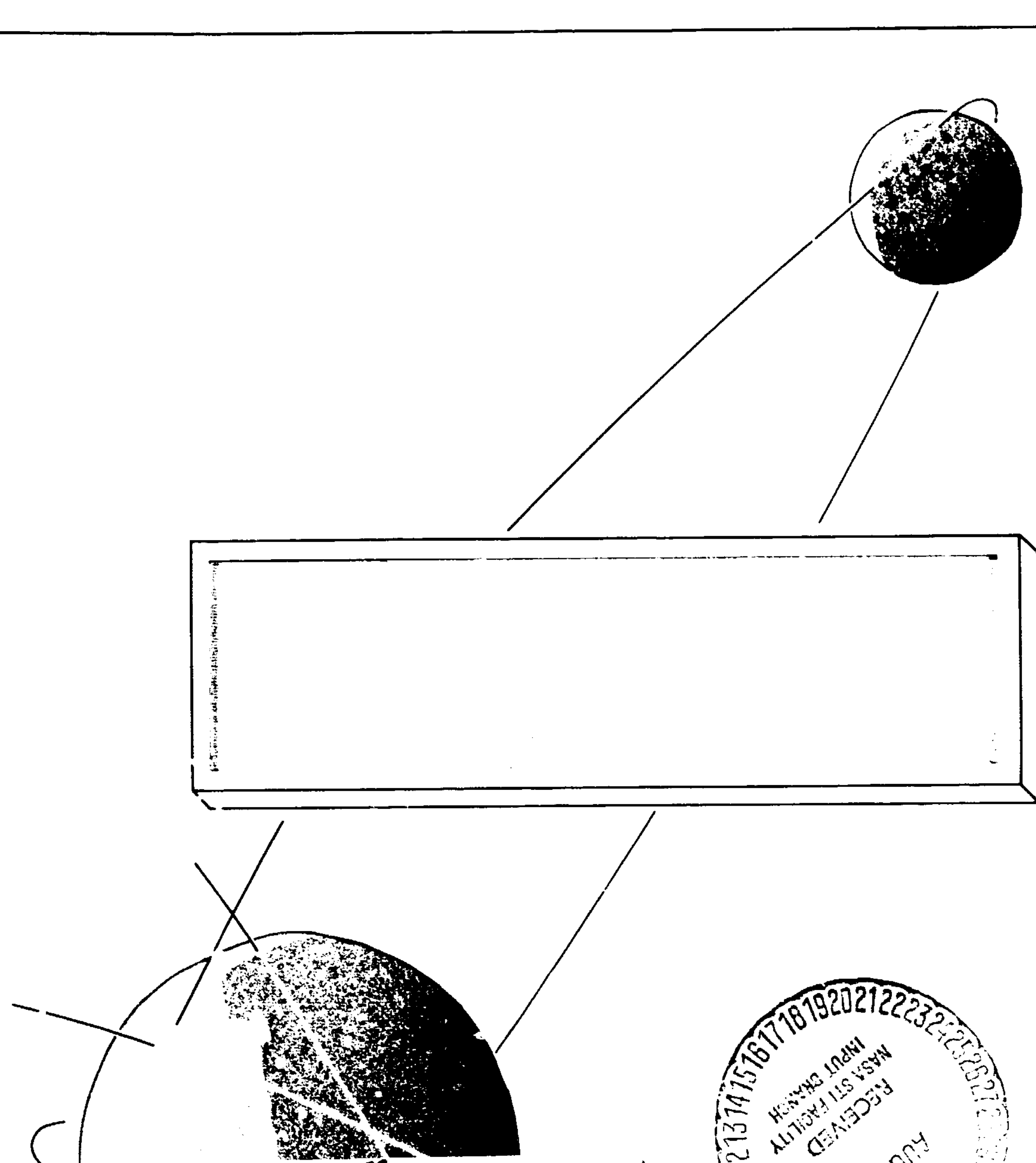

ENGINEERING MECHANICS RESEARCH LABORATORY

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STABILITY OF A CLASS OF COUPLED SYSTEMS[1]

W. L. Garrard[2]

J. A. Walker[3]

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2 W. L. Garrard is in the Department of Engineering Mechanics, University of Texas, Austin, Texas.

3 J. A. Walker is with the Department of Mechanical Engineering and Astronautical Sciences, Northwestern University, Evanston, Illinois, and on summer appointment with the Engineering Mechanics Research Laboratory, University of Texas, Austin, Texas.

Stability of a Class of Coupled Systems

Control problems in the area of gyrodynamics normally involve a set of coupled second-order differential equations. A special class of such systems is considered and the stability results obtained are shown to have direct application to a current problem involving sub-optimal control of artificial satellites.

