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LONGITUDINAL MASS-SPRING MODELING OF LAUNCH VEHICLES

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16. ABSTRACT

To determine longitudinal axisymmetric vibration modes on launch vehicles, simplified models must be established. As a first step, a launch vehicle can be idealized by a chain of masses, helical springs, and liquid-filled containers. However, the realistic mass-spring representation of the liquid propellants supported by the elastic container wall is the problem. In this report, application of Galerkin's method to the equations of motion of the above simplified system is proposed, using the container-liquid modes with special normalization factors as coordinate functions. This method, identical with the Ritz method in this case, is favorable in several aspects. First, the resulting model is based on the energy law. The use of the container-liquid modes, however, results in accurate representation of significant structural and inertial characteristics of the container-liquid system. The special normalization of the coordinate functions converts the container-liquid systems directly in lumped parameter systems, having a number of degrees of freedom equal to the number of coordinate functions used. In this report, the containers are assumed cylindrical membranes with flat, rigid bottoms. For simplicity, typical tandem parts of vehicles, consisting of only one or two solid masses and springs and one liquid-filled container, are examined. If one-term approaches are considered, the resulting model can be interpreted as an actual spring-mass model. Because of the rapid decrease of the equivalent modal masses (with increasing mode number), restriction to one-term approaches does not mean considerable loss of accuracy. Comparison of analytical results with experiments provides satisfactory agreement. In this way, the use of large matrices in the final eigenvalue problem can be avoided.

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DEFINITION OF SYMBOLS

Symbol	Definition
$A_j(t)$, $a_j(t)$	Displacement of the j^{th} modal mass
c_j	Dimensionless j^{th} modal factor defined by equation (17)
D	Membrane stiffness defined by equation (30)
E	Young's modulus of elasticity
$E_{eff.}$	Effective Young's modulus of elasticity of ring-stiffened membrane
f	Frequency
f_1, f_2, f_3	Frequencies
g	Acceleration due to gravity
h	Liquid height
$I_0(x)$	Modified Bessel function of order 0
$I_1(x)$	Modified Bessel function of order 1
i	$= 1, 2, 3, \ldots$
j	$= 1, 2, 3, \ldots$
K_i, $i = 1, 2$	Spring constants
$K^{(j)}$, $\bar{K}^{(j)}$, K_{ij}	Spring constants defined by equations (43)
k_j	Spring constant of the j^{th} mode defined by equation (21)

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
k_0	Spring constant defined by equation (32)
l	Length of the cylindrical membrane
$M_i, i = 1, 2$	Solid masses
M	Mass defined by equation (46)
m_j	j^{th} modal mass defined by equation (20)
m_T	Total liquid mass
N_θ, N_z	Membrane forces of the cylinder wall
n	Index
$p(r, z, t)$	Liquid pressure
$p_j(r, z, t)$	Liquid pressure created by the j^{th} mode of vibration
R	Cylinder radius
r	Cylindrical coordinate
t	Time
$u(z, t)$	Axial membrane displacement
W	Weight
W_1, W_2, W_3	Three different versions of Wood's model defined in Appendix B
$w(z, t)$	Radial membrane displacement
$w_j(z, t)$	Radial membrane displacement created by the j^{th} mode of vibration
$\bar{w}_j(z)$	Radial membrane displacement of the j^{th} mode, dimensionless

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
$x_i = x_i(t)$, $i = 1, 2$	Displacements of the solid masses M_i, $i = 1, 2$
z	Cylindrical coordinate (container fixed)
δ_{ij}	$= \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ Kronecker's symbol
δ	Membrane thickness
θ	Cylindrical coordinate
κ_j	Defined by equation (25)
κ^*	Quantity defined by equation (42)
λ	Parameter defined by equation (47)
λ_j	Quantity defined by equation (7)
ν	Poisson's ratio
ξ	Argument of Bessel functions
ρ	Liquid mass density
$\phi(r, z, t)$	Velocity potential of the liquid
$\left.\begin{array}{l} \phi_j(r, z, t) \\ \bar{\phi}_j(r, z) \end{array}\right\}$	Velocity potentials of the j^{th} mode of vibration
ω_j	Circular frequency of the j^{th} mode of vibration defined by equation (11)
ω_0	Circular frequency defined by equations (46)

LONGITUDINAL MASS-SPRING MODELING OF LAUNCH VEHICLES

SUMMARY

To determine longitudinal axisymmetric vibration modes of launch vehicles, simplified models must be established. As a first step, a launch vehicle can be idealized by a chain of masses, helical springs, and liquid-filled containers. However, the realistic mass-spring representation of the liquid propellants supported by the elastic container wall is the problem. In this report, application of Galerkin's method to the equations of motion of the above simplified system is proposed, using the container-liquid modes with special normalization factors as coordinate functions. This method, identical with the Ritz method in this case, is favorable in several aspects. First, the resulting model is based on the energy law. The use of the container-liquid modes, however, results in accurate representation of significant structural and inertial characteristics of the container-liquid system. The special normalization of the coordinate functions converts the container-liquid systems directly in lumped parameter systems, having a number of degrees of freedom equal to the number of coordinate functions used. In this report, the containers are assumed cylindrical membranes with flat, rigid bottoms. For simplicity, typical tandem parts of vehicles, consisting of only one or two solid masses and springs and one liquid-filled container, are examined. If one-term approaches are considered, the resulting model can be interpreted as an actual spring-mass model. Because of the rapid decrease of the equivalent modal masses (with increasing mode number), restriction to one-term approaches does not mean considerable loss of accuracy. Comparison of analytical results with experiments provides satisfactory agreement. In this way the use of large matrices in the final eigenvalue problem can be avoided.

INTRODUCTION

The "POGO" oscillations of launch vehicles are of primary interest. POGO oscillations result from a closed-loop interaction between one of the axisymmetric longitudinal vibration modes and the propellant system. Hence, the POGO problem has led to increased interest in the accurate determination of longitudinal axisymmetric modes and frequencies of vehicles. However, knowledge of the elastic and inertial characteristics of the structural components is required before the modes and frequencies can be calculated. Because of the complexity of launch vehicles, adequate simplified mass and stiffness models must be established for calculation of the desired modes. Usually only a few of the fundamental modes are of interest; thus, the structure can be idealized by taking only significant dynamic properties into account. In general, the representation of the solid masses is not as critical as that of the liquid propellants. From experience it is known that the solid masses can be lumped at certain points of the vehicle axis and that the elasticity of the structure can be idealized by helical springs between the mass points. In this way, the solid structure is simplified by a chain of springs and masses. The liquid masses inside the elastic container must be treated differently. The liquid propellant masses constitute a high percentage of the overall vehicle masses throughout much of the power flight and, coupled with the solid vehicle masses and springs, may generate the fundamental modes of the complete vehicle. Thus, the correct comprehension of the liquid behavior inside the elastic container is of extreme importance. There is, however, no direct way to replace the liquid and the elastic container wall by lumped solid masses and springs. The problem of interaction between liquid and elastic container wall, a so-called hydroelastic problem, is governed by the laws of fluid dynamics and elasticity. Its analytical formulation represents a boundary value problem. To ensure proper representation and clear understanding of the liquid-container interactions, this problem must be solved.

Nevertheless, one feels intuitively that the liquid masses supported by the elastic walls act like spring-mass systems coupled to the solid vehicle structure. Thus, it is understandable that engineers like to simulate the dynamic behavior of the liquid inside the elastic tanks by single spring-mass systems. Models of this kind have existed for a long time. Some of them are described in publications [1-5]. These models are based on the following assumptions:

1. The equivalent vibrating solid mass equals or can be larger [3] than the total liquid mass.

2. The liquid velocity field inside the tank, generated by the accelerated container motion, is neglected.

3. Bending stresses and inertia of the wall are neglected.

In Appendix A a short description of these models is given, and their usefulness is shown by comparison with test results.

Another kind of approach is presented in Archer [6] and Rubin [7]. Based on orthotropic shell theory, a finite element technique is utilized to construct the total launch vehicle stiffness and mass matrices by dividing the structure into (axisymmetric) shell components, fluid components, and spring-mass components. The stiffness and mass matrices for the complete vehicle are obtained by superposition of the stiffness and mass matrices of the individual components, which are computed using a Rayleigh-Ritz approach.

This paper is concerned with the modal analysis of vehicles, based on spring-mass modeling. However, the method proposed is quite different from that noted at the beginning. Analysis and model are based on the equations of motion of the reduced vehicle, consisting of a chain of masses, springs, and liquid-filled containers. Application of Galerkin's method reduces these equations to a system of finite ordinary differential equations that can be interpreted as the equations of motion of a lumped parameter system, provided properly selected coordinate functions are used. This is explained briefly in the following paragraph.

The equations of motion contain a finite number of ordinary differential equations (describing the motion of the solid mass-spring system) and a number of partial differential equations governing the liquid-filled container. The latter represent distributed systems having infinite degrees of freedom. To replace these equations by a number of ordinary differential equations, Galerkin's technique using the vibration modes of the partially liquid-filled container as coordinate functions is applied. In doing so, one obtains a finite number of coupled ordinary differential equations for each container and, collectively, a finite number of equations of motion of the entire vehicle. These equations can be written in matrix form displaying stiffness and mass matrix. In this way, the problem of free vibrations is reduced to a matrix eigenvalue problem.

In the following sections it is shown that the container-liquid equations can be considered the equations of motion of lumped mass systems, provided the coordinate functions are normalized by specific factors. If this is the case, Galerkin's approach converts the container-liquid systems directly into equivalent lumped mass systems, whereby the number of masses (degrees of freedom) equals the chosen number of coordinate functions. In this way, relations evolve between the masses of the solid structure and those masses representing the liquid.

Galerkin's method using the container-liquid modes as coordinate functions is also favorable from other aspects. Galerkin's approach, identical with the Ritz method in this case, ensures proper adherence to the law of energy, while the use of the modes results in accurate representation of significant structural and inertial properties of the container-liquid systems. The entire vehicle is then represented by a lumped parameter model, established without any assumptions violating the laws of fluid dynamics and elasticity.

