D)$. Since $L^2(D)$ is its own dual the following inclusion relation holds

$$C_o^\infty(D) \subset L^2(D) \subset \mathcal{D}'(D)$$

The second property of distributions which it is useful to exploit is the unique specification of the derivative of a distribution. If D is a region in n-dimensional Euclidean space and z is the n-tuple $(z_1, z_2, \ldots, z_n)$, $\partial F/\partial z_i$ for some $F \epsilon \mathcal{D}'(D)$ is uniquely specified by

$$\frac{\partial F(\phi)}{\partial z_i} = \int_D \frac{\partial f}{\partial z_i}(z)\phi(z)dz = -\int_D f(z)\frac{\partial \phi}{\partial z_i}(z)dz = f(-\frac{\partial \phi}{\partial z_i}) \;\forall \phi \epsilon C_o^\infty(D)$$

What, in effect, has been achieved is the ability to specify a meaningful expression for the operation of differentiation of any distribution, or, more to the point, differentiation is defined for all elements of $L^2(D)$.

This generalized approach to differentiation can be extended to more complicated differential operators. Introducing the following notation, we let

$$q = (q_1, q_2, \ldots q_n) \qquad |q| = \sum_{i=1}^{n} q_i \tag{2.2.1}$$

where q_i is a positive integer for $i=1,2,\ldots n$, and defining the differential operator

$$D^q = D_1^{q_1} D_2^{q_2} \ldots D_n^{q_n}, \quad \text{with } D_i = \frac{\partial}{\partial z_i} \tag{2.2.2}$$

then for each $F \epsilon \mathcal{D}'(D)$

$$D^q F(\phi) = (-1)^{|q|} F(D^q \phi) \;\forall \phi \epsilon C_o^\infty(D)$$

Let us make the following definition

<u>Definition 2.1</u>: The subset of $\mathcal{D}'(D)$, denoted by $H^m(D)$, with the property

$$H^m(D) = \{F \epsilon \mathcal{D}'(D) \;:\; F \epsilon L^2(D),\; D^q F \epsilon L^2(D) \;\forall q,\; |q| \leq m\}$$

is called the Sobolev space of order m, with m an integer.

Moreover, defining the following inner product for F, $G \epsilon H^m(D)$

$$\langle F, G \rangle_{H^m(D)} = \sum_{|q| \leq m} \langle D^q F, D^q G \rangle_{L^2(D)}$$

$H^m(D)$ can be shown to be complete in the topology induced by this inner product (see Ref. 17). Chapter 4), and, therefore, $H^m(D)$ is a Hilbert space.

The usefulness of the Sobolev space $H^m(D)$ can be understood when it is recalled exactly what are the useful properties of finite-dimensional state spaces. First, any finite dimensional space is complete and any operator (matrix) on this space is everywhere defined. The differential operator D^q, as described above, is everywhere defined on $C_0^\infty(D)$, the space of infinitely differentiable functions with compact support in D. Unfortunately, there is no norm topology for which this space has the completeness property of finite dimensional vector spaces. The second useful property of finite dimensional spaces is the fact that all linear operators on these spaces are closed. If $L^2(D)$ is taken to be the space on which D^q is defined (in this case only densely defined), D^q is not a closed operator. If $\overline{D}^q$ is the closure of D^q on $L^2(D)$, then the domain of $\overline{D}^q$ would contain non-differentiable functions. By the artifice of introducing distributions we are able to define the derivative even for non-differentiable functions, and it is easily seen that the non-differentiable functions in the domain of the closure of D^q are those functions F in $L^2(D)$ for which D^qF is in $L^2(D)$. More succinctly, the domain of the closure of D^q is $H^m(D)$ for some m. With the inner product defined above for $H^m(D)$ the Sobolev space of order m has the very useful property of completeness. Thus, it is seen that Sobolev spaces fill the bill as candidates for distributed parameter state spaces.

2.3 DIFFERENTIAL OPERATORS

With the introduction of Sobolev spaces as the prototype of a state space for distributed parameter systems, it remains to be discussed what exactly are the properties of spatial differential operators, which play the role in distributed parameter systems which matrices play in lumped parameter systems. Some of these properties were touched on in the preceding section as part of the justification of the usefulness of Sobolev spaces. It was shown, in essence, that a differential operator of order m is everywhere defined (with the aid of distribution theory) and closed on $H^m(D)$. This is, however, all that linear differential operators have in common with linear operators in finite dimensional spaces.

The first property which characterizes differential operators is the fact that they are not bounded. This, aside from the infinite dimensionality, is the single most complicating factor in distributed parameter systems. It causes difficulty in proving existence of solutions to partial differential equations, and, in contradistinction to finite dimensional systems, necessitates that great pains must be taken in characterizing these solutions, as will be seen in Sections 7 and 8 of this chapter.

The particular type of differential operators which will be considered, as indicated somewhat by the operator D^q in the preceding section are those of linear form, composed of partial derivatives with respect to each component of the spatial variable and of a specified order m. Embellishing the notation of Section 2, let us introduce the real functions $a_q(z)$, where q is the n-tuple defined by (2.2.1), and define the formal differential operator A, of order m

$$A = \sum_{|q| \leq m} a_q(z) D^q \tag{2.3.1}$$

where D^q is the differential operator described in Eq. 2.2.2 and the notation $\sum_{|q| \leq m}$ signifies the composite summation

$$\sum_{|q| \leq m} = \sum_{\substack{q \\ |q|=0}} + \sum_{\substack{q \\ |q|=1}} + \dots \sum_{\substack{q \\ |q|=m}}$$

with D^o representing the zeroth order differential, or identity, operator. The particular nature of the functions $a_q(z)$ will be clarified in the discussions on coercivity and ellipticity.

Just as the formal differential operator A has been defined, it is a straightforward matter to define the formal adjoint of A, denoted by A^+, as

$$A^+ = \sum_{|q| \leq m} (-1)^{|q|} D^q a_q(z) \tag{2.3.2}$$

In general, the formal adjoint A^+ does not equal the adjoint operator A^*, where A^* satisfies

$$\langle x, Ay \rangle_{H^m(D)} = \langle A^* x, y \rangle_{H^m(D)}$$

Indeed, it can be shown, by means of Green's Formula, that

$$\langle x, Ay \rangle_{H^m(D)} = \langle A^+ x, y \rangle + C$$

where the constant C depends on conditions at the boundary ∂D. In the case of Dirichlet boundary conditions, which will be discussed in Section 2.5, $C=0$ and the formal adjoint A^+ equals the adjoint A^*.

We shall now discuss what is meant by an elliptic differential operator, and we shall subsequently define the properties of coercivity and strong ellipticity of elliptic differential operators, which properties will play an important role in the optimization results of Chapter IV. If the functions $a_q(\cdot)$ are required to be essentially bounded functions, or, equivalently, are elements of the space $L^\infty(D)$, then an m^{th} order differential operator of the form specified in Eq. 2.3.1 is said to be elliptic (see Ref. 18, p. 1704) if the inequality

$$\sum_{|q|=m} a_q(z)\, \zeta^q \neq 0 \quad \text{for all} \quad \zeta \in R^n,\ z \in D$$

is satisfied. Note that this is a condition on the highest order term of the differential operator, i.e., the terms containing partial derivatives of order m. If we restrict our attention to elliptic differential operators which contain only even order partial derivatives, we define the concept of coercivity in the following manner:

Definition 2.2: If A is an elliptic differential operator of the form

$$A = \sum_{|q| \leq 2p} a_q(z) D^q$$

where $a_q(z) = 0$ if $|q| \neq 2k$, for $k=0, 1, \ldots, p$, then A is said to be coercive if the inequality

$$(-1)^k \sum_{|q|=2k} a_q(z)\zeta^q \leq -\alpha \sum_{|q|=2k} \zeta^q \tag{2.3.3}$$

is satisfied for some $\alpha > 0$, for $k=0,1,\ldots p$, and for all $\zeta \epsilon R^n$ and $z \epsilon D$.

This concept of coercivity arises from the use of this term by J. L. Lions (Ref. 15, p.22) to describe the property of operators more commonly referred to as "negative definiteness", namely the condition

$$\langle Ax, x\rangle_{H^m(D)} \leq - \alpha \|x\|^2_{H^m(D)}$$

for some $\alpha > 0$ and for all $x \epsilon H^m(D)$. It might be noted that just as negative definiteness of a matrix implies that the eigenvalues of the matrix lie on the negative real axis, the spectrum of a coercive operator is a subset of the left half-plane.

The condition for strong ellipticity is not as stringent, and a strongly elliptic operator is defined by:

Definition 2.3: If A is an elliptic differential operator of even order $2p$, then A is said to be strongly elliptic if the inequality

$$(-1)^p \sum_{|q|=2p} a_q(z)\, \zeta^q \leq - \alpha \sum_{|q|=2p} \zeta^q \tag{2.3.4}$$

is satisfied for some $\alpha > 0$, and for all $\zeta \epsilon R^n$ and $z \epsilon D$.

Note that, unlike in the Inequality 2.3.3 for the coercive operator case, the summation in Inequality 2.3.4 is taken over only the highest order terms of the operator A. All of the terms of a given order in the coerciv operator case must satisfy this type of inequality. Thus, coercivity implies strong ellipticity, but the converse does not hold.

To illustrate coercive and strongly elliptic operators, let us consider the second order differential operator defined on some subset D of R^2

$$A_1 = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} - 2$$

The coefficients of this operator satisfy Inequality 2.3.3 for $k=0$ and $k=1$ if we choose $\alpha=1$, implying that this operator is coercive (and, of course, strongly elliptic). If, on the other hand, we consider the differential operator

$$A_2 = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + 2$$

we see that Inequality 2.3.3 is satisfied for $k=1$, but is not satisfied for $k=0$, and, thus, the operator A_2 is not coercive. However, Inequality 2.3.4 is satisfied, which implies that A_2 is a strongly elliptic operator.

The fact that we do not consider more general operators than those described above is a reflection of the state of knowledge concerning differential operators and the fact that there are many physical systems of interest whose mathematical models have spatial differential operators falling within these categories.

2.4 SYSTEM EQUATIONS--PARABOLIC AND HYPERBOLIC

The purpose of this section is to tie together the concepts discussed in the preceding two sections--namely, state, state space, and system differential operators--and arrive at a description of a distributed parameter system in the form of one or more partial differential equations. This, of course, is in direct analogy with the equations of state in finite dimensional systems. The only ingredient missing up to now is the time variable.

Let us consider functions $x(t)$ defined on $t \epsilon [0, T]$ and having values in the Sobolev space $H^m(D)$, defined in Section 2 of this chapter, that is, $x(t) \epsilon H^m(D) \forall t \epsilon [0, T]$. Just as was done in Section 2, these vector functions $x(t)$ may be considered as points of a function space. Since emphasis has been placed on considering Hilbert spaces as state spaces, the space $L^2(0, T; H^m(D))$ is defined:

Definition 2.4: If $x(t) \epsilon H^m(D)$ for all $t \epsilon [0, T]$, then the square integrable Sobolev space-valued functions are

$$L^2(0, T; H^m(D)) = \{x(\cdot) : x(t) \epsilon H^m(D), \forall t \epsilon [0, T] \text{ and } \int_0^T \|x(t)\|^2_{H^m(D)} dt <$$

Note that this is a Hilbert space with inner product

$$\langle x(\cdot), y(\cdot) \rangle_{L^2(0, T; H^m(D))} = \int_0^T \langle x(t), y(t) \rangle_{H^m(D)} dt$$

Since it is desired to represent physical distributed parameter systems, it is essential to be able to characterize partial differentiation by time. With the discussion of the distribution theoretic results in Section 2 the tools are on hand to make this a straightforward procedure. If we consider the space of infinitely differentiable Sobolev space-valued functions with compact support in $[0, T]$ and its corresponding dual space of distributions, which, for convenience, may be denoted by $\mathcal{D}[\overline{0, T}]$, then the following Sobolev space of Sobolev space-valued functions may be defined (see Ref. 15, p. 115).

Definition 2.5: $W(0, T)$ is the set of Sobolev space-valued functions defined on $[0, T]$ with the property

$$W(0, T) = \{x(\cdot) : x(\cdot) \epsilon L^2(0, T; H^m(D)) \; ; \; \frac{d}{dt} x(\cdot) \epsilon L^2(0, T; H^m(D))\}$$

This, as might be expected, is a Hilbert space with inner product

$$\langle x(\cdot), y(\cdot) \rangle_{W(0, T)} = \langle x(\cdot), y(\cdot) \rangle_{L^2(0, T; H^m(D))} + \left\langle \frac{dx(\cdot)}{dt}, \frac{dy(\cdot)}{dt} \right\rangle_{L^2(0, T, H^m(D))}$$

We are now in a position to describe partial differential equations by ordinary differential equations in Sobolev space-valued functions. Two types of partial differential equations are considered--parabolic and hyperbolic. Parabolic equations are of the form:

$$\frac{\partial x(t,z)}{\partial t} = A\,x(t,z) + f(t,z)^{*} \qquad (2.4.1)$$

where A is an elliptic partial differential operator in the spatial variable z as described in Section 3. If $x(t,z)$, $t\epsilon[0,T]$, $z\epsilon D$ is assumed to be the element $x(t)\epsilon W(0,T)$, Eq. 2.4.1 has the equivalent formulation as the ordinary differential equation in $L^2(0,T;H^m(D))$

$$\frac{d}{dt}\,x(t) = A\,x(t) + f(t) \qquad (2.4.2)$$

where $\qquad f(\cdot)\epsilon L^2(0,T;\ L^2(D))$

As might be expected from knowledge of the finite-dimensional problem, an initial condition must be given so as to specify an exact solution of Eq. 2.4.2. If the initial data is given by $x(0,z)=x_o(z)$ where $x_o(z)$ has the representation $x_o\epsilon H_o^m(D)$, then Eq. 2.4.2 has the initial condition

$$x(0) = x_o \qquad (2.4.3)$$

As an example of a parabolic equation, consider the single degree of freedom heat diffusion equation

$$\frac{\partial x(t,z)}{\partial t} = \mu\,\frac{\partial^2 x(t,z)}{\partial z^2} \quad ; \qquad \mu = \text{constant}$$

where, of course, the operator A is $\frac{\partial^2}{\partial z^2}$, an elliptic operator, and $x(t,z)$ is a temperature distribution.

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* This description of distributed parameter systems is, of course, not complete without the specification of boundary conditions, which will be discussed in the next section.

Before discussing hyperbolic equations it is necessary to extend the previously defined state space to a two-dimensional form consisting of column vectors $(x_1(t), x_2(t))$, where, for each value of $t\epsilon[0,T]$, $x_i(t)\epsilon H^m(D)$, $i=1,2$. A more general spatial differential operator must also be defined, namely, a 2×2 matrix the elements of which are spatial differential operators as described in Section 2.3. The particular matrix operator to be discussed is:

$$\underset{\sim}{\mathcal{A}} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \tag{2.4.4}$$

where A is as defined above and I is the identity operator on $H^m(D)$.

We are now in a position to describe second order hyperbolic equations in terms of the state variables and state spaces of Section 2.2 Hyperbolic equations are of the form

$$\frac{\partial^2 x(t,z)}{\partial t^2} = A\,x(t,z) + f(t,z) \tag{2.4.5}$$

where A is elliptic. If $x(t,z)$ and $\frac{\partial x(t,z)}{\partial t}$ $t\epsilon[0,T]$, $z\epsilon D$ are element of $W(0,T)$, Eq. 2.4.5 has the first order vector ordinary differential equation representation:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underset{\sim}{\mathcal{A}}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \tag{2.4.6}$$

where $x_1(t) = x(t)$ $\qquad x_2(t) = \frac{dx(t)}{dt}$, and $f(\cdot)$ is assumed to be an element of $L^2(0,T;H^m(D))$.

Once again, initial conditions are required and this time they take the form of a 2-vector

$$\underline{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_o \\ \dot{x}_o \end{bmatrix} = \underline{x}_o \tag{2.4.7}$$

with x_o and $\dot{x}_o$ each being elements of $H^m(D)$ and representing the initial data $x(0,z)$ and $\frac{\partial}{\partial t}x(t,z)\,|\,t=0$, respectively.

An example of a hyperbolic equation is the equation which governs the magnitude of longitudinal vibrations in a rigid beam

$$\frac{\partial^2 x(t,z)}{\partial t^2} = \mu \frac{\partial^2 x(t,z)}{\partial z^2} \qquad \mu = \text{constant}$$

where $x(t,z)$ is the transverse deflection of the point z in the beam, at time t. The operator A is again the elliptic operator $\frac{\partial^2}{\partial z^2}$. It should be stressed that the operator A is elliptic in both parabolic and hyperbolic equations.

These two classes of partial differential equations, though not general enough to describe all linear distributed parameter systems, describe a great number of physical systems, and, such being the case, are worthy of being the equations of state considered in a system theoretic and, subsequently, control theoretic development. All of the elements analogous to system description in lumped parameter systems--namely, state, state-space, system operator, and state equation--have been introduced. One subject, boundary conditions, which are indigenous to distributed, but not in lumped, parameter systems, remains to be discussed in Section 5.

2.5 BOUNDARY CONDITIONS

This section is devoted to the discussion of boundary conditions to partial differential equations. This is a slight deviation from the stated purpose of this chapter--the development of a system theoretic approach parallel to that of finite dimensional systems--but one which is necessary for the sake of completeness. It will be shown that boundary conditions can be treated within the framework of the system theoretic notions developed in the preceding sections. Dirichlet and Neumann boundary conditions will be defined and, for the case of Dirichlet boundary conditions, the compatibility with the state space conditions already given will be demonstrated in detail.

If the differential operator A in either the parabolic system (2.4.2) and (2.4.3) or the hyperbolic system (2.4.6) and (2.4.7) is of order m, then the Dirichlet boundary conditions, defined on the boundary ∂D of the region D, are

$$x(t)\Big|_{\partial D} = \frac{\partial x}{\partial n}(t)\Big|_{\partial D} = \dots = \frac{\partial^{m-1} x(t)}{\partial n^{m-1}}\Bigg|_{\partial D} = 0 \;\forall\; t \in [0, T] \tag{2.5.1}$$

where n denotes the normal to boundary ∂D and $\frac{\partial^k}{\partial n^k}$ is the k^{th} derivative normal to, and directed to the exterior of, the boundary. As an example of Dirichlet boundary conditions, let us consider the heat equation defined on the unit circle in R^2, that is, the equation

$$\frac{\partial x(t,z)}{\partial t} = \mu\left[\frac{\partial^2 x(t,z)}{\partial z_1^2} + \frac{\partial^2 x(t,z)}{\partial z_2^2}\right] \quad ; \quad \mu = \text{constant}$$

where the spatial variable z is the vector $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$; and the spatial domain D is

$$D = \{z \in R^2 : z_1^2 + z_2^2 < 1\}$$

The boundary ∂D is, of course,

$$\partial D = \{z \in R^2 : z_1^2 + z_2^2 = 1\}$$

The Dirichlet boundary conditions tell us that

$$x(t,z)\Big|_{z \in \partial D} = 0$$

or, the temperature on the circle $z_1^2 + z_2^2 = 1$ is required to be 0. Moreover, since the order of the system, m, is 2, we have the remaining condition

$$\left.\frac{\partial x(t,z)}{\partial n}\right|_{z\in\partial D}=\left.\frac{\partial x(t,z)}{\partial z_1}\right|_{z\in\partial D}\cos\theta+\left.\frac{\partial x(t,z)}{\partial z_2}\right|_{z\in\partial D}\sin\theta=0,\ \forall\ \theta\in[0,2\pi]$$

which tells us that the component of the gradient normal to the unit circle must be 0, i.e., no heat flow outward through the boundary.

In order to incorporate this within the framework of the theory discussed in Section 2.2, we must first develop the concept of Sobolev spaces of negative and fractional orders. The Sobolev space of negative order $H^{-m}(D)$ can simply be looked upon as the dual space of the Sobolev space of positive order $H^m(D)$, or $(H^m(D))' = H^{-m}(D)$. Fractional order Sobolev spaces are defined by means of Fourier analysis. If z is the spatial variable, which is an element of R^n, then the Fourier transform of $x(z)$, $\mathcal{F}x(\zeta)$ is

$$\mathcal{F}\ x(\zeta)\ =\int_D \exp(2\pi j(\zeta\cdot z))x(z)dz \tag{2.5.2}$$

where $(\zeta\cdot z)$ is the usual vector inner product on R^n. It is shown that the Fourier transform of the differential operator D^q operating on $x(z)$, $\mathcal{F}\ D^q x(\zeta)$ is of the form

$$\mathcal{F}\ D^q x(\zeta)\ =\ (2\pi j)^{|q|}\zeta^q\mathcal{F}x(\zeta),\ \forall\ x\in L^2(D) \tag{2.5.3}$$

where ζ^q is the product defined in Section 2. This results in an alternative definition of the Sobolev space $H^m(D)$, namely

$$H^m(D)\ =\ \{x : \zeta^q\mathcal{F}x\in L^2(D)\ \forall\ q\ \text{ with }\ |q|\le m\}$$

or, equivalently,

$$H^m(D)\ =\ \{x : (1+|\zeta|^2)^{m/2}\mathcal{F}x(\zeta)\in L^2(D)\} \tag{2.5.4}$$

There is no restriction in allowing m to be any real number in Expression (2.5.4), rather than requiring it to be a whole number in Section 2. Thus, we have arrived at the specification of fractional order Sobolev spaces. These are again Hilbert spaces with inner product given by

$$(x, y)_{H^m(D)} = ((1+|\zeta|^2)^{m/2}\mathcal{F}x, (1+|\zeta|^2)^{m/2}\mathcal{F}y)_{L^2(D)}$$

The theorem of the trace, stated and proved by Lions and Magenes,[1] yields the information that the normal derivatives $\left.\frac{\partial^k x}{\partial n^k}\right|_{\partial D}$ given in Eq. 2.5.1 are elements of the fractional Sobolev spaces $H^{m-k-1/2}(\partial D)$, $0 \leq k \leq m-1$, if $x \epsilon H^m(D)$, and the transformation $x \to \{\left.\frac{\partial^k x}{\partial n^k}\right|_{\partial D}, 0 \leq k \leq m-1\}$ is a continuous linear surjection of $H^m(D)$ onto the product space $\prod_{k=0}^{m-1} H^{m-k-1/2}(\partial D)$. The kernel of this transformation, that is, the space of $x \epsilon H^m(D)$ for which $x\big|_{\partial D} = \left.\frac{\partial x}{\partial n}\right|_{\partial D} = \ldots = \left.\frac{\partial^{m-1} x}{\partial n^{m-1}}\right|_{\partial D} = 0$, is the closure of the space of infinitly differentiable function of compact support in D, $C_o^\infty(D)$, in the norm of $H^m(D)$. Dunford and Schwartz (see Ref. 2.1, p. 1652) denote this closure as $H_o^m(D)$, so that we have shown that we can represent this closure of $C_o^\infty(D)$ in the following manner:

$$H_o^m(D) = \{x \epsilon H^m(D) : \left.\frac{\partial^k x}{\partial n^k}\right|_{\partial D} = 0, \quad 0 \leq k \leq m-1\} \qquad (2.5.5)$$

Since $H_o^m(D)$ is a closed subspace of $H^m(D)$, and therefore a Hilbert space (with the inner product of $H^m(D)$), it may just as easily be considered as a candidate for a state space, in the sense of Section 2.2, as $H^m(D)$. Thus, the additional consideration of Dirichlet boundary conditions does not divert our course from that of developing a system theory analogous to that of finite dimensional systems.

The Neumann boundary value problem is associated with a second order elliptic operator of the form

$$A = -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial z_i}\left(a_{ij}(z)\frac{\partial}{\partial z_j}\right) + a_o(z) \qquad (2.5.6)$$

with the coercivity property, Inequality 2.3.3, requiring that there exist an $\alpha > 0$ such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(z)\zeta_i\zeta_j \geq \alpha(\zeta_1^2 + \zeta_2^2 + \ldots + \zeta_n^2) \quad ; \quad \text{for all } \underline{\zeta}\epsilon R^n \text{ and } z\epsilon D$$

and

$$a_o(z) \geq \alpha > 0 \quad ; \quad \text{for all } z\epsilon D$$

The Neumann boundary condition relative to A is

$$\left.\frac{\partial x}{\partial \nu_A}\right|_{\partial D} = g$$

where $\frac{\partial x}{\partial \nu_A} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial x}{\partial z_j}\cos(n, z_j)$, n is the normal to the exterior of ∂D at $z\epsilon\partial D$, and therefore $\cos(n, z_j)$ is the j^{th} direction cosine; g is a specified function. Since by the theorem of the trace, discussed above for the Dirichlet problem $\left.\frac{\partial x}{\partial \nu_A}\right|_{\partial D}$ must be an element of $H^{-1/2}(\partial D)$, so must it be true that $g\epsilon H^{-1/2}(\partial D)$. In this case the kernel of the transformation $x \rightarrow \left.\frac{\partial x}{\partial \nu_A}\right|_{\partial D} - g$ is not so readily identifiable as was the case for the Dirichlet transformation, however, direct use of this kernel itself will not cause too many analytical stumbling blocks.

Ellipticity of the system operator for both types of boundary conditions is required to prove existence and uniqueness of solutions for either parabolic or hyperbolic systems.[15, 20] The property of strong ellipticity will be used to derive a very useful system theoretic result in Section 2.7 and optimization results in Chapter IV.

2.6 SEMIGROUPS OF OPERATORS

This section and the final two sections of this chapter are devoted to semigroups of operators and the systems which generate them. It will complete the system theoretic description of distributed parameter systems by giving the distributed parameter analog to transition matrices and variation of constants formulae of finite dimensional system theory. In this section we shall consider semigroups of operators defined on a general Banach space $\mathcal{X}$ with range in $\mathcal{X}$. It will be useful in the sequel to consider these operators as elements of a space of operators, $\mathcal{E}(\mathcal{X})$, the space of endomorphisms on the Banach space $\mathcal{X}$. Let us make the following definition:

Definition 2.6: A mapping $\Phi(t) : [0,\infty] \to \mathcal{E}(\mathcal{X})$, denoted by $\{\Phi(t)\}_{t\in[0,\infty)}$, is called a one-parameter semigroup of endomorphisms with parameter $t\in[0,\infty)$, if for all $t_1, t_2 \in [0,\infty)$

$$\Phi(t_1+t_2) = \Phi(t_1)\Phi(t_2) \tag{2.6.1}$$

Equation 2.6.1 is called the semigroup property, and the set of operators $\{\Phi(t)\}_{t\in[0,\infty)}$ with the semigroup property will be referred to as a semigroup of operators as a matter of convenience.

Two different types of semigroups of operators may be considered, depending on the manner in which $\Phi(t)$ converges as t approaches zero. The convergence may be uniform in the operator topology of $\mathcal{E}(\mathcal{X})$, or more specifically, $\lim_{t\to 0} \|\Phi(t) - \Phi(0)\| = 0$, where the norm is the usual induced operator norm on $\mathcal{X}$. For this case of uniform convergence the procedure of characterizing the semigroup of operators is quite straightforward and stands as a direct analogy to the description of the matrix $e^{\underline{A}t}$ in finite dimensional systems. The other type of semigroup to be considered is one in which the convergence as t approaches zero is strong, or $\lim_{t \to 0} \|\Phi(t)x - \Phi(0)x\| = 0 \; \forall \; x \in \mathcal{X}$. With strong convergence the analysis is much less straightforward.

The Banach space of endomorphisms on $\mathcal{X}$, $\mathcal{E}(\mathcal{X})$, is a Banach algebra, and Hille and Phillips (Ref. 21, p. 283) show that for any Banach algebra $\mathcal{B}$ and uniformly continuous $f(t) : [0,\infty) \to \mathcal{B}$ such that

$$f(t_1+t_2) = f(t_1)f(t_2) \quad , \quad \text{for } t_1, t_2 \in [0,\infty)$$

then $f(t)$ must be of the form

$$f(t) = I + \sum_{n=1}^{\infty} \frac{t^n}{n!} a^n \qquad (2.6.2)$$

where I is the unit (identity) element of the Banach algebra $\mathcal{B}$ and a is some unique element of $\mathcal{B}$. The series is absolutely convergent for all $t \in [0,\infty)$. This result can be specialized, of course, to the Banach algebra of prime interest, namely, $\mathcal{B} = \mathcal{E}(\mathcal{X})$. Any uniformly convergent semigroup of operators $\{\Phi(t)\}_{t \in [0,\infty)}$ can be represented by the expression

$$\Phi(t) = \exp(tA) \qquad (2.6.3)$$

where A is a bounded operator in $\mathcal{E}(\mathcal{X})$ and the exponential expression follows from Eq. 2.6.2.

An important relation exists between the resolvent of the operator A and the Laplace transformation of the semigroup. The <u>resolvent</u> is the operator $R(\lambda; A) = (\lambda I - A)^{-1}$ defined for all values of λ for which the inverse exists. It can be shown (see, for example, Ref. 21, p. 338) that the resolvent operator is the Laplace transform of the semigroup operator

$$R(\lambda; A) = \int_0^{\infty} e^{-\lambda t}\Phi(t)\,dt$$

for all λ with $|\lambda| \geq \|A\|$, and, moreover, as might be expected from knowledge of Laplace transform theory,

$$\Phi(t) = \frac{1}{2\pi j}\int_{\Gamma} e^{\lambda t} R(\lambda; A)\,d\lambda$$

where Γ is a closed path surrounding the spectrum of A in the clockwise sense.

The operator A is called the infinitesimal generator of the semigroup and special note should be taken of the fact that it is bounded. It is also important to note that every bounded operator in $\mathcal{E}(\mathcal{X})$ is the infinitesimal generator of a uniformly convergent semigroup of operators. This leads to the conclusion that unbounded (or, more particularly, differential) operators do not generate uniformly convergent semigroups of operators, so that attention naturally becomes focused on strongly convergent semigroups of operators.

In order to characterize strongly convergent semigroups of operators, we first make the following definition:

Definition 2.7: The infinitesimal operator A_o of a semigroup $\{\Phi(t)\}_{t\in[0,\infty)}$ is defined by

$$A_o x = \lim_{\eta\to 0} A_\eta x \tag{2.6.4}$$

where $$A_\eta = \frac{1}{\eta}[\Phi(\eta) - I]$$

whenever the limit in (2.6.4) exists.

The set of $x \in \mathcal{X}$ for which the limit exists is simply called the domain of A_o, $Do(A_o)$. We would ideally like to achieve an exponential characterization of the semigroup as in the uniform case, but in this case the candidate for infinitesimal generator, A_o, is not bounded (the domain of A_o is not necessarily all of $\mathcal{X}$) so that an exponential expression involving A_o would be meaningless.* Aid in this dilemma lies in the fact that the operators A_η given in Definition 2.7 are bounded operators so that we might expect the exponential solution we desire to be some kind of limit of exponential expressions involving the A_η's. It is shown (see Ref. 22 , p. 401) that a limiting exponential solution does exist, namely

$$\Phi(t)x = \lim_{\eta \to 0} \exp(tA_\eta)x \quad \forall \quad x \in Do(A_o) \tag{2.6.5}$$

where the convergence is uniform with respect to t in every finite interval $[0, s]$. So every strongly convergent semigroup has the characterization (2.6.5).

The most important question of all, at least for our purposes, is under what conditions will an unbounded operator A be the infinitesimal generator of a strongly convergent semigroup of operators? The Hille-Yosida theorem (Ref.21,p.363) tells us that a sufficient condition for a closed linear operator A to be the infinitesimal generator of a semigroup $\{\Phi(t)\}_{t\in[0,\infty)}$ such that $\|\Phi(t)\| \leq M$ is that the domain of A be dense in $\mathcal{X}$ and the following inequality holds:**

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* Despite this, we shall use infinitesimal generator and infinitesimal operator interchangeably.

** The Inequality 2.6.6 is a sufficient condition for the inverse Laplace transform of $R(\lambda; A)$ to exist. This inverse transform is the semigroup $\{\Phi(t)\}_{t\in[0,\infty)}$.

$$\|(\lambda I-A)^{-1}\| \leq M\lambda^{-n} \quad \text{for} \quad \lambda > 0 \qquad \text{and} \quad n=1,2,3,\ldots \qquad (2.6.6)$$

We now have the tools to determine whether the spatial differential operators of Section 3 of this chapter are infinitesimal generators of semigroups. This is the direct concern of the next section.

2.7 DIFFERENTIAL OPERATORS AS INFINITESIMAL GENERATORS OF SEMIGROUPS

This section relates what has been stated about semigroups of operators with the characterization of solutions to partial differential equations. For systems containing elliptic operators of even order Dunford and Schwartz[18] establish the connection by giving the necessary conditions for the differential operator to be the infinitesimal generator of a semigroup of operators and showing that the solution of the unforced parabolic equation associated with the spatial differential operator at time t is simply the operator $\Phi(t)$ operating on the initial data.

To qualify for infinitesimal generator the system operator A, given by Eq. 2.3.1, must satisfy a condition which is a slight modification of the condition for strong ellipticity given in Section 2.3, namely,

$$(-1)^{m/2} \sum_{|q|=m} a_q(z)\,\zeta^q \leq 0 \quad \text{for all} \quad \zeta\epsilon R^n,\ z\epsilon D \qquad (2.7.1)$$

Additional restrictions must be placed on the state space to be considered. First, let us define two restrictions A_1 and A_2 of the operator A which have the following properties:

$$\begin{aligned} Do(A_1) &= C_o^{\infty}(D) \quad ; \quad A_1 x = Ax \ \forall\ x\epsilon C_o^{\infty}(D) \\ Do(A_2) &= H^m(D) \quad ; \quad A_2 x = Ax \ \forall\ x\epsilon H^m(D) \end{aligned} \qquad (2.7.2)$$

We now define the extension A_3 of A_1 which has the property

$$\mathrm{Do}(A_3) = H^m(D) \cap H_o^{m/2}(D) \quad ; \quad A_3 x = A_2 x \;\forall\; x \in \mathrm{Do}(A_3) \tag{2.7.3}$$

where $H_o^{m/2}(D)$ is specified in Section 5. Note that the problem of Dirichlet has entered with the introduction of the space $H_o^{m/2}(D)$.

With these assumptions on the operator A and on the state space to be considered, Dunford and Schwartz [18] prove a theorem, stated in detail in Appendix A, which, in summary, yields the following results:

(1) A_3 is the infinitesimal generator of a semigroup of bounded operators $\{\Phi(t)\}_{t\in[0,\infty)}$

(2) If $x_o \in \mathrm{Do}(A_3)$ is the initial condition for the equation $\dot{x} = Ax$, then the solution is $x(t) = \Phi(t)x_o$

It is clear that the differential operator $\partial^2/\partial z^2$, which is the system operator for both the one-dimensional heat equation and the transverse beam vibration equation, satisfies the Inequality 2.7.1, and, therefore, by the result of this section, is an operator having a restriction A_3 which is the infinitesimal generator of a semigroup of operators.

Note that the condition for strong ellipticity, Inequality 2.3.4, is the condition for strict inequality in Expression 2.7.1. If $\{\Phi(t)\}_{t\in[0,\infty)}$ is the semigroup of operators generated by a strongly elliptic operator, following the above procedure, then the bounded operator $\Phi(t)$ has the exponential bound

$$\|\Phi(t)\| \leq Me^{-\lambda t}$$

where M and λ are positive constants.

Thus, we are able to characterize the solutions to unforced partial differential equations with the aid of a distributed parameter equivalent of the finite dimensional transition matrix. It remains to characterize solutions of the forced partial differential equation.

2.8 VARIATION OF CONSTANTS FORMULA FOR DISTRIBUTED PARAMETER SYSTEMS

We are now in a position to characterize solutions of distributed parameter systems described by forced parabolic and hyperbolic partial differential equations. This characterization will take an analogous form to the variation of constants formula familiar in finite dimensional system theory and will complete the system theoretic description for distributed parameter systems.

Phillips[23] proves a theorem, stated specifically in Appendix B, which yields a variation of constants formula for the parabolic system described by Eqs. 2.4.2 and 2.4.3. The necessary assumption is that the system operator A is the infinitesimal generator of a semigroup of operators $\{\Phi(t)\}_{t\in[0,\infty)}$ as described in the preceding section. The result is that the solution of the equation

$$\dot{x}(t) = Ax(t) + f(t) \quad ; \quad x(0) = x_o$$

is
$$x(t) = \Phi(t)x_o + \int_0^t \Phi(t-\sigma)\, f(\sigma)\,d\sigma \tag{2.8.1}$$

The only requirement on $f(t)$ for this characterization to be valid is that $f(t)$ be <u>strongly continuously differentiable</u>.* Of course, from the arguments of Section 7, the initial condition x_o must be in the domain of the operator A_3 defined in that section.