The system considered is of the form

$$\ddot{\underline{y}} + [D + S(\underline{y}, \dot{\underline{y}}, t)](a\dot{\underline{y}} + b\underline{y}) = 0 \qquad (1)$$

where $\underline{y}$ is an n-vector, D is a constant symmetric $n \times n$ matrix, S is a skew-symmetric $n \times n$ matrix whose elements may be functions of $\underline{y}$, $\dot{\underline{y}}$, and t, and a and b are constants.[1]

Theorem: If $a > 0$, $b > 0$, $S + S' = 0$, and $a^2 D - bI_n > 0$, then the equilibrium at the origin of (1) is globally asymptotically stable.[2]

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[1] It is assumed, as usual, that solutions of (1) do indeed exist.

[2] I_n represents the $n \times n$ unit matrix.

<u>Proof</u>: Let $\underline{x}_1 = \underline{y}$, $\underline{x}_2 = \dot{\underline{y}}$, and write (1) in the form

$$\dot{\underline{x}}_1 = \underline{x}_2$$

$$\dot{\underline{x}}_2 = - [D + S(\underline{x}_1 , \underline{x}_2 , t)](a\underline{x}_2 + b\underline{x}_1) \quad (2)$$

Consider the scalar function

$$V = 2ab\underline{x}_1'D\underline{x}_1 + 2b\underline{x}_1'\underline{x}_2 + a\underline{x}_2'\underline{x}_2 \quad (3)$$

having the derivative according to (2)

$$\dot{V} = - 2\underline{x}_2'[a^2D-bI_n]\underline{x}_2 - 2b^2\underline{x}_1'D\underline{x}_1 \quad (4)$$

Since (3) may also be written as

$$V = \frac{b}{a}\, \underline{x}_1'[2a^2D - bI_n]\underline{x}_1 + a(\underline{x}_2' + \frac{b}{a}\, \underline{x}_1')(\underline{x}_2 + \frac{b}{a}\, \underline{x}_1)$$

the theorem follows immediately [1].

The theorem can be applied to a problem in sub-optimal attitude control of an artificial satellite using motor driven inertia wheels as a source of control torque. The angular motion of a satellite whose principle moments of inertia are equal is determined by

$$\ddot{\underline{\theta}} + H\dot{\underline{\theta}} + \dot{\underline{h}} = 0 \qquad (5)$$

where $H = \begin{vmatrix} 0 & h_3 & h_2 \\ -h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{vmatrix}$, $\underline{\theta} = \begin{vmatrix} \theta_1 \\ \theta_2 \\ \theta_2 \end{vmatrix}$, $\underline{h} = \begin{vmatrix} h_1 \\ h_2 \\ h_3 \end{vmatrix}$

and h_i = the angular momentum of the i^{th} inertia wheel divided by the principle moment of inertia of the satellite.

$\underline{\theta}$ = the angular displacement (assumed small) of the satellite relative to some non-rotating orthogonal reference frame whose origin is located at the center of mass of the satellite.

The vector $\dot{\underline{h}}$ represents the control effort used to regulate the attitude of the satellite. This vector is to be selected in feedback form as a function of the current values of $\underline{\theta}$, $\dot{\underline{\theta}}$, and $\underline{h}$, and must drive $\underline{\theta}$ and $\dot{\underline{\theta}}$ to zero while producing an "approximate minimum" of the integral performance index

$$J = \int_0^\infty (q\underline{\theta}'\theta + p\dot{\underline{\theta}}'\dot{\underline{\theta}} + \underline{h}'\underline{h})dt \qquad (6)$$

where p and q are positive constants.

The $\underline{h}$ vector is time-varying and its functional form cannot be specified a priori. In order to determine an adequate control, a method for sub-optimal control proposed in [2] may be applied. This method yields

$$\underline{\dot{h}} = a\underline{\dot{\theta}} + b\underline{\theta} + \frac{b}{a} H\underline{\theta} \tag{7}$$

where $a = \sqrt{p+2\sqrt{q}}$ and $b = \sqrt{q}$.

This control, however, is acceptable only if (5) is then asymptotically stable. Combining (5) and (7),

$$\underline{\ddot{\theta}} + [I_3 + \frac{1}{a} H](a\underline{\dot{\theta}} + b\underline{\theta}) = 0 \tag{8}$$

and the theorem then assures that (9) is asymptotically stable for all $p > 0$, $q > 0$, regardless of the unknown time-varying matrix H. By (7) it is also clear that $\underline{\dot{h}} \to \underline{0}$ as $t \to \infty$.

References

1. W. Hahn, "Theory and Application of Liapunov's Direct Method," Prentice Hall, Inc., Englewood Cliffs, N.J., 1963.

2. W. L. Garrard, N. H. McClamroch, and L. G. Clark, "An Approach to Sub-Optimal Feedback Control of Non-Linear Systems." International Journal of Control (to appear).