Thus, studying this model, one obtains understanding of the vibration problem at hand. First, one recognizes that the resulting model, in general, does not represent an actual spring-mass model. If interpreted in this way it would display "springs" having negative constants. However, the point is, if one-term approaches are considered, the container-liquid models represent actual spring-mass systems which agree, as far as the arrangement of masses and springs are concerned, with the model mentioned at the beginning, although the amount of mass and the spring constants are different. In this way, the entire vehicle is represented by a lumped spring-mass model. As shown below, the restriction to one-term approaches does not mean considerable loss in accuracy. One fact should be emphasized. To apply the approach suggested, the container-liquid modes must be known. This seems to be a disadvantage of the approach. However, in general, one cannot expect to be able to handle the vibration problem of a complex structure without being able to handle the vibration problems of its single parts.

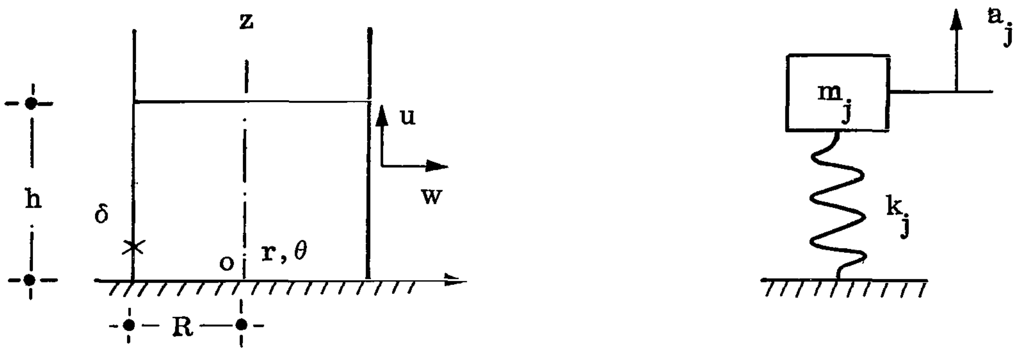
To realize the model outlined above, specialization of the coordinate functions by normalizing factors is necessary. This procedure is based on the well-known principle of "Mechanical Analogy." From the vibration modes, simple mechanical one-degree-of-freedom systems can be derived that respond in the same fashion as the modes, having the same energy and producing the same support forces. In Graham and Rodriguez [8] and Schmitt [9, 10], this principle is applied to the lateral sloshing motion of liquid in a rigid container. Because of the special nature of this case, where the potential energy stems from the liquid surface elevation, a pendulum analogy results. Pozhalostin [11] contains a discussion of this principle, based on the longitudinal vibrations of an elastic container partially filled with liquid. In all these cases, forced vibrations were considered to derive the analog mechanical systems.

Thus, the single container-liquid modes must be studied. From the above remarks it can be concluded that any axisymmetric mode of vibration is equivalent to a single spring-mass system resting on the container bottom. The amount of mass and spring constant are determined by the following requirements: potential and kinetic energy of the spring-mass system equal, at any time, those of the container-liquid system, vibrating in that particular mode. In addition, the resulting liquid pressure force on the container bottom and the force exerted there by the mechanical spring are equal. Consequently, the particular mode used must be normalized to correspond with the container geometry and liquid height. In the following section, the equivalent single spring-mass systems of the modes of a partially liquid-filled cylindrical membrane with flat, rigid bottom are derived. It is easy to confirm that the sum of all "modal masses" determined as outlined above is smaller than the total liquid mass. Furthermore, one realizes the rapid decrease of the modal masses with increasing mode number.

Using these normalized modes as coordinate functions, the Galerkin approach can be performed. Because of the rapid decrease of the modal masses, only a few degrees of freedom must be added to obtain a realistic model of the container-liquid system. In this way, the use of large matrices in the final eigenvalue problem can be avoided. In the third section, the equations of motion of typical tandem parts of vehicles consisting of solid masses, springs, and a partially liquid-filled cylindrical membrane are derived. In the following sections, Galerkin's method is applied to two of these cases: partially liquid-filled cylindrical membrane with a top mass; partially liquid-filled membrane resting on a single spring-mass system. Two special cases were numerically solved and compared with test-results performed by Southwest Research Institute, San Antonio, Texas, under Contracts NAS8-11045 and NAS8-20329. As seen in the last two sections, the agreement is satisfactory. The solutions of these special cases render possible the spring-mass modeling of any chain of solid springs and masses and liquid-filled cylindrical membranes.

EQUIVALENT SINGLE SPRING-MASS MODELS OF THE VIBRATION MODES

The axisymmetric free vibrations of partially liquid-filled cylindrical membranes with flat, rigid bottoms are considered. The membrane is referred to a cylindrical system of coordinates r, θ, z (Fig. 1a).



a) Coordinate System; Displacements

b) Equivalent Single Spring-Mass System of the j^{th} Mode.

FIGURE 1. PARTIALLY LIQUID-FILLED CYLINDRICAL MEMBRANE

The following assumptions are made:

1. The wall inertia is neglected.

2. The membrane is free at the top.

3. The liquid is considered inviscid and the flow irrotational.

4. The pressure created by the surface wave is neglected.

From assumptions 1 and 2, the equation of motion of the membrane [13-15] follows as

$$\frac{E\delta}{R^2}\, w(z, t) = p(R, z, t) \quad . \tag{1}$$

From assumption 3, it follows that the liquid velocity inside the tank can be derived from a potential $\phi(r, z, t)$, which is the solution of Laplace equation,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \begin{matrix} 0 \le r \le R \\ 0 \le z \le h \end{matrix} \quad . \tag{2}$$

The liquid pressure caused by the velocity potential is then given by

$$p(r, z, t) = -\rho \frac{\partial \phi(r, z, t)}{\partial t} \quad . \tag{3}$$

The boundary conditions are

$$z = 0 \quad , \quad \frac{\partial \phi}{\partial z} = 0 \tag{4}$$

$$r = R \quad , \quad \frac{\partial \phi}{\partial r} = \frac{\partial w}{\partial t} \tag{5}$$

$$z = h \quad , \quad \phi = 0 \quad . \tag{6}$$

Condition (6) follows from assumption 4 and equation (3).

As well-known [14, 15], equation (2) and the boundary conditions (4) through (6) represent a boundary value problem with the following solution:

$$\left.\begin{aligned} \phi_j(r, z, t) &= \dot{a}_j(t)\ \bar{\phi}_j(r, z) \\ \bar{\phi}_j(r, z) &= c_j\, R\, I_0(\lambda_j \frac{r}{h}) \cos(\lambda_j \frac{z}{h}) \\ \lambda_j &= \frac{(2j-1)\pi}{2} \quad , \quad j = 1, 2, 3, \ldots \end{aligned}\right\} \tag{7}$$

(Proof by substitution.) The quantity $\dot{a}_j(t)$ has the dimension of a velocity and is determined by initial conditions. The dimensionless constant, c_j, will be determined in the following. According to equations (3) and (7) the liquid pressure created by $\phi_j(r, z, t)$ follows as

$$p_j(r, z, t) = -\rho \frac{\partial \phi_j(r, z, t)}{\partial t} = -\rho\, \ddot{a}_j(t)\ \bar{\phi}_j(r, z) \quad . \tag{8}$$

It can be concluded easily that the solution (7) fulfills boundary conditions (4) and (6).

From boundary condition (5) and equation (7) the radial membrane displacement follows as

$$\left.\begin{aligned} w_j(z,t) &= a_j(t)\,\bar{w}_j(z) \\ \bar{w}_j(z) &= \frac{\partial\bar{\phi}_j(R,z)}{\partial r} = c_j\lambda_j\frac{R}{h}\,I_1\!\left(\lambda_j\frac{R}{h}\right)\cos\left(\lambda_j\frac{z}{h}\right) \\ & \qquad j = 1,\ 2,\ 3,\ \ldots \end{aligned}\right\} \tag{9}$$

Substitution of equations (7), (8), and (9) into equation (1) yields

$$\rho\bar{\phi}_j(R,z)\,\ddot{a}_j + \frac{E\delta}{R^2}\,\bar{w}_j(z)\,a_j = 0 \quad , \tag{10}$$

or

$$-\frac{\ddot{a}_j}{a_j} = \omega_j^2 = \frac{E\delta}{R^3\rho}\,\frac{\lambda_j\frac{R}{h}\,I_1\!\left(\lambda_j\frac{R}{h}\right)}{I_0\!\left(\lambda_j\frac{R}{h}\right)} \quad . \tag{11}$$

Equations (7), (9), and (11) represent the j^{th} solution of the eigenvalue problem at hand: the free vibration of the partially liquid-filled cylindrical membrane having a flat, rigid bottom. Since the problem is homogeneous, the solution is determined only up to the constant factor c_j as shown by equations (7) and (9). As noted in the introduction, the j^{th} vibration mode is equivalent to a single spring-mass system resting on the container bottom and having, at any time, the same kinetic energy, potential energy, and reaction force on the container bottom. From the latter three conditions, the amount of mass and the spring constant of the mass-spring system and the modal factor c_j can be determined.