* That is, $\lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$ exists in the strong topology of $L^2(0,T;L^2(D))$.

A similar result can be achieved for forced hyperbolic systems of the type represented by vector Eqs. 2.4.6 and 2.4.7. Fattorini[5] shows that the solution to this hyperbolic system can be written in a form similar to that of Eq. 2.8.1 by first introducing the two strongly continuous operator-valued functions $\Phi_1(t)$ and $\Phi_2(t)$. $\Phi_1(t)$ is the operator function which is obtained by writing the solution $x(t)$ of Eq. 2.4.6 with $f=0$ and with boundary condition $\underline{x}_o = \begin{bmatrix} x_o \\ 0 \end{bmatrix}$ in the form

$$x(t) = \Phi_1(t)\, x_o \tag{2.8.3}$$

Let us denote the solution of Eq. 2.4.6 with $f=0$ and with initial condition $\underline{x}_o = \begin{bmatrix} 0 \\ x_o \end{bmatrix}$ as $v(t)$ and write $v(t)$ in the form

$$v(t) = \Phi_2(t) x_o \tag{2.8.4}$$

It is clearly seen that $\Phi_1(t)$ and $\Phi_2(t)$ are related as

$$\Phi_2(t)x_o = \int_0^t \Phi_1(\sigma) x_o d\sigma$$

Now, if f is twice continuously differentiable and if $\underline{x}_o$ is $\underline{0}$, the solution of (2.4.6) can be shown to be

$$x(t) = \int_0^t \Phi_2(t-\sigma) f(\sigma) d\sigma \tag{2.8.5}$$

Combining (2.8.3), (2.8.4), and (2.8.5) with the general initial condition given by (2.4.7) the solution of (2.4.6) is given by

$$x(t) = \Phi_1(t)x_o + \Phi_2(t)\dot{x}_o + \int_0^t \Phi_2(t-\sigma) f(\sigma) d\sigma \tag{2.8.6}$$

It is quite reasonably asked whether the operator-valued functions $\Phi_1(t)$ and $\Phi_2(t)$ are semigroups or not, and, if so, how are they generated?

The answer is that they are not exactly semigroups, but are very closely related to them. Fattorini[5] proves that the resolvent of A is not the Laplace transform of either $\Phi_1(t)$ or $\Phi_2(t)$, but the following relation exists between the Laplace transforms of $\Phi_1(t)$ and $\Phi_2(t)$ and the operator $R(\lambda^2; A) = (\lambda^2 I - A)^{-1}$:

$$\begin{aligned} \int_0^\infty e^{-\lambda t}\Phi_1(t)x(dt &= \lambda R(\lambda^2; A)x \\ & \qquad\qquad\qquad x \in Do(A_3) \qquad (2.8.7) \\ \int_0^\infty e^{-\lambda t}\Phi_2(t)xdt &= R(\lambda^2; A)x \end{aligned}$$

Moreover, there exist constants K and $\omega < \infty$ such that $\|\Phi_1(t)\| \leq Ke^{\omega t}$, $\|\Phi_2(t)\| \leq Ke^{\omega t}$. The variable λ^2 appears in the resolvent expression because we are dealing with a second order time derivative in the system equation.

The value of having variation of constants formulae like Eqs. 2.8.1 and 2.8.6 does <u>not lie</u> in having exact specification of solutions to partial differential equations, but in having a specific form of the solution will become extremely useful in the optimization results presented in Chapter IV.

CHAPTER III

FORMULATION OF THE CONTROL PROBLEM

3.1 INTRODUCTION

This chapter is concerned with the mathematical description of the problems which will be solved in Chapters IV and V. For both parabolic and hyperbolic systems the state regulator problem will be introduced. The set of admissible controls will be defined and the quadratic cost criterion will be specified. This cost criterion will be shown to be analogous to the quadratic cost criterion customarily specified for a finite dimensional system. In addition, the restriction of the class of controls to those which are applied at a finite number of points within the spatial domain is considered and the subsequent modification of the cost criterion will be specified.

In Section 3.2 of this chapter precise descriptions of both the system and the control space are given. This will correspond to the state equation description for finite dimensional systems in the form $\underline{\dot{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$. Conditions on the distributed parameter analog of the $\underline{B}$ matrix are specified.

Section 3.3 is concerned with the remainder of the formulation of the distributed parameter state regulator problem--namely, the introduction of and justification for a meaningful quadratic cost criterion for the systems described in Section 3.2.

Section 3.4 contains the restriction of the set of controls to a finite dimensional space as described above--a restriction to be called the pointwise control problem.

3.2 THE SPACE OF CONTROLS

The general parabolic and hyperbolic systems to be considered are given by Eqs. 2.4.2 and 2.4.3 for the parabolic case and by Eqs. 2.4.6 and 2.4.7 for the hyperbolic case. Moreover, we shall restrict ourselves to the case of Dirichlet boundary conditions given by Eq. 2.5.1. This will result in the consideration of the Hilbert space $H_o^m(D)$, rather than $H^m(D)$, for the state space for the system as described in Section 2.5. This is not a severe restriction and does not fundamentally affect the generality of the results, since, as was mentioned in Section 2.5, other types of boundary value problems can be placed within a Hilbert space framework similar to that of $H_o^m(D)$ in the Dirichlet problem. The system operator A is assumed to be either coercive or the infinitesimal generator of a semigroup of operators as discussed in Chapter II.

With these assumptions we enter into a discussion of the form of the forcing function $f(\cdot)$ appearing in both Eqs. 2.4.2 and 2.4.6, which we rewrite

$$\frac{dx}{dt} = Ax(t) + f(t) \tag{2.4.2}$$

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathcal{A}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \tag{2.4.6}$$

Note that $f(\cdot)$ is required to lie in the function space $L^2(0, T;\ L^2(D))$.*
In order to put the forcing term in a form which appears more commonly in system theoretic notation, let us introduce the control $u(t)$, where,

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* The exact requirement is that for each instant of time t, $f(t)$ must be an element of $H_o^{m'}(D) = H^{-m}(D)$. Since $L^2(D) \subset H^{-m}(D)$ there is no great loss of generality and the attractiveness of using $L^2(D)$ is overwhelming.

for each instant t, we require $u(t)$ to be an element of a Hilbert space U, the control space. Moreover, let us assume that $u(\cdot)\epsilon L^2(0,T;U)$. To complete the description of the forcing term we define $B(t)$ as a bounded linear operator defined, for each value of $t\epsilon[0,T]$, on the control space U with range in $L^2(D)$, or, in more convenient notation $B(t)\epsilon \mathcal{L}(U;L^2(D))\forall t\epsilon[0,T]$. We now make the identification of the forcing term $f(t)$ as

$$f(t) = B(t)u(t) \quad , \forall\, t\epsilon[0,T] \tag{3.2.1}$$

The parabolic system now becomes:

$$\frac{dx(t)}{dt} = Ax(t) + B(t)u(t),\ x(0) = x_o \epsilon H_o^m(D) \tag{3.2.2}$$

And if the "vector" operator $\underline{\mathcal{B}}(t)$ is defined to be

$$\underline{\mathcal{B}}(t) = \begin{bmatrix} 0 \\ B(t) \end{bmatrix}$$

then the hyperbolic system is represented by

$$\frac{d\underline{x}(t)}{dt} = \underline{\mathcal{A}}\ \underline{x}(t) + \underline{\mathcal{B}}(t)u(t) \quad ; \quad \underline{x}(0) = \underline{x}_o \quad , x_{oi}\epsilon H_o^m(D) \tag{3.2.3}$$

One further assumption must be made--$B(t)u(t)$ is assumed to be a strongly (in $L^2(0,T;L^2(D))$), continuously differentiable function of t. This assumption will enable us to use the variation of constants formula given in Section 2.8 (Eq. (2.8.1)).

With the control u defined and the manner in which u enters the parabolic and hyperbolic systems clarified, we proceed to the formulation of quadratic optimization problems for these systems in Section 3

3.3 QUADRATIC CRITERIA FOR PARABOLIC AND HYPERBOLIC SYSTEMS

In this section quadratic cost criteria weighing the state and the control introduced in the previous section are presented for systems

(3.2.2) and (3.2.3). These criteria will be seen to be directly analogous to finite dimensional quadratic cost criteria. The choice of a quadratic cost criterion is sometimes motivated by practical considerations. In many applied distributed parameter control problems of interest it is not feasible to consider driving the system to a fixed final state distribution, but rather one would have deviations from a desired distribution be damped out by the control system. Thus, a weighted sum of the deviation of the state from the desired distribution and the magnitude of the control is the necessary type of criterion. The particular choice of a quadratic cost criterion also stems from the hindsight that what yielded such elegant results as linear feedback control laws in the lumped optimal control theory should yield at least some fraction of the same in distributed optimal control theory.

As a preliminary to the specification of a quadratic cost criterion for the problems under consideration we make the following definitions:

Definition 3.1: The bounded linear operator $Q(t)$, defined for all $t \in [0, T]$ on the Sobolev space $H_o^m(D)$ with range in $H_o^m(D)$, is called the state weighting operator. $Q(t)$ is assumed to be a self-adjoint positive operator, that is, $Q(t)$ has the properties:

1. $$Q(t) = Q^*(t) \quad , \forall t \in [0, T]$$

2. $$\langle x, Q(t)x \rangle_{H_o^m(D)} \geq 0 \quad \forall\, x \in H_o^m(D), \quad \forall\, t \in [0, T]$$

The form $\langle x, Q(t)x \rangle_{H_o^m(D)}$ corresponds, for each $t \in [0, T]$, to a positive weighted average over the spatial domain D. This spatial weighting is, of course, imbedded in the Hilbert space notation. It is seen that $Q(t)$ corresponds directly to the positive semidefinite state weighting matrix $\underline{Q}(t)$ for the finite dimensional state regulator problem treated by Athans and Falb (Ref. 24, Chapter 9). We make the further definition:

Definition 3.2: The bounded linear operator $R(t)$, defined for all $t \epsilon [0, T]$ on U with range in U, is called the control weighting operator. $R(t)$ is assumed to be a self-adjoint strictly positive operator, or

1. $R(t) = R^*(t) \quad , \forall \; t \epsilon [0, T]$

2. $\langle u, R(t)u \rangle_U \geq \alpha \|u\|_U^2 \quad$, for some $\alpha > 0, \forall \; u \epsilon U, \forall \; t \epsilon [0, T]$

Once again, this corresponds to the positive definite control weighting matrix $\underline{R}(t)$ for the finite dimensional problem and the form $\langle u, R(t)u \rangle_U$ is a weighted average over the spatial domain D. It is sometimes desirable to add a penalization cost for deviation of the state distribution at the final time T from the desired distribution. For this case we have:

Definition 3.3: The bounded linear operator F, defined on $H_o^m(D)$ with range in $H_o^m(D)$, is called the terminal state weighting operator. F is assumed to be a self-adjoint positive operator, or

1. $F = F^*$

2. $\langle x, Fx \rangle_{H_o^m(D)} \geq 0 \quad \forall \; x \epsilon H_o^m(D)$

Not surprisingly, the operator F corresponds to the terminal state weighting matrix $\underline{F}$ in the finite dimensional regulator problem and, of course, the form $\langle x, Fx \rangle_{H_o^m(D)}$ is a weighted average over the spatial domain D.

If we denote the desired state distribution as $x_d(t) \epsilon H_o^m(D)$, we may now state the cost criterion for parabolic systems as:

$$J=\int_0^T [\langle(x(t)-x_d(t)), Q(t)(x(t)-x_d(t)\rangle_{H_o^m(D)}$$

$$+ \langle u(t), Ru(t)\rangle_U] dt + \langle (x(T)-x_d(T)), F(x(T)-x_d(T))\rangle_{H_o^m(D)} \tag{3.3.1}$$

where $x(t)$ is the solution of (3.2.2) with the control sequence $u(t)\epsilon U$, $t\epsilon[0,T]$, specified. The optimal control problem may then be defined:

Definition 3.4: The optimal control problem for the system (3.2.2) is to determine the control $u^*(t)$, $t\epsilon[0,T]$, with $u^*(t)\epsilon U$ for all $t\epsilon[0,T]$ such that, if $x^*(t)$ is the solution of (3.2.2) with $u(t)=u^*(t)$, the functional J in (3.3.1) is minimized. The minimizing control $u^*(t)$, $t\epsilon[0,T]$, is called the optimal control (if it exists).

As an example of an optimal control problem for a parabolic system, let us consider the heat equation, given in Section 2.4, with the control $u(t)$ entering in a forcing term. Assuming Dirichlet boundary conditions for this problem, we have:

$$\dot{x} = Ax(t)+B(t)u(t) \quad ; \quad x(0) = x_o\epsilon H_o^2 (D)$$

where A is the operator defined on $H_o^2(D)$, corresponding to the spatial differential operator $\mu\frac{\partial^2}{\partial z^2}$. Let us choose $x_d(t) = 0$ and the cost criterion to be such that we penalize mean square deviation of the state trajectory from zero and total expended control energy, that is, we choose a criterion of the form:

$$J = \int_0^T \left[\int_D x^2(t,z)dz + r\int_D u^2(t,z)dz\right]dt \quad ; \quad r\epsilon R^1, \ r > 0$$

This can be put within the framework of the optimal control problem specified in Definition 3.4 if we choose $Q(t)$, $R(t)$, and F to be:

1. $Q(t)$ is the identity operator on $H_o^2(D)$, which can be written as the integral operator

$$Q(t)x(t) = I_{H_o^2(D)}x(t) = \int_D \delta(z-\zeta)x(t,\zeta)d\zeta \quad , \quad \forall \; x(t) \in H_o^2(D)$$

where $\delta(z-\zeta)$ is Dirac delta function.

2. $R(t)$ is the identity operator on U, multiplied by the scalar r, or

$$R(t)u(t) = rI_U u(t) = r\int_D \delta(z-\zeta)u(t,\zeta)d\zeta \quad , \quad u(t)\in U$$

3. F is the zero operator

With these choices of $Q(t)$, $R(t)$, and F the cost criterion of Eq. 3.3.1 is seen to be the desired cost criterion.

The preceding discussion must be modified somewhat to achieve the definition of the control problem for hyperbolic systems. As a preliminary to this modification, let us consider a general 2×2 matrix operator $\underline{\mathcal{M}}$ whose elements M_{ij} are bounded linear operators on a Hilbert space H. $\underline{\mathcal{M}}$ operates on the two dimensional vector $\underline{x}$ the components of which are elements of H. It is useful to define the inner product $\langle \underline{x}, \underline{\mathcal{M}}\,\underline{x} \rangle = \underline{x}'\underline{\mathcal{M}}\,\underline{x}$ as

$$\underline{x}'\underline{\mathcal{M}}\,\underline{x} = \sum_{i=1}^{2}\sum_{j=1}^{2} \langle x_i, M_{ij}x_j \rangle_H \tag{3.3.2}$$

We are now in a position to make the modifications of Definitions 3.1, 3.2, and 3.3 to fit the hyperbolic case, beginning with the definition of the state weighting matrix operator:

Definition 3.5: The 2×2 matrix $\mathcal{Q}(t)$, the elements of which, $Q_{ij}(t)$, are bounded linear operators defined, for all $t \epsilon [0,T]$, on $H_o^m(D)$ with range in $H_o^m(D)$, is called the state weighting matrix operator. $\mathcal{Q}(t)$ is assumed to be a symmetric positive semidefinite matrix with self-adjoint elements, or

1. $Q_{ij}(t) = Q_{ji}(t) \quad , \quad \forall\, t \epsilon [0,T] \quad , \quad i,j = 1,2$

2. $\underline{x}'\mathcal{Q}(t)\underline{x} = \sum_{i=1}^{2} \sum_{j=1}^{2} < x_i, Q_{ij}(t)x_j >_{H_o^m(D)} \geq 0 \quad , \quad \forall\, x_i \epsilon H_o^m(D),$

$i = 1,2 \quad , \quad \forall\, t \epsilon [0,T]$

3. $Q_{ij}(t) = Q_{ij}^*(t) \quad , \quad \forall\, t \epsilon [0,T]$

There is no need to modify Definition 3.2 for the cost weighting operator, since the control space is the same for both parabolic and hyperbolic systems. However, the terminal state weighting operator of Definition 3.3 must be modified as follows:

Definition 3.6: The 2×2 matrix $\mathcal{F}$, the elements of which, F_{ij}, are bounded linear operators defined on $H_o^m(D)$ with range in $H_o^m(D)$, is called the terminal state weighting matrix operator. $\mathcal{F}$ is assumed to be symmetric positive semidefinite with self-adjoint elements,

1. $F_{ij} = F_{ji} \quad , \quad i,j = 1,2$

2. $\underline{x}'\mathcal{F}\underline{x} = \sum_{i=1}^{2} \sum_{j=1}^{2} < x_i F_{ij} x_j \geq 0 \quad \forall\, x_i \epsilon H_o^m(D) \quad , \quad i=1,2$

3. $F_{ij} = F_{ij}^* \quad i,j = 1,2$

If we now denote the desired state vector as $\underline{x}_d(t)$, the cost criterion for parabolic systems is given by:

$$J = \int_0^T [(\underline{x}(t)-\underline{x}_d(t))'\mathcal{Q}(t)(\underline{x}(t)-\underline{x}_d(t)) + \langle u(t), R(t)u(t)\rangle_U]dt$$

$$+ (\underline{x}(T)-\underline{x}_d(T))'\mathcal{F}(\underline{x}(T)-\underline{x}_d(T)) \qquad (3.3.3)$$

where $\underline{x}(t)$ is the solution of (3.2.3) with the control sequence $u(t)\epsilon U$, $t\epsilon[0,T]$, specified. Just as in the case for parabolic systems, the optimal control problem is similarly defined for hyperbolic systems.

Definition 3.7: The optimal control problem for the system (3.2.3) is to determine the control $u^*(t), t\epsilon[0,T]$ such that, if $\underline{x}^*(t)$ is the solution of (3.2.3) with $u(t) = u^*(t)$, the functional J in (3.3.3) is minimized. The minimizing control $u^*(t)$, $t\epsilon[0,T]$, is called the optimal control (if it exists).

As an example of an optimal control problem for a hyperbolic system, we consider the forced equation for longitudinal vibrations in a rigid beam, the unforced version of which is given in Section 2.4. Assuming Dirichlet boundary conditions, the equation may be written in the form of Eq. 3.2.3, namely:

$$\frac{d\underline{x}(t)}{dt} = \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ A & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ B(t) \end{bmatrix} u(t) \quad ; \quad \underline{x}(0) = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \quad x_{01}, x_{02} \epsilon H_o^2(D)$$

where A is the operator on $H_o^2(D)$ corresponding to the differential operator $\mu \frac{\partial^2}{\partial z^2}$. In this example let us choose $\underline{x}_d(t) = \underline{0}$ and have the cost criterion penalize both the mean square derivation of the trajectories $x_1(t)$ and $x_2(t)$ from zero as well as the expended control energy, or

$$J = \int_0^T \left[\int_D (x_1^2(t,z) + r_1 x_2^2(t,z))\,dz + r_2 \int_D u^2(t,z)\,dz \right] dt$$

with r_1 and r_2 both positive real numbers. It is seen that the choice of

$$\mathcal{Q}(t) = \begin{bmatrix} I_{H_o^2(D)} & 0 \\ 0 & r_1 I_{H_o^2(D)} \end{bmatrix}$$

and

$$R(t) = r_2 I_U$$

where $I_{H_o^2(D)}$ and I_U are the identity operators specified in the example following Definition 3.4, puts the cost criterion of Eq. 3.3.3 in the above desired form.

The parabolic control problem defined in Definition 3.4 will be studied in great detail in Chapter 4, whereas the hyperbolic control problem of Definition 3.7 will be briefly discussed in Chapter VI. Important special cases of these problems are discussed in the next section of this chapter.

3.4 THE POINTWISE CONTROL PROBLEM

The optimal control problems defined in the preceding section will be specialized in this section to consider the case where the control does not enter into the system in a distributed fashion, but rather control energy enters the system at a fixed number of "points" within the spatial domain of the system. The justification of the use of this type of pointwise control is on physical grounds. For many physical distributed parameter systems it is next to impossible to drive the system by application of a control distribution. For instance, in the rigid beam considered in the preceding section, the control energy would enter much more realistically as forces at various discrete points along the length of the beam, rather than a "perfect" distribution of force defined at every point of the beam. Another example is the membrane of a drum. Here the distributed displacement of the tympanic membrane is achieved through the approximately pointwise control of the impulsively applied beating of the drumsticks. In both of these cases the analysis would become terribly complicated if the control were modeled by a distribution on the spatial domain. Since it is more likely that one would approximate the distributed control in many physical systems by a finite number of lumped controls, this would motivate the a priori use of non-distributed controls and the subsequent optimization problem in terms of these controls. Moreover, it seems more likely that the analytic specification of an optimal control distribution would be much more difficult than the specification of an optimal control vector. In essence, the pointwise control problem is a hybrid of pure distributed parameter control and finite dimensional control.

If we suppose that control is applied at the k points $z_i, i=1,2,\ldots,k$, the control space U to be considered is k-dimensional Euclidean space R^k, or, in other words, the control defined in Section 2 is assumed to be a k-vector $\underline{u}$. On first thought, it would be desirable mathematically to have the forcing term of Eqs. 3.2.2 and 3.2.3 be of the form

$$B(t)\,\underline{u}(t) = \sum_{i=1}^{k} \delta(z-z_i)b_i(t)u_i(t) \qquad (3.4.1)$$

where $\delta(z-z_i)$ is the Dirac δ-function defined on the spatial domain D, and $b_i(t)$, $i=1,2,\ldots k$, are bounded continuous functions of time. Equation 3.4.1 reflects "true" pointwise control, that is, finite control energy really enters at the set of control points $\{z_i\}_{i=1}^{k}$. Unfortunately, expressions of the form of the right hand side of Eq. 3.4.1 cannot be elements of $L^2(D)$ for each $t\epsilon[0,T]$, because the Dirac δ-function is not square integrable. Since it is required in Section 2 that the forcing term be an element of $L^2(D)$ for all $t\epsilon[0,T]$, we must abandon hope of using "true" pointwise control.

The next logical step is to assume that control action takes place over a small volume surrounding each of the control points z_i. This actually gives a more accurate picture of the procedure of applying pointwise control over a spatial domain, since it is a mathematical fiction to consider control applied at a single point. The physical justification of this assumption can be seen by considering the examples given above. In the rigid beam, any device which applies force at a "point" of the beam cannot apply this force over a region of the beam of zero width. There must be some small length of the beam over which the force is actually applied. In the case of the drum, the vibration of the membrane is not caused by excitation of a point of the membrane with zero area, but by excitation of a small area corresponding to the area of the tip of the drumstick. Both of these cases represent a valid approximation to the pointwise control problem, since the "volumes" surrounding the control points are sufficiently small compared to the "volume" of the spatial region D.

This pointwise control approximation is achieved through the introduction of the following B operator:

<u>Definition 3.8</u>: The pointwise control operator $B_o(t)$, defined for all $t\epsilon[0,T]$ on R^k, is described by

$$B_o(t)\,\underline{u}(t) = \sum_{i=1}^{k} \chi_i(z) b_i(t) u_i(t) \quad \forall\, \underline{u}(t)\epsilon R^k \tag{3.4.2}$$

where $\chi_i(z)$ is the characteristic function of the set $E_i \subset D$ which includes the control point z_i as described above. This characteristic function is given by

$$\chi_i(z) = \begin{cases} 1 & \text{if } z \in E_i \\ 0 & \text{if } z \notin E_i \end{cases}$$

The functions $b_i(t)$ are assumed to be bounded on $[0,T]$. Note that, according to the assumption in Section 2 of this chapter, $\underline{u}(\cdot) \in L^2(0,T;R^k)$.

In order to show that the form $B_o(t)\underline{u}(t)$ is an element of $L^2(0,T;L^2(D))$, and, therefore, satisfies the required condition to be a forcing term for Eqs. 3.2.2 and 3.2.3, we prove the following lemma:

Lemma 3.1: For each $t \in [0,T]$, $B_o(t)$ is a bounded linear operator with domain R^k and range in $L^2(D)$. Moreover, the function $f(\cdot)$, where $f(t) = B_o(t)\underline{u}(t)$, $\forall t \in [0,T]$, is an element of $L^2(0,T;L^2(D))$.

Proof:

$$\int_D [B_o(t)\underline{u}(t)]^2 dz = \int_D \Big(\sum_{i=1}^{k} \chi_i(z) b_i(t)\, u_i(t)\Big)^2 dz$$

$$= \sum_{i=1}^{k} \int_D \chi_i^2(z)\, b_i^2(t) u_i^2(t) dz = \sum_{i=1}^{k} b_i^2(t) u_i^2(t) \int_D \chi_i^2(z) dz$$

Since $\int_D \chi_i^2(z)dz = \int_D \chi_i(z)dz = \mu(E_i)$, the Lebesque measure of the set E_i, and since this must be less than the Lebesque measure of the domain D, we have

$$\int_D [B_o(t)\underline{u}(t)]^2 dz \le \mu(D) \sum_{i=1}^{k} b_i^2(t) u_i^2(t) = \mu(D)\, \|\underline{B}(t)\underline{u}(t)\|^2_{R^k}$$

where $\underline{B}(t)$ is the $k \times k$ diagonal matrix with $B_{ii}(t) = b_i(t)$, $i = 1, 2, \ldots k$. If $\|\underline{B}(t)\|_{R^k}$ is the induced matrix norm of $\underline{B}(t)$, it follows that $B_o(t)$ is a bounded linear operator from R^k into $L^2(D)$ and

$$\|B_o(t)\|_{L^2(D)} \leq \mu^{1/2}(D)\ \|\underline{B}(t)\|_{R^k}\ , \quad \text{for all } t\epsilon[0,T] \qquad (3.4.3)$$

To show that $f(\cdot)$ is an element of $L^2(0,T;\ L^2(D))$, we write

$$\|f(\cdot)\|^2_{L^2(0,T;L^2(D))} = \int_0^T \|f(t)\|^2_{L^2(D)}\ dt$$

$$= \int_0^T \|B_o(t)\underline{u}(t)\|^2_{L^2(D)}\ dt$$

which, by Inequality 3.4.3, can be written

$$\|f(\cdot)\|^2_{L^2(0,T;L^2(D))} \leq \mu(D)\int_0^T \|\underline{B}(t)\|^2_{R^k}\ \|\underline{u}(t)\|^2_{R^k}\ dt$$

$$< \mu(D)\ \|\underline{B}(\cdot)\|^2_{L^2(0,T;R^k)}\ \|\underline{u}(\cdot)\|^2_{L^2(0,T;R^k)}$$

where the last inequality is obtained by the use of Schwarz' inequality. By the assumed boundedness of the functions $b_i(t)$, $i=1,\ldots,k$ and the assumption that $\underline{u}(\cdot)\epsilon L^2(0,T;R^k)$, we obtain

$$\|f(\cdot)\|^2_{L^2(0,T;L^2(D))} < \infty$$

implying that $f(\cdot)\epsilon L^2(0,T;L^2(D))$.

Since the pointwise operator $B_o(t)$ operating on controls $\underline{u}$ in the control space R^k qualifies as a forcing term for systems (3.2.2) and (3.2.3), it remains to formulate the optimal control problem for this case. Since the state space remains unchanged neither the state weighting operator $Q(t)$ nor the terminal state weighting operator F must be modified in the parabolic system case. The same holds true for their counterparts $\mathcal{L}(t)$ and $\mathcal{F}$ in the hyperbolic case. The control space is the finite dimensional space R^k so that the control weighting operator is changed accordingly:

Definition 3.9: The $k \times k$ possibly time-varying matrix $\underline{R}(t)$, defined for all $t \in [0, T]$ on R^k with range in R^k, is called the pointwise control weighting matrix. $\underline{R}(t)$ is assumed to be symmetric and positive definite for all $t \in [0, T]$.

We are now in a position to specify the quadratic cost criterion for parabolic systems with pointwise control as follows:

$$J = \int_0^T [\langle (x(t) - x_d(t)), Q(t)(x(t) - x_d(t)) \rangle_{H_o^m(D)} + \underline{u}'(t)\underline{R}(t)\underline{u}(t)]\,dt$$
$$+ \langle (x(T) - x_d(T)), F(x(T) - x_d(T)) \rangle_{H_o^m(D)} \tag{3.4.4}$$

where $x(t)$ is the solution of (3.2.2) with $B(t) = B_o(t)$ and the control sequence $\underline{u}(t) \in R^k$, $t \in [0, T]$, specified. The pointwise optimal control problem for parabolic systems may now be stated as:

Definition 3.10: The optimal control problem for the system (3.2.2) with $B(t) = B_o(t)$ and $U = R^k$ is to determine the control $\underline{u}^*(t)$, $t \in [0, T]$, with $\underline{u}^*(t) \in R^k$ for all $t \in [0, T]$, such that, if $x^*(t)$ is the solution of (3.2.2) with $B(t) = B_o(t)$ and $\underline{u}(t) = \underline{u}^*(t)$, the functional J in (3.4.4) is minimized. The minimizing $\underline{u}^*(t)$, $t \in [0, T]$, is called the pointwise optimal control (if it exists).

The discussion of pointwise controls will be tabled until Chapter V, where the optimal pointwise control problem for parabolic systems will be solved. The pointwise control problem for hyperbolic systems has not been introduced for the reason that study of this problem will not yield any more insight into the nature of pointwise control than is obtained through the study of pointwise controls for parabolic systems alone.

CHAPTER IV

OPTIMAL CONTROL OF PARABOLIC SYSTEMS

4.1 INTRODUCTION

The purpose of this chapter is to solve the optimal control problem for parabolic systems as specified in Definition 3.4 of the preceding chapter. The first concern of this chapter will be to show that unique solutions of the optimal control problem exist in both the case where the system operator is coercive and the case where the system operator is the infinitesimal generator of a semigroup of operators. Next, necessary conditions for optimality will be discussed and a feedback solution for the optimal control will be derived for both types of system operators. In this chapter we shall also treat the solution of the parabolic optimal control problem defined on an infinite time interval, and we shall derive an integro-differential equation the solution of which specifies the form of the optimal feedback control law.

In Section 4.2 the existence and uniqueness of solutions of the optimal control problems for both types of system operators is demonstrated. With existence and uniqueness guaranteed, we derive, in Section 4.3, the necessary conditions for optimality in the coercive system operator case and show, in Section 4.4, that these necessary conditions imply the existence of a feedback form in which the feedback operator is seen to satisfy a nonlinear operator equation of the Riccati type. The minimum value of the cost criterion will also be shown to be directly expressible in terms of this feedback operator.

Section 4.5 is concerned with the proof that if a bounded solution of the operator equation discussed above exists in the case where the system operator is the infinitesimal generator of a semigroup of operators then the feedback form derived for the optimal control in the coercive case is also optimal in this case. This leads naturally to the proof in Section 4.6 that a bounded solution of the Riccati operator equation does indeed exist.

Section 4.7 contains the discussion of the parabolic control problem defined on the infinite time interval $(0,\infty)$.

In Section 4.8 it is shown that the Riccati operator equation is equivalent to a nonlinear partial integro-differential equation.

4.2 EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this section we examine the question of existence and uniqueness of solutions of the optimal control problem stated in Definition 3.4. We shall show that for an elliptic operator, either coercive or strongly elliptic, the optimal control problem for parabolic systems, as discussed in Section 3 of the preceding chapter, has a unique solution. Lions[15] provides the machinery for demonstrating this by giving a general existence and uniqueness theorem for controls minimizing a certain cost functional. This is then shown to cover existence and uniqueness of optimal controls in the parabolic control problem. Lions does not consider terminal-time cost in his cost criterion, so that any modification of the results due to the slightly more general inclusion of terminal-time cost will be indicated.

As a preliminary to the discussion of existence and uniqueness of optimal controls let us make the following definitions:

Definition 4.1: A coercive bilinear form $\Pi(u,v)$ is a mapping of $U \times U$ into the reals for which there exists a $c > 0$ such that

$$\Pi(u,u) \geq c \ \|u\|^2 \qquad \forall\, u \in U$$

Definition 4.2: The bilinear form $\Pi(u,v)$ is said to be symmetric if

$$\Pi(u,v) = \Pi(v,u) \qquad \forall\, u,v \in U$$

Definition 4.3: The bilinear form $\Pi(u,v)$ is said to be continuous if it is a continuous function of each of its arguments.

Now, having introduced the bilinear form $\Pi(u,v)$, let us consider the cost functional

$$J(u) = \Pi(u,u) - 2L(u) \quad , \quad u \in U \tag{4.2.1}$$

where L is a bounded linear functional defined on U. The existence and uniqueness of a control u^* which minimizes J in (4.2.1) is provided by the following theorem.

Theorem 4.1: If $\Pi(u,v)$ is a continuous, symmetric, coercive bilinear form, then there exists a unique $u^* \in U$ such that

$$J(u^*) = \inf_{u \in U} J(u)$$

Existence is proved by defining a sequence approaching the infimum, showing it is bounded, and extracting a subsequence which has a weak limit in U. Since $\Pi(v,v)$ is lower semicontinuous and $L(v)$ is continuous in the weak topology of U it is seen that the weak limit in U

is the minimizing element u^*. Uniqueness follows directly from the strict convexity of the function $\Pi(v, v)$. Details have been omitted, but are readily available in Ref. 15.

In order to proceed to the discussion of existence and uniqueness of the solution to the parabolic optimal control problem, the question of existence and uniqueness of solutions of the parabolic equation (3.2.2) must be considered. This existence and uniqueness question for the case of coercive elliptic system operators is best answered through the use of another result of Lions' which will also be used to obtain necessary conditions for optimality in the following section.

<u>Theorem 4.2:</u> If $\Pi(u,v)$ satisfies the hypotheses of Theorem 4.1, then $J(u)$ has a minimum value $J(u^*)$ if and only if u^* satisfies the equation*

$$\Pi(u^*, v) = L(v) \quad , \quad \forall v \in U \tag{4.2.2}$$

The proof of this theorem is due to Lions; since it is essential to the optimization results of Section 3 of this chapter, it is presented in Appendix C for the sake of completeness.

To show how this result yields the answer to the existence and uniqueness question in parabolic equations, consider the bilinear form

$$\Pi(x, y) = -\langle Ax, y\rangle_{H_o^m(D)} \quad ; \quad x, y \in H_o^m(D) \tag{4.2.3}$$

where $-A$ is assumed to be a coercive operator, satisfying the

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* If the control u^* is required to lie in some convex constraint set $\Omega \subset U$, the equation which u^* must satisfy becomes

$$\Pi(u^*, v-u^*) \geq L(v-u^*) \quad , \quad \forall v \in \Omega$$

Inequality 2.3.3. Hence, the bilinear form $\Pi(x,y)$ in (4.2.3) is coercive. If we let the linear form $L(v)$ in Eq. 4.2.1 be the inner product

$$L(y) = \langle Bu, y \rangle_{H_o^m(D)} \; ; \quad \forall y \in H_o^m(D) \tag{4.2.4}$$

$H_o^m(D)$ is a Hilbert space and, by a well-known result in elementary Hilbert space theory (see Ref. 25, p. 80), any bounded linear functional on $H_o^m(D)$ is an inner product of y with some element in the dual space of $H_o^m(D)$. Since $L^2(D)$ is contained in this dual space, and since Bu is in $L^2(D)$ by our assumption in Section 3.2, then Expression 4.2.4 is a valid linear form on $H_o^m(D)$.

The hypotheses of Theorem 4.2 are thus satisfied and, therefore, we have the result that there exists a unique $x \epsilon H_o^m(D)$ such that

$$-\langle Ax, y \rangle_{H_o^m(D)} = \langle Bu, y \rangle_{H_o^m(D)} \quad \forall y \epsilon H_o^m(D) \tag{4.2.5}$$

For Eq. 4.2.5 to hold for all $y \epsilon H_o^m(D)$ it must be true that there is a unique $x \epsilon H_o^m(D)$ satisfying the equation

$$Ax + Bu = 0 \tag{4.2.6}$$

Thus, we have demonstrated existence and uniqueness of a solution to Eq. 4.2.6. Needless to say, we have not proved existence and uniqueness of solutions of the parabolic equation (3.2.2), i.e., $\dot{x} = Ax + Bu$. However, Lions uses the procedure demonstrated above, with a few analytic embellishments to account for time evolution, to prove that there exists a unique solution of the parabolic equation

$$\dot{x} = Ax(t) + B(t)u(t) \quad , \quad x(0) = x_o$$

Just as important, Lions shows that the mapping $u(\cdot)\to x(\cdot)$ from $L^2(0,T;U)\to L^2(0,T;H_o^m(D))$ is continuous. The continuity of this mapping plays a role in the application of Theorem 4.1 to the parabolic control problem of Definition 3.4.

It remains to show that solutions of parabolic equations with elliptic operators satisfying Condition 2.7.1 exist, are unique, and depend continuously on the control as was shown for coercive operators. Proving existence and uniqueness is trivial in this case, since the hypotheses of the theorems given in Appendices A and B are satisfied and the solution of Eq. 3.2.2 is uniquely given by

$$x(t) = \Phi(t)x_o + \int_0^t \Phi(t-\sigma)u(\sigma)\,d\sigma \quad , \quad x_o \epsilon Do(A_3) \tag{4.2.7}$$

where $\{\Phi(t)\}_{t\epsilon[0,\infty]}$ is the semigroup of operators with infinitesimal generator A_3 as defined in Section 2.7. Since we wish in addition to show that the solution depends continuously on the control we state and prove the following theorem:

<u>Theorem 4.3</u>: If $x(t)$ is the solution of the parabolic equation (3.2.2) given by Eq. 4.2.7, then the mapping $u(\cdot)\to x(\cdot)$ of $L^2(0,T;U)$ into $L^2(0,T;H_o^m(D))$ is continuous.

<u>Proof</u>: Suppose $u_1(\cdot)$ and $u_2(\cdot)$, defined for all $t\epsilon[0,T]$, are elements of $L^2(0,T;U)$ and $x_1(\cdot)$ and $x_2(\cdot)$, defined for all $t\epsilon[0,T]$ are elements of $L^2(0,T;H_o^m(D))$ given by:

$$x_i(t) = \Phi(t)x_o + \int_0^t \Phi(t-\sigma)u_i(\sigma)\,d\sigma \quad , \quad t\epsilon[0,T], \quad i=1,2 \tag{4.2.8}$$

Forming the difference $x_1(t)-x_2(t)$ and taking the norm squared on $L^2(0,T;H_o^m(D))$, we deduce that

$$\|x_1(\cdot)-x_2(\cdot)\|^2_{L^2(0,T;H_o^m(D)}$$

$$= \int_0^T \|x_1(t)-x_2(t)\|^2_{H_o^m(D)} dt$$

$$= \int_0^T \|\int_0^t \Phi(t-\sigma)B(\sigma)[u_1(\sigma)-u_2(\sigma)]d\sigma\|^2_{H_o^m(D)} dt \quad (4.2.9)$$

$$\leq \int_0^T \left[\int_0^t \|\Phi(t-\sigma)B(\sigma)[u_1(\sigma)-u_2(\sigma)]\|_{H_o^m(D)} d\sigma\right]^2 dt$$

where the inequality follows from a generalization of the triangle inequality for normed spaces. Since $\Phi(t-\sigma)$ and $B(\sigma)$ are bounded linear operators we may write the inequality

$$\|\Phi(t-\sigma)B(\sigma)[u_1(\sigma)-u_2(\sigma)]\|_{H_o^m(D)}$$

$$\leq \|\Phi(t-\sigma)B(\sigma)\|_{H_o^m(D)} \|u_1(\sigma)-u_2(\sigma)\|_U \quad (4.2.10)$$

so that Inequality 4.2.9 can be written

$$\|x_1(\cdot)-x_2(\cdot)\|^2_{L^2(0,T;H_o^m(D))}$$

$$\leq \int_0^T \left[\int_0^t \|\Phi(t-\sigma)B(\sigma)\|_{H_o^m(D)} \|u_1(\sigma)-u_2(\sigma)\|_U d\sigma\right]^2 dt \quad (4.2.11)$$

It should be noted that since $\|u_1(\cdot)-u_2(\cdot)\|_U$ is an element of $L^2(0,t)$ for all $t\in[0,T]$, then the inner integral is, in effect, an integral operator on $L^2(0,t)$ with kernel $k(t,\sigma) = \|\Phi(t-\sigma)B(\sigma)\|_{H_o^m(D)}$. If this kernel is square-summable, that is, if

$$\int_0^t \|\Phi(t-\sigma)B(\sigma)\|^2_{H_o^m(D)} d\sigma < \infty$$

for all $t\in[0,T]$, then by application of Schwartz' inequality it can be shown that (see Ref. 22, p. 148)

$$\left[\int_0^t \|\Phi(t-\sigma)B(\sigma)\|_{H_o^m(D)} \|u_1(\sigma)-u_2(\sigma)\|_U d\sigma\right]^2$$

$$\leq \int_0^t \|\Phi(t-\sigma)B(\sigma)\|^2_{H_o^m(D)} d\sigma \int_0^t \|u_1(\sigma)-u_2(\sigma)\|^2_U d\sigma \qquad (4.2.12)$$

$$\leq \int_0^t \|\Phi(t-\sigma)B(\sigma)\|^2_{H_o^m(D)} d\sigma \; \|u_1(\cdot)-u_2(\cdot)\|^2_{L^2(0,T;U)}$$

holds for all $t\in[0,T]$. Now, by the uniform boundedness principle, which is stated in Appendix D, $\Phi(t)$ is uniformly banded over $[0,T]$. Let us denote this bound by $\|\Phi(t)\| \leq M$. Moreover, $B(\sigma)$ is uniformly bounded on $[0,T]$, with $\|B(\sigma)\| \leq b$. Thus, it follows that

$$\int_0^t \|\Phi(t-\sigma)B(\sigma)\|^2_{H_o^m(D)} d\sigma \leq M^2 b^2 t \qquad (4.2.13)$$

is finite for all $t\epsilon[0,T]$, and the substitution of Inequalities 4.2.12 and 4.2.13 into Inequality 4.2.11 yields the result that

$$\|x_1(\cdot)-x_2(\cdot)\|^2_{L^2(0,T;H_o^m(D))}$$

$$\leq \int_0^T M^2 b^2 t \|u_1(\cdot)-u_2(\cdot)\|^2_{L^2(0,T;U)} dt$$

$$\leq \frac{M^2 b^2}{2} T^2 \|u_1(\cdot)-u_2(\cdot)\|^2_{L^2(0,T;U)}$$

or, equivalently, that

$$\|x_1(\cdot)-x_2(\cdot)\|_{L^2(0,T;H_o^m(D))}$$

$$\leq \frac{MbT}{\sqrt{2}} \|u_1(\cdot)-u_2(\cdot)\|_{L^2(0,T;U)}$$

which implies that the mapping $u(\cdot)$ into $x(\cdot)$ from $L^2(0,T;U)$ into $L^2(0,T;H_o^m(D))$ is continuous.

It may also be shown that the solution at the final time T also depends continuously on the control. This is also necessary for the application of Theorem 4.1 to the parabolic optimal control problem.

Theorem 4.4: If $x(T)$ is the solution, at the terminal time, of the parabolic equation (3.2.2) given by Eq. 4.2.7, then the mapping $u(\cdot)\rightarrow x(T)$ of $L^2(0,T;U)$ into $H_o^m(D)$ is continuous.

Proof: With $u_1(\cdot)$, $u_2(\cdot)$, $x_1(\cdot)$, and $x_2(\cdot)$ defined as in the proof of Theorem 4.3, we have the following expression:

$$\|x_1(T)-x_2(T)\|^2_{H_o^m(D)} = \left\|\int_0^T \Phi(T-\sigma)B(\sigma)[u_1(\sigma)-u_2(\sigma)]d\sigma\right\|^2_{H_o^m(D)}$$

$$\leq \left[\int_0^T \| \Phi(T-\sigma) B(\sigma)[u_1(\sigma) - u_2(\sigma)] \|_{H_o^m(D)} d\sigma \right]^2$$

$$\leq \left[\int_0^T \| \Phi(T-\sigma) B(\sigma) \|_{H_o^m(D)} \| u_1(\sigma) - u_2(\sigma) \|_U \, d\sigma \right]^2$$

and, proceeding as in the proof of Theorem 4.3, we use the square summability of the kernel $\|\Phi(T-\sigma)B(\sigma)\|_{H_o^m(D)}$ to deduce that

$$\|x_1(T) - x_2(T)\|_{H_o^m(D)} \leq MbT^{1/2} \|u_1(\cdot) - u_2(\cdot)\|_{L^2(0,T;U)}$$

which implies that the transformation $u(\cdot)$ into $x(T)$ from $L^2(0,T;U)$ into $H_o^m(D)$ is continuous.

To summarize what has been done so far in this section: unique solutions have been shown to exist for the parabolic equation (3.2.2) with either coercive operators or elliptic infinitesimal generators of semi-groups as system operator. In addition, these solutions have been shown to depend continuously on the control u. With this as a foundation, we may use Theorem 4.1 to extend Lions' results to include the case of terminal cost in the following manner:

Theorem 4.5: The optimal control problem for parabolic systems as specified in Definition 3.4 has a unique solution $u^*(\cdot) \epsilon L^2(0,T;U)$.

Proof: First, let us introduce the notation $x^u(t)$ to denote the solution of parabolic equation (3.2.2) on $[0,T]$ with the control $u(\cdot)\epsilon L^2(0,T;U)$. Likewise $x^v(t)$ denotes the solution corresponding to $v(\cdot)\epsilon L^2(0,T;U)$. The cost criterion, Eq. 3.3.1, for the parabolic control problem can be written in the form

$$J(u) = \langle x^u - x_d, Q(x^u - x_d) \rangle_{L^2(0,T;H_o^m(D))} + \langle u, Ru \rangle_{L^2(0,T;U)} \tag{4.2.15}$$

$$+ \langle x^u(T) - x_d(T), F(x^u(T) - x_d(T)) \rangle_{H_o^m(D)}$$

which, in turn, can be rewritten as

$$J(u) = \Pi(u,u) - 2L(u) + \langle x_d, Qx_d \rangle_{L^2(0,T;H_o^m(D))} \tag{4.2.16}$$

$$+ \langle x_d(T), Fx_d(T) \rangle_{H_o^m(D)}$$

where we define the bilinear form $\Pi(u,v)$ to be

$$\Pi(u,v) \triangleq \langle x^u, Qx^v \rangle_{L^2(0,T;H_o^m(D))} + \langle x^u(T), Fx^v(T) \rangle_{H_o^m(D)}$$

$$+ \langle u, Rv \rangle_{L^2(0,T;U)}$$

$$L(v) \triangleq \langle Qx_d, x^v \rangle_{L^2(0,T;H_o^m(D))} + \langle Fx_d(T), x^v(T) \rangle_{H_o^m(D)}$$

Since the last two terms of Eq. 4.2.16 are independent of u, minimizing the cost functional $J'(u)$

$$J'(u) = \Pi(u,u) - 2L(u)$$

is equivalent to minimizing the original cost functional $J(u)$. We now note that $\Pi(u,v)$ is symmetric since

$$\Pi(v,u) = \langle x^v, Qx^u \rangle_{L^2(0,T;\, H_o^m(D))} + \langle x^v(T), Fx^u(T) \rangle_{H_o^m(D)} + \langle v, Ru \rangle_{L^2(0,T;U)}$$

$$= \langle Q^* x^v, x^u \rangle_{L^2(0,T;H_o^m(D))} + \langle F^* x^v(T), x^u(T) \rangle_{H_o^m(D)} + \langle R^* v, u \rangle_{L^2(0,T;U)} = \Pi(u,v)$$

by the self-adjointness of the operators $Q(t)$, $R(t)$ and F and by the symmetry of the inner products on $L^2(0,T;H_o^m(D))$, $L^2(0,T;U)$, and $H_o^m(D)$.

Next we note that $\Pi(u,v)$ is a coercive bilinear form because

$$\Pi(u,u) \geq \alpha \, \|u\|_U^2 \;, \quad \alpha > 0 \;\; \forall u(\cdot) \in L^2(0,T;U)$$

by the positivity of the operators $Q(t)$ and F and by the strict positivity of $R(t)$.

Also, $\Pi(u,v)$ is continuous, since by Theorem 4.3 $x^u(\cdot)$ is continuous in $u(\cdot)$ on $L^2(0,T;H_o^m(D))$ and by Theorem 4.4 $x^u(T)$ is continuous in $u(\cdot)$ on $H_o^m(D)$.

The hypotheses of Theorem 4.1 are thus satisfied, so that there exists a unique $u^*(\cdot) \in L^2(0,T;U)$ such that

$$J'(u^*) = \inf_{u(\cdot) \in L^2(0,T;U)} J'(u)$$

or, equivalently, there exists a unique solution of the parabolic optimal control problem given in Definition 3.4.

It should be noted that in the above proof continuity was discussed in terms of the strong topology of $L^2(0,T;H_o^m(D))$, whereas Lions' requires continuity in the strong topology of $W(0,T)$ which is defined in Chapter II, Section 4. This more stringent continuity requirement is not necessary, however, since the cost criterion involves only $x(t)$ and not $\dot{x}(t)$, so that behavior of derivatives of solutions in terms of u is beside the point.

We have shown in this section that unique solutions exist to the parabolic control problem for both coercive system operators and elliptic system operators which are the infinitesimal generators of semigroups. It remains to characterize these optimal solutions for both types of system operators.

4.3 DERIVATION OF NECESSARY CONDITIONS--COERCIVE CASE

Since, in the preceding section, it has been shown that a unique optimal control exists, we shall derive in this section what precise conditions that optimal control must satisfy in the case of parabolic systems with coercive system operators. Again, this derivation is due formally to Lions,[15] but his results are extended to include the case of terminal-time cost in the cost criterion.

For convenience, let us rewrite Eq. 3.2.2

$$\frac{dx(t)}{dt} = A\,x(t) + B(t)u(t) \quad ; \quad x(0) = x_o \tag{4.3.1}$$

Recall from the preceding section that the solution of the parabolic optimal control problem, namely, the optimal control u^*, must minimize the cost functional

$$J(u) = \Pi(u,u) - 2L(u) \tag{4.3.2}$$

where

$$\Pi(u,v) \triangleq \langle x^u, Qx^v \rangle_{L^2(0,T;H_o^m(D))} + \langle x^u(T), Fx^v(T) \rangle_{H_o^m(D)} + \langle u, Rv \rangle_{L^2(0,T;U)}$$

$$L(v) \triangleq \langle Qx_d, x^v \rangle_{L^2(0,T;H_o^m(D))} + \langle Fx_d(T), x^v(T) \rangle_{H_o^m(D)}$$

Now, by Theorem 4.2, the optimal control must satisfy

$$\Pi(u^*, v) = L(v) \quad \text{for all } v(\cdot)\in L^2(0,T;U)$$

or, equivalently,

$$\Pi(u^*, v-u^*) = L(v-u^*) \forall v(.)\in L^2(0,T;U) \qquad (4.3.3)$$

Further, let us introduce the adjoint equation

$$\frac{dp(t)}{dt} = -A^* p(t) - Q[x(t) - x_d(t)] \quad * \qquad (4.3.4)$$

$$p(T) = F[x(T) - x_d(T)]$$

$p(t)$ is called the costate, and, by changing the time variable from t to $T-t$ and realizing that A^* is coercive if A is, the results of the preceding section tell us that a unique solution $p(\cdot)\in L^2(0,T;H_o^m(D))$** exists for Eq. 4.3.4. Let us denote the solution $p(\cdot)$ due to the application of control $u(\cdot)\in L^2(0,T;U)$ as p^u. Forming the inner product on $L^2(0,T;H_o^m(D))$ with $x^v - x^u$ we obtain

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* The standard asterisk notation for adjoint operators is used here. This should not be confused with the equally standard use of the asterisk superscript to denote such optimal quantities as $u^*(t)$, $x^*(t)$, and $p^*(t)$.

** Actually, it must be true (and it can be shown) that $p(.)\in W(0,T)$. This is necessary since we wish to take $L^2(0,T;H_o^m(D))$ inner products with $\frac{dp(.)}{dt}$.

$$\left\langle \frac{dp^u}{dt}, x^v - x^u \right\rangle_{L^2(0,T;H_o^m(D))} = -\left\langle A^* p^u, x^v - x^u \right\rangle_{L^2(0,T;H_o^m(D))}$$

(4.3.5)

$$-\left\langle Q[x^u - x_d], x^v - x^u \right\rangle_{L^2(0,T;H_o^m(D))}$$

Evaluating the left-hand side of Eq. 4.3.5,

$$\left\langle \frac{dp^u}{dt}, x^v - x^u \right\rangle_{L^2(0,T;H_o^m(D))} = \int_0^T \left\langle \frac{dp^u}{dt}(t), x^v(t) - x^u(t) \right\rangle_{H_o^m(D)} dt$$

$$= \left\langle p^u(T), x^v(T) - x^u(T) \right\rangle_{H_o^m(D)} - \int_0^T \left\langle p^u(t), \frac{d}{dt}(x^v(t) - x^u(t)) \right\rangle_{H_o^m(D)} dt$$

But since $p^u(T) = F[x^u(T) - x_d(T)]$ as seen in Eq. 4.3.4,

$$\left\langle \frac{dp^u}{dt}, x^v - x^u \right\rangle_{L^2(0,T;H_o^m(D))} = \left\langle F[x^u(T) - x_d(T)], x^v(T) - x^u(T) \right\rangle_{H_o^m(D)}$$

$$-\left\langle p^u, \frac{d}{dt}(x^v - x^u) \right\rangle_{L^2(0,T;H_o^m(D))}$$

The first term on the right-hand side of Eq. 4.3.5 can be written

$$\left\langle A^* p^u, x^v - x^u \right\rangle_{L^2(0,T;H_o^m(D))} = \left\langle p^u, A(x^v - x^u) \right\rangle_{L^2(0,T;H_o^m(D))}$$

(4.3.7)

Combining (4.3.5), (4.3.6), and (4.3.7) and letting $u = u^*$ we obtain

$$-\left\langle Q[x^{u^*} - x_d], x^v - x^{u^*} \right\rangle_{L^2(0,T;H_o^m(D))}$$

(4.3.8)

$$= -\left\langle p^u, \left(\frac{d}{dt} - A\right)(x^v - x^{u^*}) \right\rangle_{L^2(0,T;H_o^m(D))}$$

$$+ \left\langle F[x^{u^*}(T) - x_d(T)], x^v(T) - x^{u^*}(T) \right\rangle_{H_o^m(D)}$$

$$= -\langle p^{u*}, B(v-u*)\rangle_{L^2(0,T;H_o^m(D))} + \langle F[x^{u*}(T)-x_d(T)], x^v(T)-x^{u*}(T)\rangle_{H_o^m(D)}$$

Now from Eq. 4.3.3

$$\Pi(u*, v-u*) - L(v-u*)$$

$$= \langle Q[x^{u*}-x_d], x^v-x^{u*}\rangle_{L^2(0,T;H_o^m(D))}$$

$$+ \langle F[x^{u*}(T)-x_d(T)], x^v(T)-x^{u*}(T)\rangle_{H_o^m(D)}$$

$$+ \langle Ru*, v-u*\rangle_{L^2(0,T;U)} = 0 \qquad (4.3.9)$$

Combining Eqs. 4.3.8 and 4.3.9 we obtain

$$-\langle p^{u*}, B(v-u*)\rangle_{L^2(0,T;H_o^m(D))}$$

$$= \langle Ru*, v-u*\rangle_{L^2(0,T;U)} \quad \forall v(\cdot) \in L^2(0,T;U)$$

or, equivalently,

$$-\langle B*p^{u*}, v-u*\rangle_{L^2(0,T;U)}$$

$$= \langle Ru*, v-u*\rangle_{L^2(0,T;U)} \quad \forall v(\cdot) \in L^2(0,T;U) \qquad (4.3.10)$$

Since equality must hold in Eq. 4.3.10 for all elements $v(\cdot) \in L^2(0,T;U)$, it must be true that

$$-B*(t)\, p^{u*}(t) = R(t)u*(t) \qquad (4.3.11)$$

is satisfied by the optimal control $u*$. Moreover, since $R(t)$ is assumed to be strictly positive in Definition 3.2 it has an inverse for all $t \in [0,T]$ and so Eq. 4.3.11 reduces to

$$u*(t) = -R^{-1}(t)B*(t)p^{u*}(t) \qquad (4.3.12)$$

It might reasonably be asked, at this point, why the above derivation does not hold as well for parabolic systems with system operators which are the infinitesimal generators of a semigroup of operators. The answer lies in the fact that for this class of parabolic systems the costate Eq. 4.3.4 cannot be shown to have a solution $p(\cdot)\epsilon W(0,T)$, that is, although $p(\cdot)$ is an element of $L^2(0,T;H_o^m(D))$, we cannot show that $\frac{dp}{dt}(\cdot)$ is an element of $L^2(0,T;H_o^m(D))$. Thus, the inner product with $\frac{dp}{dt}(\cdot)$ in Eq. 4.3.5 would be meaningless in this case. This inability to express the necessary conditions for optimality in the form derived above for this class of systems will be circumvented in Section 5, however.

Let us summarize the results of this section:

<u>If</u> $u^*(\cdot)\epsilon L^2(0,T;U)$ <u>is the optimal control for the problem specified in Definition 3.4, then it is necessary that there exists a unique costate</u> $p^*(\cdot)$ <u>such that</u>:

$$u^*(t) = -R^{-1}(t)B^*(t)p^*(t)$$

where $p^*(\cdot)\epsilon L^2(0,T;H_o^m(D))$ satisfies the equation

$$\frac{dp^*(t)}{dt} = -\overline{A^*p^*(t) - Q}[x^*(t)-x_d(t)] \quad ; \quad p^*(T) = F[x^*(T) - x_d(T)]$$

and $x^*(\cdot)\epsilon L^2(0,T;H_o^m(D))$ satisfies the equation

$$\frac{dx^*(t)}{dt} = Ax^*(t) + B(t)u^*(t) \quad ; \quad x^*(0) = x_o$$

4.4 DECOUPLING AND THE RICCATI OPERATOR EQUATION

In this section the necessary conditions derived in the preceding section are shown to yield the fact that there exists a feedback form of the optimal control given by Eq. 4.3.12. The optimal feedback operator will be defined and will be shown to satisfy a nonlinear operator differential equation of the Riccati type. Bounded, positive and self-adjoint solutions to this equation will be shown to exist. Moreover, an optimal cost function will be defined and shown to be simply related to the optimal feedback operator. The results of this section are due to Lions (see Ref. 15, pp. 147-157) with slight modifications and an extension to include the terminal-time cost.

If we consider the system of equations:

$$\frac{dx^*}{dt} = Ax^*(t) - B(t)R^{-1}(t)B^*(t)p^*(t)$$

$$t \in (s, T) \quad ; \quad 0 < s < T \tag{4.4.1}$$

$$\frac{dp^*}{dt} = -A^*p^*(t) - Q[x^*(t) - x_d(t)]$$

$$x^*(s) = h, \; h \in H_o^m(D) \quad \text{and} \quad p(T) = F[x^*(T) - x_d(T)]$$

This system admits a unique solution pair $(x^*(\cdot), p^*(\cdot)) \in W(s,T) X W(s,T)$, where $W(s,T)$ is the space $W(0,T)$ defined in Section 2.4 with s taking the place of the lower limit 0. This fact is easily seen if the cost criterion of the preceding section is defined on the time interval (s,T) instead of $[0,T]$ and the same straightforward procedure of deriving necessary conditions is used. Lions shows that the transformation $h \rightarrow \{x^*(\cdot), p^*(\cdot)\}$ is continuous from $H_o^m(D)$ into $W(s,T) X W(s,T)$, and that the transformation $h \rightarrow p^*(s)$ is continuous from $H_o^m(D)$ into $H_o^m(D)$, this latter result following from the fact that $h \rightarrow p^*(s)$ is a composite transformation composed of $h \rightarrow \{x^*(\cdot), p^*(\cdot)\}$, $\{x^*(\cdot), p^*(\cdot)\} \rightarrow p^*(T)$, and the transformation which relates the "initial" value $p^*(T)$ to the solution $p^*(s)$ of the adjoint equation in (4.4.1), all of which transformations are continuous in their range spaces. The result of the continuity of the transformation $h \rightarrow p^*(s)$ is that $p^*(s)$ can be written in the form

$$p^*(s) = K(s)h + g(s) \tag{4.4.2}$$

where $K(s)\in\mathcal{L}(H_o^m(D); H_o^m(D))$ and $g(s)\in H_o^m(D)$ for all $s\in(0,T)$. Since s is any arbitrary time in $(0,T)$ and h is the evaluation of $x^*(s)$, then Eq. 4.4.2 tells us that

$$p^*(t) = K(t)x^*(t) + g(t) \qquad \forall t\in[0,T] \tag{4.4.3}$$

where $x^*(.)$ and $p^*(\cdot)$ are the solutions of the state and adjoint equations, respectively, given, in summary, at the end of Section 3 of this chapter. It should be noted that $K(s)$ in Eq. 4.4.2 is given by $p^*(s)=K(s)h$ in System 4.4.1 with $x_d(t)=0$, and likewise $g(s)$ is given by $p^*(s)=g(s)$ with $h=0$. In the sequel, we shall, for convenience, drop the asterisk superscript notation for the optimal quantities $x^*(\cdot)$ and $p^*(\cdot)$.

The operator $K(s)$ can be shown to be self-adjoint by considering the scalar product $\langle K(s)h_1, h_2\rangle_{H_o^m(D)}$ where h_1 and h_2 are initial conditions for System 4.4.1, with $x_d(t)=0$ for all $t\in[0,T]$, which result in the solution pairs $\{x_1(\cdot), p_1(\cdot)\}$ and $\{x_2(\cdot), p_2(\cdot)\}$, respectively:

$$\begin{aligned} 0 &= \int_s^T \left\langle \frac{dp_1}{dt} + A^*p_1 + Qx_1, x_2\right\rangle_{H_o^m(D)} dt \\ &= \langle p_1(T), x_2(T)\rangle_{H_o^m(D)} - \langle p_1(s), x_2(s)\rangle_{H_o^m(D)} \\ &\qquad - \left\langle p_1, \frac{dx_2}{dt} - Ax_2\right\rangle_{L^2(s,T;H_o^m(D))} \\ &\qquad + \langle Qx_1, x_2\rangle_{L^2(s,T;H_o^m(D))} \end{aligned} \tag{4.4.4}$$

$$= \langle Fx_1(T), x_2(T)\rangle_{H_o^m(D)} - \langle K(s)h_1, h_2\rangle_{H_o^m(D)}$$

$$+ \langle p_1, B(t)R^{-1}B^*(t)p_2\rangle_{L^2(s,T;H_o^m(D))}$$

$$+ \langle Qx_1, x_2\rangle_{L^2(s,T;H_o^m(D))} = 0$$

and since the operators F, $B(t)R^{-1}(t)B^*(t)$, and $Q(t)$ are all self-adjoint, $K(s)$ is self-adjoint.

To show that $K(s)$ is a positive operator, let us define the cost of starting at time s in System 4.4.1 with $x_d(t) = 0$ and with initial state $h \epsilon H_o^m(D)$ and control $u(\cdot) \epsilon L^2(s, T; U)$ as $J_s^h(u)$. If $u^*(\cdot) \epsilon L^2(s, T; U)$ is the optimal control for this problem then $u^*(\cdot)$ satisfies the necessary condition

$$B^*(t)p(t) + R(t)u^*(t) = 0 \qquad t \epsilon (s, T) \tag{4.4.5}$$

If $h_1 = h_2 = h$ (it follows that $x_1(\cdot) = x_2(\cdot)$ and $p_1(\cdot) = p_2(\cdot)$), the last equality of (4.4.4) may be written

$$\langle K(s)h, h\rangle_{H_o^m(D)} = \langle Qx, x\rangle_{L^2(s,T;H_o^m(D))}$$

$$+ \langle p, BR^{-1}B^*p\rangle_{L^2(s,T;H_o^m(D))}$$

$$+ \langle Fx(T), x(T)\rangle_{H_o^m(D)} \tag{4.4.6}$$

But, by virtue of Eq. 4.4.5,

$$\int_s^T \langle R(t)u^*(t), u^*(t)\rangle_U \, dt = \int_s^T \langle B^*(t)p(t), R^{-1}(t)B^*(t)p(t)\rangle_U \, dt$$

$$= \int_s^T \langle p(t), B(t)R^{-1}(t)B^*(t)p(t)\rangle_{H_o^m(D)} \, dt = \langle p, BR^{-1}B^*p\rangle_{L^2(s,T;H_o^m(D))} \tag{4.4.7}$$

so that $\langle K(s)h, h\rangle_{H_o^m(D)}$ is the optimal cost starting at time s with initial state h, or

$$\langle K(s)h, h\rangle = J_s^h(u^*) \geq 0$$

proving the positivity of $K(s)$.

The boundedness of $K(s)$ follows from the fact that the transformations $h \to x(\cdot)$, $h \to p(\cdot)$, and $h \to x(T)$ are continuous in the strong topologies of their range spaces, so that

$$\|x(\cdot)\|_{L^2(s,T;H_o^m(D))} \leq c_1 \|h\|_{H_o^m(D)}, \quad \|p(\cdot)\|_{L^2(s,T;H_o^m(D))}$$
$$\leq c_2 \|h\|_{H_o^m(D))}$$

and $\|x(T)\|_{H_o^m(D)} \leq c_3 \|h\|_{H_o^m(D)}$. From Definitions 3.1, 3.2, and 3.3 it is seen that the operators $Q(t)$, $R(t)$, and F are bounded so that

$\langle Qx, x\rangle_{L^2(s,T;H_o^m(D))} \leq M_1 c_1^2 \|h\|^2_{H_o^m(D)}$, $\langle p, BR^{-1}B^*p\rangle_{L^2(s,T;H_o^m(D))}$ $\leq M_2 c_2^2 \|h\|^2_{H_o^m(D)}$, and $\langle x(T), Fx(T)\rangle_{H_o^m(D)} \leq M_3 c_3^2 \|h\|^2_{H_o^m(D)}$. This implies that

$$\langle K(s)h, h\rangle_{H_o^m(D)} = J_s^h(u^*)$$
$$\leq (M_1 c_1^2 + M_2 c_2^2 + M_3 c_3^2) \|h\|^2_{H_o^m(D)} = c\|h\|^2_{H_o^m(D)} \qquad (4.4.8)$$

proving boundedness of the operator $K(s)$.

It will now be shown that $K(s)$ is the solution of a nonlinear equation of the Riccati type and $g(s)$ is the solution of a linear equation. Using Eq. 4.4.3 we can rewrite the system given in the summary of the preceding section as

$$\dot{x}(t) = Ax(t) - B(t)R^{-1}(t)B^*(t)[K(t)x(t) + g(t)]$$

and $$\frac{dp}{dt} = \frac{d}{dt}[K(t)x(t) + g(t)] = \dot{K}(t)x(t) + K(t)\dot{x}(t) + \dot{g}(t)$$

$$= -A^*[K(t)x(t) + g(t)] - Q(t)x(t) + Q(t)x_d(t) \tag{4.4.9}$$

and $$p(T) = K(T)x(T) + g(T) = Fx(T) - Fx_d(T) \quad ; \quad x(0) = x_o$$

The two equations in (4.4.9) can be combined to yield

$$[\dot{K}(t) + K(t)A + A^*K - K(t)B(t)R^{-1}(t)B^*(t)K(t) + Q(t)]x(t)$$

$$= [-\dot{g}(t) - A^*g(t) + K(t)B(t)R^{-1}(t)B^*(t)g(t) + Q(t)x_d(t)] \tag{4.4.10}$$

since $x(t)$ is arbitrary in the sense that it depends on an arbitrary choice of x_o, the only way equality can be achieved in (4.4.10) is if the terms in each of the two square brackets sum to zero, or, equivalently, if the following two differential equations are satisfied

$$\dot{K}(t) = -K(t)A - A^*K(t) + K(t)B(t)R^{-1}(t)B^*(t)K(t) - Q(t)$$

$$K(T) = F \tag{4.4.11}$$

and $$\dot{g}(t) = -A^*g(t) + K(t)B(t)R^{-1}(t)B^*(t)g(t) + Q(t)x_d(t)$$

$$g(T) = -Fx_d(T) \tag{4.4.12}$$

We may summarize in part what has been shown above by stating the following theorem:

<u>Theorem 4.6</u>: The optimal control $u^*(\cdot) \in L^2(0, T; U)$ for the parabolic control problem specified in Definition 3.4 is given by the feedback form:

$$u^*(t) = -R^{-1}(t)B^*(t)[K(t)x(t) + g(t)] \tag{4.4.13}$$

where $K(t)$, $t\epsilon[0,T]$ is the bounded, positive self-adjoint solution of Eq. 4.4.11 and $g(t)$ is the solution of Eq. 4.4.12.

It remains to determine the relationship between the cost of starting at the initial state h at time s and the operator $K(s)$ and function $g(s)$ given above. This is stated as:

Theorem 4.7: The value of the parabolic cost criterion attained by System 4.4.1 (the optimal system over (s,T)) is given by the expression:

$$J = \langle K(s)h, h\rangle_{H_o^m(D)} + 2\langle g(s), h\rangle_{H_o^m(D)} + \phi(s)$$

where $K(s)$ is the solution of (4.4.11), $g(s)$ is the solution of (4.4.12) and $\phi(s)$ is the solution of

$$\dot{\phi}(t) = -\langle x_d(t), Q(t)x_d(t)\rangle_{H_o^m(D)} + \langle g(t), B(t)R^{-1}(t)B^*(t)g(t)\rangle_{H_o^m(D)} \tag{4.4.14}$$

$$\phi(T) = \langle Fx_d(T), x_d(T)\rangle$$

Proof: From Eq. 4.4.1 we may write*

$$\int_s^T \langle Q(x-x_d), x-x_d\rangle dt = \int_s^T \langle -\frac{dp}{dt} - A^*p, x-x_d\rangle dt$$
$$= \int_s^T \langle -\frac{dp}{dt} - A^*p, x\rangle dt - \int_s^T \langle -\frac{dp}{dt} - A^*p, x_d\rangle dt \tag{4.4.15}$$

---

*All inner products are defined on $H_o^m(D)$ unless otherwise specified.