Next, the law of conservation of energy of the liquid-filled membrane, vibrating in the j^{th} mode, is examined. Multiplication of equation (10) by $\partial w_j(z,t)/\partial t$ (given by equation (9)) integration over the wetted membrane wall and over a certain time interval gives

$$\frac{2R\pi}{2}\rho\int_0^h \bar{\phi}_j(R,z)\,\bar{w}_j(z)\,\dot{a}_j^2 + \frac{2R\pi}{2}\frac{E\delta}{R^2}\int_0^h [\bar{w}_j(z)]^2 dz\, a_j^2 = \text{const.} \tag{12}$$

Equation (12) expresses the law of conservation of energy of the system membrane-liquid, vibrating in the j^{th} mode. It can also be assumed to be the energy law of a single spring-mass system having the mass

$$m_j = 2R\pi\rho \int_0^h \bar{\phi}_j(R, z)\,\bar{w}_j(z)\,dz = c_j^2\,\lambda_j \frac{R}{h}\, I_0(\lambda_j \frac{R}{h})\, I_1(\lambda_j \frac{R}{h})\, m_T \qquad (13)$$

and the spring constant

$$k_j = 2R\pi \frac{E\delta}{R^2} \int_0^h [\bar{w}_j(z)]^2 dz = \frac{1}{2}\,[c_j \lambda_j I_1(\lambda_j \frac{R}{h})]^2 \;\; \frac{2R\pi\, E\delta}{h} \quad . \qquad (14)$$

The equation of motion of the single spring mass system (Fig. 1b) follows then from equations (12) through (14) as

$$m_j \ddot{a}_j + k_j a_j = 0 \quad . \qquad (15)$$

Because it is assumed that the spring of this system rests on the container bottom and that the spring force equals the force exerted there by the liquid, it follows

$$-k_j a_j = 2\pi \int_0^R p_j(r, o, t)\, r dr$$

or, under consideration of equations (7), (8), and (15), ([16], p. 484)

$$m_j = -2\pi\rho \int_0^R \bar{\phi}_j(r, o)\, r dr = -c_j \frac{2}{\lambda_j} I_1(\lambda_j \frac{R}{h})\, m_T \quad . \qquad (16)$$

Thus, from equations (13) and (16) one obtains

$$c_j = -\frac{2}{\lambda_j} \quad \frac{1}{\lambda_j \frac{R}{h} I_0(\lambda_j \frac{R}{h})} \quad . \qquad (17)$$

Now equations (7), (9), (13), (14), and (17) result in

$$\bar{\phi}_j = -\frac{2h}{\lambda_j^2} \frac{I_0(\lambda_j \frac{r}{h})}{I_0(\lambda_j \frac{R}{h})} \cos(\lambda_j \frac{z}{h}) \tag{18}$$

$$\bar{w}_j = -\frac{2}{\lambda_j} \frac{I_1(\lambda_j \frac{R}{h})}{I_0(\lambda_j \frac{R}{h})} \cos(\lambda_j \frac{z}{h}) \tag{19}$$

$$m_j = \frac{4}{\lambda_j^2} \frac{I_1(\lambda_j \frac{R}{h})}{\lambda_j \frac{R}{h} I_0(\lambda_j \frac{R}{h})} m_T \tag{20}$$

$$k_j = 2\left[\frac{I_1(\lambda_j \frac{R}{h})}{\lambda_j \frac{R}{h} I_0(\lambda_j \frac{R}{h})}\right]^2 \frac{2R\pi\, E\delta}{h} \quad ; \; j = 1, 2, 3, \ldots \tag{21}$$

Equations (15), (20) and (21) define completely the equivalent single spring-mass system of the j^{th} vibration mode. From equations (20) and (21) also follows the circular frequency as given by equation (11).

To determine the spring-mass model numerically, the function,

$$\frac{I_1(x)}{I_0(x)} \; , \; 0 \le x \le \infty \qquad , \tag{22}$$

must be evaluated. As shown in Appendix A, continued fractions and asymptotic expansions prove to be useful for that purpose.

Figure 2 is the modal mass ratio (20),

$$\frac{m_j}{m_T} \; , \; j = 1, 2, 3 \qquad ,$$

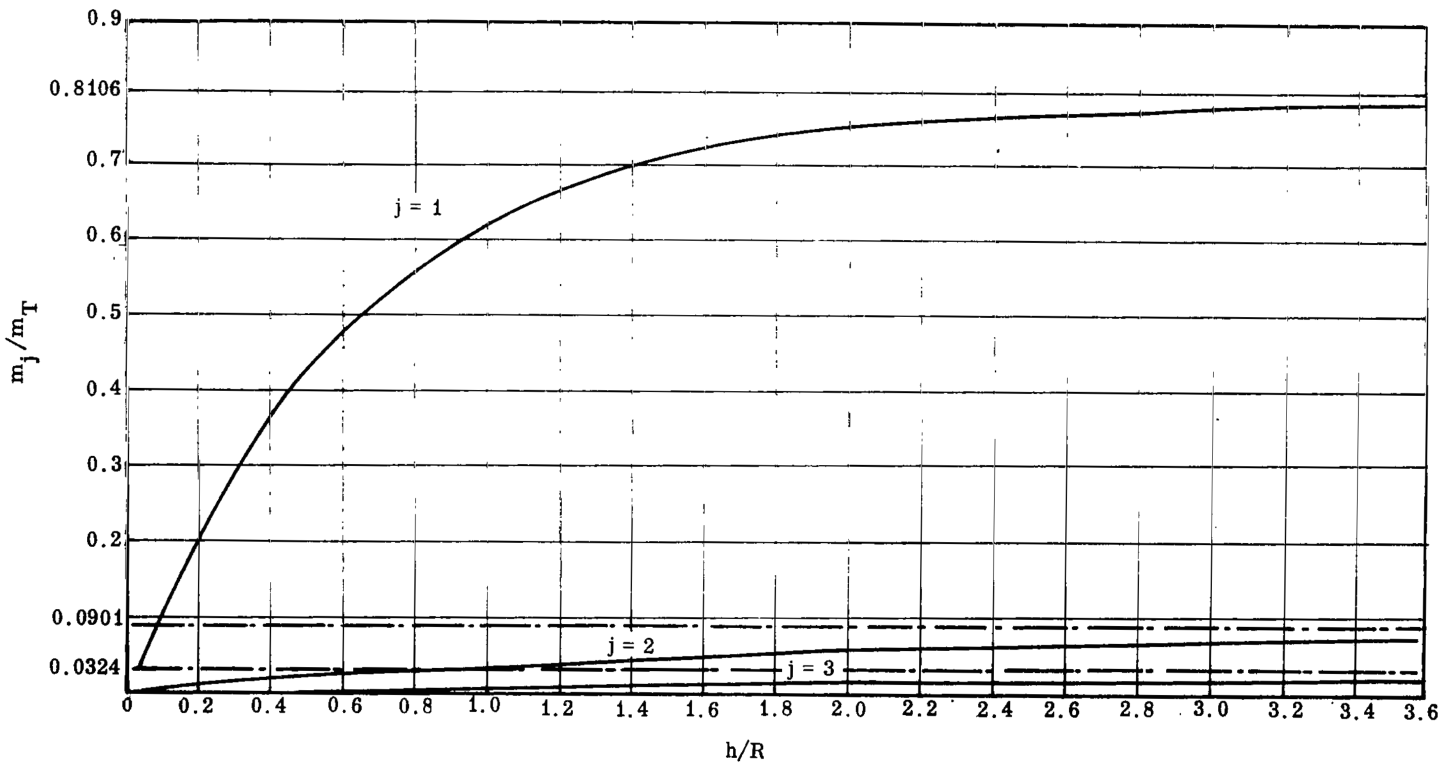


FIGURE 2. MODAL MASS RATIO m_j/m_T VERSUS h/R

versus h/R. It manifests the importance of the first mass m_1 and the rapid decrease of the higher modal masses.

From equation (A-1), it follows

$$\lim_{h \to \infty} \frac{I_1(\lambda_j \frac{R}{h})}{\lambda_j \frac{R}{h} I_0(\lambda_j \frac{R}{h})} = \frac{1}{2} \quad , \tag{23}$$

whereby the limes 1/2 also represents the maximum value. Thus, from equation (20) it can be concluded

$$\lim \sup \frac{m_j}{m_T} = \frac{8}{(2j-1)^2 \pi^2} \quad ,$$

hence

$$\lim \sup \frac{m_1}{m_T} = 0.8106$$

$$\lim \sup \frac{m_2}{m_T} = 0.0901$$

$$\lim \sup \frac{m_3}{m_T} = 0.0324 \quad .$$

The above values, also in Figure 2, demonstrate again the degree of significance of the single modal masses.

A point of primary interest may now be touched upon. The amount of the j^{th} modal mass is given by equation (20). From this equation and the third equation (7), the sum of all modal masses follows as

$$\sum_{j=1}^{\infty} m_j = \frac{16}{\pi^2} m_T \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} \frac{I_1(\lambda_j \frac{R}{h})}{\lambda_j \frac{R}{h} I_0(\lambda_j \frac{R}{h})} \quad . \tag{24}$$

Now, the question arises whether

$$\sum_{j=1}^{\infty} m_j \lesseqgtr m_T \quad .$$

From equations (23) and (24) it follows ([16], p. 808)

$$\lim_{h\to\infty} \frac{1}{m_T} \sum_{j=1}^{\infty} m_j = \frac{8}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = 1 \quad .$$

Because 1/2 represents the maximum value of the function in equation (23), which never can be reached, one concludes

$$\sum_{j=1}^{\infty} m_j < m_T \quad .$$

The following integrals should be mentioned for later use:

$$\left.\begin{aligned} &\frac{1}{R} \int_0^h \bar{w}_j(z)\,dz = (-1)^j \kappa_j \\ &\kappa_j = \frac{2}{\lambda_j} \frac{I_1(\lambda_j \frac{R}{h})}{\lambda_j \frac{R}{h} I_0(\lambda_j \frac{R}{h})} \end{aligned}\right\} \tag{25}$$