Integrating the first term by parts, we obtain

$$\int_s^T \left\langle -\frac{dp}{dt} - A^* p, x \right\rangle dt = \langle p(s), x(s) \rangle - \langle p(T), x(T) \rangle$$

$$+ \int_s^T \langle p, \dot{x} - Ax \rangle dt$$

(4.4.16)

$$= \langle K(s)h + g(s), h \rangle - \langle F(x(T) - x_d(T), x(T) \rangle$$

$$- \int_s^T \langle p, BR^{-1}B^* p \rangle dt$$

Now, the second term in Eq. 4.4.16 may be written

$$-\langle F(x(T)-x_d(T)), x(T) \rangle = -\langle F(x(T)-x_d(T)), x(T)-x_d(T) \rangle$$

(4.4.17)

$$-\langle F(x(T)-x_d(T)), x_d(T) \rangle$$

Moreover, from Eq. 4.4.7 we see that

$$-\int_s^T \langle p, BR^{-1}B^* p \rangle dt = -\int_s^T \langle u(t), R(t)u(t) \rangle_U \, dt$$

So that, by virtue of the equation for the cost functional J (Eq. 3.3.1) and Eqs. 4.4.16 and 4.4.17, Eq. 4.4.15 may be written

$$J = \langle K(s)h, h \rangle + \langle g(s), h \rangle - \langle F(x(T) - x_d(T), x_d(T) \rangle$$

(4.4.18

$$+ \int_s^T \left\langle \frac{dp}{dt} + A^* p, x_d \right\rangle dt$$

Examining the last term of Eq. 4.4.18,

$$\int_s^T \left\langle \frac{dp}{dt} + A^* p,\ x_d \right\rangle dt = -\int_s^T \langle Q(x-x_d), x_d \rangle dt$$

$$= \int_s^T \langle Qx_d, x_d \rangle dt - \int_s^T \langle Qx, x_d \rangle dt \tag{4.4.19}$$

But, by the differential equation for g, Eq. 4.4.12,

$$-\int_s^T \langle Qx, x_d \rangle dt = -\int_s^T \langle Qx_d, x \rangle dt$$

$$= -\int_s^T \langle \dot{g} + A^* g - KBR^{-1}B^* g, x \rangle dt$$

$$= \langle g(s), x(s) \rangle - \langle g(T), x(T) \rangle$$

$$+ \int_s^T \langle g, \dot{x} - Ax + BR^{-1}B^* Kx \rangle dt$$

Since $g(T) = -Fx_d(T)$ and $\dot{x}-Ax+BR^{-1}B^*Kx = BR^{-1}B^*g$, we obtain

$$-\int_s^T \langle Qx, x_d \rangle dt = \langle g(s), h \rangle + \langle Fx_d(T), x(T) \rangle$$

$$-\int_s^T \langle g, BR^{-1}B^* g \rangle dt \tag{4.4.20}$$

Combining Eqs. 4.4.18, 4.4.19, and 4.4.20 yields

$$J = \langle K(s)h, h \rangle + 2\langle g(s), h \rangle - \langle F(x(T)-x_d(T)), x_d(T) \rangle$$
$$+ \langle Fx_d(T), x(T) \rangle$$

$$+ \int_s^T \langle Qx_d, x_d \rangle dt - \int_s^T \langle g, BR^{-1}B^* g \rangle dt$$

$$= \langle K(s)h, h\rangle + 2\langle g(s), h\rangle + \langle Fx_d(T), x_d(T)\rangle$$

$$+ \int_s^T \langle Qx_d, x_d\rangle dt - \int_s^T \langle g, BR^{-1}B^*g\rangle dt$$

$$= \langle K(s)h, h\rangle + 2\langle g(s), h\rangle + \phi(s)$$

where $\phi(s)$ satisfies Eq. 4.4.14.

Thus, we have achieved what was set out to be done in this section. The optimal control for the parabolic optimal control problem was shown to be a linear feedback control with a positive, bounded and self-adjoint feedback operator. It was also shown, by means of Theorem 4.7, that the optimal cost is related to this feedback operator. It is important to note that the existence of this operator $K(t)$ is guaranteed by the strong continuity of the transformation $h \to \{x(\cdot), p(\cdot)\}$ in $W(0,T) \times W(0,T)$ Once existence is guaranteed, it is a trivial matter to determine what equation the operator $K(t)$ must satisfy. In the case of the operators treated in the next section, namely, the elliptic operators which are infinitesimal generators of semigroups, the transformation $h \to \{x(\cdot), p(\cdot)\}$ cannot be proved continuous, so that existence of the optimal feedback operator $K(t)$ must be proved through other means.

4.5 NECESSARY CONDITIONS FOR OPTIMALITY--INFINITESIMAL GENERATOR CASE

The necessary conditions for optimality have been derived for the case of coercive system operators and have been summarized at the end of Section 3 of this chapter; the resulting feedback form of the optimal control has been given in Section 4. The existence of the feedback operator $K(t)$ and of the related vector function $g(t)$ and scalar function

$\phi(t)$ are a consequence of the strong continuity in $W(0,T) \times W(0,T)$ of the transformation of the initial conditions into the solution $\{x(\cdot), p(\cdot)\}$ of the canonical system of Eqs. 4.4.1. Such a canonical system may be defined in the case where the system operator in the parabolic system Eq. 3.2.2 is a strongly elliptic operator satisfying Inequality 2.7.1, or, equivalently, is the infinitesimal generator of a semigroup of operators. However, in this case, the transformation from the initial conditions to the canonical solution set $\{x(\cdot), p(\cdot)\}$ can only be shown to be strongly continuous in $L^2(0,T;H_o^m(D)) \times L^2(0,T;H_o^m(D))$. Although this continuity feature was enough to guarantee existence and uniqueness of optimal controls for this type of system operator (as was shown in Section 2 of this chapter), it is not enough to guarantee the existence and uniqueness of a bounded, self-adjoint positive feedback operator $K(t)$ and the related functions $g(t)$ and $\phi(t)$.

It is the purpose of this section and of the following section to show that the optimal control for parabolic systems with this class of operators has precisely the same feedback form given for parabolic systems with coercive system operators, namely, Eq. 4.4.13. In this section, we shall show that <u>if a bounded solution $K(t)$ exists for the Riccati operator equation (4.4.11), then the feedback form in Eq. 4.4.13 is the optimal control for parabolic systems with system operators which are infinitesimal generators of semigroups of operators</u>. It will also be shown that the optimal cost function for parabolic systems with coercive system operators, given in Theorem 4.7, is also the optimal cost function for this class of systems. In Section 6 the important question of existence and uniqueness of bounded solutions to the Riccati operator equation will be considered.

Let us, for convenience, rewrite the cost criterion (3.3.1)

$$J = \left\langle x-x_d, Q(x-x_d)\right\rangle_{L^2(0,T;H_o^m(D))} + \left\langle u, Ru\right\rangle_{L^2(0,T;U)}$$
$$+ \left\langle x(T)-x_d(T), F(x(T)-x_d(T))\right\rangle_{H_o^m(D)} \qquad (3.3.1)$$

We shall show that the minimum value which J can attain is

$$J_{min} = \left\langle K(0)x(0), x(0)\right\rangle_{H_o^m(D)} + 2\left\langle g(0), x(0)\right\rangle + \phi(0) \qquad (4.5.1)$$

where $K(t)$, $g(t)$, and $\phi(t)$ are given by Eqs. 4.4.11, 4.4.12, and 4.4.14, respectively. Let us make the assumption that Eq. 4.4.11 has a bounded solution $K(t)$ defined on $[0,T]$. To show that this implies the existence of $g(t)$ and $\phi(t)$ we prove the following lemma:

<u>Lemma 4.1</u>: If a bounded solution $K(t)$ of Eq. 4.4.11 exists for all $t\epsilon[0,T]$, then a unique solution $g(t)$ of Eq. 4.4.12 and, consequently, a unique solution $\phi(t)$ of Eq. 4.4.14 exist in the case of parabolic systems with system operators which satisfy Inequality 2.7.1.

<u>Proof</u>: Rewriting Eq. 4.4.12

$$\frac{dg}{dt} = -A^* g(t) + K(t)B(t)R^{-1}(t)B^*(t)g(t) + Q(t)x_d(t)$$
$$g(T) = -Fx_d(T) \qquad (4.4.12)$$

Let us make the transformation $t \to T-s$, so that Eq. 4.4.12 becomes

$$\frac{dg}{ds} = A^* g(s) - K(s)B(s)R^{-1}(s)B^*(s)g(s) - Q(s)x_d(s)$$

$$g(0) = -Fx_d(0)$$

Since the operator A is the infinitesimal generator of a strongly continuous semigroup of operators $\{\Phi(t)\}_{t\in[0,T]}$ as described in Section 7 of Chapter II, the adjoint operator A^* is the infinitesimal generator of the strongly continuous* semigroup of operators $\{\Phi^*(t)\}_{t\in[0,T]}$ as shown by Hille and Phillips (see Ref. 21, p. 426). We may thus apply the variation of constants formula, Eq. 2.8.1 to obtain

$$g(s) = \alpha(s) - \int_0^s \Phi^*(s-\sigma)K(\sigma)B(\sigma)R^{-1}(\sigma)B^*(\sigma)g(\sigma)\,d\sigma \qquad (4.5.2)$$

where $\alpha(s) = \Phi^*(s)g(0) - \int_0^s \Phi^*(s-\sigma)Q(\sigma)x_d(\sigma)\,d\sigma$

To show that a solution $g(s)$, $s\in[0,T]$ exists for Eq. 4.5.2 we shall apply the well-known Picard method of successive approximations (see Ref. 26, p.6). Form the sequence of Sobolev-space valued functions $\{g_i\}_{i=0}^{\infty}$ defined by:

$$g_o(s) = \alpha(s)$$

$$g_{i+1}(s) = \alpha(s) - \int_0^s \Phi^*(s-\sigma)K(\sigma)B(\sigma)R^{-1}(\sigma)B^*(\sigma)g_i(\sigma)\,d\sigma$$

We would like to show that the sequence $\{g_i\}_{i=0}^{\infty}$ converges in $L^2(0,T;H_o^m(D))$ to some limit $g(\cdot)$. The convergence of this sequence depends on the convergence of the infinite series

$$h(s) = \sum_{i=0}^{\infty} g_{i+1}(s) - g_i(s)$$

*Strong continuity is a result of the reflexivity of the space $L^2(0,T;H_o^m(D))$ on which $\Phi^*(t)$ is defined.

The N^{th} partial sum of this series has the form

$$h_N(s) = \sum_{i=0}^{N} (g_{i+1}(s) - g_i(s)) = g_{N+1}(s) - g_o(s)$$

and, so, the series converges in $L^2(0,T;H_o^m(D))$ if and only if the sequence converges in $L^2(0,T;H_o^m(D))$. The series will converge if $\sum_{i=0}^{\infty} \|g_{i+1}(\cdot) - g_i(\cdot)\|_{L^2(0,T;H_o^m(D))}$ converges.

Now,

$$g_{i+1}(s) - g_i(s) = -\int_0^s \Phi^*(s-\sigma)K(\sigma)B(\sigma)R^{-1}(\sigma)B^*(\sigma)[\,g_i(\sigma) - g_{i-1}(\sigma)\,]\,d\sigma$$

with the result that for $i \geq 1$

$$\|g_{i+1}(s) - g_i(s)\|_{H_o^m(D)}$$

$$\leq \int_0^s \|\Phi^*(s-\sigma)K(\sigma)B(\sigma)R^{-1}(\sigma)B^*(\sigma)\|_{H_o^m(D)} \|g_i(\sigma) - g_{i-1}(\sigma)\|_{H_o^m(D)} d\sigma$$

By an argument similar to that appearing in the proof of Theorem 4.3 we may write the above inequality as

$$\|g_{i+1}(s) - g_i(s)\|_{H_o^m(D)} \leq c\int_0^s \|g_i(\sigma) - g_{i-1}(\sigma)\|_{H_o^m(D)}$$

Moreover, since

$$\|g_1(s) - g_o(s)\|_{H_o^m(D)}$$

$$\leq \int_0^s \|\Phi^*(s-\sigma)K(\sigma)B(\sigma)R^{-1}(\sigma)B^*(\sigma)\| \; \|a(\sigma)\| d\sigma$$

and since $a(s)$, given in Eq. 4.5.2, is bounded in norm on the finite interval $[0, T]$ by $\|\alpha(s)\| \leq a$

$$\|g_1(s) - g_o(s)\|_{H_o^m(D)} \leq acs$$

From this it can be shown by induction that

$$\|y_{i+1} - y_i\|_{H_o^m(D)} \leq a \frac{(ct)^{i+1}}{(i+1)!}$$

and, thus,

$$\|y_{i+1} - y_i\|_{L^2(0,T;H_o^m(D))} \leq \frac{a}{c} \frac{(cT)^{i+2}}{(i+2)!}$$

so that

$$\sum_{i=0}^{\infty} \|y_{i+1} - y_i\|_{L^2(0,T;H_o^m(D))} \leq \frac{a}{c}[e^{cT} - (1 + cT)]$$

and convergence is guaranteed for T finite. Thus, existence of a solution $g(t)$ of Eq. 4.4.12 is proved. Uniqueness will not be proved, but it is a straightforward matter to modify Bellman's uniqueness proof (see Ref. 26, p. 8). The existence and uniqueness of $\phi(s)$ follows directly from the existence and uniqueness of $g(s)$.

We shall now show that the optimal control is given by Eq. 4.4.13 and the optimal cost function is given by Eq. 4.5.1.

<u>Theorem 4.8</u>: If a bounded self-adjoint operator solution $K(t)$ of Eq. 4.4.11 exists, then the optimal control is given by

$$u^*(t) = -R^{-1}(t)B^*(t)[K(t)x(t) + g(t)] \qquad (4.4.13)$$

and the optimal cost function is given by

$$J_{min} = \langle K(0)x(0), x(0) \rangle_{H_o^m(D)} + 2 \langle g(0), x(0) \rangle_{H_o^m(D)} + \phi(0)$$

Proof: Since $g(t)$ and $\phi(t)$ exist by Lemma 4.1, let us write the identity

$$\int_0^T \frac{d}{dt} [\langle K(t)x(t), x(t) \rangle_{H_o^m(D)} + 2 \langle g(t), x(t) \rangle_{H_o^m(D)} + \phi(t)]\, dt$$

$$- [\langle K(t)x(t), x(t) \rangle_{H_o^m(D)} + 2 \langle g(t), x(t) \rangle_{H_o^m(D)} + \phi(t)]_0^T = 0 \tag{4.5.3}$$

Performing the differentiation inside the integral and using Eqs. 4.4.11, 4.4.12, and 4.4.14 to eliminate $\dot{K}(t)$, $\dot{g}(t)$, and $\dot{\phi}(t)$, respectively, Eq. 4.5.3 becomes

$$\int_0^T \langle K(t)B(t)R^{-1}(t)B^*(t)[K(t)x(t) + 2g(t)],\ x(t) \rangle_{H_o^m(D)}\, dt$$

$$+ 2 \int_0^T \langle B^*(t)[K(t)x(t) + g(t)], u(t) \rangle_U\, dt \tag{4.5.4}$$

$$+ \int_0^T \langle B(t)R^{-1}(t)B^*(t)g(t), g(t) \rangle_{H_o^m(D)}\, dt$$

$$- \int_0^T \langle Q(t)(x(t) - x_d(t)), x(t) - x_d(t) \rangle_{H_o^m(D)}\, dt$$

$$- [\langle K(t)x(t), x(t) \rangle_{H_o^m(D)} + 2 \langle g(t), x(t) \rangle_{H_o^m(D)} + \phi(t)]_0^T = 0$$

If Eq. 4.5.4 is added to the cost criterion, Eq. 3.3.1, and if the equality

$$\langle K(T)x(T), x(T)\rangle_{H_o^m(D)} + 2\langle g(T), x(T)\rangle_{H_o^m(D)} + \phi(T)$$

$$= \langle F(x(T)-x_d(T)), x(T)-x_d(T)\rangle_{H_o^m(D)}$$

is used, then we obtain the result that

$$J = \langle K(0)x(0), x(0)\rangle_{H_o^m(D)} + 2\langle g(0), x(0)\rangle + \phi(0)$$

$$+ \int_0^T \langle K(t)B(t)R^{-1}(t)B^*(t)[K(t)x(t)+2g(t)], x(t)\rangle_{H_o^m(D)} dt$$

$$+ 2\int_0^T \langle B^*(t)[K(t)x(t)+g(t)], u(t)\rangle_U dt$$

$$+ \int_0^T \langle R(t)u(t), u(t)\rangle_U dt$$

$$+ \int_0^T \langle B(t)R^{-1}(t)B^*(t)g(t), g(t)\rangle_{H_o^m(D)} dt$$

This, in turn, can be written in the form

$$J = \langle K(0)x(0), x(0)\rangle_{H_o^m(D)} + 2\langle g(0), x(0)\rangle_{H_o^m(D)} + \phi(0) \tag{4.5.5}$$

$$+ \int_0^T \langle R(t)[R^{-1}(t)B^*(t)(K(t)x(t)+g(t))+u(t)],$$

$$R^{-1}(t)B^*(t)(K(t)x(t)+g(t)+u(t)\rangle_U dt$$

Since $R(t)$ is a strictly positive operator, the integral term must be greater than or equal to zero, the latter occurring if the control $u(\cdot)$ is chosen to be

$$u(t) = -R^{-1}(t)B^*(t)[K(t)x(t)+g(t)]$$

and the minimizing value of the cost criterion is

$$J_{min} = \langle K(0)x(0), x(0) \rangle_{H_o^m(D)} + 2 \langle g(0), x(0) \rangle_{H_o^m(D)} + \phi(0)$$

Thus, we have shown, in the case of parabolic systems with system operators which are infinitesimal generators of semigroups, that the assumption of the existence of a positive self-adjoint solution $K(t)$ of the Riccati operator equation yields precisely the same results for the characterization of the optimal control and the optimal cost function as were obtained in Section 4 for the case of coercive system operators. All of this motivates a vital question, namely, under what circumstances, (if any), do solutions of the Riccati operator equation exist for the class of system operators under consideration? This question is treated in the next section of this chapter.

4.6 EXISTENCE OF SOLUTIONS OF THE RICCATI OPERATOR EQUATION

In this section it will be shown that bounded, positive, self-adjoint solutions $K(t)$ of the Riccati operator equation (4.4.11) exist in the case where the system operator A is the infinitesimal generator of a semigroup of operators. This will be achieved by using an extension of the method of quasilinearization (see Ref. 27, p. 19) used by D. Kleinman[28] to prove the existence of a solution of the matrix Riccati equation for finite dimensional systems. In brief, this method consists of demonstrating existence and uniqueness of a solution of an auxiliary linear operator equation and using this equation in an iterative fashion to prove existence of a solution of the Riccati operator equation.

If the differential operator A is assumed to satisfy the inequality, Eq. 2.7.1, then the operator A_3, defined in Section 2.7, is the infinitesimal generator of the strongly continuous semigroup of operators $\{\Phi(t)\}_{t\epsilon[0,T]}$, and the solution of the parabolic equation

$$\dot{x}(t) = Ax(t) + B(t)u(t) \quad ; \quad x(0) = x_o \epsilon D_o(A_3)$$

may be written as

$$x(t) = \Phi(t)x_o + \int_0^t \Phi(t-\sigma)B(\sigma)u(\sigma)\,d\sigma \tag{4.6.1}$$

where $B(\sigma)\epsilon\mathcal{L}(U;L^2(D))$ and $u(\sigma)\epsilon U$. We shall have need in this section to discuss solutions of the linear operator equation

$$\frac{dV(t)}{dt} = -V(t)A - A^*V(t) - W(t) \quad ; \quad V(T) = F \tag{4.6.2}$$

where $V(t)$ and $W(t)$ are assumed to be bounded linear operators on $[0,T]$ with domain equal to $D_o(A_3)$ and F is the bounded self-adjoint terminal state-weighting matrix defined in Definition 3.3. For this equation we state the following lemma:

Lemma 4.2: The solution of the linear equation (4.6.2) is given uniquely by

$$V(t) = \Phi^*(T-t)F\Phi(T-t) + \int_t^T \Phi^*(\sigma-t)W(t)\Phi(\sigma-t)\,d\sigma \tag{4.6.3}$$

Proof: If A is the infinitesimal generator of a semigroup of operators, as described above, then its adjoint A^* is also the infinitesimal generator of a semigroup $\{S(t)\}_{t\epsilon[0,T]}$ of operators, and it is easily seen that this semigroup has the property

$$\frac{dS(t)}{dt}x_o = A^*S(t)x_o \qquad \forall x_o \epsilon Do(A_3) \qquad (4.6.4)$$

Moreover, since $S(t)$ is bounded on $[0, T]$, we have

$$\frac{dS^*(t)}{dt}x_o = \left(\frac{dS(t)}{dt}\right)^* x_o = S^*(t)Ax_o \qquad \forall x_o \epsilon Do(A_3) \qquad (4.6.5)$$

It should also be noted that, from the discussion in the proof of Lemma 4.1, $S(t) = \Phi^*(t)$ and $S^*(t) = \Phi(t)$. If we let x_o be an arbitrary element of $Do'A_3)$, Eq. 4.6.3 may be written

$$V(t)x_o = S(T-t)FS^*(T-t)x_o + \int_t^T S(\sigma-t)W(t)S^*(\sigma-t)x_o \, d\sigma$$

Differentiating this expression we obtain

$$\begin{aligned}\dot{V}(t)x_o &= \dot{S}(T-t)FS^*(T-t)x_o + S(T-t)F\dot{S}^*(T-t)x_o \\ &\quad -S(0)W(t)S^*(0)x_o - \int_t^T \dot{S}(\sigma-t)W(t)S^*(\sigma-t)x_o d\sigma \\ &\quad -\int_t^T S(\sigma-t)W(t)\dot{S}^*(\sigma-t)x_o d\sigma \\ &= -A^*[S(T-t)FS^*(T-t) + \int_t^T S(\sigma-t)W(t)S^*(\sigma-t)d\sigma]x_o \\ &\quad -[S(T-t)FS^*(T-t) + \int_t^T S(\sigma-t)W(t)S^*(\sigma-t)d\sigma]Ax_o \\ &\quad -W(t)x_o\end{aligned}$$

or, $\dot{V}(t)x_o = -V(t)Ax_o - A^*V(t)x_o - W(t)x_o$

Since x_o is arbitrary, the Eq. 4.6.3 holds under the assumption that $Do(V) = Do(A_3)$, demonstrating existence and uniqueness.

We shall want to use this lemma to prove the existence of the auxiliary equation discussed in the introductory paragraph of this section. Let us state and prove the following:

Theorem 4.9: If $L(t)$ is a bounded positive self-adjoint operator, defined on $[0, T]$, then there exists a unique positive, self-adjoint solution $V_L(t)$ of the equation

$$\dot{V}(t) = -V(t)[A-B(t)R^{-1}(t)B^*(t)L(t)] - [A^*-L(t)B(t)R^{-1}(t)B^*(t)]V(t) \tag{4.6.6}$$
$$- L(t)B(t)R^{-1}(t)B^*(t)L(t) - Q(t) \quad ; \quad V(T) = F$$

Moreover, if in the parabolic control problem specified by Definition 3.4 we let $x_d(t) = 0$ on $[0, T]$ and require the control to be of the form

$$u(t) = -R^{-1}(t)B^*(t)L(t)x(t) \tag{4.6.7}$$

then the cost criterion has the value

$$\underline{J} = \langle V_L(0)x_o, x_o \rangle_{H_o^m(D)}$$

Proof: We shall prove the existence of the solution $V_L(t)$ in much the same way that existence was proved for $g(t)$ in Lemma 4.1, that is, by means of successive approximations. For any arbitrary $x_o \in Do(A_3)$, $V_L(t)x_o$ must satisfy, according to Lemma 4.2, the following equation:

$$V_L(t)x_o = \Phi^*(T-t)F\Phi(T-t)x_o$$
$$+\int_t^T \Phi^*(\sigma_1-t)[L(\sigma_1)B(\sigma_1)R^{-1}(\sigma_1)B^*(\sigma_1)L(\sigma_1)+Q(\sigma_1)]\Phi(\sigma_1-t)x_o d\sigma_1 \tag{4.6.8}$$

(contd. on next page)

$$-\int_t^T \Phi^*(\sigma_1 - t)[V_L(\sigma_1)B(\sigma_1)R^{-1}(\sigma_1)B^*(\sigma_1)L(\sigma_1)$$
$$+L(\sigma_1)B(\sigma_1)R^{-1}(\sigma_1)B^*(\sigma_1)V_L(\sigma_1)]\Phi(\sigma_1 - t)x_o d\sigma_1$$

If we define $M(t)$ to be the bounded operator

$$M(t) = \Phi^*(T-t)F\Phi(T-t)$$
$$+\int_t^T \Phi^*(\sigma_1 - t)[L(\sigma_1)B(\sigma_1)R^{-1}(\sigma_1)B^*(\sigma_1)L(\sigma_1)+Q(\sigma_1)]$$
$$\Phi(\sigma_1 - t)d\sigma_1 \quad (4.6.9)$$

then Eq. 4.6.8 can be written in the form

$$V_L(t)x_o = M(t)x_o$$
$$-\int_t^T \Phi^*(\sigma_1 - t)[V_L(\sigma_1)B(\sigma_1)R^{-1}(\sigma_1)B^*(\sigma_1)L(\sigma_1)$$
$$+L(\sigma_1)B(\sigma_1)R^{-1}(\sigma_1)B^*(\sigma_1)V_L(\sigma_1)]\Phi(\sigma_1 - t)x_o d\sigma$$

Forming the sequence $\{V_L^i(t)\}_{i=0}^{\infty}$, where

$$V_L^o(t) = M(t), \text{ and}$$
$$V_L^{i+1}(t)x_o = M(t)x_o$$
$$-\int_t^T \Phi^*(\sigma_1 - t)[V_L^i(\sigma_1)B(\sigma_1)R^{-1}(\sigma_1)B^*(\sigma_1)L(\sigma_1)$$
$$+L(\sigma_1)B(\sigma_1)R^{-1}(\sigma_1)B^*(\sigma_1)V_L^i(\sigma_1)]\Phi(\sigma_1 - t)x_o d\sigma_1$$
$$\text{for } i \geq 1$$

we obtain

$$(V_L^{i+1}(t) - V_L^i(t))x_o$$
$$= -\int_t^T \Phi^*(\sigma_1 - t)[(V_L^i(\sigma_1) - V_L^{i-1}(\sigma_1))B(\sigma_1)R^{-1}(\sigma_1)B^*(\sigma_1)L(\sigma_1)]\Phi(\sigma_1 - t)x_o d\sigma_1$$

(4.6.10)

(contd. on next page)

$$-\int_t^T \Phi^*(\sigma_1-t)[\,L(\sigma_1)B(\sigma_1)R^{-1}(\sigma_1)B^*(\sigma_1)(V_L^i(\sigma_1)-V_L^{i-1}(\sigma_1))\,]\,\Phi(\sigma_1-t)x_o\,d\sigma_1$$

From Eq. 4.6.10 and by the self-adjointness of $L(\sigma_1)$, $B(\sigma_1)$, and $R^{-1}(\sigma_1)$ we see that

$$\|(V_L^{i+1}(t) - V_L^i(t))x_o\|_{H_o^m(D)}$$

$$\leq 2\int_t^T \|\Phi^*(\sigma-t)L(\sigma)B(\sigma)R^{-1}(\sigma)B^*(\sigma)\|_{H_o^m(D)} \|(V_L^i(\sigma)-V_L^{i-1}(\sigma))\Phi(\sigma-t)x_o\|_{H_o^m(D)}\,d\sigma$$

$$\leq 2\,k\int_t^T \|(V_L^i(\sigma)-V_L^{i-1}(\sigma))\Phi(\sigma-t)x_o\|_{H_o^m(D)}\,d\sigma$$

since $\Phi^*(\sigma_1)$, $L(\sigma_1)$, $B(\sigma_1)$, and $R^{-1}(\sigma_1)$ are bounded operators. We can, in turn, write

$$\|(V_L^i(\sigma_1)-V_L^{i-1}(\sigma_1))\Phi(\sigma_1-t)x_o\|_{H_o^m(D)}$$

$$\leq 2k\int_{\sigma_1}^T \|(V_L^{i-1}(\sigma_2)-V_L^{i-2}(\sigma_2))\Phi(\sigma_1-t)\Phi(\sigma_2-\sigma_1)x_o\|_{H_o^m(D)}\,d\sigma_2 \tag{4.6.11}$$

But, by the semigroup property (2.6.1),

$$\Phi(\sigma_1-t)\Phi(\sigma_2-\sigma_1) = \Phi(\sigma_2-t)$$

so that we may continue this iterative substitution to obtain

$$\|(V_L^{i+1}(t)-V_L^i(t))x_o\|_{H_o^m(D)}$$

$$\leq 2^i k^i\int_t^T\int_{\sigma_1}^T \cdots \int_{\sigma_{i-1}}^T \|(V_L^1(\sigma_i)-V_L^o(\sigma_i))\Phi(\sigma_i-t)x_o\|_{H_o^m(D)}\,d\sigma_1\,d\sigma_2\ldots d\sigma_i$$

$$\leq 2^i k^i \int_t^T \int_{\sigma_1}^T \cdots \int_{\sigma_{i-1}}^T \int_{\sigma_i}^T$$

$$\|\Phi^*(\sigma_{i+1}-\sigma_i)[M(\sigma_{i+1})B(\sigma_{i+1})R^{-1}(\sigma_{i+1})B^*(\sigma_{i+1})L(\sigma_{i+1})$$

$$+ L(\sigma_{i+1})B(\sigma_{i+1})R^{-1}(\sigma_{i+1})B^*(\sigma_{i+1})M(\sigma_{i+1})]$$

$$\Phi(\sigma_{i+1}-\sigma_i)\Phi(\sigma_i-t)x_o\| d\sigma_1 \ldots d\sigma_{i+1}$$

$$\leq 2^{i+1} k^i \int_t^T \int_{\sigma_1}^T \cdots \int_{\sigma_{i-1}}^T \int_{\sigma_i}^T$$

$$\|\Phi^*(\sigma_{i+1}-\sigma_i)\| \; \|M(\sigma_{i+1})B(\sigma_{i+1})R^{-1}(\sigma_{i+1})B^*(\sigma_{i+1})L(\sigma_{i+1})\|$$

$$\|\Phi(\sigma_{i+1}-t)\| \; \|x_o\| d\sigma_1 \ldots d\sigma_{i+1}$$

As before, it can be shown that $\Phi(t)$, $\Phi^*(t)$, $M(t)$, $B(t)$, $R^{-1}(t)$, and $L(t)$ are uniformly bounded on the finite interval $[0,T]$, so that

$$\|(V_L^{i+1}(t)-V_L^i(t))x_o\|_{H_o^m(D)}$$

$$\leq 2^{i+1} k^i \int_t^T \int_{\sigma_1}^T \cdots \int_{\sigma_i}^T G\|x_o\|_{H_o^m(D)} \, d\sigma_1 \ldots d\sigma_{i+1}$$

$$\leq 2^{i+1} k^i G\|x_o\| \int_t^T \int_{\sigma_1}^T \cdots \int_{\sigma_i}^T d\sigma_1 \ldots d\sigma_{i+1}$$

$$= \frac{2^{i+1} k^i G(T-t)^{i+1}}{(i+1)\,!} \|x_o\|_{H_o^m(D)}$$

From this we determine the $L^2(0,T;H_o^m(D))$ norm to be

$$\|(V_L^{i+1}(\cdot)-V_L^i(\cdot))x_o\|_{L^2(0,T;H_o^m(D))}$$

$$\leq \frac{2^{i+1}k^iG\;T^{i+2}}{(i+2)\,!}\;\|x_o\|_{H_o^m(D)} = \frac{G}{2k^2}\;\frac{(2kT)^{i+2}}{(i+2)!}\;\|x_o\|_{H_o^m(D)}$$

From this we can see that the infinite series

$\sum_{i=0}^{\infty}\|(V_L^{i+1}(\cdot)-V_L^i(\cdot))x_o\|_{L^2(0,T;H_o^m(D)}$ converges and is less than or equal to $\frac{G}{2k}[e^{2kT}-(1+2KT)]\;\|x_o\|_{H_o^m(D)}$ implying that we have convergence of $\{V_L^i(\cdot)\}_{i=0}^{\infty}$ to the solution $V_L(\cdot)$ in the $L^2(0,T;H_o^m(D))$ sense. Thus, the existence of a solution to Eq. 4.6.6 is proved. Once again, as in the case of Lemma 4.1, the proof of uniqueness is a straightforward procedure and will be omitted. The self-adjointness of $V_L(t)$ can be deduced from the fact that its adjoint also satisfies Eq. 4.6.6, so that, by uniqueness, $V_L(t) = V_L^*(t)$.

To prove the second part of the theorem, we examine the cost criterion for the parabolic control problem in the case where the desired state trajectory, $x_d(t)$, is zero. The cost criterion becomes

$$J = \langle Qx,x\rangle_{L^2(0,T;H_o^m(D))} + \langle Ru,u\rangle_{L^2(0,T;U)} + \langle Fx(T),x(T)\rangle_{H_o^m(D)} \qquad (4.6.12)$$

If, in the identity

$$\langle V_L(0)x_o, x_o\rangle_{H_o^m(D)} - \langle V_L(T)x(T), x(T)\rangle_{H_o^m(D)}$$

$$= \int_0^T [\langle \dot{V}_L(t)x(t), x(t)\rangle_{H_o^m(D)}$$

$$+ \langle V_L(t)\dot{x}(t), x(t)\rangle_{H_o^m(D)} + \langle V_L(t)x(t), \dot{x}(t)\rangle_{H_o^m(D)}]\, dt$$

we replace $\dot{V}_L(t)$ and $V_L(T)$ by their specifications in Eq. 4.6.6 and use the value of $\dot{x}$ from the system equation, we obtain

$$\langle V_L(0)x_o, x_o\rangle_{H_o^m(D)} - \langle Fx(T), x(T)\rangle_{H_o^m(D)}$$

$$= -\int_0^T \langle Q(t)x(t), x(t)\rangle_{H_o^m(D)}\, dt$$

$$- \int_0^T \langle L(t)B(t)R^{-1}(t)B^*(t)L(t)x(t), x(t)\rangle_{H_o^m(D)}\, dt$$

$$+ \int_0^T \langle [V_L(t)B(t)R^{-1}(t)B^*(t)L(t)x(t) \tag{4.6.13}$$

$$+ L(t)B(t)R^{-1}(t)B^*(t)V_L(t)x(t) + V_L(t)B(t)u(t)], x(t)\rangle_{H_o^m(D)}\, dt$$

$$+ \int_0^T \langle V_L(t)x(t), B(t)u(t)\rangle_{H_o^m(D)}\, dt$$

Now, using the fact that the control $u(t)$ satisfies Eq. 4.6.7, the second integral in the right hand side of Eq. 4.6.13 can be identified as

$$-\int_0^t \langle L(t)B(t)R^{-1}(t)B^*(t)L(t)x(t), x(t)\rangle_{H_o^m(D)}\, dt$$

$$= -\langle Ru, u\rangle_{L^2(0,T;U)}$$

and the last two integrals on the right-hand side of Eq. 4.6.13 sum to zero, so that

$$J = \langle Qx, x\rangle_{L^2(0,T;H_o^m(D))} + \langle Ru, u\rangle_{L^2(0,T;U)}$$

$$+ \langle Fx(T), x(T)\rangle_{H_o^m(D)} = \langle V_L(0)x_o, x_o\rangle_{H_o^m(D)}$$

proving the second part of the theorem. Since, from the above discussion, the system could have been started at any time $t\epsilon[0,T]$ with initial state x_o, it must be true that

$$J = \langle Qx, x\rangle_{L^2(t,T;H_o^m(D))} + \langle Ru, u\rangle_{L^2(t,T;U)}$$

$$+ \langle Fx(T), x(T)\rangle_{H_o^m(D)} = \langle V_L(t)x_o, x_o\rangle$$

and since $J \geq 0$ for all initial states $x_o \epsilon Do(A_3)$, it follows that

$$\langle V_L(t)x_o, x_o\rangle \geq 0 \qquad \forall\, t\epsilon[0,T],\ \forall\, x_o \epsilon Do(A_3)$$

demonstrating the positivity of the solution $V_L(t)$ and completing the proof of Theorem 4.9.

Corollary 4.1: If, in the parabolic control problem, we specify $\iota=0$, the solution of Eq. 4.6.6 becomes

$$V_o(t) = \Phi^*(T-t)F\Phi(T-t) + \int_t^T \Phi^*(\sigma-t)Q(\sigma)\Phi(\sigma-t)\,d\sigma \tag{4.6.14}$$

Proof: Equation 4.6.14 results from $L(t)$ being the zero operator in Eq. 4.6.8.

We shall now generate a sequence of positive, self-adjoint linear operator functions $\{V_n(t)\}_{n=0}^{\infty}$, $t\epsilon[0,T]$, each with domain equal to $Do(A_3)$. This sequence will be generated by the method of successive approximations in Eq. 4.6.6. The elements in the sequence will be shown to be bounded and monotonically decreasing in a sense which will be defined. It will then be shown that the sequence has a limit and this limit is the solution of the Riccati operator equation.

The sequence $\{V_n(t)\}$ is given by

$$V_o(t) = 0$$

$$\dot{V}_{n+1}(t) = -V_{n+1}(t)[A-B(t)R^{-1}(t)B^*(t)V_n(t)] - [A^*-V_n(t)B(t)R^{-1}(t)B^*(t)]V_{n+1}(t) \tag{4.6.15}$$

$$-V_n(t)B(t)R^{-1}(t)B^*(t)V_n(t) - Q(t) \quad ; \quad V_{n+1}(T) = F$$

It should be noted that Eq. 4.6.15 is equivalent to Eq. 4.6.6 with $V_L(t) = V_{n+1}(t)$ and $L(t) = V_n(t)$. One of the properties we would like to show that the sequence of operator functions possesses is that of monotonically decreasing positivity. We make this concept precise in the following definition:

Definition 4.4: If P_1 and P_2 are both positive linear operators, then P_1 is said to be greater than or equal to P_2, denoted $P_1 \geq P_2$ if the linear operator $(P_1 - P_2)$ is a positive operator.

We shall now state and prove a lemma which will be useful in the characterization of the sequence of operators:

Lemma 4.3: If X_1 and X_2 are self-adjoint linear operators, then

$$X_1BR^{-1}B^*X_1 = X_1BR^{-1}B^*X_2 + X_2BR^{-1}B^*X_1 - X_2BR^{-1}B^*X_2 + N \tag{4.6.16}$$

where N is some positive operator.

Proof: Equation 4.6.16 follows from the fact that

$$(X_1 - X_2)\, BR^{-1}B^*(X_1 - X_2) \geq 0$$

which implies that

$$X_1BR^{-1}B^*X_1 \geq X_1BR^{-1}B^*X_2 + X_2BR^{-1}B^*X_1 - X_2BR^{-1}B^*X_2$$

and this implies that there must exist a positive operator N such that Eq. 4.6.16 holds.

The desired properties of the sequence $\{V_n(t)\}_{n=0}^{\infty}$ are now stated in the following theorem:

Theorem 4.10: The elements of $\{V_n(t)\}_{n=0}^{\infty}$ are bounded, and the sequence is monotonically decreasing for $n \geq 1$, in the sense that

$$V_1(t) \geq V_2(t) \geq V_3(t) \geq \dots\dots$$

Proof: (By induction) $V_1(t)$ is given by the expression in Eq. 4.6.14, which is clearly bounded and positive. According to the sequence generation formula (4.6.15):

$$\dot{V}_2(t) = -V_2(t)A_1(t) - A_1^*(t)V_2(t) - V_1(t)B(t)R^{-1}(t)B^*(t)V_1(t) - Q(t) \tag{4.6.17}$$

where

$$A_1(t) = A - B(t)R^{-1}(t)B^*(t)V_1(t)$$

The self-adjoint solution $V_2(t)$ exists and is positive by Theorem 4.9 (with $V_2(t) = V_L(t)$ and $L(t) = V_1(t)$). We may write the following differential equation for $\delta V_2(t) = V_1(t) - V_2(t)$

$$\begin{aligned}
\delta \dot{V}_2 &= -(V_1(t) - V_2(t))A - A^*(V_1(t) - V_2(t)) - V_2(t)B(t)R^{-1}(t)B^*(t)V_1(t) \\
&\quad - V_1(t)B(t)R^{-1}(t)B^*(t)V_2(t) + V_1(t)B(t)R^{-1}(t)B^*(t)V_1(t) \\
&= -\delta V_2(t)[A - B(t)R^{-1}(t)B^*(t)V_1(t)] \\
&\quad -[A^* - V_1(t)B(t)R^{-1}(t)B^*(t)]\,\delta V_2 \\
&\quad - V_1(t)B(t)R^{-1}(t)B^*(t)V_1(t)
\end{aligned} \tag{4.6.18}$$

with terminal condition $\delta V_2(T) = 0$
for which a positive solution $\delta V_2(t)$ for all $t \epsilon [0, T]$ exists by Theorem 4.9 (with $V_L(t) = \delta V_2(t)$, $L(t) = V_1(t)$, $Q(t) = 0$, and $F = 0$). Since both $V_1(t)$ and $V_2(t)$ are positive this means that

$$V_1(t) \geq V_2(t) \qquad \forall\, t \epsilon [0, T]$$

Moreover, since $V_1(t)$ is bounded, then $V_2(t)$ must also be bounded by the following argument:

Since $V_2(t)$ is positive and self-adjoint, we can apply the generalized Schwartz inequality (see Ref. 22 , p.262)

$$|\langle V_2(t)x, y\rangle|^2 \leq \langle V_2(t)x, x\rangle\langle V_2(t)y, y\rangle$$

and obtain

$$\|V_2(t)\| = \sup_{\substack{\|x\|\neq 0 \\ \|y\|\neq 0}} \frac{|\langle V_2(t)x, y\rangle|}{\|x\|\ \|y\|} \leq \left[\sup_{\|x\|\neq 0} \frac{\langle V_2(t)x, x\rangle}{\|x\|^2} \cdot \sup_{\|y\|\neq 0} \frac{\langle V_2(t)y, y\rangle}{\|y\|^2}\right]^{1/2}$$

$$\leq \left[\sup_{\|x\| \neq 0} \frac{\langle V_1(t)x, x \rangle}{\|x\|^2} \cdot \sup_{\|y\| \neq 0} \frac{\langle V_1(t)y, y \rangle}{\|y\|^2} \right]^{1/2} \leq \|V_1(t)\|$$

hence, proving the boundedness of $V_2(t)$.

To continue the induction process we assume that $V_{n-1}(t)$ is bounded and $V_{n-1}(t) \geq V_n(t)$. We must prove that $V_n(t) \geq V_{n+1}(t)$. Now, $V_n(t)$ satisfies the equation

$$\dot{V}_n(t) = -V_n(t)A_{n-1}(t) - A^*_{n-1}(t)V_n(t) - V_{n-1}(t)B(t)R^{-1}(t)B^*(t)V_{n-1}(t) - Q(t)$$

with $\qquad A_{n-1}(t) = A - B(t)R^{-1}(t)B^*(t)V_{n-1}(t)$

We may apply Lemma 4.3 to the next to last term of this equation to obtain the following equation for $V_n(t)$:

$$\dot{V}_n(t) = -V_n(t)A_{n-1}(t) - A^*_{n-1}(t)V_n(t) - Q(t) - V_{n-1}(t)B(t)R^{-1}(t)B^*(t)V_n(t)$$
$$-V_n(t)B(t)R^{-1}(t)B^*(t)V_{n-1}(t) + V_n(t)B(t)R^{-1}(t)B^*(t)V_n(t) - N(t)$$

where $N(t)$ is some bounded, self-adjoint positive operator. By a rearrangement of terms similar to that done to obtain Eq. 4.6.18 this can be shown to be

$$\dot{V}_n(t) = -V_n(t)A_n(t) - A^*_n(t)V_n(t) - Q(t) - V_n(t)B(t)R^{-1}(t)B^*(t)V_n(t) - N(t)$$

with $\qquad A_n(t) = A - B(t)R^{-1}(t)B^*(t)V_n(t)$

But, since a positive self-adjoint $V_{n+1}(t)$ exists by Theorem 4.9 and in addition satisfies the equation

$$\dot{V}_{n+1}(t) = -V_{n+1}(t)A_n(t) - A^*_n(t)V_{n+1}(t) - Q(t) - V_n(t)B(t)R^{-1}(t)B^*(t)V_n(t)$$

then the differential equation for the difference linear operator

$$\delta V_{n+1}(t) = V_n(t) - V_{n+1}(t)$$

is given by

$$\delta\dot{V}_{n+1}(t) = \dot{V}_n(t) - \dot{V}_{n+1}(t) = -\delta V_{n+1}(t)A_n(t) - A_n(t)\delta V_{n+1}(t) - N(t);\ \delta V_{n+1}(T) = 0 \tag{4.6.19}$$

The proof that the solution of this equation is positive for all $t \epsilon [0,T]$ is given in Appendix E so that we know that

$$\delta V_{n+1}(t) \geq 0 \qquad \forall\ t \epsilon [0,T]$$

which implies that

$$V_n(t) \geq V_{n+1}(t) \qquad \forall\ t \epsilon [0,T]$$

and $V_{n+1}(t)$ is bounded on this interval by the same argument which proved $V_2(t)$ bounded, completing the proof of Theorem 4.10.

It remains to achieve the stated aim of this section, that is, the demonstration of the existence of the optimal feedback operator $K(t)$. This will be done by showing that the sequence $\{V_n(t)\}_{n=0}^{\infty}$ has a limit and this limit is precisely $K(t)$. We state this as a theorem:

Theorem 4.11: A self-adjoint, positive, bounded solution $V_\infty(t)$ of the Riccati operator equation (4.4.11) exists and is the limit (in $H_o^m(D)$) of the sequence of operators $\{V_n(t)\}_{n=1}^{\infty}$ defined by Eq. 4.6.15.

Proof: We shall show that $\lim_{n \to \infty} V_n(t)x$ exists for all $x \epsilon H_o^m(D)$ and that the limit satisfies the Riccati equation (4.4.11).

For any $x_o \epsilon H_o^m(D)$, we have from Theorem 4.10 that

$$\langle V_1(t)x_o, x_o \rangle_{H_o^m(D)} \geq \langle V_2(t)x_o, x_o \rangle_{H_o^m(D)} \geq \cdots\cdots$$

Moreover, Theorem 4.10 also implies that each of the elements of the sequence $\{\langle V_n(t)x_o, x_o \rangle_{H_o^m(D)}\}_{n=1}^{\infty}$ is bounded below by 0. Since any monotonically decreasing sequence of real numbers which is bounded below has a limit, we know that $\lim_{n\to\infty} \langle V_n(t)x_o, x_o \rangle_{H_o^m(D)}$ exists for all $x_o \in H_o^m(D)$.

To show that the limit is of the form $\langle V_\infty(t)x_o, x_o \rangle_{H_o^m(D)}$ we use a well-known theorem (see Ref. 29, p. 189) on linear operators in a general Hilbert space H, which states that a monotone decreasing sequence of positive, self-adjoint operators $\{V_n\}_{n=1}^{\infty}$ has a limit V_∞ in the sense that

$$V_\infty x = \lim_{n\to\infty} V_n x \qquad , \text{ for all } x \in H$$

Using this result and Theorem 4.10, with the Hilbert space $H = H_o^m(D)$, we may conclude that

$$V_\infty(t)x = \lim_{n\to\infty} V_n(t)x \quad , \text{ for all } x \in H_o^m(D) \quad \text{and } t \in [0, T]$$

Now that we have shown that $V_\infty(t)$ exists, we shall show that it satisfies the Riccati operator equation (4.4.11). Integrating both sides of Eq. 4.6.15 from t to T we obtain

$$V_{n+1}(t) - F = \int_t^T \{V_{n+1}(\sigma) A - V_{n+1}(\sigma) B(\sigma) R^{-1}(\sigma) B^*(\sigma) V_n(\sigma) + A^* V_{n+1}(\sigma)$$

$$- V_n(\sigma) B(\sigma) R^{-1}(\sigma) B^*(\sigma) V_{n+1}(\sigma) + V_n(\sigma) B(\sigma) R^{-1}(\sigma) B^*(\sigma) V_n(\sigma) - Q(\sigma)\} d\sigma$$

Taking the limit as n approaches infinity,

$$V_\infty(t) - F = \int_t^T \{V_\infty(\sigma) A - V_\infty(\sigma) B(\sigma) R^{-1}(\sigma) B^*(\sigma) V_\infty(\sigma) + A^* V_\infty(\sigma) - Q(\sigma)\} \; d\sigma \tag{4.6.20}$$

Equation 4.6.20 shows that $V_\infty(t)$ is continuous in t and differentiable, so that by differentiating (4.6.20), we obtain

$$\dot{V}_\infty(t) = -V_\infty(t) A - A^* V_\infty(t) + V_\infty(t) B(t) R^{-1}(t) B^*(t) V_\infty(t) - Q(t) \; ; \quad V_\infty(T) = F$$

showing that $V_\infty(t)$ satisfies the Riccati operator equation. $V_\infty(t)$ is clearly self-adjoint and can be shown to be bounded by application of the generalized Schwartz inequality in precisely the same fashion as was done in the proof of Theorem 4.10

We have thus satisfied the hypotheses of Theorem 4.8 and can identify the operator $V_\infty(t)$ with the optimal feedback operator $K(t)$. It should be noted that although the results appear to be the same in both the case where the system operator is coercive and the case where the system operator is the infinitesimal generator of a semigroup of operators, there is a difference. In the coercive case the results hold for all initial states in the state space $H_o^m(D)$, whereas, in the latter case, the results hold only for initial states in the domain of A_3. This is not as restrictive as it might seem, since the domain of A_3 is dense in $H_o^m(D)$, and, thus, in the latter case, $K(t)x$ can be defined for $x \epsilon H_o^m(D)$ and $x \notin Do(A_3)$ by letting

$$\overline{K(t)}x = \lim_{n \to \infty} K(t)x_n$$

where $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in $Do(A_3)$ converging to x. Since $K(t)$ has been shown to be bounded, the sequence $\{K(t)x_n\}_{n=1}^{\infty}$ has a limit.

With the matter of existence of solutions to the Riccati operator equation resolved, let us briefly consider the problem of actually solving the Riccati operator equation. Since the space $H_o^m(D)$ is a separable Hilbert space, there exists a basis $\{\phi_i\}_{i=1}^{\infty}$ where each ϕ_i in the basis is an element of $H_o^m(D)$, such that any element $x \epsilon H_o^m(D)$ has the unique representation

$$x = \sum_{j=1}^{\infty} x_j \phi_j$$

where the coefficients x_j are given by

$$x_j = \langle x, \phi_j \rangle_{H_o^m(D)} \quad ; \quad j=1,2,\ldots$$

Thus, we may consider an element $x \epsilon H_o^m(D)$ to be alternatively represented by the infinite dimensional vector $\underline{x}$, with j^{th} component x_j. If L is any linear operator from $H_o^m(D)$ into $H_o^m(D)$, we have for $x \epsilon H_o^m(D)$,

$$Lx = L \sum_{j=1}^{\infty} x_j \phi_j = \sum_{j=1}^{\infty} x_j L\phi_j$$

Now, $L\phi_j$ is an element of $H_o^m(D)$ so that

$$L\phi_j = \sum_{i=1}^{\infty} L_{ij} \phi_i$$

where

$$L_{ij} = \langle L\phi_j, \phi_i \rangle_{H_o^m(D)} \quad ; \quad i=1,2,\ldots \quad j=1,2,\ldots$$

Thus, Lx may be represented by $\underline{L}\underline{x}$, where $\underline{L}$ is the infinite matrix with ij^{th} element L_{ij}. Similarly, since U is a separable Hilbert space, with basis $\{\psi_i\}_{i=1}^{\infty}$, each element $u \epsilon U$ may be considered to be an infinite dimensional vector $\underline{u}$, and the control operator $B(t)$ may be considered to be the infinite matrix $\underline{B}(t)$ with ij^{th} element

$$B_{ij}(t) = \langle B(t)\psi_j, \phi_i \rangle_{H_o^m(D)}$$

Let us, for the purpose of illustration assume that $R(t) = I$, the identity operator. We may now rewrite the Riccati operator equation as the <u>infinite dimensional</u> matrix Riccati equation

$$\dot{\underline{K}}(t) = -\underline{K}(t)\underline{A}-\underline{A}'\underline{K}(t) + \underline{K}(t)\underline{B}(t)\underline{B}'(t)\underline{K}(t) - \underline{Q}(t) \quad ; \quad \underline{K}(T) = \underline{F}$$

where all of the matrices in this expression are uniquely determined in the fashion prescribed above.

As an example of this procedure let us consider the scalar heat equation (see Section 2.4), with coefficient $\mu=1$, on the domain $D=(0,1)$

$$\frac{\partial x(t,z)}{\partial t} = \frac{\partial^2 x(t,z)}{\partial z^2} + u(t,z) \quad ; \quad x(0,z) = x_o(z)$$

with boundary conditions

$$x(t,0) = x(t,1) = 0$$

Let us choose the cost functional to be.

$$J = \int_0^T [\ \|x(t)\|^2_{H_o^2(D)} + \|u(t)\|^2_{H_o^2(D)}\ ]\,dt + \|x(T)\|^2_{H_o^m(D)}$$

which corresponds to choosing the operators $Q(t)$, $R(t)$, and F to be the identity operator on $H_o^2(D)$; which has the infinite matrix representation

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & \underline{0} \\ & & 1 & \\ & \underline{0} & & \ddots \end{bmatrix}$$

The Sobolev space under consideration is $H_o^2(D)$, and we shall choose the countable basis $\left\{ \frac{2}{1+n^2\pi^2+n^4\pi^4} \sin n\pi z \right\}_{n=1}^{\infty}$. It is easily seen that this is an orthonormal set in the $H_o^2(D)$ norm. Using this basis, the operator $A = \frac{\partial^2}{\partial z^2}$ has the matrix representation $\underline{A}$ the ij^{th} element of which is given by

$$A_{ij} = \left\langle \frac{\partial^2 \phi_j(z)}{\partial z^2},\ \phi_i(z) \right\rangle_{H_o^m(D)}$$

$$= \left\langle -j^2\pi^2\phi_j(z),\ \phi_i(z) \right\rangle_{H_o^m(D)} = -i^2\pi^2\delta_{ij}$$

where δ_{ij} is the Kronecker delta. Thus $\underline{A}$ is the infinite diagonal matrix

$$\underline{A} = \begin{bmatrix} -\pi^2 & & & & \\ & -4\pi^2 & & & \underline{0} \\ & & -9\pi^2 & & \\ & & & -16\pi^2 & \\ & \underline{0} & & & \ddots \end{bmatrix}$$

and the system partial differential equation may be written in the form

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{u}(t)$$

and the Riccati operator differential equation may be written as

$$\dot{\underline{K}} = -\underline{K}\begin{bmatrix} -\pi^2 & & & \\ & -4\pi^2 & & \underline{0} \\ & & -9\pi^2 & \\ & \underline{0} & & \ddots \end{bmatrix} - \begin{bmatrix} -\pi^2 & & & \\ & -4\pi^2 & & \underline{0} \\ & & -9\pi^2 & \\ & \underline{0} & & \ddots \end{bmatrix}\underline{K}$$

$$+ \underline{K}^2 - \begin{bmatrix} 1 & & & \\ & 1 & & \underline{0} \\ & & 1 & \\ & \underline{0} & & \ddots \end{bmatrix} ; \quad \underline{K}(T) = \begin{bmatrix} 1 & & & \\ & 1 & & \underline{0} \\ & & 1 & \\ & \underline{0} & & \ddots \end{bmatrix}$$

It is possible to truncate these matrices and solve the resulting finite dimensional matrix differential equation for an approximate value of the $\underline{K}(t)$ matrix. Approximations of this type will be discussed, in a slightly different context, when we consider modal analytic solutions in Chapter V. An alternative way of determining optimal feedback solutions will be presented in Section 4.8.

4.7 INFINITE TIME SOLUTIONS

In the parabolic control problems considered in the preceding sections, it has been assumed that the terminal time T is finite. In this section we examine the behavior of optimal solutions when T approaches infinity. It will be shown that an optimal solution and a solution of the Riccati operator equation on $[0,\infty)$ exist in the case where the system operator is coercive. These results are due to Lions.[15] Moreover, it will be shown that, under the assumption of complete controllability, a solution of the Riccati operator equation exists on $[0,\infty)$ in the case where the system operator is the infinitesimal generator of a semigroup of operators. For both types of system operators it will be shown that the Riccati operator equation becomes an algebraic time-invariant operator equation when the operators $B(t)$, $Q(t)$, and $R(t)$ are assumed to be time-invariant.

Let us assume that the operators $B(t)$, $Q(t)$, and $R(t)$ are uniformly bounded on $[0,\infty)$. Moreover, let us also assume that there is no terminal-time-weighting cost, i.e., $F=0$. Existence and uniqueness of the optimal control on the infinite time interval are guaranteed in the case of coercive system operators, since $\Pi(u,v)$ in Theorem 4.1 is still a symmetric, coercive bilinear form continuous in u and v even though u and v are controls defined on the infinite interval $[0,\infty)$. Theorem 4.2 still holds and tells us that $u(t)=-R^{-1}(t)B^{*}p(t)$ for all $t\in[0,\infty)$. It remains to show that the costate is well defined on the infinite interval, and that it can be written as

$$p(t) = K(t)x(t) + g(t), \qquad \forall\, t\in[0,\infty)$$

Lions[Ref 15, p. 181] shows that, if the desired state $x_d(\cdot)\in L^2(0,\infty;H_o^m(D))$ then the costate equation

$$\frac{dp}{dt} = -A^*p(t) - Q(t)[x(t) - x_d(t)]$$

has a unique solution $p_\infty(\cdot)\epsilon W(0,\infty)$. If s is any fixed time in $[0,\infty)$ and $x(s) = h\epsilon H_o^m(D)$ the argument in Section 4.4 can be duplicated to show that the transformation $h \rightarrow p(s)$ is continuous from $H_o^m(D)$ into $H_o^m(D)$, so that we can write

$$p_\infty(s) = K_\infty(s)h + g_\infty(s)$$

or, by using the fact that $s\epsilon[0,\infty)$ is arbitrary, we conclude that

$$p_\infty(t) = K_\infty(t)x_\infty(t) + g_\infty(t) \qquad \forall t\epsilon[0,\infty)$$

where $x_\infty(t)$ is the solution of the system equation on $[0,\infty)$. The remaining arguments of Section 4.4 are extended without difficulty to show that $K_\infty(t)$ satisfies the Riccati operator equation

$$\dot{K}_\infty(t) = -K_\infty(t)A - A^*K_\infty(t) + K_\infty(t)B(t)R^{-1}(t)B^*(t)K_\infty(t) - Q(t) \qquad (4.7.1)$$

and $g_\infty(\cdot)$ is an element of $W(0,\infty)$ which satisfies

$$\dot{g}_\infty(t) = -A^*g_\infty(t) + K_\infty(t)B(t)R^{-1}(t)B^*(t)g_\infty(t) + Q(t)x_d(t) \qquad (4.7.2)$$

and the minimum cost using the optimal control $u_\infty^*(t) = -R^{-1}(t)B^*(t)[K_\infty(t)x(t) + g_\infty(t)]^*$ on the interval $[s,\infty)$ is given by

$$J = \langle K_\infty(s)x_\infty(s), x_\infty(s)\rangle_{H_o^m(D)} + 2\langle g_\infty(s), x_\infty(s)\rangle_{H_o^m(D)} \qquad (4.7.3)$$

Under the assumption that $B(t) = B$, $Q(t) = Q$, and $R(t) = R$ Lions(15, p. 183) shows that the transformation $h \rightarrow p(s)$ is independent of s, so that we may write

$$p(s) = K_\infty h + g_\infty(s)$$

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* Note that since $x_\infty(\cdot)\epsilon W(0,\infty)$ and $g_\infty(\cdot)\epsilon W(0,\infty)$, then $\lim_{t\rightarrow\infty} u^*(t) = 0$.

where K_∞ is the solution of the time-invariant algebraic operator equation

$$K_\infty A + A^*K_\infty - K_\infty BR^{-1}B^*K_\infty + Q = 0 \qquad (4.7.4)$$

and $g_\infty(t)$ is the solution of the time-invariant differential equation

$$\dot{g}_\infty(t) = -A^*g_\infty(t) + K_\infty BR^{-1}B^*g_\infty(t) + Qx_d(t)$$

So it is seen that the important point of continuity of the transformation of the initial conditions to the costate carries through in a straightforward manner and enables us to obtain results for the infinite terminal time case similar to those obtained in Section 4.4 for the finite terminal time case. Once again, this continuous transformation cannot be defined in the case where the system operator is the infinitesimal generator of a semigroup of operators. Moreover, the infinite time version of Theorem 4.3, namely, the fact that the mapping $u(\cdot)\to x(\cdot)$ of $L^2(0,\infty;U)$ into $L^2(0,\infty;H_o^m(D))$ is continuous, cannot be proved, since inequalities in the proof do not hold on the infinite time interval. This results in an inability to use Theorem 4.5 to prove existence of a unique optimal control in this case. In order to prove existence of an optimal control we must make use of the concept of complete controllability and proceed by limiting arguments to the characterization of the optimal control.

Let us make the following definition:*

Definition 4.5: The parabolic system

$$\dot{x} = Ax + Bu \quad ; \qquad x(t) = x_o \epsilon H_o^m(D)$$

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* This is just the application of the standard definition of complete controllability (see Ref. 24, p.200) to parabolic systems.

is said to be completely controllable if for every $t \epsilon [0,\infty)$ and for every $x_o \epsilon H_o^m(D)$, there exists a time $t_1 \geq t$ and a control $u_o(\cdot)$ defined on $[t, t_1]$ such that the state at t_1, $x(t_1) = 0$. Another way to interpret this definition is that if the state at any time t_1 can be represented by a linear transformation on the control $u(\cdot)$ defined on $[t, t_1]$, namely,

$$x(t_1) = L_{t_1} u + \phi(t_1)$$

then we would like to find a control u_o and a time t_1 such that

$$L_{t_1} u_o = -\phi(t_1)$$

if $\phi(t_1)$ is arbitrary, this requires that the range of L_{t_1}, denoted $R(L_{t_1})$, be all of $H_o^m(D)$ for complete controllability. But, since $\overline{R(L_{t_1} L_{t_1}^*)} = \overline{R(L_{t_1})}$, we require $\overline{R(L_{t_1} L_{t_1}^*)} = H_o^m(D)$, or, in other words, we require $L_{t_1} L_{t_1}^*$ to be invertible. Clearly, in the case under consideration,

$$L_{t_1} u = \int_t^{t_1} \overline{\Phi(t_1 - \sigma)} B(\sigma) u(\sigma) d\sigma \qquad \forall u \epsilon L^2(t, t_1; U)$$

and

$$(L_{t_1}^* z)(t) = B^*(t) \Phi^*(t_1 - t) z \qquad \forall z \epsilon H_o^m(D)$$

so that the operator $L_{t_1} L_{t_1}^*$ is

$$L_{t_1} L_{t_1}^* = \int_t^{t_1} \Phi(t_1 - \sigma) B(\sigma) B^*(\sigma) \Phi^*(t_1 - \sigma) d\sigma \tag{4.7.6}$$

Thus, complete controllability is equivalent to finding a time t_1 for which $L_{t_1} L^*_{t_1}$ in (4.7.6) is invertible.

With the introduction of the concept of controllability we shall now be able to deal with the problem of proving the existence of an optimal control and the convergence of the feedback operator when the terminal time approaches infinity. Toward this purpose, let us denote $K(t, T; F)$ to be the optimal feedback operator for the control problem with cost criterion

$$J = \langle Qx, x\rangle_{L^2(t, T; H_o^m(D))} + \langle Ru, u\rangle_{L^2(t, T; U)} + \langle Fx(T), x(T)\rangle_{H_o^m(D)}$$

Let us also denote $J(x, t, T, u(\cdot))$ to be the cost of starting at time t with $x(t) = x$ and applying the control $u(\cdot)$ on $[t, T]$. Now, if the system is controllable, then there exists a time t_1 such that the control

$$u_o(\tau) = -B^*(\tau)T(\tau-t)(L_{t_1}L^*_{t_1})^{-1}x \quad ; \tau\epsilon[t, t_1] \tag{4.7.7}$$

results in the desired transfer to 0. This is easily seen by using Eqs. 4.7.6 and 4.7.7 in the parabolic system equation with initial condition $x(t) = x$. Moreover, by using Eqs. 4.7.6 and 4.7.7, it can also be shown that

$$\int_t^{t_1} \langle u_o(\tau), u_o(\tau)\rangle_U d\tau = \langle (L_{t_1}L^*_{t_1})^{-1}x, x\rangle_{H_o^m(D)}$$

and, since $R(t)$ is uniformly bounded with $\|R(t)\| \leq r$

$$\int_t^{t_1} \langle R(\tau)u_o(\tau), u_o(\tau)\rangle_U d\tau \leq r\langle (L_{t_1}L^*_{t_1})^{-1}x, x\rangle_{H_o^m(D)}.$$

Thus, we can obtain an upper bound on the cost due to application of control $u_o(\cdot)$ on $[t, t_1]$. If we assume that $x_d=0$, we can obtain an upper bound for the trajectory cost using control $u_o(\cdot)$ by showing that

$$\int_t^{t_1} \langle Q(\tau)x(\tau), x(\tau)\rangle_{H_o^m(D)} dt = \langle c(t,t_1)x, x\rangle_{H_o^m(D)}$$

where

$$c(t,t_1) = \int_t^{t_1} [I-(L_\tau L_\tau^*)(L_{t_1} L_{t_1}^*)^{-1}]^* \Phi^*(\tau-t)Q(\tau)\Phi(\tau-t)[I-(L_\tau L_\tau^*)(L_{t_1} L_{t_1}^*)^{-1}]\, d\cdot$$

The precise form of this expression is messy, but the important fact to note is that since t_1 is finite, $c(t,t_1)$ is positive and bounded. We can now find a bound on $K(t,T;0)$, $T \geq t_1$, which is independent of T, namely, by application of the control $\tilde{u}_o(\cdot)$ on $[t,T]$ where

$$\tilde{u}_o(\tau) = \begin{cases} u_o(\tau) & t \leq \tau \leq t_1 \\ 0 & \tau \geq t_1 \end{cases}$$

we obtain

$$J(x,t,T;\tilde{u}_o(\cdot)) = J(x,t,t_1,u_o(\cdot)) \leq \langle c(t,t_1) + r(L_{t_1} L_{t_1}^*)^{-1}x, x\rangle_{H_o^m(D)}$$

But, since

$$\langle K(t,T;0)x, x\rangle = \min_{u(\cdot)\in L_2(t,T;U)} J(x,t,T,u(\cdot))$$

we obtain

$$K(t,T;0) \leq c(t,t_1) + r(L_{t_1} L_{t_1}^*)^{-1}$$

noting that the bound is independent of T.

It is now a fairly straightforward matter to prove that the infinite time solution exists and that

$$\lim_{T\to\infty} K(t,T;0) = K(t)$$

where $K(t)$ is the solution of the Riccati equation on the infinite interval. Indeed, the arguments used are exactly parallel to those used by Kleinman[28, pp. 41-46] in the case of finite dimensional systems, so that they will not be stated in great detail.

If $T_2 > T_1$ we have, by the principle of optimality, that

$$\langle K(t,T_2;0)x,x\rangle_{H_o^m(D)} = \min_{u(\cdot)\in L^2(t,T_2;U)} J(x,t,T_2,u(\cdot))$$

$$= \min_{u(\cdot)\in L^2(t,T_2;U)} \Big[J(x,t,T_1,u(\cdot)) + \int_{T_1}^{T_2} \langle Q(\tau)x(\tau),x(\tau)\rangle_{H_o^m(D)}\, d\tau$$

$$+ \langle Ru,u\rangle_{L^2(T_1,T_2;U)}\Big]$$

$$\geq \langle K(t,T_1;0)x,x\rangle_{H_o^m(D)}$$

$$+ \min_{u(\cdot)\in L^2(T_1,T_2;U)} \Big[\int_{T_1}^{T_2} \langle Q(\tau)x(\tau),x(\tau)\rangle d\tau + \langle Ru,u\rangle_{L^2(T_1,T_2;U)}\Big]$$

since the second term on the right hand side of the inequality is positive it must be true that

$$K(t,T_1;0) \leq K(t,T_2;0)$$

If we form a sequence of terminal times $\{T_i\}_{i=1}^{\infty}$ with $T_{i+1} \geq T_i$ and $\lim_{i\to\infty} T_i = \infty$, we know that the sequence $\{\langle K(t,T_i;0)x,x\rangle_{H_o^m(D)}\}_{i=1}^{\infty}$ is monotonically increasing. However, we have also shown that this sequence is bounded, independent of T_i, by $\langle [c(t,t_1) + r(L_{t_1}L_{t_1}^*)^{-1}]x,x\rangle_{H_o^m(D)}$, which implies that the sequence converges for any fixed x and t. By arguments similar to those used in the proof of Theorem 4.11 in the preceding section, we can write this limit as $\langle K(t,\infty;0)x,x\rangle_{H_0^m(D)}$. Let us denote $K(t,\infty;0)$ as, simply, $K_\infty(t)$. $K_\infty(t)$ can be shown to be the solution of the Riccati operator equation on $[t,\infty)$ by proving that for all $t_a \in [t,\infty)$, $K_\infty(t)=K(t,t_a;K_\infty(t_a))$, which is the solution of the Riccati equation on $[0,t_a]$ with terminal-time weighting operator $K_\infty(t_a)$. Using the fact that the solution of this equation is continuous in the terminal condition $K_\infty(t_a)$ we obtain, for $t_a \le t_b$

$$K_\infty(t) = \lim_{t_b\to\infty} K(t,t_b;0) = \lim_{t_b\to\infty} K(t,t_a;K(t_a,t_b;0))$$

$$= K(t,t_a;\lim_{t_b\to\infty} K(t_a,t_b;0)) = K(t,t_a,K_\infty(t_a))$$

The proof that the optimal control, $u^*(\cdot)\in L^2(t,\infty;U)$, for the infinite terminal time parabolic control problem with $x_d=0$ is given by $u^*(\tau) = -R^{-1}(\tau)B^*(\tau)K_\infty(\tau)x(\tau)$ for all $\tau\in[t,\infty)$ and that the minimum cost is $J = \langle K_\infty(t)x,x\rangle_{H_0^m(D)}$ is exactly the same as that given by Kleinman [28, Theorem 5] for the finite dimensional case, so it will be omitted.

The demonstration of the fact that if $B(t)=B$, $Q(t)=Q$, and $R(t)=R$, then $K_\infty(t) = K_\infty$, satisfying the algebraic operator equation (4.7.4), is precisely the same as that used by Lions in the coercive system operator case.

The above results were obtained under the assumption that the desired state trajectory $x_d(t)=0$. Let us now assume that this is not the case and $x_d(t)$ is the solution of the equation

$$\dot{x}_d(t) = Gx_d(t) \quad ; \quad x_d(0) = x_{do} \qquad (4.7.8)$$

where G is a linear spatial differential operator which satisfies the conditions of Section 2.7 and is, therefore, the infinitesimal generator of a semigroup of operators, and x_{d_o} is some arbitrary element of $H_o^m(D)$. If we now consider the error function

$$e(t) = x(t) - x_d(t)$$

The cost functional (3.3.1), with F=0, can be written

$$J = \langle Qe, e\rangle_{L^2(0,T;H_o^m(D))} + \langle Ru, u\rangle_{L^2(0,T;U)} \tag{4.7.9}$$

where T is finite. We now state the following lemma:

Lemma 4.4: The control which minimizes the cost functional (4.7.9) is of the form

$$u^*(t) = -R^{-1}B^*[K(t)x(t) - S(t)x_d(t)] \tag{4.7.10}$$

and the minimum cost function is given by

$$J(x, x_d, t) = \langle K(t)x, x\rangle_{H_o^m(D)} - 2\langle S(t)x_d, x\rangle_{H_o^m(D)} + \langle P(t)x_d, x_d\rangle \tag{4.7.11}$$

where K(t) is the solution of the Riccati operator equation (4.4.11) with K(T) = 0, S(t) is the solution of the operator equation

$$\dot{S}(t) = -A^*S(t) - S(t)G + K(t)BR^{-1}B^*S(t) - Q \tag{4.7.12}$$

with S(T)=0, and P(t) is the solution of the operator equation

$$\dot{P}(t) = -G^*P(t) - P(t)G + S(t)BR^{-1}B^*S(t) - Q \tag{4.7.13}$$

with P(T) = 0.

Proof: Using the identity

$$\int_0^T \frac{d}{dt}\Big[\langle K(t)x(t), x(t)\rangle_{H_o^m(D)} - 2\langle S(t)x_d(t), x(t)\rangle_{H_o^m(D)} + \langle P(t)x_d(t), x_d(t)\rangle_{H_o^m(D)}\Big]\, dt$$

$$= -[\langle K(0)x(0),x(0)\rangle - 2\langle S(0)x_d(0),x(0)\rangle_{H_o^m(D)} + \langle P(0)x_d(0),x_d(0)\rangle_{H_o^m(D)}]$$

and following exactly the same procedure that was used in the proof of Theorem 4.8, the following expression is obtained for the cost:

$$J = \langle K(0)x(0),x(0)\rangle_{H_o^m(D)} - 2\langle S(0)x_d(0),x(0)\rangle_{H_o^m(D)} + \langle P(0)x_d(0),x_d(0)\rangle_{H_o^m(D)}$$
$$+ \int_0^T \langle R[R^{-1}B^*(K(t)x(t)-S(t)x_d(t)) + u(t)], [R^{-1}B^*(K(t)x(t)-S(t)x_d(t)) + u(t)]\rangle_U \, dt \qquad (4.7.14)$$

Since the last term in Eq. 4.7.14 is nonnegative, the cost is minimized if and only if this last term is 0, and this is achieved if and only if the optimal control is given by Eq.4.7.10 Moreover, the minimum value of the cost, starting at time t with $x(t)=x$ and $x_d(t)=x_d$, is given by Eq. 4.7.11.