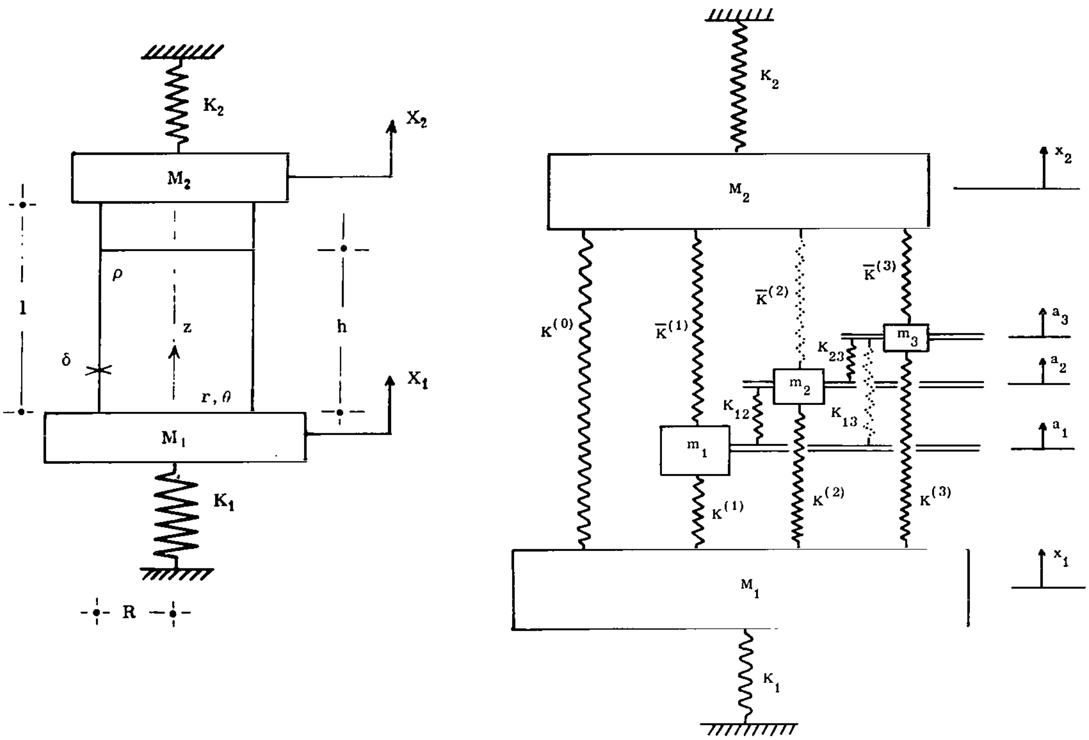
$$2R\pi\rho \int_0^h (h-z)\,\bar{w}_j(z)\,dz = -m_j \quad , \tag{26}$$

where $\bar{w}_j(z)$ and m_j are given by equations (19) and (20).

EQUATIONS OF MOTION OF TYPICAL TANDEM PARTS OF VEHICLES

Figure 3 shows a tandem part of a vehicle consisting of two solid masses, springs, and a partially liquid-filled cylindrical membrane.

To obtain the equations of motion, the system shall be divided into subsystems for which the equations of dynamic equilibrium can easily be set up.



a) TANDEM PART OF A VEHICLE CONSISTING OF TWO SOLID MASSES, TWO SPRINGS, AND A PARTIALLY LIQUID-FILLED CONTAINER.

b) EQUIVALENT LUMPED PARAMETER SYSTEM.

FIGURE 3.

In doing so, the membrane is crossed perpendicular to the axis somewhere between the masses M_1, M_2. The equilibrium conditions of the lower and upper part are given by

$$M_1 \ddot{x}_1 + K_1 x_1 - 2R\pi N_z + 2\pi \int_0^R p(r, o, t)\, r dr = 0 \tag{27}$$

$$M_2 \ddot{x}_2 + K_2 x_2 + 2R\pi N_z = 0 \quad , \tag{28}$$

where

$$x_1 = x_1(t), \; x_2 = x_2(t) \quad .$$

If the wall inertia is neglected (assumption 1), the equilibrium condition of a membrane ring element one inch high is given by

$$N_\theta = \begin{cases} R \cdot p(R\ z\ t) & 0 \le z < h \\ 0 & h \le z \le 1 \end{cases} \quad . \tag{29}$$

The membrane forces N_z, N_θ can be replaced by the displacements [13] as follows:

$$\left.\begin{aligned} N_z &= D\left(\frac{\partial u}{\partial z} + \nu \frac{w}{R}\right) \\ N_\theta &= D\left(\frac{w}{R} + \nu \frac{\partial u}{\partial z}\right) \\ u &= u(z, t)\ ;\ w = w(z, t)\ ;\ D = \frac{E\delta}{1-\nu^2} \quad . \end{aligned}\right\} \tag{30}$$

Substitution of equation (30) into equations (27) through (29) results in the three equations of motion containing the four unknown quantities u, w, x_1, x_2.

To reduce the number of unknowns, u shall be eliminated. Starting from equation (30), the displacements can be expressed as follows:

$$\left.\begin{aligned} E\delta \frac{\partial u}{\partial z} &= N_z - \nu N_\theta \\ E\delta \frac{w}{R} &= N_\theta - \nu N_z \quad . \end{aligned}\right\} \tag{31}$$

Taking into account the independence of N_z on z in $\langle 0, z \rangle$, as indicated by equations (27) and (28), and the fact that $N_\theta = 0$ in $\langle h, 1 \rangle$, one obtains from equations (31), by integration over the intervals $\langle o, 1 \rangle$ and $\langle o, h \rangle$, respectively,

$$E\delta(x_2 - x_1) = 1N_z - \nu \int_0^h N_\theta \, dz$$

$$E\delta \int_0^h \frac{w}{R} \, dz = -\nu h N_z + \int_0^h N_\theta \, dz \quad ,$$

and finally

$$\left.\begin{aligned} N_z &= \frac{k_0}{2R\pi}\,(x_2 - x_1 + \nu \int_0^h \frac{w}{R}\,dz) \\ k_o &= \frac{2R\pi\ E\delta}{l(1-\nu^2 \frac{h}{l})} \quad . \end{aligned}\right\} \tag{32}$$

From equations (27) through (29), (31) (second equation), and (32), the equations of motion (now independent of u) follows as

$$\left.\begin{aligned} &(K_1+k_0)\,x_1 - \nu k_0 \int_0^h \frac{w}{R}\,dz - k_0 x_2 = -M_1\ddot{x}_1 - 2\pi \int_0^R p(r,o,t)\,r dr \\ &-\nu k_0 x_1 + 2R\pi E\delta\,\frac{w}{R} + \nu^2 k_0 \int_0^h \frac{w}{R}\,dz + \nu k_0 x_2 = 2R^2\pi p(R,z,t) \\ &-k_0 x_1 + \nu k_0 \int_0^h \frac{w}{R}\,dz + (K_2+k_0)\,x_2 = -M_2\,\ddot{x}_2 \quad , \end{aligned}\right\} \tag{33}$$

where

$$x_1 = x_1(t)\ ;\ x_2 = x_2(t)\ ;\ w = w(z,t)\ ;\ 0 \le z \le h \quad .$$

Under assumptions 3 and 4 of the preceding section, the liquid flow is governed by a velocity potential $\phi(r, z, t)$. In accordance with equation (3), the pressure can be expressed by

$$p(r,z,t) = -\rho\left[\frac{\partial\phi(r,z,t)}{\partial t} - (h-z)\,\ddot{x}_1\right] \quad . \tag{34}$$

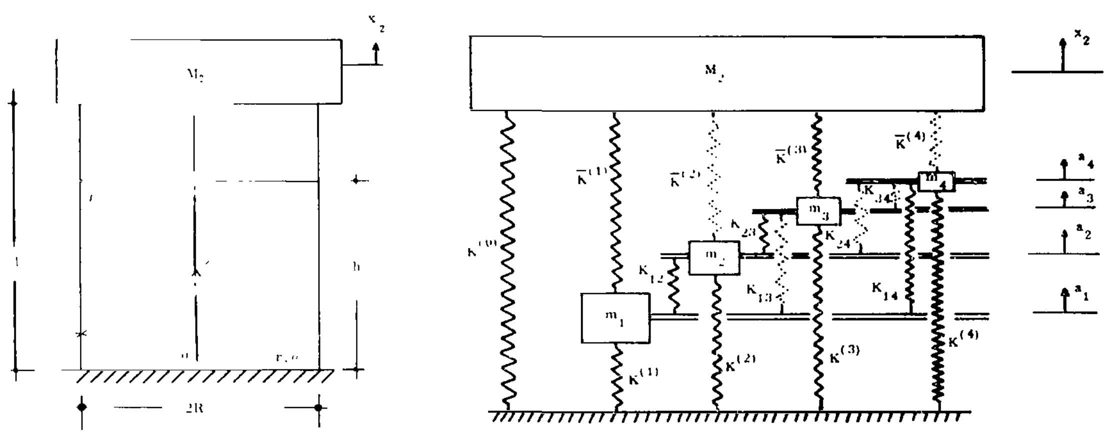
To solve equation (33) by Galerkin's approach, using the container-liquid modes of the preceding section, one has to assume, in accordance with equations (3), (7) through (9), (9)

$$
\left.
\begin{aligned}
\phi(r, z, t) &= \sum_{j=1}^{n} [\dot{A}_j(t) - \dot{x}_1(t)]\bar{\phi}_j(r, z) \\
w(z, t) &= \sum_{j=1}^{n} [A_j(t) - x_1(t)]\bar{w}_j(z)
\end{aligned}
\right\} \qquad (35)
$$

where $\bar{\phi}_j(r, z, t)$ and $\bar{w}_j(z)$ are given by equations (18) and (19). Then the well-known procedure can be performed.

Before Galerkin's method is applied to the system of equations (33), two simpler systems will be examined.

1) Figure 4 is a cylindrical membrane partially filled with liquid and fixed on the lower end having a top mass.



a) PARTIALLY LIQUID-FILLED CYLINDRICAL MEMBRANE HAVING A TOP MASS

b) EQUIVALENT LUMPED PARAMETER SYSTEM

FIGURE 4.

The equations of motion follow from the second and third equations (33) by setting

$$\left.\begin{aligned} x_1 &= 0 \\ K_2 &= 0 \end{aligned}\right\} . \qquad (36)$$

One obtains

$$\left.\begin{aligned} 2R\pi E\delta \frac{w}{R} + \nu k_0 (x_2 + \nu \int_0^h \frac{w}{R} dz) &= 2R^2\pi p(R, z, t) \\ k_0(x_2 + \nu \int_0^h \frac{w}{R} dz) &= -M_2 \ddot{x}_2 \end{aligned}\right\} . \qquad (37)$$

From equations (34) and (36), it follows

$$p(r, z, t) = -\rho \frac{\partial \phi(r, z, t)}{\partial t} . \qquad (38)$$

2) Figure 5 is a partially liquid-filled cylindrical membrane resting on a single spring mass system.

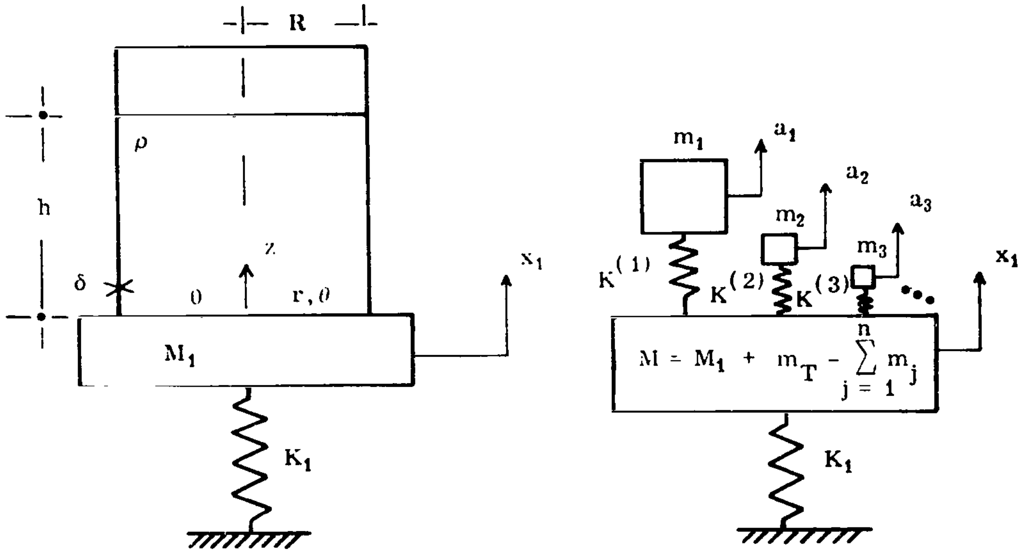
The equations of motion follow from the system of equation (33) by setting

$$\left.\begin{aligned} K_2 &= 0 \\ M_2 &= 0 \end{aligned}\right\} . \qquad (39)$$

One obtains

$$\left.\begin{aligned} K_1 x_1 &= -M_1 \ddot{x}_1 - 2\pi \int_0^R p(r, o, t)\, r dr \\ E\delta \frac{w}{R} &= R p(R, z, t) \end{aligned}\right\} . \qquad (40)$$

The liquid pressure is given by equation (34).



a) PARTIALLY LIQUID-FILLED CYLINDRICAL MEMBRANE RESTING ON A SINGLE SPRING-MASS SYSTEM

b) EQUIVALENT LUMPED SPRING-MASS SYSTEM

FIGURE 5.

Equation (39) expresses the fact that the upper end of the membrane is free. Thus, no axial membrane forces are acting during vibration of the system. Consequently, equations (40) do not contain any terms stemming from Poisson effects.

In the following, Galerkin's method using the coordinate functions of equations (18) and (19) will be applied in equations (33). From the resulting linear system of ordinary differential equations, which represent the dynamic equilibrium conditions of the mass points involved, the lumped mass-spring model sketched by Figure 3b will be derived.

Equations (34) and (35) must be substituted into equations (33), and the specified integrations must be carried out. Then the second equation (33) must be multiplied by $\bar{w}_i(z)$, $(i = 1, 2 \ldots n)$ and integrated over $\langle o, h \rangle$; whereby equations (13), (14), (16), (25), and (26) must be taken into account. Diagonalizing of the mass matrix requires simple intermediate operations, which are not shown. One obtains finally

$$
\begin{bmatrix}
k_0 + K_2 & -\nu k_0 \kappa_1 & \cdots & (-1)^n \nu k_0 \kappa_n & -k_0 \kappa^* \\
-\nu k_0 \kappa_1 & k_1 + \nu^2 k_0 \kappa_1^2 & \cdots & (-1)^{1+n} \nu^2 k_0 \kappa_1 \kappa_n & -(k_1 - \nu k_0 \kappa^* \kappa_1) \\
\nu k_0 \kappa_2 & -\nu^2 k_0 \kappa_2 \kappa_1 & \cdots & (-1)^{2+n} \nu^2 k_0 \kappa_2 \kappa_n & -(k_2 + \nu k_0 \kappa^* \kappa_2) \\
 & & & & \\
(-1)^n \nu k_n & (-1)^{n+1} \nu^2 k_0 \kappa_n \kappa_1 & \cdots & k_n + \nu^2 k_0 \kappa_n^2 & -\left[k_n + (-1)^n \nu k_0 \kappa^* \kappa_n\right] \\
-k_0 \kappa^* & -(k_1 - \nu k_0 \kappa^* \kappa_1) & \cdots & -\left[k_n + (-1)^n \nu k_0 \kappa^* k_n\right] & K_1 + k_0 \kappa^{*2} + \sum_{j=1}^{n} k_j
\end{bmatrix}
\begin{bmatrix}
x_2 \\ A_1 \\ A_2 \\ \vdots \\ A_n \\ x_1
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
M_2 & 0 & 0 & & & 0 \\
0 & m_1 & 0 & & & 0 \\
0 & & m_2 & & & 0 \\
 & & & \ddots & & \vdots \\
 & & & & m_n & \\
0 & 0 & 0 & \cdots & & M_1 + m_T - \sum_{j=1}^{n} m_j
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_2 \\ \ddot{A}_1 \\ \ddot{A}_2 \\ \vdots \\ \ddot{A}_n \\ \ddot{x}_1
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0
\end{bmatrix} , \qquad (41)
$$

where

$$
\left.\begin{matrix} m_j \\ k_j \\ \kappa_j \end{matrix}\right\} \quad j = 1,\ 2 \ldots n
$$

and

k_0

are given by equations (20), (21), (25), and (32).

Furthermore,

$$\kappa^* = 1 + \nu \sum_{j=1}^{n} (-1)^j \kappa_j \quad . \tag{42}$$

Because the mass matrix of equation (41) is diagonal, any of the equations represents the dynamic equilibrium condition of the particular point mass appearing in that equation. The stiffness matrix contains no zero element; thus, one concludes that all masses are connected with one another by springs. Figure 3b is a spring-mass system having such an arrangement of masses and springs. Massless rigid bars drawn in Figure 3b serve for connection of all masses with one another by the vertical springs. The stiffness matrix of this system can easily be determined as

$$\begin{bmatrix} K_2 + K^{(0)} + \sum_{j=1}^{n} \bar{K}^{(j)} & -\bar{K}^{(1)} & \cdots & -\bar{K}^{(n)} & -K^{(0)} \\ -\bar{K}^{(1)} & K^{(1)} + \bar{K}^{(1)} + \sum_{j=2}^{n} K_{ij} & \cdots & -K_{1n} & -K^{(1)} \\ -\bar{K}^{(2)} & -K_{21} & \cdots & -K_{2n} & -K^{(2)} \\ \vdots & \vdots & & \vdots & \vdots \\ -\bar{K}^{(n)} & -K_{n1} & \cdots & K^{(n)} + \bar{K}^{(n)} + \sum_{j=1}^{n-1} K_{nj} & -K^{(n)} \\ -K^{(0)} & -K^{(1)} & \cdots & -K^{(n)} & K_1 + K^{(0)} + \sum_{j=1}^{n} K^{(j)} \end{bmatrix} .$$

Comparison of the elements of this matrix with those of the stiffness matrix of equation (41) results in a system of linear equations having the unique solution

$$\left.\begin{aligned}
&\bar{K}^{(j)} = (-1)^{j-1}\, \nu k_0 \kappa_j \\
&K_{ij} = K_{ji} = (-1)^{i+j-1}\, \nu^2 k_0 \kappa_i \kappa_j \qquad ; \qquad i \neq j \\
&K^{(0)} = k_0 \kappa^* \\
&K^{(j)} = k_j + (-1)^j\, \nu k_0 \kappa^* \kappa_j \qquad\qquad i,j = 1, 2, \ldots n
\end{aligned}\right\} \tag{43}$$

From these equations it can be seen that the "spring" constants $\bar{K}^{(j)}$ and K_{ij} are negative if j and i+j, respectively, are even numbers. The "springs" referring to these constants are dotted in Figure 3b. Consequently, the lumped system of Figure 3b represents an actual spring-mass system only if j = 1 (one-term Galerkin approach). Because of the predominance of the first modal mass, restriction to the one-term approach will be sufficient in most cases.

As already mentioned, the arrangement of masses and springs resulting from the one-term Galerkin approach is the same as that of Wood's model, (Appendix B). Of course, the amount of the equivalent liquid mass and the spring constant differ considerably.

PARTIALLY LIQUID-FILLED CYLINDRICAL MEMBRANE HAVING A MASS ON TOP

The free vibrations of this system, shown in Figure 4a, are governed by equations (37) and (38). As in the previous case, application of Galerkin's method results in a system of ordinary differential equations of motion having stiffness and mass matrix as

$$\begin{bmatrix} k_0 & -\nu k_0\kappa_1 & \cdots\cdots & (-1)^n \nu k_0 \kappa_n \\ -\nu k_0\kappa_1 & k_1+\nu^2 k_0\kappa_1^2 & \cdots\cdots & (-1)^{1+n}\nu^2 k_0\kappa_1\kappa_n \\ \nu k_0\kappa_2 & -\nu^2 k_0\kappa_2\kappa_1 & \cdots\cdots & (-1)^{2+n}\nu^2 k_0\kappa_2\kappa_n \\ \\ (-1)^n \nu k_0\kappa_n & (-1)^{n+1}\nu^2 k_0\kappa_n\kappa_1 & \cdots\cdots & k_n+\nu^2 k_0\kappa_n^2 \end{bmatrix}$$

$$\begin{bmatrix} M_2 & 0 & 0 & & 0 \\ 0 & m_1 & 0 & & 0 \\ 0 & 0 & m_2 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & m_n \end{bmatrix}$$

This result can also be obtained directly from equations (36) and (41). The equivalent lumped parameter system is represented by Figure 4b.