Let us consider the special case where the controlled system has exactly the same dynamics as the system which is "tracked," that is, let us suppose $G=A$. Eq. 4.7.12 becomes

$$\dot{S}(t) = -A^*S(t) - S(t)A + K(t)BR^{-1}B^*S(t) - Q \qquad (4.7.15)$$

It can be shown, by use of Theorem 4.9, that Eq. 4.7.15 has a unique, positive, self-adjoint solution. Since $S(t)=K(t)$ satisfies this equation, it must be the unique solution. Similarly, it can be shown that the solution of (4.7.13), in this case, is $P(t)=K(t)$, so that the optimal control, from Eq. 4.7.10, is given by

$$u^*(t) = -R^{-1}B^*K(t)[x(t)-x_d(t)] = -R^{-1}B^*K(t)e(t)$$

and the minimum cost, from Eq. 4.7.11, is given by

$$J(x,x_d,t) = \left\langle K(t)[x(t)-x_d(t)], [x(t)-x_d(t)]\right\rangle_{H_o^m(D)}$$

$$= \left\langle K(t)e(t), e(t)\right\rangle_{H_o^m(D)} = J(e,t)$$

These results are intuitively satisfying in that, when we assume that $G=A$, the error $e(t)$ satisfies the same dynamical equation as the state $x(t)$, so that minimization of the cost functional (4.7.9) should yield precisely the same equations, in terms of $e(t)$, for the optimal control and minimum cost function as were obtained, in terms of $x(t)$, for the optimal control problem with $x_d(t)=0$.

Let us consider the behavior of the optimal solution when the terminal-time T is infinite and the operators G and A are unequal. We have shown that the feedback operator is K_∞, the bounded solution of the time-invariant algebraic operator equation (4.7.4). This implies that $S(t)$ is the solution of the time-invariant operator differential equation

$$\dot{S}(t) = -[A^*-K_\infty BR^{-1}B^*]S(t) - S(t)G - Q \qquad (4.7.16)$$

As a preliminary to writing a solution to this equation, let us take note of the fact (see Ref. 21, p. 389) that if an operator L is the infinitesimal generator of a semigroup of operators, and if N is a bounded operator, then the operator $L+N$, defined on the domain of L, is the infinitesimal generator of a semigroup of operators. Thus, the operator $A^*-K_\infty BR^{-1}B^*$ is the infinitesimal generator of a semigroup, which we shall denote $\{\Phi_1(t)\}_{t\in[0,\infty]}$. We have already assumed that

G is the infinitesimal generator of a semigroup of operators, which we denote $\{\Psi(t)\}_{t\in[0,\infty]}$. By virtue of Lemma 4.2, we can write the solution of Eq. 4.7.16 on the infinite time interval as

$$S(t) = \int_t^\infty \Phi_1(\sigma-t)Q\Psi(\sigma-t)\,d\sigma \tag{4.7.17}$$

and the solution of Eq. 4.7.13 as

$$P(t) = \int_t^\infty \Psi^*(\sigma-t)[Q-S(\sigma)BR^{-1}B^*S(\sigma)]\Psi(\sigma-t)\,d\sigma \tag{4.7.18}$$

It might reasonably be asked at this point whether time-invariant operator solutions S_∞ and P_∞ exist to Eqs. 4.7.16 and 4.7.13, respectively. A time-invariant operator solution S_∞ of Eq. 4.7.16 must satisfy the algebraic operator equation

$$[A^* - K_\infty BR^{-1}B^*]S_\infty + S_\infty G = -Q \tag{4.7.19}$$

If we consider the finite dimensional version of this equation, namely the matrix equation

$$\underline{A}\,\underline{X} + \underline{X}\,\underline{B} = \underline{C} \tag{4.7.20}$$

where $\underline{A}$, $\underline{B}$, $\underline{C}$, and $\underline{X}$ are real $n\times n$ matrices, we may make use of a well-known result (see Ref. 30, p. 231) and conclude that a necessary and sufficient condition for Eq. 4.7.20 to have a solution $\underline{X}$ is that $\lambda_i + \mu_j \neq 0$ for all $i,j = 1,2,\ldots n$, where λ_i and μ_j are the i^{th} and j^{th} eigenvalues of $\underline{A}$ and $\underline{B}$, respectively.

Let us try to generalize this result so as to obtain a condition for the existence of a solution S_∞ to Eq. 4.7.19. If L is any (bounded or unbounded) linear operator defined on $H_o^m(D)$, then, since $H_o^m(D)$ is a subspace of $L^2(D)$ and therefore has a basis which is a subset of an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $L^2(D)$, we may define the infinite matrix $\underline{L}$ with ij^{th} element

$$L_{ij} = \langle Le_i, e_j \rangle_{H_o^m(D)}$$

Of course, the matrix $(\lambda\underline{I}-\underline{L})$ will fail to have an inverse if and only if λ is an element of the spectrum of L. Using this procedure, Eq.4.7.19 can be represented by an infinite matrix equation. If we let the operators $(A^*-K_\infty BR^{-1}B^*)$, S_∞, G, and Q be represented by the infinite matrices $\underline{A}$, $\underline{S}_\infty$, $\underline{G}$, and $\underline{Q}$, respectively, then we may rewrite Eq. 4.7.19 as

$$\underline{A}\,\underline{S}_\infty + \underline{S}_\infty\,\underline{G} = -\underline{Q} \tag{4.7.21}$$

for which we may state the following existence lemma:

Lemma 4.5: Equation 4.7.21 has a solution $\underline{S}_\infty$ if and only if 0 is not an element of $\sigma(A^*-K_\infty BR^{-1}B^*) \oplus \sigma(G)$,* where $\sigma(L)$ denotes the spectrum of the operator L.

Proof: We shall prove this lemma by generalizing the concept of Kronecker products (Ref.30, p.227) to infinite dimensional matrices. By this means we may write the matrix equation (4.7.21) as the equation

$$(\underline{A} \times \underline{I} + \underline{I} \times \underline{G}')\underline{s}_\infty = \underline{q} \tag{4.7.22}$$

where $\underline{I}$ is the infinite dimensional identity matrix, $\times$ denotes the Kronecker product, and $\underline{s}_\infty$ and $\underline{q}$ are countably infinite vectors composed of the elements of $\underline{S}_\infty$ and $\underline{Q}$,

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* The symbol $\oplus$ denotes direct sum, that is, if H_1 and H_2 are two subsets of a vector space V, then

$$H_1 \oplus H_2 = \{x \in V : x = x_1 + x_2 \text{ for some } x_1 \in H_1 \text{ and } x_2 \in H_2\}$$

respectively. The manner in which the elements of $\underline{s}_\infty$ and $\underline{q}$ are chosen is as follows:

If $\underline{X}$ is an infinite dimensional matrix, we successively choose the elements of the vector $\underline{x}$ by following the indicated path through the array

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots & \\ X_{31} & X_{32} & X_{33} & \cdots & \\ X_{41} & \cdots & \cdots & \cdots & \\ \vdots & \vdots & \vdots & & \end{bmatrix}$$

It is easily seen that the spectrum of the matrix $(\underline{A}\times\underline{I}+I\times\underline{G}')$ is the direct sum of the spectrum of $\underline{A}$ and the spectrum of $\underline{G}$, or

$$\sigma(\underline{A}\times I + I\times\underline{G}') = \sigma(A^*-K_\infty BR^{-1}B^*) \oplus \sigma(G)$$

which implies that Eq. 4.7.22 has a vector solution $\underline{s}_\infty$ if and only if 0 is not an element of this direct sum. Existence of a vector solution $\underline{s}_\infty$ of Eq. 4.7.22 is, of course, equivalent to the existence of a matrix solution $\underline{S}_\infty$ of Eq. 4.7.21.

We are now in a position to state and prove the following lemma concerning the existence of a bounded solution to Eq. 4.7.19:

<u>Lemma 4.6</u>: If $A^*-K_\infty BR^{-1}B^*$ and G are strongly elliptic operators, as described in Section 2.3, then Eq. 4.7.19 has a bounded operator solution S_∞.

Proof: The strong ellipticity of $A^*-K_\infty BR^{-1}B^*$ and G implies that the spectra of $A^*-K_\infty BR^{-1}B^*$ and G are contained in the left-half of the complex plane and do not include the origin of the complex plane. This would imply that the direct sum $\sigma(A^*-KBR^{-1}B^*) \oplus \sigma(G)$ is contained in the left-half plane and 0 is not an element of the direct sum. Thus, by Lemma 4.5, matrix equation (4.7.21) has a solution $\underline{S}_\infty$, and the corresponding operator S_∞ is a solution of the operator equation (4.7.19). The boundedness of S_∞ is a consequence of the fact that

$$S_\infty = \lim_{t\to\infty} S(t)$$

where $S(t)$ is the solution of the operator differential equation (4.7.16) given explicitly by expression (4.7.17). We can show that $S(t)$ is uniformly bounded for $t\in[0,\infty)$ by using the fact that $A^*-K_\infty BR^{-1}B^*$ and G are strongly elliptic and, thus, are infinitesimal generators of semigroups of operators $\{\Phi_1(t)\}_{t\in[0,\infty)}$ and $\{\Psi(t)\}_{t\in[0,\infty)}$, respectively, which have the property that*

$$\|\Phi_1(t)\| \leq M_1 e^{-\lambda_1 t}$$

and

$$\|\Psi(t)\| \leq M_2 e^{-\lambda_2 t}$$

where M_1, M_2, λ_1, and λ_2 are positive constants. Using Eq. 4.7.17 we may now write

* See Section 2.7.

$$\|S(t)\| \leq \int_t^{\infty} \|\Phi_1(\sigma - t)\| \; \|Q\| \; \|\Psi(\sigma - t)\| d\sigma$$

$$\leq M_1 M_2 \|Q\| e^{(\lambda_1 + \lambda_2)t} \int_t^{\infty} e^{-(\lambda_1 + \lambda_2)\sigma} d\sigma$$

$$= \frac{M_1 M_2}{\lambda_1 \lambda_2} \|Q\|$$

which implies that S_∞ is bounded.

The existence of a time-invariant operator solution P_∞ to Eq. 4.7.13 hinges upon the existence of a solution to the algebraic operator equation

$$G^* P_\infty + P_\infty G = S_\infty B R^{-1} B^* S_\infty - Q \tag{4.7.23}$$

It is quite clear that under the assumptions of Lemma 4.6, a bounded solution P_∞ exists.

We shall return to the discussion of infinite terminal-time problems in Section 5.3, where the case of pointwise control will be considered.

4.8 Derivation of the Riccati Integro-differential Equation

In the preceding sections of this chapter we have shown that a bounded, positive, self-adjoint, optimal feedback operator exists, (1) in the case where the system operator is coercive, and, (2) with the additional assumption of complete controllability, in the case where the system operator is the infinitesimal generator of a semigroup of operators. The optimal feedback operator $K(t)$ is the solution of the Riccati operator equation. Unfortunately, there are no straightforward procedures for solving operator equations directly. It is the purpose of this section to derive an equation from the Riccati operator equation which can be "solved" analytically or numerically. This will be achieved by showing that $K(t)$ can be represented by an integral operator. An integro-differential equation will then be derived for the kernel of this integral operator and an expression for the optimal cost will be specified. Under the assumption that the state-weighting cost term of the cost functional is of the form $\langle Qx, x\rangle_{L^2(0,T;L^2(D))}$ where $Q(t)$ is a bounded linear operator from $L^2(D) \to L^2(D)$, we shall prove that the optimal feedback operator $K(t)$ is a bounded linear operator from $L^2(D) \to L^2(D)$. This will enable us to specify a particularly simple form for the optimal cost function.

As an introduction to the concept of representing bounded linear operators by integral operators, let us examine $I_{C_o^\infty(D)}$ the identity operator on the space of infinitly differentiable functions with compact support in D, which space is discussed in Chapter II, Section 2. This operator can be represented by the following integral operator:

$$I_{C_o^\infty(D)} \phi = \phi(z) = \int_D \delta(z-\zeta)\phi(\zeta)d\zeta \ , \ \forall \ \phi \in C_o^\infty(D)$$

The important thing to note from this is that the kernel of the integral operator, the Dirac delta function, is a distribution on $D \times D$. Indeed, Laurent Schwartz (see Ref. 31, Theorem 1) proves that any distribution on $D \times D$ is the kernel of a continuous linear operator from $C_o^\infty(D)$ into $\mathcal{D}'(D)$, the space of distributions on D discussed in Section 2.2. As a matter of notation let us denote the kernel by $L(z,\zeta)$ and the corresponding continuous linear operator by L. Since we are interested in the possible representation of the feedback operator $K(t)$ by an integral operator, we are naturally more interested in the converse of this statement. Schwartz (Ref. 31, Theorem 2) proves that the converse is true, namely, that every continuous linear operator L from $C_o^\infty(D)$ into $\mathcal{D}'(D)$ can be represented by a unique integral operator whose kernel, $L(z,\zeta)$, is a distribution on $D \times D$.

Thus, having seen that the integral operator representation holds for bounded linear operators on $C_o^\infty(D)$ we must determine when this representation holds for bounded linear operators from $H_o^m(D)$ into $H_o^m(D)$. Once again, Schwartz provides the answer in the so-called Schwartz Kernel theorem:

<u>Theorem 4.12:</u> If H_1 and H_2 are locally convex spaces and L is a continuous linear operator from H_1 into H_2, and if the following are true:

1. $C_o^\infty(D) \subset H_i \subset H_i' \subset \mathcal{D}'(D)$; $i=1,2$
2. $C_o^\infty(D)$ is dense in $H_1 \cap H_2$

then L can be represented by a unique integral operator whose kernel $L(z,\zeta)$ is a distribution on $D \times D$.

The proof of this theorem follows from the fact that since $C_o^\infty(D) \subset H_1$ and $H_2 \subset \mathcal{D}'(D)$, then the restriction of L to $C_o^\infty(D)$ is a continuous linear operator from $C_o^\infty(D)$ into $\mathcal{D}'(D)$ and can be represented by an integral operator, which, from the fact that $C_o^\infty(D)$ is dense in $H_1 \cap H_2$, can be extended to H_1 by the continuity of the operator L.

It is not very difficult to see that the optimal linear feedback operator K(t) satisfies the hypotheses of Theorem 4.12 for all $t \in [0,T]$. For any t, K(t) is a bounded linear operator from the Hilbert space

$H_o^m(D)$ into itself, implying that $K(t)$ is continuous, since boundedness of an operator on a Hilbert space is equivalent to continuity. The space $H_o^m(D)$ clearly satisfies condition 1 of Theorem 4.12 and, since both $C_o^\infty(D)$ and $H_o^m(D)$ are dense in $L^2(D)$, $C_o^\infty(D)$ is dense in $H_o^m(D)$, satisfying condition 2. Thus, by Theorem 4.12 there exists a kernel $K(t, z, \zeta)$ such that

$$K(t)x = \int_D K(t, z, \zeta)x(\zeta)d\zeta \quad \forall x \in C_o^\infty(D) \tag{4.8.1}$$

and if $x \in H_o^m(D) \cap \overline{C_o^\infty(D)}$

$$K(t)x = \lim_{n\to\infty} \int_D K(t, z, \zeta)x_n(\zeta)d\zeta \tag{4.8.2}$$

where $\{x_n\}_{n=1}^\infty$ is a sequence in $C_o^\infty(D)$ convergent to x.

To simplify notation in the sequel let us assume Eq. 4.8.1 holds for all $x \in H_o^m(D)$ <u>with the tacit assumption</u> that Eq. 4.8.2 truly represents $K(t)$ if $x \notin C_o^\infty(D)$. It can also be shown that if $K(t)$ is continuously differentiable with respect to t, then $K(t, z, \zeta)$ is continuously differentiable with respect to t, and

$$\dot{K}(t)x = \int_D \frac{\partial K(t, z, \zeta)}{\partial t} x(\zeta)d\zeta \tag{4.8.3}$$

It will also be necessary in the sequel to consider the operator

$$L(t) = B(t)R^{-1}(t)B^*(t)$$

which is also a bounded linear operator from $H_o^m(D)$ into itself and therefore can be represented by

$$L(t)x = \int_D L(t, z, \zeta)x(\zeta)d\zeta \quad \forall x \in H_o^m(D) \tag{4.8.4}$$

Likewise, the state weighting operator $Q(t)$ and the terminal state weighting operator F can be represented as

$$Q(t)x = \int_D Q(t,z,\zeta)x(\zeta)d\zeta \quad \forall x \in H_o^m(D) \tag{4.8.5}$$

and

$$Fx = \int_D F(z,\zeta)x(\zeta)d\zeta$$

We are now prepared to apply these results of the kernel theorem to the Riccati equation

$$\dot{K}(t)x = -K(t)Ax - A^*K(t)x + K(t)L(t)K(t)x - Q(t)x \quad ; \quad K(T)x = Fx \tag{4.8.6}$$

where x is an arbitrary element of $H_o^m(D)$. Let us first note that the term $A^*K(\cdot)x$ has the representation

$$A^*K(t)x = A^*\int_D K(t,z,\zeta)x(\zeta)d\zeta = \int_D A_z^* K(t,z,\zeta)x(\zeta)d\zeta \tag{4.8.7}$$

where the subscript z in the right-hand equality denotes the fact that A^* is a differential operator in the z spatial variable, operating on $K(t,z,\zeta)$. In the case of the term $K(t)Ax$, we have

$$K(t)Ax = \int_D K(t,z,\zeta)A_\zeta x(\zeta)d\zeta$$

where the subscript ζ refers to spatial differentiation in terms of ζ. But, for fixed z and t, we can look upon $K(t,z,\zeta)$ as an element of $\mathcal{D}'(D)$, the distributions on D, so that by elementary properties of distributions, the integral can be rewritten

$$K(t)Ax = \int_D K(t,z,\zeta)A_\zeta x(\zeta)d\zeta = \int_D A_\zeta^* K(t,z,\zeta)x(\zeta)d\zeta \tag{4.8.8}$$

i.e., the kernel of the operator $K(t)A$ is $A_\zeta^*K(t,z,\zeta)$.

Using Eqs. 4.8.1 and 4.8.4 we can write the following:

$$K(t)L(t)K(t)x = \int_D K(t,z,\rho) \int_D L(t,\rho,\sigma) \int_D K(t,\sigma,\zeta)x(\zeta)d\zeta\, d\sigma\, d\rho$$

$$= \int_D \int_D \int_D K(t,z,\rho)L(t,\rho,\sigma)K(t,\sigma,\zeta)x(\zeta)\,d\zeta d\sigma d\rho \qquad (4.8.9)$$

Now, using Eqs. 4.8.3, 4.8.5, 4.8.7, 4.8.8 and 4.8.9 in the Riccati equation (4.8.6), we obtain

$$\int_D \frac{\partial K}{\partial t}(t,z,\zeta)x(\zeta)d\zeta = -\int_D A_\zeta^* K(t,z,\zeta)x(\zeta)d\zeta - \int_D A_z^* K(t,z,\zeta)x(\zeta)d\zeta$$

$$+ \int_D \int_D \int_D K(t,z,\rho)\,L(t,\rho,\sigma)K(t,\sigma,\zeta)x(\zeta)\,d\zeta d\sigma d\rho$$

$$- \int_D Q(t,z,\zeta)x(\zeta)d\zeta \quad ; \quad \int_D K(T,z,\zeta)x(\zeta)d\zeta = \int_D F(z,\zeta)x(\zeta)d\zeta$$

Since this equation must be true for all $x \epsilon H_o^m(D)$, it must be true that

$$\frac{\partial K}{\partial t}(t,z,\zeta) = -(A_\zeta^* + A_z^*)K(t,z,\zeta)$$
$$+ \int_D \int_D K(t,z,\rho)\,L(t,\rho,\sigma)K(t,\sigma,\zeta)\,d\sigma d\rho - Q(t,z,\zeta)$$

(4.8.10)

and $\qquad K(T,z,\zeta) = F(z,\zeta)$

Thus, we have derived an <u>integro-differential equation</u> of the Riccati type. It is quite clear that the kernel $K(t,z,\zeta)$ is symmetric in its spatial arguments, that is,

$$K(t,z,\zeta) = K(t,\zeta,z)$$

This is a direct consequence of the fact that $K(t)$ is self-adjoint (see Sections 4.4 and 4.6) for all $t \epsilon [0,T]$.

Boundary conditions may be specified for the Riccati integro-differential equation (4.8.10). If $z \epsilon \partial D$, the boundary of D, then by the transformation in Eq. 4.8.1 we obtain

$$y(t, z)\Big|_{z\epsilon\partial D} = \int_D K(t, z, \zeta)\Big|_{z\epsilon\partial D} x(t, \zeta)\, d\zeta$$

the evaluation of a function $y(t)$ in $H_o^m(D)$ at the point $z\epsilon\partial D$. But this is zero by the definition of $H_o^m(D)$, which implies that $K(t, z, \zeta)\Big|_{z\epsilon\partial D} = 0$ where ζ is any element in D. In a like manner, it can be shown that all of the Dirichlet boundary conditions hold for $K(t, z, \zeta)$, that is referring to Section 2.5,

$$K(t, z, \zeta)\Big|_{z\epsilon\partial D} = \frac{\partial K(t, z, \zeta)}{\partial n}\Big|_{z\epsilon\partial D} = \cdots\cdots = \frac{\partial^{m-1} K(t, z, \zeta)}{\partial n^{m-1}}\Big|_{z\epsilon\partial D} = 0$$

where n is the normal to the boundary ∂D. Moreover, by the symmetry of the kernel, the above boundary conditions must also hold for $\zeta\epsilon\partial D$ and z any element of D, that is

$$K(t, z, \zeta)\Big|_{\zeta\epsilon\partial D} = \frac{\partial K(t, z, \zeta)}{\partial n}\Big|_{\zeta\epsilon\partial D} = \cdots\cdots = \frac{\partial^{m-1} K(t, z, \zeta)}{\partial n^{m-1}}\Big|_{\zeta\epsilon\partial D} = 0 .$$

Now that the Riccati operator equation (4.4.11) has been transformed into an integro-differential equation, we may specify Eq. 4.4.12, the equation for $g(t)$, as an integro-differential equation:

$$\frac{\partial g(t, z)}{dt} = -A_z^* g(t, z) + \int_D \int_D K(t, z, \zeta) L(t, \zeta, \sigma) g(t, \sigma)\, d\sigma d\zeta$$

$$+ \int_D Q(t, z, \zeta) x_d(t, \zeta)\, d\zeta$$

$$g(T, z) = - \int_D F(z, \zeta) x_d(T, \zeta) d\zeta$$

Since $g(t)$ is an element of $H_o^m(D)$, the Dirichlet boundary conditions must again be satisfied.

Recall that the minimum value of the cost is given by the expression

$$J = \langle K(0)x(0), x(0) \rangle_{H_o^m(D)} + 2 \langle g(0), x(0) \rangle_{H_o^m(D)} + \phi(0) \qquad (4.8.11)$$

where $\phi(t)$ is the solution of Eq. 4.4.14 . Using the integral operator representation for $K(t)$, we may evaulate the inner products in expression (4.8.11) according to the definition in Chapter II of inner product on the Sobolev space $H_o^m(D)$, and, thus, we obtain*

$$\begin{aligned} J = &\int_D \sum_{|q| \le m} (D^q \int_D K(0, z, \zeta) x(0, \zeta)\, d\zeta)(D^q x(0, z))\, dz \\ &+ 2 \int_D \sum_{|q| \le m} D^q g(0, z)\, D^q x(0, z)\, dz \quad + \phi(0) \end{aligned} \qquad (4.8.12)$$

where $\phi(t)$ satisfies the ordinary differential equation

$$\begin{aligned} \dot{\phi}(t) = &-\int_D \sum_{|q| \le m} (D^q \int_D Q(t, z, \zeta) x(t, \zeta)\, d\zeta)(D^q x(t, z))\, dz \\ &+ \int_D \sum_{|q| \le m} D^q g(t, z)\, D^q \int_D L(t, z, \zeta) g(t, \zeta)\, d\zeta\, dz \end{aligned} \qquad (4.8.13)$$

$$\phi(T) = \int_D \sum_{|q| \le m} (D^q \int_D F(z, \zeta) x_d(T, \zeta)\, d\zeta)(D^q x_d(T, z))\, dz$$

The above expressions for the cost terms are extremely complicated. This is not surprising, however, since our state-weighting cost term $\langle Q(x-x_d), x-x_d \rangle_{L^2(0, T; H_o^m(D))}$ may be written

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* The notation D^q is described in Section 2.2.

$$< Q(x-x_d), x-x_d >_{L^2(0,T;H_o^m(D))}$$

$$= \int_0^T \int_D \sum_{|q| \leq m} (D^q \int_D Q(t,z,\zeta)(x(t,\zeta)-x_d(t,\zeta))d\zeta) D^q(x(t,z)-x_d(t,z))dz\,dt$$

Using elementary properties of distributions this expression may be re-written as

$$<Q(x-x_d), x-x_d>_{L^2(0,T;H_o^m(D))}$$

$$= \int_0^T \int_D \int_D Q(t,z,\zeta) \sum_{|q| \leq m} (-1)^{|q|} D^q(x(t,\zeta)-x_d(t,\zeta)) D^q(x(t,z)-x_d(t,z))dz\,d\zeta\,dt$$

which indicates that <u>we are, in actuality, weighting all spatial derivatives (up to order m) of the state in the quadratic cost functional</u>.

Now (assuming, for simplicity, that $x_d(t) = 0$), it might reasonably be asked: <u>can we have a state-weighting term in the cost functional of the form</u>

$$< Qx, x >_{L^2(0,T;L^2(D))}$$

$$= \int_0^T \int_D \int_D Q(t,z,\zeta)x(t,z)x(t,\zeta)dz\,d\zeta\,dt \qquad (4.8.14)$$

<u>where the operator</u> $Q(t)$ <u>is now a bounded linear operator from</u> $L^2(D)$ <u>into</u> $L^2(D)$, <u>and will this result in the existence of an optimal feedback operator</u>? Although the expression (4.8.14) has been written in the form of an inner product on $L^2(D)$, we are, in actuality, restricting x to be in the subset $H_o^m(D)$ of $L^2(D)$ in order that the system equation be satisfied. Accordingly, the inner product in $L^2(D)$ which is represented by expression (4.8.14) must be of the form

$$\int_0^T \langle Q(t)\Lambda x(t), \Lambda x(t)\rangle_{L^2(D)} dt$$

$$= \int_0^T \int_D \int_D Q(t,z,\zeta)x(t,z)x(t,\zeta)\,d\zeta\,dz\,dt \tag{4.8.15}$$

where Λ is a bounded linear operator from $H_o^m(D)$ into $L^2(D)$. The left-hand side of Eq. 4.8.15 may be written as

$$\int_0^T \langle Q(t)\Lambda x(t), \Lambda x(t)\rangle_{L^2(D)} dt$$

$$= \int_0^T \langle \Lambda^* Q(t)\Lambda x(t), x(t)\rangle_{H_o^m(D)} dt \tag{4.8.16}$$

where Λ^* is the adjoint of Λ. The operator $\Lambda^* Q(t)\Lambda$ is a bounded linear operator from $H_o^m(D)$ into $H_o^m(D)$, and, therefore, by Theorem 4.12, has the integral operator representation

$$\Lambda^* Q(t)\Lambda x(t) = \int_D Q_1(t,z,\zeta)x(t,\zeta)\,d\zeta \tag{4.8.17}$$

for some kernel distribution $Q_1(t,z,\zeta)$. Combining Eq. 4.8.15 and 4.8.16 we obtain the relation

$$\langle \Lambda^* Q(t)\Lambda x(t), x(t)\rangle_{H_o^m(D)} = \int_D \int_D Q(t,z,\zeta)x(t,z)x(t,\zeta)\,dz\,d\zeta \tag{4.8.18}$$

and, using Eq. 4.8.17, the left-hand side of Eq. 4.8.18 may be expressed as

$$\langle \Lambda^* Q(t)\Lambda x(t), x(t)\rangle_{H_o^m(D)} = \int_D \sum_{|q|\le m} D^q \int_D Q_1(t,z,\zeta)x(t,\zeta)\,d\zeta\; D^q x(t,z)\,dz$$

$$= \int_D \int_D \Big(\sum_{|q|\le m} (-1)^{|q|} D_z^q D_\zeta^q Q_1(t,z,\zeta)\Big) x(t,z)x(t,\zeta)\,dz\,d\zeta \tag{4.8.19}$$

where the operators D_z^q and D_ζ^q are the operator D^q in the z and ζ spatial variables, respectively. The last equality in (4.8.19) follows from elementary results (see Ref. 17, pp. 323-337) in distribution theory. Thus, from Eqs. 4.8.18 and 4.8.19 it is seen that the kernel $Q_1(t, z, \zeta)$ must satisfy the partial differential equation

$$\sum_{|q| \leq m} (-1)^{|q|} D_z^q D_\zeta^q Q_1(t, z, \zeta) = Q(t, z, \zeta) \tag{4.8.20}$$

Now, since $Q_1(t, z, \zeta)$ is the kernel of a bounded linear operator from $H_o^m(D)$ into $H_o^m(D)$, Theorem 4.11 implies that the Riccati operator equation (4.4.11), with $Q(t)$ taken to be $\Lambda^* Q(t)\Lambda$, has a bounded, positive, self-adjoint solution $K_1(t)$, which, by Theorem 4.12, may be represented by an integral operator with kernel $K_1(t, z, \zeta)$. Moreover, this kernel must satisfy Eq. 4.8.10, namely

$$\frac{\partial K_1(t, z, \zeta)}{\partial t} = -(A_z^* + A_\zeta^*)\, K_1(t, z, \zeta)$$

$$+ \int_D \int_D K_1(t, z, \rho) L(t, \rho, \sigma) K_1(t, \sigma, \zeta)\, d\sigma\, d\rho - Q_1(t, z, \zeta) \tag{4.8.21}$$

Note that the double integral term in Eq. 4.8.21 is in the form of an inner product on $L^2(D)$, so that we may use the same reasoning which led to Eq. 4.8.20 to state that there exists a kernel $L_1(t, z, \zeta)$, corresponding to a bounded operator $L_1(t)$ from $H_o^m(D)$ into $H_o^m(D)$, such that

$$\sum_{|q| \leq m} (-1)^{|q|} D_z^q D_\zeta^q L_1(t, z, \zeta) = L(t, z, \zeta) \tag{4.8.22}$$

Let us now perform this type of operation on the solution $K_1(t, z, \zeta)$ of Eq. 4.8.21, that is, let

$$\sum_{|q|\leq m} (-1)^{|q|} D_z^q D_\zeta^q K_1(t,z,\zeta) = K(t,z,\zeta) \tag{4.8.23}$$

It is clear that $K(t,z,\zeta)$ is the kernel of a bounded linear operator $K(t)$ from $L^2(D)$ into $L^2(D)$. We shall now proceed to determine the equation which $K(t,z,\zeta)$ must satisfy. Using the operator $\sum_{|q|\leq m} (-1)^{|q|} D_z^q D_\zeta^q$ on each term of Eq. 4.8.21, we see, first, that

$$\sum_{|q|\leq m} (-1)^{|q|} D_z^q D_\zeta^q \frac{\partial}{\partial t} K_1(t,z,\zeta) = \frac{\partial}{\partial t} K(t,z,\zeta)$$

Next, if we assume that A_z^* is a <u>constant coefficient</u> differential operator, we obtain

$$\sum_{|q|\leq m} (-1)^{|q|} D_z^q D_\zeta^q A_z^* K_1(t,z,\zeta) = A_z^* \sum_{|q|\leq m} (-1)^{|q|} D_z^q D_\zeta^q K_1(t,z,\zeta) = A_z^* K(t,z,\zeta)$$

The same result holds for the term containing A_ζ^*. Using Eq. 4.8.22 and elementary properties of distributions, we obtain

$$\sum_{|q|\leq m} (-1)^{|q|} D_z^q D_\zeta^q \int_D \int_D K_1(t,z,\rho) L(t,\rho,\sigma) K_1(t,\sigma,\zeta) d\sigma d\rho$$

$$= \int_D \int_D K(t,z,\rho) L_1(t,\rho,\sigma) K(t,\sigma,\zeta) d\sigma d\rho$$

where $L_1(t,\rho,\sigma)$ is given by Eq. 4.8.22. Using all of these results and Eq. 4.8.20 we see that $K(t,z,\zeta)$ is the solution of the Riccati integro-differential equation

$$\begin{aligned}\frac{\partial K(t,z,\zeta)}{\partial t} &= -(A_z^* + A_\zeta^*)K(t,z,\zeta) \\ &\quad + \int_D \int_D K(t,z,\rho) L_1(t,\rho,\sigma) K(t,\sigma,\zeta) d\sigma d\rho - Q(t,z,\zeta)\end{aligned} \tag{4.8.24}$$

Moreover, it may be seen that the optimal cost is given by

$$J = \langle K_1(0)x_o, x_o \rangle_{H_o^m(D)} = \sum_{|q| \le m} \int_D (D_z^q \int_D K_1(0,z,\zeta)x_o(\zeta)d\zeta)(D_z^q x_o(z))dz$$

$$= \int_D \int_D K(0,z,\zeta)x_o(z)x_o(\zeta)d\zeta dz \tag{4.8.25}$$

Thus, we have shown that, corresponding to a state-weighting operator $Q(t)$ which is a bounded linear operator from $L^2(D)$ into $L^2(D)$, there exists a bounded linear feedback operator $K(t)$ from $L^2(D)$ into $L^2(D)$, and the kernel $K(t,z,\zeta)$ of the integral representation of this operator satisfies the Riccati integro-differential equation (4.8.24).

Let us now consider the case of infinite terminal time. If we assume that $x_d(t)=0$, $B(t)=B$, $Q(t)=Q$, and $R(t)=R$ then the time-invariant optimal feedback operator K_∞ is the solution of the algebraic Riccati operator equation (4.7.4). Using the procedures of this section, we find that operator K_∞ has an integral representation with kernel $K_\infty(z,\zeta)$ which satisfies the equation

$$-(A_z^* + A_\zeta^*)K_\infty(z,\zeta) + \int_D \int_D K_\infty(z,\rho)L(\rho,\sigma)K_\infty(\sigma,\zeta)d\sigma d\rho - Q(z,\zeta) = 0 \tag{4.8.26}$$

where $L(\rho,\sigma)$ is the kernel of the integral representation of the time-invariant operator $L=BR^{-1}B^*$.

To illustrate Eq. 4.8.26, let us, once again, consider the heat equation example given at the end of Section 4.6.

$$\frac{\partial x(t,z)}{\partial t} = \frac{\partial^2 x(t,z)}{\partial z^2} + u(t,z) \quad ; \quad x(0,z) = x_o(z)$$

with the boundary conditions

$$x(t,0) = x(t,1) = 0$$

Let us choose the cost functional

$$J = \int_0^\infty \left[\int_0^1 \int_0^1 \left(\frac{4\pi^2+1}{2}\right) \sin\pi z \sin\pi\zeta \, x(t,z)x(t,\zeta)\, dz d\zeta + \int_0^1 u^2(t,z)dz \right] dt$$

which corresponds to choosing the kernel of Q to be $\left(\frac{4\pi^2+1}{2}\right)\sin\pi z \sin\pi\zeta$ and the kernel of R to be the Dirac delta function $\delta(z-\zeta)$. The optimal feedback kernel $K_\infty(z,\zeta)$ must satisfy Eq. 4.8.26, which, for this example, becomes

$$-\frac{\partial^2}{\partial z^2} K_\infty(z,\zeta) - \frac{\partial^2}{\partial \zeta^2} K_\infty(z,\zeta) + \int_0^1 K_\infty(z,\rho)K_\infty(\rho,\zeta)d\rho$$

$$= \left(\frac{4\pi^2+1}{2}\right) \sin\pi z \, \sin\pi\zeta$$

with boundary conditions

$$K_\infty(0,\zeta) = K_\infty(1,\zeta) = K_\infty(z,0) = K_\infty(z,1) = 0$$

The solution, by inspection is

$$K_\infty(z,\zeta) = \sin\pi z \sin\pi\zeta$$

so that the optimal control may be written

$$u^*(t,z) = -\int_0^1 K_\infty(z,\zeta)x(t,\zeta)d\zeta = -\sin\pi z \left[\int_0^1 \sin\pi\zeta \, x(t,\zeta)d\zeta\right].$$

CHAPTER V

OPTIMAL POINTWISE FEEDBACK CONTROL

5.1 INTRODUCTION

In this chapter we shall specialize the results obtained in Chapter IV for the parabolic optimal control problem to the pointwise optimal control problem defined in Chapter III. Section 5.2 is concerned with the actual derivation of the optimal pointwise feedback control. It will be shown that the optimal pointwise control is of a form which is, in a sense, computationally simpler than the general feedback form of the optimal control derived in Section 4.8. In Section 5.3 it will be shown that a particular choice of the state-weighting operator results in the traditional modal analytic solution. Still another choice of the state-weighting operator will be shown to result in a feedback solution of the pointwise optimal control problem under the condition that only a finite number of specific measurements, rather than the entire state, are available.