Because of the sharp decrease of the higher modal masses, as shown in Figure 2, their coupling effect is small. Thus, if higher modes of the coupled system are not asked for, consideration of only the first modal mass will, in many cases, result in sufficient accuracy. Then equations (41) reduce to the simple system

$$\begin{bmatrix} k_0 & -\nu k_0\kappa_1 \\ -\nu k_0\kappa_1 & k_1+\nu^2 k_0\kappa_1^2 \end{bmatrix} \begin{bmatrix} x_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} M_2 & 0 \\ 0 & m_1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{a}_1 \end{bmatrix} , \qquad (44)$$

having a quadratic frequency equation. From equations (42) and (43), the spring constants of this model follow as

$$\left.\begin{aligned} K^{(0)} &= k_0(1 - \nu\kappa_1) \\ \bar{K}^{(1)} &= \nu k_0 \kappa_1 \\ K^{(1)} &= k_1 - \nu k_0 \kappa_1 + \nu^2 k_0 \kappa_1^2 \end{aligned}\right\} , \qquad (45)$$

where k_1, κ_1, and k_0 are given by equations (21), (25), and (32). Equation (20) gives m_1.

To prove the accuracy of the one-term approach, analytical and experimental results using a stainless steel container with three different top masses M_2 (weight W) were compared. The results are shown in Figures 6 through 8. The agreement is satisfactory.

PARTIALLY LIQUID-FILLED MEMBRANE RESTING ON A SINGLE SPRING-MASS SYSTEM

Figure 5a is the vibrating system which is governed by equations (34) and (40). The linear system of ordinary differential equations of motion follows by application of Galerkin's technique, or it can be concluded directly from equations (39) and (41). In the latter case, one has to consider that all terms containing ν vanish.

Stiffness and mass matrix follow as

$$\begin{bmatrix} k_1 & 0 & \cdots & 0 & -k_1 \\ 0 & k_2 & \cdots & 0 & -k_2 \\ 0 & 0 & \cdots & k_n & -k_n \\ -k_1 & -k_n & \cdots & -k_n & K_1 + \sum_{j=1}^{n} k_j \end{bmatrix} ,$$

$$\begin{bmatrix} m_1 & & 0 & & 0 \\ & m_2 & 0 & & 0 \\ 0 & 0 & m_n & & 0 \\ 0 & 0 & 0 & & M_1 + m_T - \sum_{j=1}^{n} m_j \end{bmatrix} ,$$

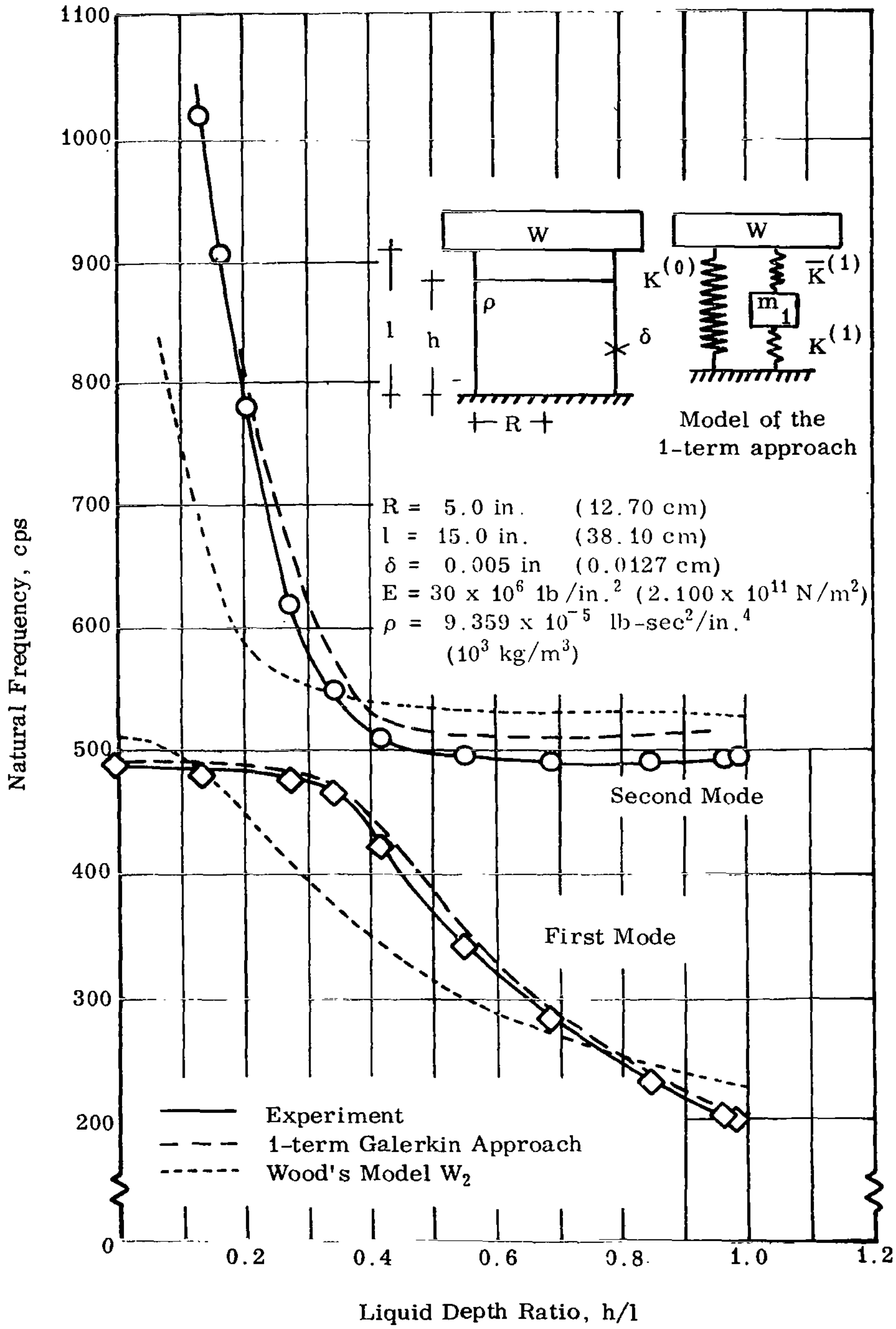


FIGURE 6. NATURAL FREQUENCIES OF PARTIALLY FILLED TANK WITH TOP MASS W = 11.48 lb (51.066 N)

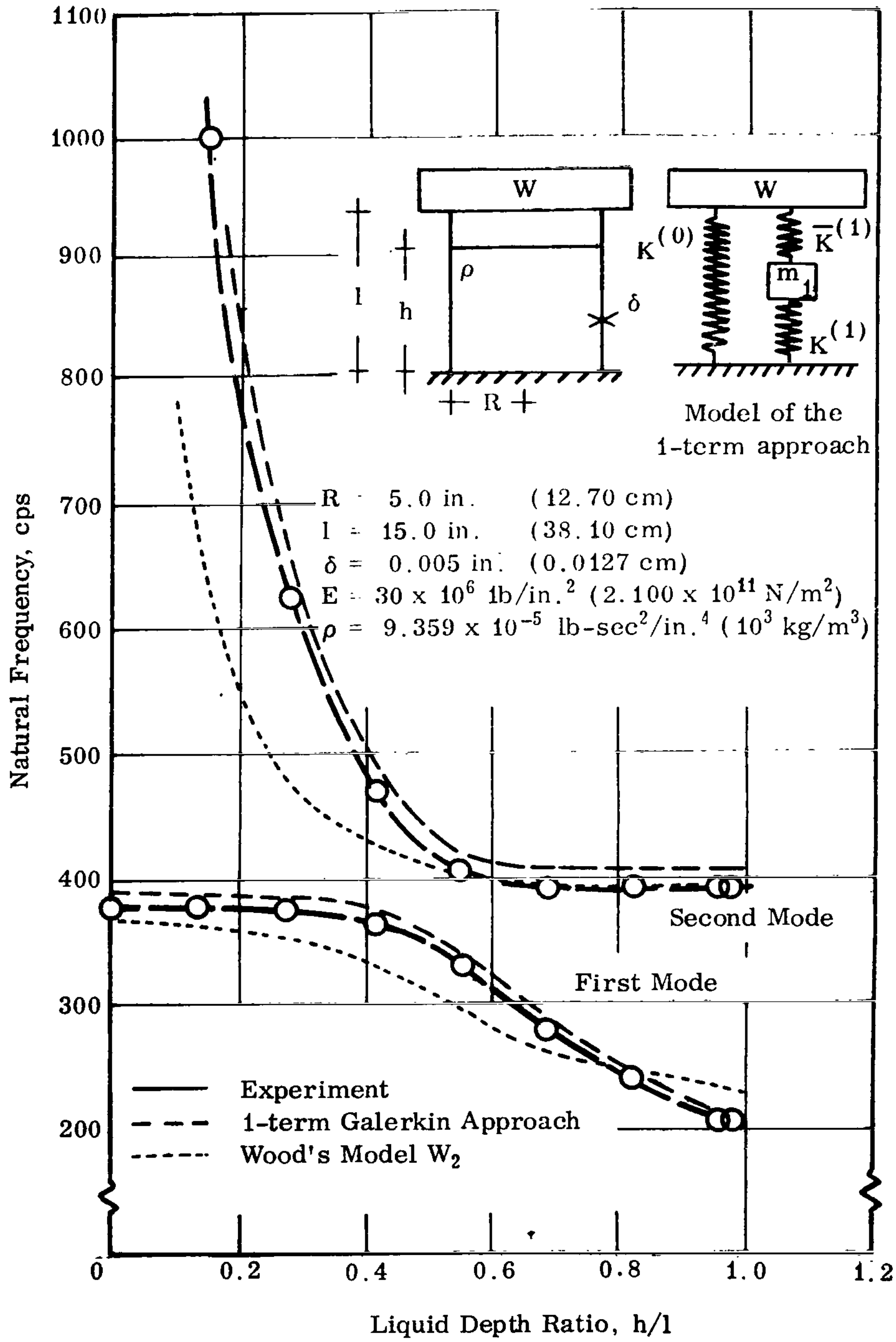


FIGURE 7. NATURAL FREQUENCIES OF PARTIALLY FILLED TANK WITH TOP MASS W = 22.32 lb (99.284 N)

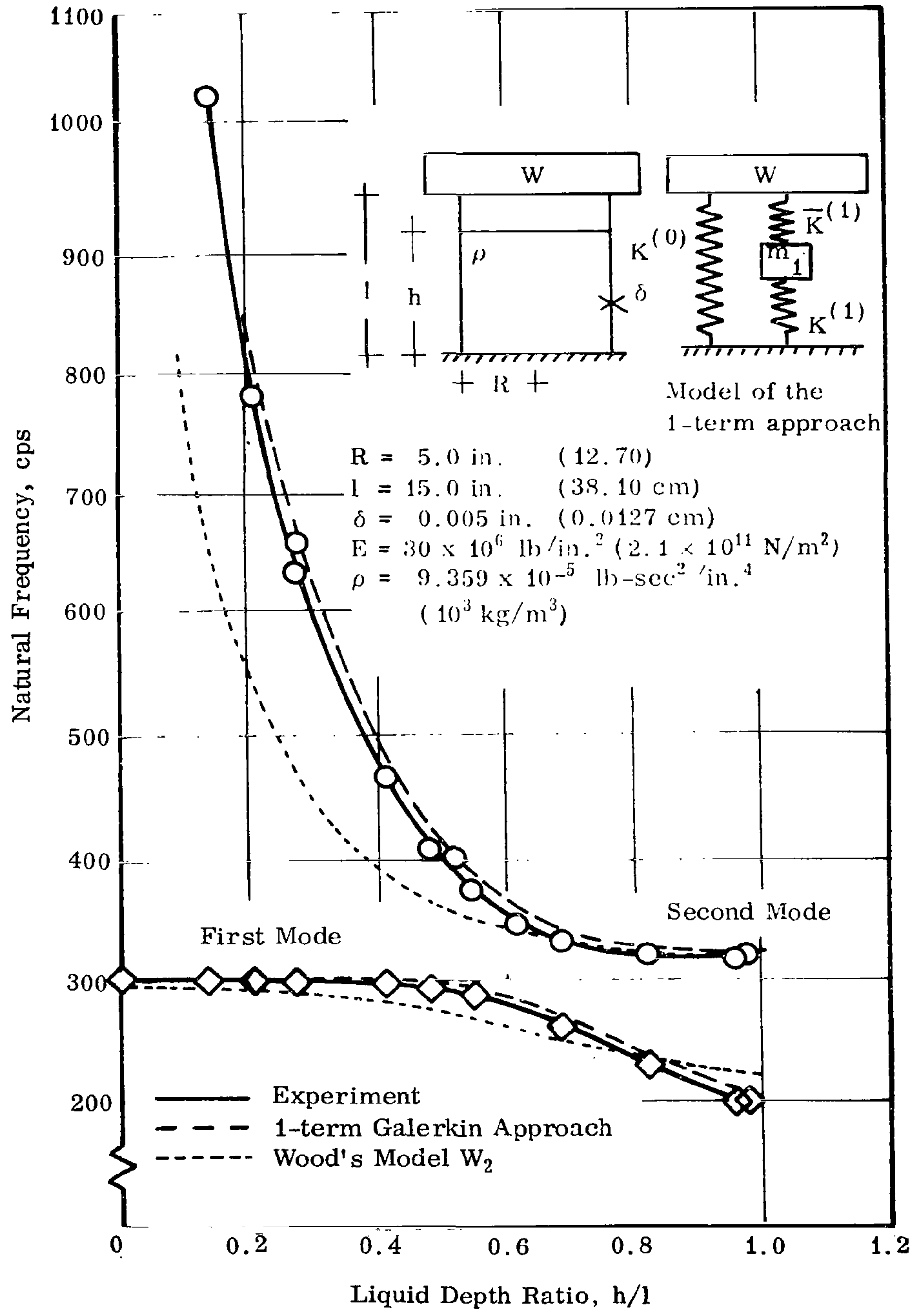


FIGURE 8. NATURAL FREQUENCIES OF PARTIALLY FILLED TANK WITH TOP MASS W = 34.53 lb (153.597 N)

where m_j and k_j $(j = 1, 2, \dots n)$ are given by equations (20) and (21).

Because all Poisson terms are zero, it follows from equations (42) and (43)

$$\bar{K}^{(j)} = K_{ij} = 0 \qquad i, j = 1, 2 \dots n$$

$$K^{(j)} = k_j \qquad j = 1, 2, 3, \dots n \qquad .$$

In this case, no "springs" with negative constants exist. Consequently, the equivalent system is an actual spring-mass system for all values of n. It is represented by Figure 5b.

Because the configuration of the nonzero elements of the stiffness matrix is relatively simple, the frequency equation can be set up immediately. For a one-term and a two-term approach these equations are

$$\lambda^2 - \left[\omega_0^2 + \omega_1^2\left(1 + \frac{m_1}{M}\right)\right]\lambda + \omega_0^2\,\omega_1^2 = 0$$

$$\lambda^3 - \left[\omega_0^2 + \omega_1^2\left(1 + \frac{m_1}{M}\right) + \omega_2^2\left(1 + \frac{m_2}{M}\right)\right]\lambda^2$$

$$+ \left[\omega_0^2\,\omega_1^2 + \omega_1^2\,\omega_2^2\left(1 + \frac{m_1}{M} + \frac{m_2}{M}\right) + \omega_0^2\,\omega_2^2\right]\lambda - \omega_0^2\,\omega_1^2\,\omega_2^2 = 0 \qquad ,$$

where

$$\left.\begin{aligned} \omega_0^2 &= \frac{K_1}{M} \\ M &= M_1 + m_t - \sum_{j=1}^{n} m_j \quad , \qquad n = 1, 2 \end{aligned}\right\} \qquad (46)$$

and

$$\lambda = (2\pi f)^2 \qquad . \qquad (47)$$

Table I is the test and analysis results from a one-term and a two-term approach at different filling heights. The first frequency resulting from the one-term approach is already as good as that of the two-term approach. Test values of the third frequency could be obtained for the highest levels only. This fact is not surprising, however, in view of the smallness of the second modal mass. The container used is a ring-stiffened Mylar cylinder with rigid bottom. The effective modulus of elasticity was determined by use of a correction factor as outlined in [12].

CONCLUSIONS

Longitudinal spring-mass models of launch vehicles can be obtained by application of Galerkin's technique. As a first step, the vehicle can be idealized by lumping its solid masses and (longitudinal) springs to a chain of lumped masses, springs, and liquid-filled containers. The equations of motion of this vehicle model consist, then, of a number of ordinary differential equations describing the motion of the solid point masses, coupled with partial differential equations governing the liquid motion inside the container. Application of Galerkin's technique using the container-liquid modes with special normalization factors reduces this system of equations to the equations of motion of a finite degree of a freedom-lumped parameter system.

In this report, the containers are assumed cylindrical membranes with flat, rigid bottoms. For simplicity, typical tandem parts of vehicles consisting of only one or two solid masses, springs, and one liquid-filled container are examined. From the analysis the following can be concluded:

1. The lumped parameter systems resulting from one-term approaches can be interpreted as actual spring-mass systems. Thereby the container-liquid system is replaced by one modal mass and three helical springs (Figs. 6 through 8). Spring constants and modal mass are given by equations (20), (21), (25), (32), and (45). Because of the rapid decrease of the equivalent modal masses, no considerable loss of accuracy results from one-term approaches. This behavior is proven by comparison of analytical results with experiments (Figs. 6 through 8, Table 1).

2. If approaches with more than one term are performed, the resulting lumped parameter system can no longer be considered an actual spring-mass model. In this case, the spring-mass arrangement that replaces the container-liquid subsystem exhibits "springs" with negative constants descending from Poisson effects. These springs are sketched with dotted lines in Figures 3 and 4. Spring constants and modal masses are given by equations (20), (21), (25), (32), (42), and (43).

TABLE I. NATURAL FREQUENCIES OF A PARTIALLY LIQUID-FILLED MYLAR CONTAINER RESTING ON A SINGLE SPRING-MASS SYSTEM. ONE-TERM APPROACH; TWO-TERM APPROACH; TEST VALUES.

$l = 11.2$ in. (28.45 cm)

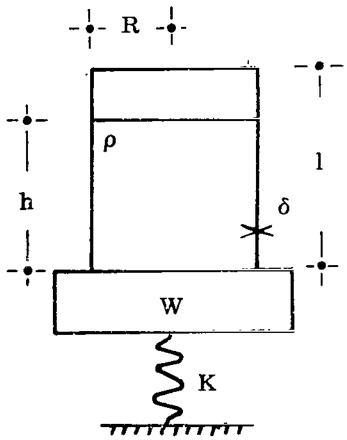
$R = 4.5$ in. (11.43 cm)

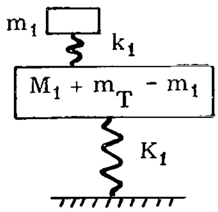
$\rho = 9.359 \times 10^5 \ \frac{\text{lb sec}^2}{\text{in.}^4} \ (10^3 \ \frac{\text{kg}}{\text{m}^3})$

$K = 11.62 \ \frac{\text{lb}}{\text{in.}} \ (2.035 \times 10^6 \ \frac{\text{N}}{\text{m}})$

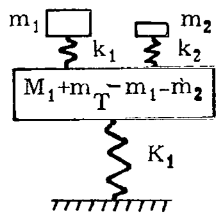
$W = 4.67$ lb (20.773 N)