5.2 DERIVATION OF THE OPTIMAL POINTWISE FEEDBACK CONTROL

In this section the pointwise control problem is solved by placing the problem within the format of Section 4.8, that is, by introducing the feedback integral operator and writing the Riccati integro-differential equation. Since the pointwise control problem is characterized, mathematically, by the control space $U=R^k$ and the pointwise control operator $B_o(t)$ defined in Section 3.4, we know, from the results of Chapter IV, that an optimal control of the form $\underline{u}^*(t)=-\underline{R}^{-1}B_o^*(t)K(t)x(t)$

exists* where $K(t)$ satisfies the Riccati operator equation (4.4.11) with $B(t)=B_o(t)$. We also know, from the results of Section 4.8, that the Riccati operator equation may be represented by the integro-differential equation (4.8.10), namely

$$\frac{\partial K}{\partial t}(t,z,\zeta) = -(A_z^* + A_\zeta^*)K(t,z,\zeta) + \int_D \int_D K(t,z,\rho)L_o(t,\rho,\sigma)K(t,\sigma,\zeta)\,d\sigma\,d\rho - Q(t,z,\zeta)$$

with

$$K(T,z,\zeta) = F(z,\zeta)$$

where $L_o(t,\rho,\sigma)$ is the kernel of the operator

$$L_o(t) = B_o(t)\underline{R}^{-1}(t)B_o^*(t)$$

Attention will be focused on the nonlinear term of the Riccati equation, in which the kernel of the operator $L_o(t)$ appears. It will be shown that in this case a simplified form of the Riccati integro-differential equation holds the solution of which leads, in an approximate sense, to a simplified form of the optimal control. The infinite time problem will also be discussed.

Recalling that the pointwise control operator $B_o(t):R^k \to L^2(D)$ is of the form:

$$B_o(t)\underline{u}(t) = \sum_{i=1}^{k} \chi_i(z)b_i(t)u_i(t) \quad , \quad \forall \underline{u}(t)\in R^k \quad , \forall t\in[0,T]$$

where $\chi_i(z)$ is, again, the characteristic function of the set $E_i \subset D$ as specified in Section 3.4. The adjoint pointwise control operator $B_o^*(t):L^2(D)\to R^k$ may be determined in the following manner:

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* We shall, for simplicity, consider the case where $x_d(t)=0$.

If $y(\cdot)\epsilon L^2(D)$ and $\underline{u}\epsilon R^k$, then

$$\langle B_o^*(t)y(\cdot),\underline{u}\rangle_{R^k} = \langle y(\cdot), B_o(t)\underline{u}\rangle_{L^2(D)}$$

$$= \int_D y(z)\sum_{i=1}^{k}\chi_i(z)b_i(t)u_i\,dz$$

$$= \sum_{i=1}^{k}\left[\int_D y(z)\chi_i(z)\,dz\right]b_i(t)u_i$$

From this we identify $B_o^*(t)y(\cdot)$ as the vector in R^k

$$B_o^*(t)y(\cdot) = \left[\begin{array}{c}\uparrow\\ b_i(t)\int_D \chi_i(z)y(z)\,dz\\ \downarrow\end{array}\right]$$

In order to obtain an equation for the feedback kernel in the form of the Riccati integro-differential equation (4.8.10) we must express the operator $L_o(t)=B_o(t)\underline{R}^{-1}(t)B_o^*(t)$ as an integral operator with kernel $L_o(t,\rho,\sigma)$. Using the dummy variable σ with the $B_o^*(t)$ operator and the dummy variable ρ with the $B_o(t)$ operator, we obtain

$$(L_o(t)y)(\rho) = B_o(t)\underline{R}^{-1}(t)\left[\begin{array}{c}\uparrow\\ b_j(t)\int_D\chi_j(\sigma)y(\sigma)\,d\sigma\\ \downarrow\end{array}\right]$$

$$= B_o(t)\left[\begin{array}{c}\uparrow\\ \sum_{j=1}^{k}R_{ij}^{-1}(t)b_j(t)\int_D\chi_j(\sigma)y(\sigma)\,d\sigma\\ \downarrow\end{array}\right]$$

$$= \sum_{i=1}^{k}\chi_i(\rho)b_i(t)\sum_{j=1}^{k}R_{ij}^{-1}(t)b_j(t)\int_D\chi_j(\sigma)y(\sigma)\,d\sigma$$

$$= \int_D\left[\sum_{i=1}^{k}\sum_{j=1}^{k}\chi_i(\rho)b_i(t)R_{ij}^{-1}(t)b_j(t)\chi_j(\sigma)\right]y(\sigma)\,d\sigma$$

so that the kernel $L_o(t, \rho, \sigma)$ is given by

$$L_o(t, \rho, \sigma) = \sum_{i=1}^{k} \sum_{j=1}^{k} \chi_i(\rho) b_i(t) R^{-1}_{ij}(t) b_j(t) \chi_j(\sigma) \tag{5.2.1}$$

The nonlinear term in Eq. 4.8.10 may now be written as

$$\int_D \int_D K(t, z, \rho) L_o(t, \rho, \sigma) K(t, \sigma, \zeta) d\sigma d\rho$$

$$= \int_D \int_D K(t, z, \rho) \sum_{i=1}^{k} \sum_{j=1}^{k} \chi_i(\rho) b_i(t) R^{-1}_{ij}(t) b_j(t) \chi_j(\sigma) K(t, \sigma, \zeta) d\sigma d\rho$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} (b_i(t) \int_D \chi_i(\rho) K(t, z, \rho) d\rho) R^{-1}_{ij}(t) (b_j(t) \int_D \chi_j(\sigma) K(t, \sigma, \zeta) d\sigma) \tag{5.2.2}$$

Let us define the vector function $\underline{k}(t, z)$ to be

$$\underline{k}(t, z) = \begin{bmatrix} \uparrow \\ b_i(t) \int_D \chi_i(\rho) K(t, z, \rho) d\rho \\ \downarrow \end{bmatrix} ; t \epsilon [0, T] \quad , \quad z \epsilon D$$

Using this vector function, we may rewrite Eq. 5.2.2 as

$$\int_D \int_D K(t, z, \rho) L_o(t, \rho, \sigma) K(t, \sigma, \zeta) d\sigma d\rho = \underline{k}'(t, z) \underline{R}^{-1}(t) \underline{k}(t, \zeta)$$

and the Riccati integro-differential equation (4.8.10) for the pointwise control problem becomes:

$$\frac{\partial K}{\partial t}(t, z, \zeta) = -(A^*_z + A^*_\zeta) K(t, z, \zeta) + \underline{k}'(t, z) \underline{R}^{-1}(t) \underline{k}(t, \zeta) - Q(t, z, \zeta) \tag{5.2.3}$$

The expression for the optimal pointwise control is

$$\underline{u}^*(t) = -\underline{R}^{-1}(t) B^*_o(t) K(t) x(t)$$

$$= -\underline{R}^{-1}(t) B_o^*(t) \int_D K(t, z, \zeta) x(t, \zeta) d\zeta$$

$$= -\underline{R}^{-1}(t) \left[b_i(t) \int_D \chi_i(z) \int_D K(t, z, \zeta) x(t, \zeta) d\zeta dz \right]$$

$$= -\underline{R}^{-1}(t) \left[\int_D [b_i(t) \int_D \chi_i(z) K(t, z, \zeta) dz] x(t, \zeta) d\zeta \right]$$

$$= -\underline{R}^{-1}(t) \int_D \underline{k}(t, \zeta) x(t, \zeta) d\zeta \qquad (5.24)$$

We shall now introduce an approximation by the use of the assumption in Section 3.4 that control action takes place over "volumes" in D which are very small compared to D itself. In other words it may be assumed that each of the sets $\{E_i\}_{i=1}^k$ containing the points $\{z_i\}_{i=1}^k$ has measure* $\mu(E_i) \leq \epsilon$, where ϵ is very small compared to $\mu(D)$. Let us also assume that the control operator coefficients $b_i(t)$, $i=1, \ldots k$ are of the order of $1/\epsilon$, that is, let us assume that

$$b_i(t) = \frac{1}{\epsilon} \beta_i(t)$$

The physical motivation for this assumption lies in the fact that unless the control coefficients were of the order of $1/\epsilon$ then any finite amount of control would enter the system with magnitude of order ϵ, and, under the assumption that ϵ is very small, would have no effect on the system. If the control coefficients $b_i(t)$ are of order $1/\epsilon$, then we shall see that finite control energy results in a forcing term of the same order of magnitude in the system equation.

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*μ is the Lebesgue measure in R^n.

If $K(t, z, \zeta)$ is sufficiently smooth and ϵ is chosen small enough then we have approximation

$$b_i(t) \int_D \chi_i(\rho) K(t, z, \rho) d\rho \simeq \beta_i(t) K(t, z, z_i)$$

which holds for $i=1,2,\ldots k$. What, in effect, has been done here is to assume that the control coefficients are approximately impulsive in the spatial variable. Note that this assumption was invalid in the rigorous proofs of existence of optimal controls and existence of solutions of the Riccati operator equation. However, at this juncture, the assumption is valid because we are simply trying to solve approximately an equation which we already know has a solution.

As a consequence of the above approximation the vector function $\underline{k}(t, z)$ can be approximated by

$$\underline{k}(t,z) \simeq \hat{\underline{k}}(t, z)$$

where

$$\hat{\underline{k}}(t, z) = \begin{bmatrix} & \uparrow \\ \beta_i(t) & K(t, z, z_i) \\ & \downarrow \end{bmatrix} \tag{5.2.5}$$

With this approximation the Riccati integro-differential equation becomes

$$\frac{\partial K}{\partial t}(t, z, \zeta) = -(A_z^* + A_\zeta^*) K(t,z,\zeta) + \hat{\underline{k}}'(t, z) \underline{R}^{-1}(t) \hat{k}(t, \zeta) - Q(t, z, \zeta) \tag{5.2.6}$$

What is more interesting is that if the above approximation is used in Eq. 5.2.4 the expression for the optimal control becomes

$$\underline{u}^*(t) = -\underline{R}^{-1}(t) \int_D \hat{\underline{k}}(t, \zeta) x(t, \zeta) d\zeta$$

$$= -\begin{bmatrix} & & \uparrow \\ \sum_{j=1}^{k} \underline{R}_{ij}^{-1}(t) & \int_D \beta_j(t) K(t, \zeta, z_j) x(t, \zeta) d\zeta \\ & & \downarrow \end{bmatrix}$$

which implies that it is necessary to determine the k functions $K(t, z, z_i)$, $i=1,2,\ldots k$ in order to completely specify the optimal feedback control. If one were to consider the computational requirements, then the computation of these k functions would be simpler than the computation of the entire feedback kernel, that is, the computation of $K(t, z, \zeta)$ for all values of both spatial arguments in $D \times D$.

To summarize the above results, let us examine the structure of the feedback control system. The state distribution $x(t, z)$ is fed back through k devices which take a weighted spatial average of the state distribution. The weighting function in the i^{th} averaging device is $K(t, z, z_i)$, $i=1,2,\ldots k$, and the output is a function of time which may be denoted $y_i(t)$, $i=1,2,\ldots k$. The k-vector $\underline{y}(t)$, with i^{th} component $y_i(t)$, is then transformed to the optimal control

$$\underline{u}^*(t) = -\underline{R}^{-1}(t)\underline{B}(t)\underline{y}(t)$$

where $\underline{B}(t)$ is the diagonal matrix with i^{th} diagonal element $B_{ii}(t) = \beta_i(t)$, the i^{th} control coefficient. Let us illustrate the system thus obtained by means of a block diagram in which we use the conventions "$\rightarrow$" to indicate the flow of a scalar quantity, "$\Longrightarrow$" a k-vector, and "$\rightsquigarrow$" a distributed quantity. The optimal closed-loop system is represented in Fig. 1. It is interesting to note from this feedback structure that if we were able to measure $\underline{y}(t)$ directly, that is, if we had k measuring devices which average the state distribution with weighting functions $K(t, z, z_i)$, $i=1,2,\ldots k$, then we would feed back the measurements, rather than the entire state distribution, in order to construct the optimal feedback control. This leads to a question which is somewhat analogous to the inverse problem of finite dimensional control theory, namely, if we have k measuring devices of the form

$$y_i(t) = \int_D m_i(t, z)x(t, z)dz \quad ; \quad i=1,2,\ldots k$$

where $\{m_i(t)\}_{i=1}^{k}$ is an arbitrary set of weighting functions, then does there exist a state-weighting kernel $Q_1(t, z, \zeta)$ such that the solution $K(t, z, \zeta)$ of Eq. 5.2.6 with $Q(t,z,\zeta)=Q_1(t, z, \zeta)$ satisfies the property

$$K(t, z, z_i) = m_i(t, z)$$

for all $t\epsilon[0, T]$, for all $z\epsilon D$, and $i=1,2,\ldots k$? Looked upon another way, obtaining a solution for the set of functions $\{K(t,z,z_i)\}_{i=1}^{k}$ enables one to design appropriate instruments with weighting functions equal to $K(t, z, z_i)$. This measurement question will be treated in Section 5.3, where a particular class of measurement weighting functions will be considered.

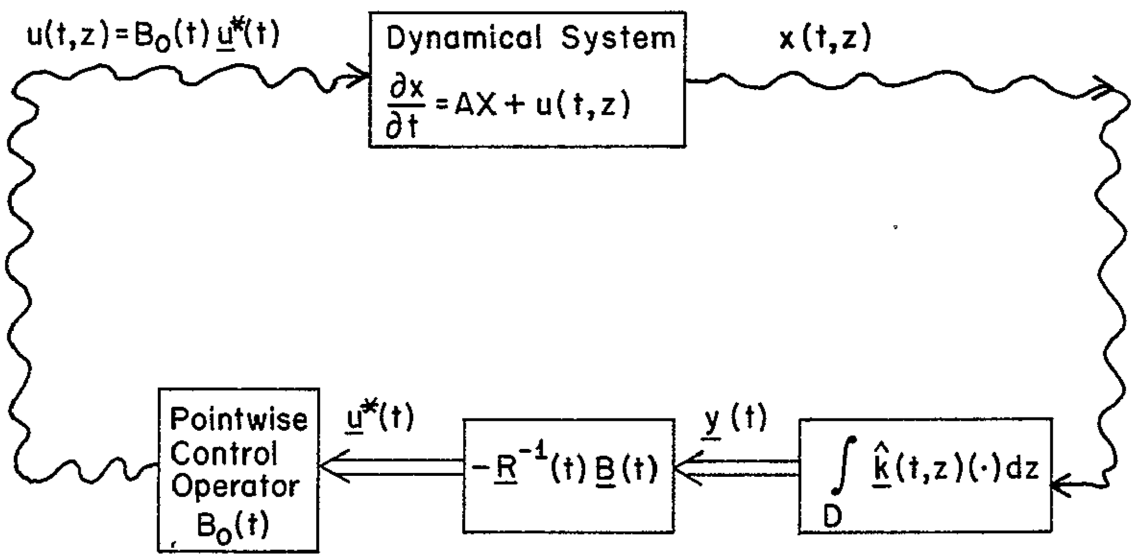

Fig. 1 Optimal Closed-Loop Pointwise Control System

Let us consider the pointwise control problem on the infinite time interval. If the system operator A is coercive then, according to Section 4.7, an optimal control exists on the infinite time interval.

Moreover, if the system is time invariant, the feedback kernel is $K(t, z, \zeta) = K(z, \zeta)$ and $\hat{\underline{k}}(t, z)$ defined in (5.2.5) becomes

$$\hat{\underline{k}}(t, z) = \hat{\underline{k}}(z) = \left[\beta_i \; K(z, z_i) \updownarrow \right]$$

so that we obtain the following time-invariant Riccati equation:

$$-(A_z^* + A_\zeta^*)K(z, \zeta) + \hat{\underline{k}}'(z)\underline{R}^{-1}\hat{\underline{k}}(\zeta) - Q(z, \zeta) = 0 \tag{5.2.7}$$

Under the assumption that the system operator A is the infinitesimal generator of a semigroup of operators, we know, from Section 4.7, that complete controllability is sufficient in order to guarantee the existence of an infinite time solution. In the case of pointwise control, the condition for complete controllability, namely, the invertibility of $L_{t_1}L_{t_1}^*$ given in Eq. 4.7.6, becomes the determination of a time t_1 such that the following is invertible

$$L_{t_1}L_{t_1}^* = \int_t^{t_1} \Phi(t_1-\sigma)B_o(\sigma)B_o^*(\sigma)\Phi^*(t_1-\sigma)d\sigma$$

which, for any $x \epsilon H_o^m(D)$, reduces to

$$L_{t_1}L_{t_1}^* x = \int_t^{t_1} \Phi(t_1-\sigma)B_o(\sigma)\left[\beta_i(\sigma) \int_D \chi_i(\zeta)(\Phi^*(t_1-\sigma)x)(\zeta)d\zeta \updownarrow \right]$$

$$= \int_t^{t_1} \Phi(t_1-\sigma)y(\sigma)d\sigma \tag{5.2.8}$$

---

* $(\Phi^*(t_1-\sigma)x)(\zeta)$ is the evaluation of $\Phi^*(t_1-\sigma)x\epsilon H_o^m(D)$ at the point $\zeta\epsilon D$.

where $y(\sigma) \epsilon H_o^m(D)$ is given by

$$y(\sigma, z) = \sum_{i=1}^{k} \chi_i(z)\beta_i^2(\sigma) \int_D \chi_i(\zeta)(\Phi^*(t_1-\sigma)x)(\zeta)d\zeta$$

Note that $y(\sigma, z)$ is zero everywhere except on the sets E_i, on which $y(\sigma, z)$ has the values

$$y(\sigma, z) = \beta_i^2(\sigma) \int_D \chi_i(\zeta)(\Phi^*(t_1-\sigma)x)(\zeta)d\zeta \quad \forall z \epsilon E_i$$

The question of invertibility of (5.2.8) for some time t_1 is still open. It can be seen that a necessary condition for invertibility of expression (5.2.8) is that there exists a subset of $(0, \infty)$ with positive measure upon which the operator $\Phi(t)$ transforms the subspace of functions with support on $\bigcup_{i=1}^{k} E_i$ onto all of $H_o^m(D)$. Otherwise, there would be no chance for the operator given by (5.2.8) to have its range space equal to $H_o^m(D)$ for some time t_1, which is equivalent to invertibility. Since controllability is still a matter of open research, we shall assume it, where necessary, in the sequel.

In any case, if controllability is assumed, then infinite time solutions exist in the case where the system operator is the infinitesimal generator of a semigroup of operators and the time-invariant Riccati integro-differential equation (5.2.7) holds in this case as well. The optimal control, in this case, is given by the time-invariant linear feedback control law

$$\underline{u}^*(t) = -\underline{R}^{-1} \int_D \hat{\underline{k}}(z)x(t, z)\,dz \tag{5.2.9}$$

and the optimal cost function is

$$J = \int_D \int_D K(z, \zeta)x(t, z)x(t, \zeta)d\zeta dz$$

As an example of a pointwise control problem let us, once again, consider the scalar heat equation

$$\frac{\partial x(t,z)}{\partial t} = \frac{\partial^2 x(t,z)}{\partial z^2} + B_o\underline{u}(t) \quad ; \quad x(0,z) = x_o(z)$$

with boundary conditions

$$x(t,0) = x(t,1) = 0$$

where

$$B_o\underline{u}(t) = \sum_{i=1}^{k} \chi_i(z)u_i(t)$$

If we choose a cost criterion of the form

$$J = \int_0^{\infty}\left[\int_0^1\int_0^1 (2\pi^2 + \sum_{i=1}^{k} \sin^2\pi z_i)\sin\pi z\sin\pi\zeta x(t,z)x(t,\zeta)\,d\zeta dz + \underline{u}'(t)\underline{u}(t)\right] dt$$

where $\{z_i\}_{k=1}^{k}$ is the set of control points, the Riccati integro-differential equation (5.2.7) becomes

$$-\frac{\partial^2}{\partial z^2}K(z,\zeta) - \frac{\partial^2}{\partial \zeta^2}K(z,\zeta) + \hat{\underline{k}}'(z)\underline{k}(\zeta) = (2\pi^2 + \sum_{i=1}^{k}\sin^2\pi z_i)\sin\pi z\sin\pi\zeta$$

for which

$$K(z,\zeta) = \sin\pi z\,\sin\pi\zeta$$

is a solution satisfying the boundary conditions

$$K(0,\zeta) = K(1,\zeta) = K(z,0) = K(z,1) = 0$$

Since $\hat{\underline{k}}(z)$ is of the form

$$\hat{\underline{k}}(z) = \sin\pi z \begin{bmatrix} \uparrow \\ \sin\pi z_i \\ \downarrow \end{bmatrix}$$

The optimal control, from Eq. 5.2.9, is

$$\underline{u}^*(t) = \begin{bmatrix} \uparrow \\ \sin\pi z_i \\ \downarrow \end{bmatrix} \int_0^1 \sin\pi z x(t,z)\,dz$$

We shall consider a special class of solutions of Eq. 5.2.7 in the next section which enable us to compare our results with those obtained by using the modal analytic approach.

5.3 THE INFINITE TIME PROBLEM AND MODAL ANALYTIC SOLUTIONS

In order to obtain a better physical understanding of the nature of the optimal pointwise feedback control obtained in the preceding section, we shall relate these results to the results obtained through the application of the techniques of modal analysis. We shall show that a particular choice of the form of the kernel $Q(z,\zeta)$ in Eq. 5.2.7 results, under certain conditions, in the transformation of the integro-differential equation (5.2.7) into an algebraic matrix equation. It will be shown that this finite dimensional Riccati equation is associated with the finite modal approximation of the optimal control problem under consideration. Placement of the control points will be shown to have a direct effect on the existence of an optimal modal solution. The optimal solution for an illustrative example will be studied.

For convenience let us rewrite the Riccati integro-differential equation for the time-invariant feedback kernel associated with infinite terminal-time pointwise control problem

$$-(A_z^* + A_\zeta^*)K(z,\zeta) + \hat{\underline{k}}'(z)\underline{R}^{-1}\hat{\underline{k}}(\zeta) - Q(z,\zeta) = 0 \qquad (5.3.1)$$

where $\hat{\underline{k}}(z)$ is the k-vector whose i^{th} component is

$$\hat{k}_i(z) = \beta_i K(z, z_i)$$

the set $\{z_i\}_{i=1}^{k}$, once again, being the control points in D.

As a preliminary to showing that the optimal modal solution can be deduced from Eq. 5.3.1, let us consider the case where no control is applied to the system. We can show the cost of starting at

$x \in H_o^m(D)$ at time t to be

$$\int_D \int_D x(z) K(z,\zeta) x(\zeta) d\zeta dz = \int_t^{\infty} \left[\int_D \int_D x(\sigma, z) Q(z,\zeta) x(\sigma,\zeta) d\zeta dz \right] d\sigma \qquad (5.3.2)$$

where $x(\sigma, z)$ is the evaluation at the point $z \in D$ of the element $x(\sigma) \in H_o^m(D)$ which satisfies

$$\dot{x}(\sigma) = A\, x(\sigma) \quad ; \quad x(t) = x$$

and where $K(z,\zeta)$ is the solution of the linear equation

$$-(A_z^* + A_\zeta^*) K(z,\zeta) - Q(z,\zeta) = 0 \qquad (5.3.3)\ .$$

Let us suppose that the system operator A has a countable spectrum $\{\lambda_i\}_{i=1}^{\infty}$. The eigenfunctions $\{v_i(z)\}_{i=1}^{\infty}$ of the adjoint operator A^* satisfy the equation

$$A_z^* v_i(z) = \lambda_i v_i(z)$$

for $i = 1, 2, \ldots$. If we choose the kernel of the state-weighting operator to be

$$Q(z,\zeta) \triangleq \underline{v}'(z) \underline{Q}\, \underline{v}(\zeta) \qquad (5.3.4)$$

where $\underline{Q}$ is an $n \times n$ positive definite constant matrix and $\underline{v}(z)$ is the n-vector whose i^{th} component is the eigenfunction $v_i(z)$, the state-weighting operator will still satisfy the requirement of positivity, since

$$\langle Q, x, x \rangle_{L^2(D)} = \int_D \int_D x(z) \underline{v}'(z) \underline{Q}\, \underline{v}(\zeta) x(\zeta) d\zeta dz$$

$$= \left[\int_D x(z) \underline{v}'(z) dz \right] \underline{Q} \left[\int_D \underline{v}(\zeta) x(\zeta) d\zeta \right]$$

$$= \underline{x}' \underline{Q}\, \underline{x} \geq 0$$

where $\underline{x}$ is the n-vector whose i^{th} component is $\int_D x(z) v_i(z) dz$. Note that the operator Q is not strictly positive since there exist nonzero vectors $x \epsilon H_o^m(D)$ which are orthogonal to the subspace generated by the first n eigenfunctions, resulting in $\langle Qx, x \rangle_{L^2(D)} = 0$. Note that if we allow n to approach infinity the kernel $Q_\infty(z, \zeta)$ of a positive operator is obtained. The precise nature of this limiting procedure will be discussed when the concepts of modal approximation are treated later on in this section. We may now state the following theorem:

Theorem 5.1: If $Q(z, \zeta)$ is given by Eq. 5.3.4, then the optimal feedback* kernel for the zero-control case is given by

$$K(z, \zeta) = \underline{v}'(z)\underline{K}\underline{v}(\zeta) \tag{5.3.5}$$

where $\underline{K}$ is the $n \times n$ positive definite solution matrix of the matrix equation

$$\underline{\Lambda}\underline{K} + \underline{K}\underline{\Lambda} = -\underline{Q} \tag{5.3.6}$$

with $\underline{\Lambda}$ defined to be the diagonal $n \times n$ matrix with i^{th} diagonal element $\Lambda_{ii} = \lambda_i$.

Proof: Using Eq. 5.3.5 in Eq. 5.3.3 and using the linearity of A^* we obtain

$$A_z^* \underline{v}'(z)\underline{K}\underline{v}(\zeta) + \underline{v}'(z)\underline{K}A_\zeta^*\underline{v}(\zeta) = -\underline{v}'(z)\underline{Q}\,\underline{v}(\zeta)$$

Since the elements of $\underline{v}(z)$ are eigenfunctions of A_z^*, this equation becomes

$$\underline{v}'(z)\underline{\Lambda}\,\underline{K}\underline{v}(\zeta) + \underline{v}'(z)\underline{K}\underline{\Lambda}\,\underline{v}(\zeta) = -\underline{v}'(z)\underline{Q}\underline{v}(\zeta)$$

If a solution of this equation is to exist for all $z, \zeta \epsilon D$, then it must be true that the matrix $\underline{K}$ satisfies Eq. 5.3.6. Moreover, since $K(z, \zeta)$ must be the kernel of a positive operator on $H_o^m(D)$, the matrix $\underline{K}$ must be positive definite. It is a well-known fact that if the matrix $\underline{Q}$ is positive definite

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* The term "feedback" is used loosely here, since we are applying no control.

and the matrix $\underline{\Lambda}$ has all its eigenvalues in the left half-plane, then a positive definite solution of Eq. 5.3.6 exists. The matrix $\underline{Q}$ is positive definite by assumption and, since the spectra of both coercive and strongly elliptic system operators lie in the left half-plane, the eigenvalues of $\underline{\Lambda}$ lie in the left half-plane, so that a positive definite matrix solution $\underline{K}$ to Eq. 5.3.6 exists. Thus, $K(z,\zeta)$, given by Eq. 5.3.5, is the kernel of a positive operator which is the solution of Eq. 5.3.3. By the uniqueness* of positive solutions of Eq. 5.3.3 this kernel is optimal.

We can conclude from this theorem that <u>in the zero-control case the cost function depends only on the first</u> n <u>mode coefficients of the initial state</u> $x \in H_o^m(D)$. This can be shown by evaluating the cost function

$$J = \langle Kx, x\rangle_{L^2(D)} = \int_D\int_D x(z)K(z,\zeta)x(\zeta)d\zeta dz = \int_D x(z)\underline{v}'(z)dz\,\underline{K}\int_D x(\zeta)\underline{v}(\zeta)d\zeta$$

$$= \underline{x}'\underline{K}\underline{x}$$

where $\underline{x}$ is the n-vector whose i^{th} component is the i^{th} mode coefficient $x_i = \int_D x(z)v_i(z)dz$.

A natural question to ask at this juncture would be : <u>does a solution of the form (5.3.5) exist for the system with pointwise control when the kernel</u> $Q(z,\zeta)$ <u>is again given by expression (5.3.4)</u>? We shall show that under certain circumstances such a solution exists for the optimal feedback kernel and that the solution is directly related to the finite modal approximation of the original system.

If the control is a k-vector, with k not necessarily equal to n (the dimension of the vector $\underline{v}(z)$) the substitution of Eq. 5.3.5 into

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* Uniqueness follows from the uniqueness of the limit in Theorem 4.11.

the pointwise Riccati integro-differential equation (5.3.1) yields the equation

$$-(A_z^*+A_\zeta^*)\underline{v}'(z)\underline{K}\,\underline{v}(\zeta) + \underline{m}'(z)\underline{R}^{-1}\underline{m}(\zeta) - \underline{v}'(z)\underline{Q}\,\underline{v}(\zeta) = 0 \tag{5.3.7}$$

where $\underline{m}(z)$ is the k-vector whose i^{th} component is

$$m_i(z) = \beta_i\underline{v}'(z)\underline{K}\,\underline{v}\,(z_i) \qquad i=1,2,\ldots k$$

We can write the vector $\underline{m}(z)$ in the form

$$\underline{m}(z) = \underline{B}\,\underline{V}\,\underline{K}\,\underline{v}(z) \tag{5.3.8}$$

where $\underline{B}$ is the diagonal $k\times k$ matrix whose i^{th} diagonal element is $B_{ii} = \beta_i$ and $\underline{V}$ is the $k\times n$ matrix with ij^{th} element $V_{ij} = v_j(z_i)$. Using Eq. 5.3.8 in Eq. 5.3.7, we see that the Riccati integro-differential equation (5.3.1) has a positive solution of the form $K(z,\zeta) = \underline{v}'(z)\underline{K}\underline{v}(\zeta)$ if there exists a positive definite solution $\underline{K}$ of the algebraic matrix Riccati equation

$$-\underline{\Lambda}\,\underline{K} - \underline{K}\,\underline{\Lambda} + \underline{K}\,\underline{V}'\underline{B}\underline{R}^{-1}\underline{B}\underline{V}\underline{K} - \underline{Q} = 0 \tag{5.3.9}$$

where $\underline{\Lambda}$ is again the diagonal $n\times n$ matrix of eigenvalues. We know that this Riccati equation is associated with the following finite-dimensional optimization problem:

<u>Given the n-dimensional system</u>

$$\underline{\dot{x}} = \underline{\Lambda}\,\underline{x}(t) + \underline{V}'\underline{B}\,\underline{u}(t) \quad ; \quad \underline{x}(0) = x_o \tag{5.3.10}$$

<u>Determine the control</u> $\underline{u}^*(t)\epsilon R^k$ <u>which minimizes the cost functional</u>

$$J = \int_0^\infty [\underline{x}'(t)\underline{Q}\underline{x}(t) + \underline{u}'(t)\underline{R}\,\underline{u}(t)]\,dt \quad . \tag{5.3.11}$$

Thus, from our knowledge of the finite-dimensional state regulator, we know that Eq. 5.3.9 has a positive definite solution $\underline{K}$ if the system (5.3.10) is completely controllable.* We consider the standard test for complete controllability in time-invariant, finite dimensional, linear systems (see Athans and Falb, 24 p.205), namely, if $\underline{G}$ is the $n \times (nk)$ matrix defined by

$$\underline{G} \triangleq \left[\underline{V}'\underline{B} \;\vdots\; \underline{\Lambda}\,\underline{V}'\underline{B} \;\vdots\; \underline{\Lambda}^2\underline{V}'\underline{B} \;\vdots\; \cdots \;\vdots\; \underline{\Lambda}^{n-1}\underline{V}'\underline{B} \right] \tag{5.3.12}$$

then the system (5.3.10) is completely controllable if and only if

$$\operatorname{rank} \underline{G} = n$$

The system given by Eq. 5.3.10 is interesting in another respect. It is precisely the n-mode modal analytic approximation of the original distributed parameter system given in Eq. 3.2.2 with $B = B_o$, the pointwise control operator. This can be seen by considering the modal decomposition of the forcing term $B_o\underline{u}(t)$:

$$\begin{aligned}
\int_D \underline{v}(z) B_o \underline{u}(t)\,dz &= \int_D \underline{v}(z) \sum_{i=1}^{k} \chi_i(z) b_i u_i(t)\,dz \\
&= \sum_{i=1}^{k} b_i u_i(t) \int_D \underline{v}(z)\chi_i(z)\,dz \simeq \sum_{i=1}^{k} \beta_i u_i(t) \underline{v}(z_i) \\
&= \underline{V}'\underline{B}\,\underline{u}(t)
\end{aligned}$$

which is the forcing term in Eq. 5.3.10. The preceding is summarized as:

Theorem 5.2 If $Q(z,\zeta) = \underline{v}'(z)\underline{Q}\,\underline{v}(\zeta)$, with $\underline{Q}$ positive definite,** and if the rank of the matrix $\underline{G}$, defined in Eq. 5.3.12, is n, then there is a solution of the Riccati integro-differential equation (5.3.1) which is the kernel of a positive operator and which has the form $K(z,\zeta) = \underline{v}'(z)\underline{K}\,\underline{v}(\zeta)$, where the matrix $\underline{K}$ is the positive definite solution of the matrix Riccati Eq. 5.3.9.

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* Observability is actually sufficient for definiteness.

** Positive semi-definiteness is sufficient in this case.

Let us also note that the optimal pointwise feedback control, from Eq. 5.2.4, is given by

$$\underline{u}^*(t) = -\underline{R}^{-1}\int_D \hat{\underline{k}}(\zeta)x(t,\zeta)d\zeta$$
$$= -\underline{R}^{-1}\underline{B}\ \underline{V}\ \underline{K}\int_D \underline{v}(\zeta)x(t,\zeta)d\zeta = -\underline{R}^{-1}\underline{B}\ \underline{V}\ \underline{K}\ \underline{x}(t) \tag{5.3.13}$$

with $\quad \underline{x}(t) \triangleq \int_D \underline{v}(\zeta)x(t,\zeta)d\zeta$

i.e., $\underline{x}(t)$ is the n-vector of modal coefficients of $x(t,z)$. Moreover, the minimum cost of starting at time t with initial state $x_o \epsilon H_o^m(D)$ is given by

$$J = \int_D\int_D x_o(z)K(z,\zeta)x_o(\zeta)d\zeta dz = \int_D x_o(z)\underline{v}'(z)dz\ \underline{K}\int_D \underline{v}(\zeta)x_o(\zeta)d\zeta$$
$$= \underline{x}_o'\underline{K}\,\underline{x}_o$$

where $\underline{x}_o$ is the n-vector of modal coefficients of $x_o(z)$. Thus, we have shown that by choosing $Q(z,\zeta)$ to be of the form specified in Eq. 5.3.4, both the optimal control and the optimal cost function depend only on the first n modal coefficients of the state variable.