$E_{eff}\,\delta = 5050 \ \frac{\text{lb}}{\text{in.}} \ (884390 \ \frac{\text{N}}{\text{m}})$





one-term approach model



two-term approach model

h	f_1			f_2			f_3		
	1-Term	2-Term	Test	1-Term	2-Term	Test	1-Term	2-Term	Test
11.1	42.5	42.5	43.1	131.8	115.4	116.0		173.9	167.0
10.1	45.6	45.6	46.2	136.6	122.3	122.0		180.7	175.0
9.1	49.2	49.2	49.6	142.1	129.8	129.0		188.7	-
8.1	53.4	53.4	53.9	148.4	138.3	136.0		198.2	194.0
7.1	58.4	58.3	58.7	154.5	147.0	140.0		209.4	-
6.1	64.2	64.2	64.5	164.0	158.4	144.0		224.6	-
5.1	71.2	71.7	71.7	175.0	171.6	164.0		244.6	-
4.1	80.2	80.1	80.5	188.0	186.0	181.0		271.7	-
3.1	91.3	91.2	91.0	206.8	205.9	196.0		312.7	-
2.1	105.6	105.5	104.0	238.3	238.0	213.0		392.0	-
1.1	143.3	124.5	122.0	314.3	313.0	-		538.0	-

Although this paper deals only with cylindrical membranes having a flat, rigid bottom, the method outlined can be extended to general axisymmetric membrane or shell containers, provided the container-liquid modes are known. In reality, many tanks consist of long cylindrical parts and spherical or elliptical ends. During much of the power flight time, even if the propellant masses are still large, the container-liquid system will behave like a liquid-filled cylinder with a flat, rigid bottom. Certainly an elliptical or spherical end represents a disturbance of the liquid flow, and its influence must be investigated. However, this is only one of many other disturbances present in the container-liquid system. Especially the behavior of tanks stiffened by rings and stringers, and also the influence of anti-slosh devices, must be studied. For cylindrical tanks with stiffening rings only, a correction factor of Young's modulus can be derived so that the ring-stiffened wall is approximated by a wall of uniform thickness having a slightly greater modulus of elasticity [12].

George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Marshall Space Flight Center, Alabama 35812, December 10, 1968
124-08-05-00-62

APPENDIX A

NUMERICAL EVALUATION OF $I_1(x)/I_0(x)$

In the section on equivalent spring-mass model of vibration modes, normalized modes and equivalent masses and springs are represented in terms of $I_1(x)/I_0(x)$. For numerical evaluation, useful expressions of this function should be considered.

For argument values

$x < 10$

a continued fraction representation proves to be convenient. In [16] the following expansion is given:

$$\frac{J_1(\xi)}{J_0(\xi)} = \cfrac{\xi}{2 - \cfrac{\xi^2}{4 - \cfrac{\xi^2}{6 - \cfrac{\xi^2}{8 - \cdots}}}}$$

where $J_0(\xi)$ and $J_1(\xi)$ are the Bessel functions of orders 0 and 1 of the first kind.

Setting $\quad \xi^2 = -x^2 \quad ,$

the above continued fraction transforms into

$$\frac{I_1(x)}{I_0(x)} = \cfrac{x}{2 + \cfrac{x^2}{4 + \cfrac{x^2}{6 + \cfrac{x^2}{8 + \cdots}}}} \tag{A-1}$$

([16] p. 375 9.6.3)

For argument values

$$x \geq 10$$

asymptotic expansions of the modified Bessel functions can be used [16].

$$\frac{I_1(x)}{I_0(x)} \sim \frac{1 - \frac{4-1}{8x} + \frac{(4-1)\,(4-9)}{2!(8x)^2} - \frac{(4-1)\,(4-9)\,(4-25)}{3!(8x)^3} + \cdots}{1 + \frac{1}{8x} + \frac{1.9}{2!(8x)^2} + \frac{1.9.25}{3!(8x)^3} + \cdots} \qquad \text{(A-2)}$$

Figure A-1 shows $I_1(x)/I_0(x)$, and $\frac{I_1(x)}{xI_0(x)}$ versus x.

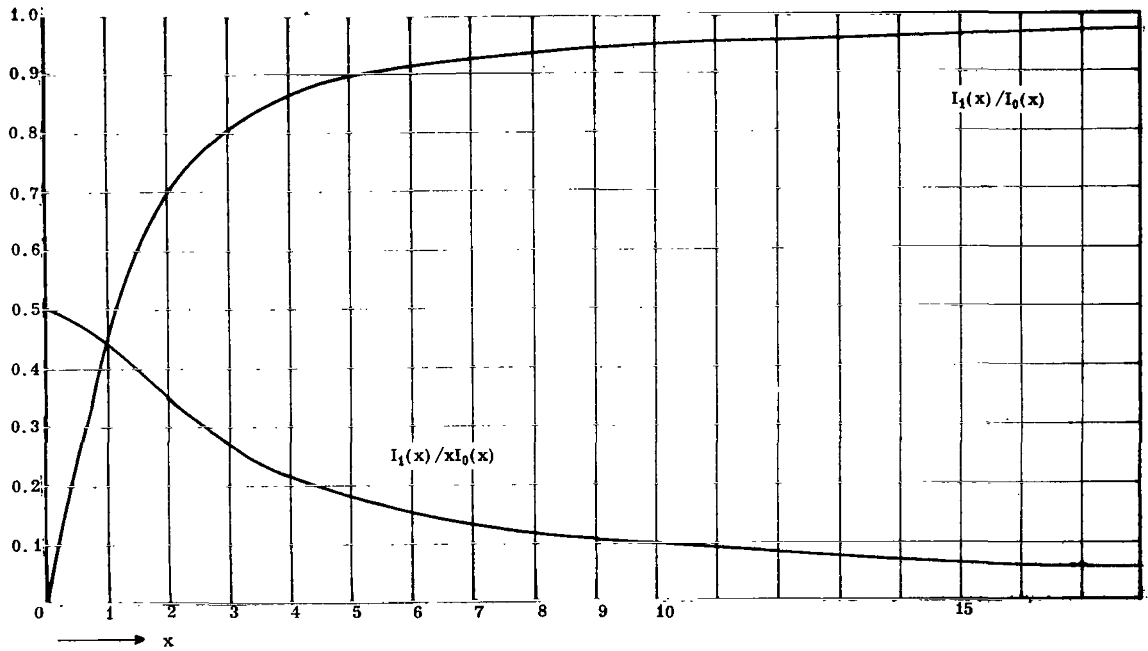


FIGURE A-1. $I_1(x)/I_0(x)$ AND $I_1(x)/xI_0(x)$ VERSUS x

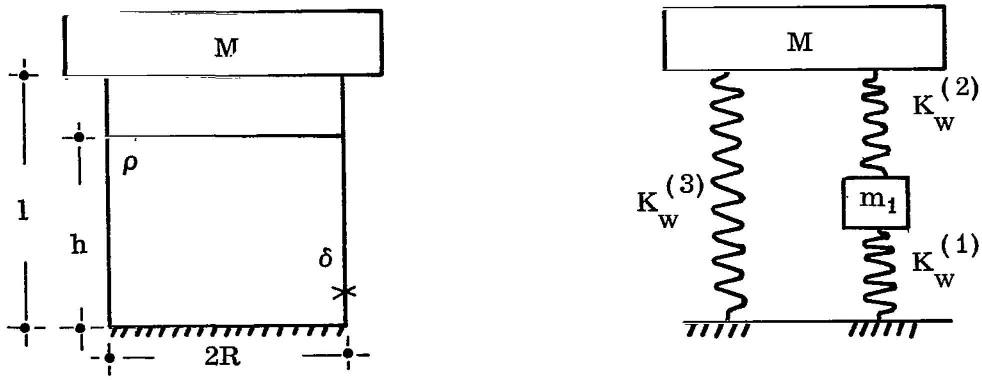
APPENDIX B

WOOD'S MODEL

Figure B-1 is Wood's model of a cylindrical tank with top mass and rigid bottom. As already mentioned, the arrangement of springs and masses agrees with that of a one-term Galerkin approach, while the amount of equivalent mass and the spring constants are different. There is, however, no unique interpretation of Wood's model. In addition to the original model, two modifications can be found in the literature on the subject. Spring constants and equivalent masses of these models, designated by W_1, W_2, and W_3, are gathered in Table B-I. Wood's model is not restricted to cylindrical tanks with rigid bottoms. Versions for tanks with flexible bulkheads and tanks stiffened by rings and stringers also exist [1, 4].

Obviously, important structural and inertial properties of the container-liquid system cannot be represented by simple models of this kind. There are, however, some other weaknesses of these models. Consider for instance the spring constants of the models W_1 and W_3. If h goes to zero, all three spring constants $K_W^{(j)}i$ $(j = 1, 2, 3)$ go to infinite, thus both frequencies of the models go to infinite. This behavior is not in agreement with test results, as shown by Figures 6 through 8. Now consider the model W_2. If M vanishes, the container-liquid frequency, determined by $K_W^{(1)}$ and m_1, goes to zero if the container length l goes to infinite. This is, however, inconsistent with theory and experience.

In Figures 6 through 8, analytical results obtained by application of Wood's model W_2 are also presented and can be compared with experimental results.



a) PARTIALLY LIQUID FILLED CYLINDRICAL TANK WITH TOP MASS

b) WOOD'S MODEL

FIGURE B-1.

TABLE B-I. SPRING CONSTANTS AND MASS OF WOOD'S MODEL

	Original Model	Modified Models	
	W_1	W_2	W_3
$K_W^{(1)}$	$\frac{2(1-\nu)}{3-2\nu^2} k_W$	$\frac{3(1-\nu)}{4-3\nu^2} k_W$	$\frac{3(1-\nu)}{4-3\nu^2} k_W$
$K_W^{(2)}$	$\frac{2\nu}{3-2\nu^2} k_W$	$\frac{3\nu}{4-3\nu^2} k_W$	$\frac{3\nu}{4-3\nu^2} k_W$
$K_W^{(3)}$	$\frac{3-2\nu}{3-2\nu^2} k_W$	$\frac{4-3\nu}{4-3\nu^2} k_W$	$\frac{4-3\nu}{4-3\nu^2} k_W$
k_W	$\frac{2R\pi\ E\delta}{h}$	$\frac{2R\pi\ E\delta}{l}$	$\frac{2R\pi\ E\delta}{h}$
m_1	$R^2\pi\ h\rho$	$\geq R^2\pi\ \rho h$	$R^2\pi\ \rho h$
ref.	[1] [2]	[3]	[5]

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