This has very interesting implications as far as the modal analytic approach is concerned. In the modal analytic approach, a system of the form (5.3.10) is obtained and a finite-dimensional cost functional of the form (5.3.11) is used. Naturally, the optimal control and optimal cost function would only depend on the finite-dimensional state variable (the modal coefficients). It is difficult to say, one way or the other, via straightforward modal analytic techniques, whether feeding back higher order modes would result in a smaller value of the cost functional. Theorem 5.2 allows us to make a definitive statement, namely: <u>if the rank of the controllability matrix $\underline{G}$ is n then we can never do any better by feeding back more than the first n modes</u>. If the rank of $\underline{G}$ is less than n, we know, from the results of Chapter IV and Section 5.2, that a positive operator kernel solution of the Riccati integro-differential equation (5.3.1) still exists, but it is not of the form $K(z,\zeta) = \underline{v}'(z)\,\underline{K}\,\underline{v}(\zeta)$, or, in other words, the optimal control and optimal cost function will depend on modes of order higher than n.

The above results allow us to make still another conclusion concerning the modal analytic approximation. If we are trying to approximate an arbitrary state-weighting kernel $Q(z,\zeta)$ by the n^{th} order approximate kernel

$$Q_n(z,\zeta) = \underline{v}_n'(z)\,\underline{Q}_n\,\underline{v}_n(\zeta) \tag{5.3.14}$$

then the positive operator Q_n represented by this kernel is less than the operator Q, represented by the kernel $Q(z,\zeta)$ in the sense of the ordering relation introduced in Section 4.6. Moreover, increasing the order of the modal approximation by one results in a more positive state-weighting operator, Q_{n+1}, that is, $Q_{n+1} \geq Q_n$, since, now, the presence of the $n+1^{th}$ mode increases the cost. We may now ask whether this results in an increase in the resulting optimal cost, or, more precisely, it is true that we have the relation

$$K_1 \leq K_2 \leq K_3 \;\ldots\ldots\ldots\ldots\ldots$$

where the operator K_n is the positive, self-adjoint solution of the Riccati operator equation

$$-A^*K - KA + KB_o\underline{R}^{-1}B_o^*K - Q_n = 0 \tag{5.3.15}$$

where B_o is the time-invariant pointwise control operator. If we consider the operators K_n and K_{n+1} and the difference of the equations of the form of Eq. 5.3.15 which they satisfy, we obtain

$$-A^*(K_{n+1}-K_n) - (K_{n+1}-K_n)A + K_{n+1}BR^{-1}B^*K_{n+1}$$
$$-K_nBR^{-1}B^*K_n - (Q_{n+1}-Q_n) = 0$$

This equation may be written in the form

$$\begin{aligned}&-(A-BR^{-1}B^*K_n)^*\delta K_n-\delta K_n(A-BR^{-1}B^*K_n)\\&\qquad -\delta K_nBR^{-1}B^*\delta K_n - (Q_{n+1}-Q_n) = 0\end{aligned} \tag{5.3.16}$$

where $\delta K_n = K_{n+1} - K_n$

Since $(Q_{n+1}-Q_n)$ is a positive operator, Theorem 4.11 implies that a positive, bounded solution δK_n of Eq. 5.3.16 exists, from which we may conclued that $K_{n+1} \geq K_n$. This result may be briefly summarized by the statement that <u>monotone approximation of the state-weighting operator results in monotone approximation of the optimal feedback operator</u>. It is difficult to prove this monotonicity property by direct modal analytic considerations, but when recourse is taken to the fact that any modal approximation of a given order n corresponds to a distributed optimization problem with state-weighting operator Q_n, the proof becomes quite simple.

This result has a bearing on the problem of determining what order modal approximation to choose. If n is chosen so that $Q_n(z,\zeta)$ is a good approximation to $Q(z,\zeta)$ in the sense that

$$\int_D\int_D[Q(z,\zeta)-Q_n(z,\zeta)]\,x(z)x(\zeta)dzd\zeta < \epsilon\int_D\int_D Q(z,\zeta)x(z)x(\zeta)dzd\zeta$$

where ϵ is a small positive number, then it is clearly seen that, by using the above procedure, the feedback kernel $K_n(z,\zeta)$ resulting from the solution of Eq. 5.3.1, with $Q(z,\zeta) = Q_n(z,\zeta)$, will satisfy the inequality

$$\int_D \int_D [K(z,\zeta) - K_n(z,\zeta)]\, x(z)x(\zeta)\,dz\,d\zeta$$

$$< \epsilon \int_D \int_D K(z,\zeta)\, x(z)x(\zeta)\,dz\,d\zeta$$

where $K(z,\zeta)$ is the optimal feedback kernel. This follows directly from solving Eq. 5.3.16 with forcing term $(Q-Q_n)$ for the difference operator $\delta K_n = K - K_n$. To summarize: an analytic procedure for determining the number of modes which will result in an approximation to the optimal cost of a particular degree of accuracy is to choose n such that the state-weighting kernel is approximated to that degree of accuracy. Let us now illustrate these ideas by means of the following example:

Example 5.1: Consider the one-dimensional heat equation with pointwise control, described by the equation

$$\frac{\partial x(t,z)}{\partial t} = \frac{\partial^2 x(t,z)}{\partial z^2} + B_o u(t) \quad , \quad 0 \leq z \leq 1$$

where B_o is the time-invariant pointwise control operator. Here, of course, the system operator A is $\partial/\partial z^2$. Let us choose the boundary conditions to be

$$x(t,0) = x(t,1) = 0$$

In this case the system operator A is self adjoint and the eigenvalues are

$$\lambda_i = -i^2\pi^2$$

with associated eigenfuctions

$$v_i(z) = \sin i\pi z$$

Let us suppose that we are using two pointwise controls, that is, k=2. Moreover, let us choose the state weighting kernel $Q(z,\zeta)$ to be

$$Q(z,\zeta) = [v_1(z) v_2(z)] \; \underline{Q} \begin{bmatrix} v_1(\zeta) \\ v_2(2) \end{bmatrix}$$

where $\underline{Q}$ is a positive definite 2×2 matrix. The matrix $\underline{V}$ is

$$\underline{V} = \begin{bmatrix} \sin\pi z_1 & \sin 2\pi z_1 \\ \sin\pi z_2 & \sin 2\pi z_2 \end{bmatrix}$$

The controllability matrix $\underline{G}$ is the 2×4 matrix

$$\underline{G} = [\underline{V}'\underline{B}_1 \,\vdots\, \underline{\Lambda}\underline{V}'\underline{B}_1]$$

$$= \begin{bmatrix} \beta_1 \sin\pi z_1 & \beta_2 \sin\pi z_2 & \lambda_1\beta_1 \sin\pi z_1 & \lambda_1\beta_2 \sin\pi z_2 \\ \beta_1 \sin\mu\pi z_1 & \beta_2 \sin 2\pi z_2 & \lambda_2\beta_1 \sin 2\pi z_1 & \lambda_2\beta_2 \sin 2\pi z_2 \end{bmatrix}$$

The first two column vectors are linearly independent for all choices of z_1 and $z_2 \in (0,1)$, since

$$\det \begin{bmatrix} \beta_1 \sin\pi z_1 & \beta_2 \sin\pi z_2 \\ \beta_1 \sin 2\pi z_1 & B_2 \sin 2\pi z_2 \end{bmatrix} = \beta_1\beta_2 [\sin\pi z_1 \sin 2\pi z_2 - \sin\pi z_2 \sin 2\pi z_1]$$

$$= 2\beta_1\beta_2 \sin\pi z_1 \sin\pi z_2 \, [\cos\pi z_2 - \cos\pi z_1]$$

which is not equal to zero for $z_1 \neq z_2$, because $\sin\pi z_1 \sin\pi z_2 > 0$ on $(0,1)$ and $\cos\pi z$ is monotonically decreasing on $(0,1)$. Thus, the rank of $\underline{G}$ is 2 and Eq. 5.3.14 has a positive definite solution. If there is only one control (k=l) we have

$$\underline{G} = \begin{bmatrix} \beta_1 \sin\pi z_1 & \lambda_1\beta_1 \sin\pi z_1 \\ \beta_1 \sin 2\pi z_1 & \lambda_2\beta_1 \sin 2\pi z_1 \end{bmatrix}$$

which has rank 2 since λ_1 and λ_2 are distinct. Note that in the two control case we do not require that λ_1 and λ_2 are distinct.

In order to actually compute an optimal solution we assign the following values:

$$\underline{B}_1 = \underline{R} = \underline{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we denote the matrix $\underline{K}$ to be

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$$

then the matrix equation (5.3.9) yields the three scalar equations

$$\begin{aligned} &2\pi^2 k_{11} + Ak_{11}^2 + 2Bk_{11}k_{12} + Ck_{12}^2 - 1 = 0 \\ &5\pi^2 k_{12} + Ak_{11}k_{12} + Bk_{12}^2 + Bk_{11}k_{22} + Ck_{12}k_{22} = 0 \qquad (5.3.17) \\ &8\pi^2 k_{22} + Ak_{12}^2 + 2Bk_{12}k_{22} + Ck_{22}^2 - 1 = 0 \end{aligned}$$

where

$$A = \sin^2 \pi z_1 + \sin^2 \pi z_2$$

$$B = \sin\pi z_1 \sin 2\pi z_1 + \sin\pi z_2 \sin 2\pi z_2$$

$$C = \sin^2 2\pi z_1 + \sin^2 2\pi z_2$$

A simplification can be achieved if we choose the control points z_1 and z_2 to lie symmetrically about the midpoint of the interval, $z=\frac{1}{2}$, that is, $z_2 = 1 - z_1$. In this case $A = 2\sin^2\pi z_1$, $B=0$, and $C = 2\sin^2 2\pi z_1$, so that we obtain as a solution of the set of Eqs. 5.3.17

$$k_{11} = \frac{-\pi^2 + \sqrt{\pi^4 + 2\sin^2\pi z_1}}{2\sin^2\pi z_1}$$

$$k_{12} = 0$$

$$k_{22} = \frac{-4\pi^2 + \sqrt{16\pi^4 + 2\sin^2 2\pi z_1}}{2\sin^2 2\pi z_1}$$

Thus, the optimal cost function is

$$
\begin{aligned}
J(x) = \int_D \int_D x(z) \Bigg[& \sin\pi z \left(\frac{-\pi^2 + \sqrt{\pi^4 + 2\sin^2 \pi z_1}}{2\sin^2 \pi z_1} \right) \sin\pi\zeta \\
& + \sin 2\pi z \left(\frac{-4\pi^2 + \sqrt{16\pi^4 + 2\sin^2 2\pi z_1}}{2\sin^2 2\pi z_1} \right) \sin 2\pi\zeta \Bigg] x(\zeta)\, d\zeta\, dz
\end{aligned}
\tag{5.3.18}
$$

And, from Eq. 5.3.13, the optimal control is

$$
\underline{u}^*(t) = \begin{bmatrix} -\dfrac{-\pi^2 + \sqrt{\pi^4 + 2\sin^2 \pi z_1}}{2\sin \pi z_1} \displaystyle\int_D \sin\pi\zeta\, x(t,\zeta)\, d\zeta \\ \\ -\dfrac{-4\pi^2 + \sqrt{16\pi^4 + 2\sin^2 2\pi z_1}}{2\sin 2\pi z_1} \displaystyle\int_D \sin 2\pi\zeta\, x(t,\zeta)\, d\zeta \end{bmatrix}
\tag{5.3.19}
$$

The resulting optimal closed-loop system is illustrated in Fig. 2, where we again use the conventions adopted in Section 5.2 for Fig. 1.

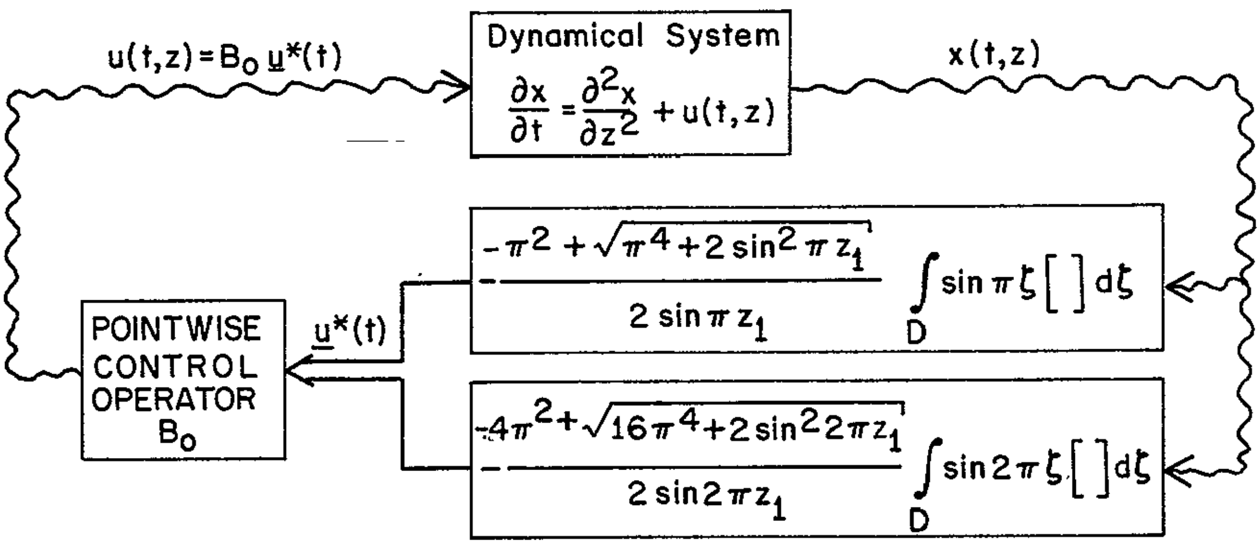

Fig. 2 Closed-Loop System for Example 5.1

Note that if we had measuring devices which yield the measurement vector

$$\underline{y}(t) = \begin{bmatrix} \int_D \sin\pi\zeta \, x(t,\zeta)\,d\zeta \\ \int_D \sin 2\pi\zeta \, x(t,\zeta)\,d\zeta \end{bmatrix}$$

then $\underline{y}(t)$ can be fed directly through the diagonal gain matrix

$$\underline{M} = \begin{bmatrix} -\dfrac{-\pi^2 + \sqrt{\pi^4 + 2\sin^2 \pi z_1}}{2\sin\pi z_1} & 0 \\ 0 & -\dfrac{-4\pi^2 + \sqrt{16\pi^4 + 2\sin^2 2\pi z_1}}{2\sin 2\pi z_1} \end{bmatrix}$$

to obtain the optimal control

$$\underline{u}^* = \underline{M}\,\underline{y}(t)$$

Clearly, the measurement does not depend on the control point location; only the gain matrix $\underline{M}$ does. This indicates there is a decoupling of the measurement and control problems in the sense that changing the control point locations does not modify the basic types of measuring devices in use. Thus, the design procedure of "trying" different control points in order to reduce some average cost does not interfere with the basic structure of the closed-loop system.

This problem of minimizing some average cost with respect to control point location can be done analytically as a parameter optimization problem. For example, if we consider the optimal cost function for example 5.1, given by Eq. 5.3.18, and take the average cost over the unit ball in $L^2(D)$, we obtain

$$J_{avg} = \frac{1}{2}\left[\frac{-\pi^2 + \sqrt{\pi^4 + 2\sin^2 \pi z_1}}{2\sin^2 \pi z_1} + \frac{-4\pi^2 + \sqrt{16\pi^4 + 2\sin^2 2\pi z_1}}{2\sin^2 2\pi z_1}\right]$$

Differentiating J_{avg} with respect to z_1, equating the result to zero and solving this equation for z_1 (hopefully, a solution exists in $(0, 1)$) results in the "optimal" control point location (end of example).

Let us now consider the question, touched on briefly in the preceding section: <u>when can a set of measurements of the form</u>

$$y_i(t) = \int_D m_i(z)x(t, z)dz \quad ; \quad i=1, 2, \ldots n$$

<u>where $\{m_i(z)\}_{i=1}^{n}$ is an, as yet, unspecified set of functions, be fed back directly to obtain the optimal pointwise control?</u> Let us assume that z is a scalar and that each measurement function $m_i(z)$ may be written as a linear combination of the elements of $\{z^{i-1}\}_{i=1}^{n}$, that is, each measurement function is a polynomial of order $n-1$. The vector $\underline{m}(z)$, with i^{th} element $m_i(z)$, can then be written

$$\underline{m}(z) = \underline{W}\underline{q}(z) \tag{5.3.20}$$

where $\underline{W}$ is an $n \times n$ matrix and $\underline{q}(z)$ is the vector

$$\underline{q}(z) = \begin{bmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{n-1} \end{bmatrix}$$

If we choose a cost criterion of the form

$$J = \int_0^{\infty} [\underline{y}'(t)\underline{y}(t) + \underline{u}'(t)\underline{R}\underline{u}(t)] \; dt$$

then $\underline{y}'(t)\underline{y}(t)$ can be written in the form

$$\underline{y}'(t)\underline{y}(t) = \int_D\int_D Q(z,\zeta)x(t, z)x(t,\zeta)\, dz d\zeta$$

where

$$Q(z,\zeta) = \underline{m}'(z)\underline{m}(\zeta) \tag{5.3.21}$$

We shall now proceed to show that under certain conditions the optimal feedback operator $K(z,\zeta)$ for this choice of $Q(z,\zeta)$ is of the form

$$K(z,\zeta) = \underline{m}'(z)\underline{K}\underline{m}(\zeta) \tag{5.3.22}$$

where $\underline{K}$ is an $n\times n$ positive definite matrix. Using Eqs. 5.3.21 and 5.3.22 in the Riccati equation 5.3.1, we obtain

$$-A_z^*\underline{m}'(z)\underline{K}\,\underline{m}(\zeta) - A_\zeta^*\underline{m}'(z)\underline{K}\,\underline{m}(\zeta) + \underline{m}'(z)\underline{K}\,\underline{Y}'\underline{B}\underline{R}^{-1}\underline{B}\underline{Y}\underline{K}\underline{m}(\zeta) - \underline{m}'(z)\underline{m}(\zeta) = 0 \tag{5.3.23}$$

where $\underline{Y}$ is a $k\times n$ matrix with $Y_{ij} = m_j(z_i)$. Using (5.3.20) we see that

$$A_z^*\,\underline{m}(z) = A_z^*\underline{W}\,\underline{q}(z) = \underline{W}\,A_z^*\,\underline{q}(z)$$

Since A_z^* is a differential operator, we may write

$$A_z^*\,\underline{q}(z) = \underline{C}\,\underline{q}(z)$$

where $\underline{C}$ is a lower triangular $n\times n$ matrix. For example, suppose

$$A_z^* = \frac{\partial^2}{\partial z^2} \quad \text{and} \quad n=4, \quad \text{then}$$

$$A_z^*\,\underline{q}(z) = \frac{\partial^2}{\partial z^2}\begin{bmatrix}1\\ z\\ z^2\\ z^3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 2\\ 6z\end{bmatrix} = \begin{bmatrix}0&0&0&0\\ 0&0&0&0\\ 2&0&0&0\\ 0&6&0&0\end{bmatrix}\begin{bmatrix}1\\ z\\ z^2\\ z^3\end{bmatrix} = \underline{C}\,\underline{q}(z)$$

If we assume that $\underline{W}$ is nonsingular, then we may write

$$A_z^*\underline{m}(z) = \underline{W}\,\underline{C}\,\underline{q}(z) = \underline{W}\,\underline{C}\,\underline{W}^{-1}\underline{m}(z)$$

Let us denote the $n\times n$ matrix $\underline{A}$ by

$$\underline{A} = \underline{W}\,\underline{C}\,\underline{W}^{-1}$$

Then the Riccati equation (5.3.23) may be written

$$\underline{m}'(z)[-\underline{A}'\underline{K} - \underline{K}\underline{A} + \underline{K}\underline{Y}'\underline{B}\underline{R}^{-1}\underline{B}\underline{Y}\underline{K} - \underline{I}]\ \underline{m}(\zeta) = 0 \qquad (5.3.24)$$

Thus, a solution $K(z,\zeta)$ of the form specified in Eq. 5.3.22 exists if and only if a positive definite solution of the matrix equation

$$-\underline{A}'\underline{K} - \underline{K}\underline{A} + \underline{K}\underline{Y}'\underline{B}\underline{R}^{-1}\underline{B}\underline{Y}\underline{K} - I = 0 \qquad (5.3.25)$$

exists. Once again, much as in the finite modal analytic case, existence hinges on the controllability of the finite dimensional system

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{Y}'\underline{B}\,\underline{u}$$

The optimal control is now given by

$$\underline{u}^*(t) = -\underline{R}^{-1}\underline{B}\underline{Y}\underline{K}\int_D \underline{m}(z)x(t,z)dz = -\underline{R}^{-1}\underline{B}\underline{Y}\underline{K}\underline{y}(t) \qquad (5.3.26)$$

that is, we have the desired result of directly feeding back the output of our measuring devices. This is a somewhat startling result in that it is impossible, in finite dimensional systems, to have optimal output feedback. This can be explained, however, by the fact that in finite dimensional systems the output is of lower dimension than the state and is written

$$\underline{y} = \underline{C}\,\underline{x}$$

where $\underline{C}$ is not a square matrix. Thus, if we tried to assume that a solution $\underline{K}$ of the matrix Riccati equation

$$-\underline{A}'\underline{K} - \underline{K}\underline{A} + \underline{K}\underline{B}\underline{R}^{-1}\underline{B}'\underline{K} - \underline{C}'\underline{C} = 0$$

were of the form $\underline{C}'\underline{K}_1\underline{C}$, where $\underline{K}_1$ satisfies the equation

$$-\underline{A}'\underline{K}_1 - \underline{K}_1\underline{A} + \underline{K}_1\underline{C}\underline{B}\underline{R}^{-1}\underline{B}'\underline{C}'\underline{K}_1 - \underline{I} = \underline{0}$$

with the resulting optimal control given by

$$\underline{u}^*(t) = -\underline{R}^{-1}\underline{B}'\underline{C}'\underline{K}_1\underline{C}\underline{x}(t) = -\underline{R}\underline{B}'\underline{C}'\underline{K}_1\underline{y}(t)$$

i.e., output feedback, we would not be able to verify that $\underline{K} = \underline{C}'\underline{K}_1\underline{C}$ is indeed a solution of the first matrix Riccati equation. This inability to satisfy the original Riccati equation occurs from the fact that

$$\underline{C}'\underline{K}_1\underline{C}\ \underline{A} \neq \underline{C}'\underline{K}_1\underline{A}\ \underline{C}$$

that is, the output matrix $\underline{C}$ and the system matrix $\underline{A}$ can never commute when $\underline{C}$ is not a square matrix. This stumbling block is avoided in our output feedback derivation, because of the fact that (1) we are using differential operators (A_z^* and A_ζ^*) and (2) the relation $A_z^* \underline{m}(z) = \underline{A}\,\underline{m}(z)$ holds, so that there is no problem in deriving a matrix Riccati equation for $\underline{K}$ given in Eq. 5.3.21 .

To illustrate this result, let us consider the following simple example:

Example 5.2: Let us, once again, consider the scalar heat equation and assume that we have a single pointwise control and a single measuring device which gives the average temperature $y(t)$ over the spatial domain $(0, 1)$, or, more specifically,

$$y(t) = \int_0^1 x(t, z)\,dz$$

If we wish to minimize

$$J = \int_0^\infty [y^2(t) + r u^2(t)]\,dt$$

then we can place this problem within the framework of the preceding result by observing that $\underline{q}(z)$ is the scalar 1, $\underline{W}$ is the scalar 1, and from the fact that

$$A_z^*\,\underline{q}(z) = \frac{\partial^2}{\partial z^2}(1) = 0$$

the matrix $\underline{A}$ is the scalar 0. The matrix $\underline{Y}$ is also the scalar 1 and the matrix $\underline{B}$ is the scalar β_1. The Riccati equation (5.3.25) thus becomes

$$\frac{\beta_1^2}{r} k^2 = 1$$

which has the "positive definite" solution

$$k = \frac{\sqrt{r}}{|\beta_1|}$$

The optimal control is given by

$$u^*(t) = -\frac{\beta_1}{|\beta_1|} \frac{1}{\sqrt{r}} \int_0^1 x(t, z)\, dz = -\frac{\beta_1}{|\beta_1|} \frac{1}{\sqrt{r}} y(t)$$

which shows that we directly feed back the average of the temperature distribution on (0, 1).

To summarize the results of this section, we have shown that by a judicious choice of the state-weighting kernel $Q(z,\zeta)$ one is able to derive the finite modal approximation to the pointwise control problem, from which it was possible to conclude the following:

1. The optimal control law over the class of control laws which feed back only the modes under consideration is optimal over the class of all feedback control laws.
2. The optimal cost for the n^{th} modal approximation monotonically increases with n.

Neither of these two conclusions can be made very easily using straightforward modal analytic techniques. In the example presented it was shown that the feedback structure of the pointwise control system can be separated into a measurement part, which is independent of control point location, and a gain part, which depends directly on control point location. We next considered the problem of having only n measurements of the state distribution available, rather than the entire state distribution, and we were able to show, under the assumption of a particular form for these measurements, that the optimal feedback control law consists of directly feeding back these measurements.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

In the preceding chapters we have considered the problem of minimizing a quadratic cost criterion in systems described by linear parabolic partial differential equations. We have shown that optimal controls exist both in the case where the system operator is coercive and in the case where the system operator is the infinitesimal generator of a semigroup of operators. The optimal control is given by a bounded linear transformation of the state of the system. The resulting optimal feedback operator was shown to be the solution of an operator differential equation of the Riccati type. By application of the Schwartz kernel theorem the feedback operator was shown to be represented by an integral operator whose kernel satisfies an integro-differential equation of the Riccati type.

Using these results for general parabolic optimal control problems, we were able to specialize to the case of pointwise control. It was shown that the optimal pointwise control is also given by a state feedback law, which, in this case, is of a simpler form than that of the distributed control case. We were also able to use the general results to derive the modal analytic approximation to the optimal pointwise control and to show that for a special class of state measuring devices the optimal pointwise control is given by a linear feedback operation on the measured quantities.

It is felt that, in addition to the results obtained for the optimal pointwise control problem, this research represents a philosophical contribution to distributed parameter control theory. The general parabolic optimal control problem was formulated in such a way as to resemble as closely as possible an analogous problem in finite dimensional control theory. This approach leads to the ability, at many junctures, to make direct extensions of finite dimensional results to corresponding distributed parameter results using only the simplest analytic tools.

There are several areas touched on in this thesis which remain open topics for research. The hyperbolic optimal control problem was introduced in Definition 3.7. The remainder of the thesis was devoted to parabolic optimal control problems, but it seems that a parallel development for hyperbolic problems would entail using the variation of constants formula (2.8.6) to eventually derive an optimal <u>matrix</u> feedback operator which is the solution to a <u>matrix</u> Riccati operator equation. Presumably, it would then be a straightforward matter to derive the optimal pointwise control for the hyperbolic case.

In example 5.1 of Section 5.3 brief mention is made of the optimal point location problem. Although done for a special case, one could, in the general case, take the optimal cost function resulting from an arbitrary set of control point locations, average the optimal cost function in order to eliminate dependence on the initial state, and then optimize the averaged cost function over the set of allowable control points.

Finally, the output feedback problem considered in Section 5.3 might be generalized to the distributed parameter analog of the finite dimensional problem solved by Levine,[32] namely, the determination of the linear feedback operation on the output which minimizes some averaged cost functional.

APPENDIX A

INFINITESIMAL GENERATOR THEOREM

If the Assumptions 2.7.1, 2.7.2, and 2.7.3 hold then the operator A_3 is the infinitesimal generator of a strongly continuous semigroup $\{\Phi(t)\}_{t\in[0,\infty)}$ defined on $H_o^m(D)$. The function on D represented by $\Phi(t)x$, namely $(\Phi(t)x)(z)$, is analytic in t and m-times differentiable in the components z_i of z for $t > 0$. Moreover, if $x_o \epsilon H_o^m(D)$ there exists a unique function $x(t,z)$ defined for $t > 0$ and $z \epsilon D$ such that

i. $x(t) \epsilon H_o^m(D) \quad , \quad \forall t \epsilon [0,\infty)$

ii. $\lim\limits_{t\to s} \|x(t) - x(s)\|_{H_o^m(D)} = 0 \quad , \quad s\epsilon[0,\infty)$

iii. $x(0) = x_o$

iv. $x(t) \epsilon \, Do(A_3) \; \forall \; t > 0$

v. $\lim\limits_{t\to s} \|A_3 x(t) - A_3 x(s)\|_{H_o^m(D)} = 0 \quad , \quad s\epsilon(0,\infty)$

vi. $\frac{\partial}{dt} x(t,z) = A\, x(t,z) \quad ; \; t\epsilon(0,\infty) \quad , \quad z\epsilon D$

vii. $x(t,z) = (\Phi(t)x_o)(z)$

APPENDIX B

VARIATION OF CONSTANTS FORMULA

R. S. Phillips[23] proves the following result: Let A be the infinitesimal generator of a strongly continuous semigroup of operators $\{\Phi(t)\}_{t\in[0,\infty)}$ and let $f(t)$ be strongly continuously differentiable on $(0,\infty)$. Then for each $x \in Do(A) \subset \mathcal{X}$ there exists a unique continuously differentiable function $y(s) : [0,\infty) \to H_o^m(D)$ such that the system

$$\dot{y}(t) = Ay(t) + f(t) \quad , \quad y(0) = x \tag{B.1}$$

has the solution

$$y(t) = \Phi(t)x + \int_0^t \Phi(t-\sigma)f(\sigma)\,d\sigma \tag{B.2}$$

The requirement of strong continuous differentiability on $f(t)$ is required in order that $y(t)$ be continuously differentiable. If we denote $g(t) = \int_0^t \Phi(\sigma)[f(t+h-\sigma) - f(t-\sigma)]\,d\sigma + \int_t^{t+h} \Phi(\sigma)f(t+h-\sigma)\,d\sigma$

The integrand in the first term is bounded and converges pointwise to zero as $h \to 0$. The integrand in the second term is bounded as $h \to 0$, so that $\|g(t+h) - g(h)\| \to 0$ as $h \to 0$, implying strong continuity of $g(t)$. Dividing $g(t+h) - g(t)$ by h, noting that $\|\frac{\Phi(t)}{h}\{[f(t+h-\sigma) - f(t-\sigma)] - f'(t-\sigma)\}\|$ converges boundedly to zero as $h \to 0$, and noting that $\|\Phi(\sigma)f(t+h-\sigma) - \Phi(t)f(0)\| \to 0$ as $h \to 0$ for $\sigma \in [t, t+h]$, we can write

$$\dot{g}(t) = \Phi(t)f(0) + \int_0^t \Phi(t-\sigma)f'(\sigma)\,d\sigma$$

and the strong continuity of $\dot{y}$ follows from the strong continuity of $f'(t)$.

It should be noted that the application in which this result is used in Chapter II, Section 8 requires that (B.2) holds only for $x \in D\alpha(A_3)$ rather than $D\alpha(A)$.

APPENDIX C

PROOF OF THEOREM 4.2

In this appendix a proof of Theorem 4.2 due to Lions[15] is presented. This theorem is used in Section 4.2 to prove existence of solutions of the parabolic system equation (3.2.2) and in Section 4.3 to derive necessary conditions for optimality in the coercive system operator case. Let us restate the theorem for convenience.

Theorem 4.2: If $\Pi(u,v)$ is a symmetric, continuous, coercive bilinear form on $U \times U$, and $L(u)$ is a linear form on U, then the cost functional $J(u) = \Pi(u,u) - 2L(u)$ has a minimum value $J(u^*)$, if and only if u^* satisfies the equation

$$\Pi(u^*, v) = L(v) \quad , \quad \forall\, v \in U \tag{C.1}$$

Proof: Suppose u^* is the minimizing element of the space U

$$J(u^*) \leq J(1-\theta)u^* + \theta\omega) \quad \forall\, \omega \in U \text{ and } \theta \in [0,1]$$

or $$\frac{1}{\theta}[J(u^* + \theta(\omega - u^*)) - J(u^*)] \geq 0$$

In the limit as $\theta \to 0$, this expression is the Frechet derivative of $J(u)$ at $u=u^*$, which implies that

$$\left\langle \left.\frac{\delta J}{\delta u}\right|_{u=u^*}, \omega - u^* \right\rangle_U = 2[\Pi(u^*, \omega - u^*) - L(\omega - u^*)] \geq 0 \tag{C.2}$$

Since (C.2) must hold for all perturbations $\omega - u^*$, both positive and negative, it must be true that

$$\Pi(u^*, \omega - u^*) = L(\omega - u^*) \qquad \text{for all} \quad \omega \epsilon U$$

Since ω is any vector in U, $\omega - u^* = v$, any vector in U, so that Eq. C.1 holds.

The "if" part of Theorem 4.2 is proved by using the convexity of $J(u)$ to show that

$$J(v) - J(u^*) \geq \frac{1}{\theta}[J(1-\theta)u^* + \theta v) - J(u^*)] \quad \forall v \epsilon U, \ \theta \epsilon [0,1]$$

which, in the limit as $\theta \to 0$, yields

$$J(v) - J(u^*) \geq \left\langle \frac{\delta J}{\delta u}\bigg|_{u=u^*}, v \right\rangle = 2[\Pi(u^*, v) - 2L(v)] = 0$$

implying

$$J(u^*) \leq J(v) \qquad \forall \ v \epsilon U$$

APPENDIX D

UNIFORM BOUNDEDNESS PRINCIPLE

In the proof of Theorem 4.3 and in other later theorems we appeal to the uniform boundedness principle to obtain a uniform bound on the operators $\Phi(t)$ and $B(t)$ on a finite interval $[0,T]$. The uniform boundedness principle, or, as it is sometime referred to, the Banach-Steinhaus theorem, is as follows:

Suppose X is a Banach space, Y is a normed linear space, and $\{\Lambda_\alpha\}$ is a collection of bounded linear operators of X into Y, where α ranges over some index set A. If it is true that

$$\sup_{\alpha \in A} \|\Lambda_\alpha x\| < \infty$$

for all x in a dense subset of X, then there exists an $M < \infty$, such that

$$\|\Lambda_\alpha\| \leq M \qquad \text{for all } \alpha \in A$$

A straightforward proof of this theorem is given by Rudin.[25] The collection of operators $\{\Phi(t)\}_{t\in[0,T]}$ and $\{B(t)\}_{t\in[0,T]}$ are collections of bounded linear operators from one Hilbert space into another, and the index set A is $[0,T]$, so that, by this theorem, we can write $\|\Phi(t)\| \leq M$ and $\|B(t)\| \leq B$.

APPENDIX E

POSITIVITY OF THE DIFFERENCE OPERATOR $\delta V_{n+1}(t)$

We show in this appendix that the solution $\delta V_{n+1}(t)$ of Eq. 4.6.19 is positive on the interval $[0, T]$. Rewriting the equation for convenience

$$\delta \dot{V}_{n+1}(t) = -\delta V_{n+1}(t)A - A^{*}\delta V_{n+1}(t) + \delta V_{n+1}(t)B(t)R^{-1}(t)B^{*}(t)V_n(t)$$

$$+V_n(t)B(t)R^{-1}(t)B^{*}(t)\delta V_{n+1}(t) - N(t) \quad ; \quad \delta V_{n+1}(T) = 0 \qquad (E.1)$$

Considering the parabolic system defined on the subinterval $[s, T]$

$$\dot{x}(t) = Ax(t) - B(t)R^{-1}(t)B^{*}(t)V_n(t)x(t); \quad x(s) = x_s \in Do(A_3) \qquad (E.2)$$

we examine the expression*

$$\langle \delta V_{n+1}(t)\, x(t), x(t) \rangle \Big|_s^T = \int_s^T \frac{d}{dt} \langle \delta V_{n+1}(t)x(t), x(t) \rangle\, dt$$

$$= \int_s^T \langle \delta \dot{V}_n(t)x(t), x(t) \rangle\, dt + \int_s^T \langle \delta V_n(t)\dot{x}(t), x(t) \rangle + \int_s^T \langle \delta V_n(t)x(t), \dot{x}(t) \rangle\, dt$$

and use Eqs. E.1 and E.2 to eliminate $\delta \dot{V}_{n+1}(t), \dot{x}(t)$, and $\delta V_{n+1}(T)$ obtaining

$$\langle \delta V_{n+1}(s)\, x_s, x_s \rangle = \int_s^T \langle N(t)x(t), x(t) \rangle\, dt \geq 0$$

since $N(t)$ is a positive operator on $[0, T]$ and proving the positivity of $\delta V_{n+1}(t)$ on $[0, T]$ since the initial time s can be any time in the interval.

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* All inner products are taken in $H_0^m(D)$.

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